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Vibro-impact dynamics of a two-degree-of freedom periodically-forced system with a clearance: Diversity and parameter matching of periodic-impact motions

G.W. Luo, X.H. Lv, Y.Q. Shi

Key Laboratory of System Dynamics and Reliability of Rail Transport Equipment of Gansu Province, Lanzhou Jiaotong University, Lanzhou 730070, PR China

A two-degree-of-freedom system with a clearance and subjected to harmonic excitation is considered. The correlative relationship and matching law between dynamic performance and system parameters are studied by multi-parameter and multi-performance co-simulation analysis. Two key parameters of the system, the exciting frequency ω and clearance δ , are emphasized to reveal the influence of the main factors on dynamic performance of the system. Diversity and evolution of periodic impact motions are analyzed. The fundamental group of impact motions is defined, which have the period of exciting force and differ by the numbers p and q of impacts occurring at the left and right constraints of the clearance. The occurrence mechanism of chattering-impact vibration of the system is studied. As the clearance δ is small or small enough, the transition from $1-p-p$ to $1-(p\bar{p}1)-(p\bar{p}1)$ motion (the fundamental group of motions, $pZ1$) basically goes through the processes as follows: pitchfork bifurcation of symmetric $1-p-p$ motion, period-doubling bifurcation of asymmetric $1-p-p$ motion, non-periodic or chaotic motions caused by a succession of period-doubling bifurcations, symmetric $1-(p\bar{p}1)-(p\bar{p}1)$ motion generated by a degeneration of chaos. As for slightly large clearance, a series of grazing bifurcations of periodic symmetrical impact motions occur with decreasing the exciting frequency so that the number p of impacts of the fundamental group of motions increases two by two. As p becomes big enough, the incomplete chattering-impact motion will appear which exhibits a chattering sequence in an excitation period followed by a finite sequence of impacts with successively reduced velocity and reaches the non-sticking region. Finally, the complete chattering-impact motion with sticking will occur with decreasing the exciting frequency ω up to the sliding bifurcation boundary. A series of singular points on the boundaries between existence regions of any adjacent symmetrical impact motions with fundamental period are found, i.e., two different saddle-node bifurcation boundaries of one of them, real-grazing and bare-grazing bifurcation boundaries of the other alternately and mutually cross themselves at the points of intersection and create inevitably two types of transition regions: narrow hysteresis and small tongue-shaped regions. A series of zones of regular periodic and subharmonic impact motions are found to exist in the tongue-shaped regions. Based on the sampling ranges of parameters, the influence of dynamic parameters on impact velocities, existence regions and correlative distribution of different types of periodic-impact motions of the system is emphatically analyzed.

1. Introduction

The clearances or constraints among mechanical components exist inevitably in mechanical systems due to the restriction on technological level in the mechanical machining process, the demand for mechanical design and assembly errors, the need for the heat-expansion and cold-contraction of some part of mechanical components or other reasons. The existence of the clearances

or motion limiting constraints is bound to bring about the vibro-impact effects which have a great influence on performance of mechanical equipments, increase noise levels and intensify fatigue and abrasion of mechanical parts. The non-smooth natures of phase trajectories caused by the impacts complicate the dynamic analysis of such systems inevitably, but they can be described theoretically and numerically by the discontinuities in good agreement with reality. It is necessary to be able accurately to model the dynamics of the mechanical systems with clearances or motion limiting constraints from the multi-parameter level, so as to utilize effectively and enlarge profitable effects by reasonable

parameter matching design, to furthest minimize the influence of unfavorable factors such as abrasion, chattering-impact, high noise levels, etc. The broad interest in revealing dynamical performance of vibro-impact systems is reflected by an ever increasing amount of investigations devoted to this aspect. Several effective methods of qualitative, numerical and experimental analyses have been proposed and their validity has been demonstrated by different vibro-impact models in the past several years. Dynamics of different types of vibratory systems with clearances or motion limiting constraints, including detailed observations about stability and bifurcations using numerical simulations and qualitative analyses, were partly reported in Refs. [1–10]. Shaw and Holmes [1] studied a single-degree-of-freedom vibro-impact system by casting the traditional approaches used for analyzing the periodic responses in the system in the form of a more general Poincaré map. The notable work showed how even a very simple impact oscillator could display the full range of typical nonlinear behavior: saddle node and flip bifurcations, multiple coexisting attracting solutions, chaos, etc. Periodic and chaotic behavior of a threshold-limited two-degree-of-freedom system was analyzed by Aidanpää and Gupta [2]. Souza and Caldas [3] put forward an algorithm of the spectrum of Lyapunov exponents of attractors of a vibro-impact system for interpreting bifurcations and chaotic transitions. Peterka and Kotera [4] studied the appearance and characteristics of intermittency chaos of an impact oscillator occurring near saddle-node bifurcation boundaries of neighboring fundamental motions with two different impacts in an excitation period. Pavlovskaja and Wiercigroch [5,6] developed an efficient semi-analytical method to study the percussive drilling principle for the impact systems with drift and accurately predicted a range of control parameters for which the best progression rates were obtained. Quinn and Bairavarasu [7] considered a three-mass collision model with compliant contacts, a two-dimensional map is derived to describe initial conditions that lead to pairwise sequences, and the variation in the final velocities is characterized in terms of the initial configuration of the system. Luo [8] determined analytically and numerically stability and bifurcation conditions for the LR model motion in an impact oscillator, for which the regions are developed in parameter space. Leine [9] presented an asymptotic approximation method for the critical restitution coefficient of a parametrically excited impact oscillator and described its dynamics by an unilaterally constrained Hill's equation. More studies on vibro-impact dynamics were summarized in Ibrahim's monograph [10]. A special feature of vibro-impact systems that might prove to be useful in the current study is the instability caused by low-velocity collisions, which is the so-called grazing effect. The pioneer work in this field was done by Nordmark [11], who developed systematic methods for investigating grazing dynamics and their attendant bifurcations, providing the results which laid the foundation for many subsequent studies. Thereafter, this work has been further expanded by discontinuity-geometry approaches of a series of impact maps, where the increasingly thorough investigations of grazing bifurcations of such maps have been reported in Refs. [12–25]. The chattering-impact vibration denotes an accumulation of impact events in finite time (or a motion period), and sticking is where the dynamics becomes finally constrained to the impact surface as the impact attenuation ends. Detailed studies of chattering-impact vibration with sticking were carried out for the vibro-impact systems in Ref. [26–35]. Rising events occurring in multi-sticking solutions are discussed by Wagg [29,30], the study on simulation and stability analysis of impacting systems with complete chattering has been further expanded by Nordmark and Piiroinen [31]. de Souza and Caldas [32] numerically investigated the basins of attraction of coexisting periodic and chaotic attractors in a gear-rattling impact model. Luo [33,34] presented an idealized,

piecewise linear system to model non-smooth vibration of gear transmission arising from impacts between the gear teeth and predicted analytically occurrence of the periodic impact-sticking motions. Hósa and Champneys [35] considered a mechanical model of a pressure relief valve and analyzed the geometry of the chattering region via the computation of several pre-images of the grazing set. The vibro-impact systems display many of the complex dynamical features near critical points of high codimension bifurcations; in particular, different types of codimension two bifurcations and parameter unfolding associated with the interaction of saddle-node and grazing bifurcations [23], double flip bifurcation [36], Hopf-flip bifurcation [37], Hopf-pitchfork bifurcation and double Neimark-Sacker bifurcation [38], codimension-two grazing bifurcations [39,40], grazing-sliding bifurcation [41], etc., were studied by numerical simulations and qualitative analyses in recent years. Several methods of experimental analyses have been developed for some impact models in Refs. [42–46]. Wiercigroch and Sin [42] performed an experimental study of base excited symmetrically piecewise linear oscillator for observing and proving symmetric and asymmetric periodic-impact motions and their attendant bifurcations. Jin and Hu [43] designed an experimental setup consisting of both vibro-impacting cantilever beams with lumped mass blocks for testing various types of vibro-impact motions such as subharmonic, superharmonic, quasi-periodic and chaotic motions, etc. Quinn [44] considered the effect of external forces during finite duration collisions using an incremental model of impact. Based on the results of the experimental system and the incremental model, the standard algebraic model of restitution is modified to include these finite duration effects. Nguyen et al. [45] designed an experimental scheme of vibro-impact rig to mimic a mole penetration through the soil, which was based on electro-mechanical interactions of a conductor with an oscillating magnetic field. Sitnikova and Pavlovskaja [46] studied bifurcations and chaotic attractors of an impact oscillator with a shape memory alloy constraint. The evolution of chaotic attractors is recorded experimentally, and changes in the structure of the attractors are shown. Along with the requirement of engineering application and the increasing of basic research on non-smooth dynamics, vibro-impact dynamics has been applied to a wide range of practical mechanical systems for finding the correlative relationship between dynamic performance and model parameters, e.g., wheel-rail impacts of railway coaches [47,48], vibrating hammer [49], pile driver [50,51], ground moling dynamics [52], mill rolls [53], the fly-wheel model of the bouncing ball [54], link mechanism [55], ultrasonic percussive drilling [56,57], the vibro-impact capsule system [58], Jeffcott rotors with bearing clearance [59–61], impact dampers [62,63], excited pendula with impacts [64], high frequency vibro-impact drilling [65], pipes conveying [66], vibro-impact interaction of ships with ice [67], gears transmission systems [32–34,68–71], etc. The dynamical models designed in Refs. [47–71] have proved to be useful for revealing dynamical performance of such practical systems and the related studies have provided much information of the fundamental nature that broadened the scope of knowledge on the non-smooth motions of mechanical systems with impacts.

It is important to note, that most studies of vibro-impact dynamics are based on single-parameter bifurcation analysis in the past several years. Very few have considered multi-performance, multi-process coupling and multi-parameter simulation analysis and collaborative optimization for research on vibro-impact dynamics. The purpose of the present study is to focus attention on multi-performance and multi-parameter simulation analysis for dynamics of a two-degree-of-freedom periodically-forced system with a clearance represented by two symmetric rigid stops. The correlative relationship between dynamic performance and system parameters is particularly studied to find their

reasonable matching law. Two key parameters of the system, the exciting frequency ω and clearance δ , are emphasized to analyze the influence of the main factors on dynamic performance of the system in Section 3, and diversity and evolution of periodic-impact motions are discussed. As for small clearance, the transition between adjacent fundamental motions basically goes through three processes with decrease in ω , which are epitomized as follows: pitchfork bifurcation of symmetric $1-p-p$ motion; period-doubling bifurcation of asymmetric $1-p-p$ motion; non-periodic or chaotic motions caused by a succession of period-doubling bifurcations; symmetric $1-(p+1)-(p+1)$ motion generated by a degeneration of chaos. As for slightly large clearance, a series of grazing bifurcations occur with decreasing the exciting frequency so that the number p of impacts of the fundamental group of motions increases two by two. As p becomes big enough, the system will exhibit an incomplete chattering sequence in an excitation period followed by a finite sequence of impacts with successively reduced velocity. Finally, complete chattering-impact motion with sticking will occur with decreasing the exciting frequency ω up to the sliding bifurcation boundary. A series of singular points on the critical lines between adjacent fundamental motions are found, at which saddle-node bifurcation boundaries of one of them and real-grazing and bare-grazing bifurcation boundaries of the other alternately and mutually cross themselves. Consequently, two types of transition regions are created inevitably by the singular points, i.e., hysteresis and small tongue-shaped regions. A series of zones of subharmonic impact motions are found in the tongue-shaped regions. Based on the sampling ranges of parameters, the influence of system parameters on impact velocities, existence regions and correlative distribution of different types of periodic-impact motions of the system is emphatically analyzed in Sections 4 and 5.

2. Mechanical model

The mechanical model of a two-degree-of-freedom periodically-forced system with a clearance represented by two symmetric rigid stops is shown in Fig. 1. The vibration displacements of the mass blocks with masses M_1 and M_2 are expressed by X_1 and X_2 , respectively. Two mass blocks are connected by a linear spring with stiffness K_1 and a linear viscous dashpot with damping constant C_1 , and the mass block M_2 is attached to the supporting base by the linear spring K_2 and linear viscous dashpot C_2 . The exciting forces acting on the mass blocks are harmonic with amplitudes P_1 and P_2 . The exciting frequency and phase angle are the same for two mass blocks, which are respectively represented by Ω and τ . The vibration amplitude of the system is restricted by a clearance. For small forcing amplitudes the system exhibits simple linear oscillation. With increase in the amplitude, the mass block M_1 eventually begins to hit the rigid stops of the clearance situated in certain distances B and $-B$ from the equilibrium position of the mass block. Consequently, the contacts of the mass block with the stops result in discontinuous changes in the motion properties of the system. The kinetic energy of the system is dissipated partially on the impact. The presence of the

Fig. 1. Schematic of a two-degree-of-freedom periodically-forced system with a clearance.

clearance and impact leads to non-smooth nature of motion trajectory of the system and complicates its dynamic analysis. The impact is generally inelastic and modeled by a coefficient of restitution with $0 < R < 1$. For the simplicity of the induction and summarization, the dynamic analysis can be considered in a dimensionless form – in the following work, we assume all parameters and variables are dimensionless, which are given by

$$\begin{aligned} \mu_m &= \frac{M_2}{M_2 + M_1}, \mu_k = \frac{K_2}{K_2 + K_1}, \mu_c = \frac{C_2}{C_2 + C_1}, \\ \zeta &= \frac{C_1}{2\sqrt{K_1 M_1}}, f_{20} = \frac{P_2}{P_2 + P_1}, \omega = \Omega \sqrt{\frac{M_1}{K_1}}, \\ \delta &= \frac{BK_1}{P_1 + P_2}, t = T \sqrt{\frac{K_1}{M_1}}, x_i = \frac{X_i K_1}{P_1 + P_2}, i = 1, 2. \end{aligned} \quad (1)$$

Eq. (1) allows us to make a conclusion that the sampling range of partial dimensionless parameters can be easily identified as $\mu_m \in (0, 1)$, $\mu_k \in (0, 1)$, $\mu_c \in (0, 1)$, $f_{20} \in [0, 1]$ and $R \in (0, 1)$. It is important to note that the selection of the dimensionless parameters and variables is technically convenient to the induction and summarization of dynamics of the system.

Between any two adjacent impacts, dynamics of the two-degree-of-freedom periodically-forced system is smooth and linear so that its motion process can be governed by the following non-dimensional equations:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\mu_m}{1-\mu_m} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2\zeta & -2\zeta \\ -2\zeta & \frac{2\zeta}{1-\mu_c} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & \frac{1}{1-\mu_k} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \\ = \begin{Bmatrix} 1-f_{20} \\ f_{20} \end{Bmatrix} \sin(\omega t + \tau), \quad (|x_1| < \delta), \end{aligned} \quad (2)$$

Immediately after the impact occurs, for $|x_1| = \delta$, a discontinuous change in the velocity of the mass block M_1 occurs and results highly in nonlinear and non-smooth properties, which is determined by the impact law

$$\dot{x}_{1A+} = -R\dot{x}_{1A-}, \quad (x_1 = \delta); \quad \dot{x}_{1\bar{A}+} = -R\dot{x}_{1\bar{A}-}, \quad (x_1 = -\delta). \quad (3)$$

where the subscript signs “-” and “+” denote the states just before and after the impact, respectively; the subscripts A and $\bar{A}$ represent respectively the impacts occurring at the right and left stops of the clearance.

We can conveniently and visually describe different types of periodic-impact motions of the system by the symbol $n-p-q$, where p and q mean the number of impacts occurring at the left $\bar{A}$ and right A constraints respectively, and n represents the number of the exciting force period in the motion period $T_1 = 2\pi\pi/\omega$. At small forcing amplitudes a periodic steady state solution of the system will simply be the solution of Eq. (2) with no impacts at all. $p=0$ and $q=0$ symbolize the impactless motion, which exists for enough large clearance δ (or slightly larger clearances and higher exciting frequencies) and behaves as that of a linear forced vibration system. The quantity $\beta=(p+q)/n$ is of theoretic and practical significance for us to discuss and research thoroughly some dynamic performance on the parameter design of the periodically-forced system with a clearance, which expresses the average impact number during the exciting period.

More generally, a map derived from Eqs. (2) and (3) is used to study the correlative relationship and matching law between dynamics and parameters of the non-smooth system. There is a design technique of the map based on the impact number occurring at a certain stop of the clearance. Consequently, an impact map can be constructed; it is also called impact Poincaré map. This means that each iterate of the map corresponds to the mass block M_1 impacting the rigid stop once. As the Poincaré section $\sigma_p = \{(x_1, \dot{x}_1, x_2, \dot{x}_2, t) \in \mathbf{R}^4 \times \mathbf{S}, x_1 = \delta, \dot{x}_1 = \dot{x}_{1A+}\}$ related to the motion state of the vibro-impact system just immediately

after the impact occurring at stop A , is chosen, we can construct the Poincaré map of the periodically-forced system with a clearance

$$X^{(i+1)} = \tilde{f}(\nu, X^{(i)}), \quad (4)$$

where $X \in \mathbf{R}^4$, $X^{(i)} = (\tau^{(i)}, \dot{x}_2^{(i)}, \dot{x}_{1A+}^{(i)}, \dot{x}_2^{(i)})^T$, $X^{(i+1)} = (\tau^{(i+1)}, \dot{x}_2^{(i+1)}, \dot{x}_{1A+}^{(i+1)}, \dot{x}_2^{(i+1)})^T$, ν are real parameters, $\nu \in \mathbf{R}^m$, $m=1$ or $2 \dots$ or 8 .

It is important to note that the Poincaré section $\sigma_n = \{(x_1, \dot{x}_1, x_2, \dot{x}_2, t) \in \mathbf{R}^4 \times \mathbf{S}, x_1 = x_{1 \min, \text{mod}}(t = 2\pi/\omega)\}$ may be also considered to construct the other map. Consequently, the impact number q of $n-p-q$ motion can be determined easily by the impact map corresponding to σ_p and number n of the exciting force periods in the motion period be ascertained by the map related to σ_n . This means that $n-p-q$ motion can be described totally by the two maps. An analytic derivation is made explicitly for the determination of the group of symmetric $n-1-1$ motions and the implicit expression of disturbed map such motions is analytically derived in Ref. [34]. A simplified stability analysis presumes the linear dependence between small differences of the system state from symmetric $n-1-1$ motion at two succeeding impacts occurring at the same stop. The stability and local bifurcations of the group of symmetric $n-1-1$ motions can be ascertained by computing and analyzing eigenvalues of Jacobian matrix of disturbed map. Correspondingly, saddle-node, period-doubling, pitchfork and Neimark-Sacker bifurcation boundaries of symmetric $n-1-1$ motions, in the double-parameter plane, are determined, near which a deeper understanding of the system behavior can be provided.

Here a special case that the impact mass contacts the rigid stop with almost zero before-velocity, i.e., the bare touching, is worth to be emphasized, which causes real or bare-grazing bifurcation of symmetric $1-p-p$ motion and creates two types of transition regions (hysteresis or tongue-shaped regions) in low exciting frequency range.

Let $\bar{F}$ represent a resultant force, which consists of spring restoring force, damping force and sinusoidal excitation acted on the mass block M_1 and is expressed by

$$\bar{F} = 2\zeta(\dot{x}_2 - \dot{x}_1) + (x_2 - x_1) + (1 - f_{20}) \sin(\omega t + \tau), \quad (5)$$

If the mass block M_1 contacts the rigid stop A with almost zero before-velocity and the resultant force $\bar{F}$ is in the positive direction, in an instant immediately after the i -th impact contact, i.e.,

$$\bar{F} = 2\zeta(\dot{x}_2^{(i)} + (\dot{x}_2^{(i)} - \delta)) + (x_2^{(i)} - \delta) + (1 - f_{20}) \sin(\omega t + \tau^{(i-1)}) \geq 0, \quad (6)$$

the mass M_1 is stuck to the stop A until the resultant force $\bar{F}$ changes its direction (sign). The change from the motion of mass block M_1 to its sticking represents a reduction in the degree of freedom of the system from 2 to 1. Accordingly, the change denotes also a reduction in the dimensions of the map from 4 to 3. The presence of the sticking phase leads to the segmentation and singularity of the map and complicates the dynamic analysis of the system. If the mass block contacts the rigid stop A with zero velocity and the resultant force $\bar{F}$ is in the negative direction immediately after the i -th impact contact, i.e.,

$$\bar{F} = 2\zeta(\dot{x}_2^{(i)} + (\dot{x}_2^{(i)} - \delta)) + (x_2^{(i)} - \delta) + (1 - f_{20}) \sin(\omega t + \tau^{(i-1)}) < 0, \quad (7)$$

the mass block M_1 spends no time in sticking to the stop, the force $\bar{F}$ immediately drives the transient-static mass block M_1 off the stop.

3. Dynamical characteristics of the periodically-forced system with symmetric rigid stops

The main aim of this paper is the characterization of dynamics of the non-smooth vibratory system to reveal the correlative

relationship and matching law between dynamic performance and system parameters. Based on the definition of dimensionless parameters and variables in Eq. (1), dynamics of the periodically-forced impact system is governed by eight dimensionless parameters: $\mu_m, \mu_k, \mu_c, \zeta, f_{20}, R, \delta$ and ω . Dynamics of the system mainly depends on the relative clearance and the exciting frequency. Two key parameters of the system, the exciting frequency ω and clearance δ , are firstly chosen to analyze the influence of the main factors on its dynamic characteristics. It is important to analyze the detailed influence of each parameter on dynamics of the system to obtain parameter sensitivity and reasonable parameter matching. It is of special interest to acquire the diversity and evolution of periodic-impact motions of the system on the basis of the sampling ranges of parameters. The value range of partial dimensionless parameters are easily ascertained by Eq. (1). The sampling ranges of some parameters are identified as $\mu_m \in (0, 1)$, $\mu_k \in (0, 1)$, $\mu_c \in (0, 1)$, $f_{20} \in [0, 1]$ and $R \in [0, 1]$. Taking the dimensionless parameters: $\mu_m = 0.5, \mu_k = 0.5, \mu_c = 0.5, \zeta = 0.1, f_{20} = 0$ and $R = 0.8$ as the criterion, we analyze the determinants of system parameters and the influence of them on dynamic performance. Existence regions of different types of periodic-impact motions of the system, associated with the criterion parameters, are presented in the (ω, δ) -parameter plane plotted in Fig. 2(a). Periodic-impact motions ascertained by numerical analyses are characterized and marked by the symbol $n-p-q$ and corresponding colors. The local amplification and detail of Fig. 2 are demonstrated in Fig. 3. Moreover, 1-CIS is used to denote complete chattering-impact periodic motion with sticking (i.e., $1-\bar{p}-\bar{p}$ motion) in Figs. 2 and 3, symmetric and asymmetric $1-1-1$ motions are respectively expressed by the symbols $1-1-1_S$ and $1-1-1_{AS}$, and PF, PB, sub and sup N-S Bif represent in proper order successively pitchfork, period doubling, supercritical and subcritical Neimark-Sacker bifurcations. As for large δ , symmetric $1-1-1$ motion can be found only around the first natural frequency. At other exciting frequencies the system exhibits $1-0-0$ impactless motion. With decrease in δ , the window of symmetric $1-1-1$ motion becomes wide for large clearance; becomes narrow for slight large clearance; moves towards higher exciting frequencies for small clearance. At small δ , complete chattering-impact and fundamental $1-p-p$ ($p \geq 2$) motions appear in low exciting frequency range, and long-periodic or chaotic motions can be observed between the existence regions of adjacent $1-p-p$ and $1-(p+1)-(p+1)$ motions. Regions of existence and stability of the impactless $1-0-0$, partial fundamental $1-p-p$ motions and complete chattering-impact motion with sticking can be obviously observed in the parameter plane of Fig. 2, which are identified by different colors, such as the impactless $1-0-0$ motion is assigned to the red color, symmetric $1-1-1$ motion to yellow, asymmetric $1-1-1$ motion to brown and $1-2-2$ motion to green, etc. It must be emphasized that the fundamental $1-p-p$ motions exhibit the properties of symmetry and asymmetry. The fundamental group of impact motions have the excitation period ($n=1$) and differ by the number of impacts $p=0, 1, 2, \dots, m, \dots$, i.e., $1-p-p$ impact motions. The practical significance is considerable that the $1-0-0$ impactless motion is included in this group, because the motion is subjected to similar transition phenomena as other $1-p-p$ impact motions. For enough large clearance, dynamic performance becomes relatively simple, and the system exhibits mainly impactless $1-0-0$ or symmetric $1-1-1$ motion in most regions of the parameter plane except small tongue-shaped regions nearby the critical line $L_0: R_{1-0-0} \cap R_{1-1-1}$ between adjacent regions of $1-0-0$ and symmetric $1-1-1$ motions; see Fig. 2. For slightly large δ , the transition from $1-p-p$ to $1-(p+1)-(p+1)$ motion arises with decreasing the exciting frequency ω up to the grazing boundary G_{1-p-p} ; see Fig. 3(c). On the grazing boundary G_{1-p-p} , the mass block M_1 of the system, associated with $1-p-p$ motion, begins to touch the stops of the clearance. It can

be regarded as the $1-(p+1)-(p+1)$ motion, however the impact velocity of the additional impact degenerates to touching. This means that a new periodic motion is characterized by $1-(p+1)-(p+1)$, in which the velocity of one of impacts increases gradually from zero with decreasing the exciting frequency ω . Crossing the boundary G_{1-p-p} , $1-(p+1)-(p+1)$ motion arises and the number of impacts of the periodic motion increases by two. Thus along the grazing boundary G_{1-p-p} both neighboring motions change continuously. A series of grazing bifurcations occur with decreasing the exciting frequency ω so that the number p of impacts of the fundamental group of motions increases one by one at every rigid stop. As p becomes big enough, the $1-p-p$ motion will exhibit the incomplete chattering-impact characteristics (i.e., a chattering sequence in an excitation period followed by a finite sequence of impacts with successively reduced velocity). The complete chattering-impact motion with sticking will occur with decreasing the exciting frequency ω up to the sliding bifurcation boundary. The lower the exciting frequency ω is, the more the number p of impacts is, the longer the sticking time t_s in the time interval of an exciting period is. Consequently, for slightly large δ the transition from $1-1-1$ motion to complete chattering-impact motion is summarized as follows:

$$\begin{aligned} \omega \downarrow : 1-\bar{p}-\bar{p} \\ \text{S Bif} \leftarrow \dots \leftarrow \text{G Bif} \leftarrow 1-\bar{p}-\bar{p} \leftarrow \text{G Bif} \leftarrow \text{GBif} \leftarrow 1-(p+1)-(p+1) \leftarrow \text{G Bif} \leftarrow 1-p-p \leftarrow \text{G Bif} \leftarrow \dots \\ \text{G Bif} \leftarrow 1-4-4 \leftarrow \text{G Bif} \leftarrow 1-3-3 \leftarrow \text{G Bif} \leftarrow 1-2-2 \leftarrow \text{G Bif} \leftarrow 1-1-1, \end{aligned} \quad (8)$$

where let $1-\bar{p}-\bar{p}$ denote complete chattering-impact motion and $1-\bar{p}-\bar{p}$ represent incomplete chattering-impact motion, S Bif and G Bif represent sliding and grazing bifurcations, respectively. This sticking chattering-impact vibration is periodic, which undergoes three processes in an exciting period in the following order: free flight, chattering-impact and sticking for the impacting mass. The periodically-forced impact system enters the sticking set via a chattering sequence followed by an infinite sequence of impacts with successively reduced velocity. Consequently, the impact mass of the system becomes constrained to the rigid stop in finite time ($t < 2\pi/\omega$). This sequence is referred to as the complete chattering-impact motion [33]. In contrast, incomplete chattering-impact motion occurs if the trajectory of the impact mass, after a finite number of impacts, reaches the non-sticking region [33]. It is important to note that chattering-impact vibration enlarges the adverse effects such as high noise levels and wear and so on.

The regular pattern of the transition, for slightly large δ , is vividly plotted in the upper half plane of Fig. 3(c) and illustrated by grazing boundaries in Fig. 4(a). The analysis describes the system behavior on the boundaries of transition regions between $1-p-p$

and $1-(p+1)-(p+1)$ motions. For slightly large δ , one can observe that the existence region of $1-p-p$ motion becomes gradually narrower and narrower with increase of the impact number of the fundamental periodic motion. This means that the existence regions of incomplete chattering-impact motions ($1-\bar{p}-\bar{p}$) are too narrow to be observed clearly. Consequently, $1-\bar{p}-\bar{p}$ motion can only be included in the blue region of $1-p-p$ ($p > 8$) motions. However, it is important to note that the existence region of complete chattering-impact periodic motions is wider than that of $1-p-p$ motion ($p \geq 3$) and corresponds to lower exciting frequencies, as seen in Fig. 3(c).

Period-doubling, saddle-node and pitchfork bifurcation boundaries are collectively referred to as stability boundaries for the convenience of the following analysis. Transition from complete to incomplete chattering-impact motion arises with increasing the exciting frequency ω up to the rising bifurcation boundary. Transition from $1-p-p$ to $1-(p-1)-(p-1)$ motion occurs with increasing the exciting frequency ω up to the saddle-node bifurcation boundary SN_{1-p-p} . For slightly large δ , the transition pattern from complete chattering-impact motion with sticking to $1-1-1$ motion is summarized as follows:

$$\begin{aligned} \omega \uparrow : 1-\bar{p}-\bar{p} \\ \text{R Bif} \rightarrow 1-\bar{p}-\bar{p} \xrightarrow{\text{SN Bif}} 1-(\bar{p}-1)-(\bar{p}-1) \\ \text{SN Bif} \rightarrow \dots \xrightarrow{\text{SN Bif}} 1-p-p \xrightarrow{\text{SN Bif}} 1-(p-1)-(p-1) \xrightarrow{\text{SN Bif}} \dots \\ \text{SN Bif} \rightarrow 1-3-3 \xrightarrow{\text{SN Bif}} 1-2-2 \xrightarrow{\text{SN Bif}} 1-1-1. \end{aligned} \quad (9)$$

In Eq. (9), R Bif and SN Bif represent rising and saddle-node bifurcations, respectively.

Saddle-node bifurcation of $1-(p+1)-(p+1)$ motion corresponds to stability boundary $SN_{1-(p+1)-(p+1)}$, on which two of impacts disappears and $1-p-p$ motion stabilizes after a jump transition. The transition between neighbor fundamental motions $1-p-p$ and $1-(p+1)-(p+1)$ is non-reversible except at the singular points of the alternate intersections of grazing and stability boundaries. Narrow hysteresis region exists between grazing bifurcation boundary G_{1-p-p} of $1-p-p$ motion and saddle-node bifurcation boundary $SN_{1-(p+1)-(p+1)}$ of $1-(p+1)-(p+1)$ motion, as seen in Fig. 4(b). This means that both neighboring $1-p-p$ and $1-(p+1)-(p+1)$ motions are stable and can coexist in the dependence on the initial conditions of the system or on the way of the system parameters change in the hysteresis zones and they transit by a jump from one to the other on the grazing and saddle-node bifurcation boundaries of these regions. One neighboring motion switches to the other on the segments of the boundaries defining the hysteresis regions. Along the segment of grazing boundary, the

Fig. 2. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance, associated with the criterion parameters: $\mu_m = 0.5$, $\mu_k = 0.5$, $\mu_c = 0.5$, $\zeta = 0.1$, $f_{20} = 0$ and $R = 0.8$ in the (ω, δ) -parameter plane. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Fig. 3. The local amplification and details of Fig. 2: (a) existence regions of 1-0-0, 2-1-1 and 3-1-1 motions in the tongue-shaped zones nearby the boundary $L: R_{1-0-0} \cap R_{1-1-1}$; (b) existence regions of 1-2-1, 2-3-2 and 3-4-4 motions in the tongue-shaped zones nearby the boundary $L: R_{1-1-1} \cap R_{1-2-2}$; (c) existence regions of $1-p-p$ motions and complete chatter-impact motions; (d) existence regions of $3-(3p+1)-(3p+1)$, $2-(2p+1)-2p$, $1-(p+1)-p$ motions in the tongue-shaped zones nearby the boundary $L: R_{1-p-p} \cap R_{1-(p+1)-(p+1)}$, $p \geq 2$; (e) local detail of Fig. 3(c), ($P1=3-13-13$, $P2=3-16-16$); (f) symmetric 1-1-1, asymmetric 1-1-1 motions and transition to 1-2-2 motion.

impact number of a periodic motion increases by two. Conversely, along the other segment the impact number of the neighboring periodic motion decreases by two.

Moreover, for slightly large clearance δ , there exist some small tongue-shaped regions nearby the critical line $L_p: R_{1-p-p} \cap R_{1-(p+1)-(p+1)}$ between adjacent regions of $1-p-p$ and $1-(p+1)-(p+1)$ motions ($p \geq 1$). Dynamics in tongue-shaped regions is more complex and emphatically introduced in the following context. With decreasing the clearance δ , the region of $1-p-p$ motion approaches the grazing boundary, where the

trajectory of 1-0-0 impactless motion of the mass block M_1 or two additional impacts of $1-p-p$ motion trajectory begin to touch the stops. On the contrary, with increasing the clearance δ , the region of $1-p-p$ motion reaches the stability boundaries associated with saddle-node bifurcation, crossing one of which $1-p-p$ motion loses its stability and transits to $1-(p-1)-(p-1)$ or $1-p-(p-1)$ motion. There exist a series of singular points X_i on the boundaries between $1-p-p$ and $1-(p+1)-(p+1)$ motions, i.e., saddle-node bifurcation boundaries $SN_{1-(p+1)-(p+1)}^{1-p-p}$ and $SN_{1-(p+1)-(p+1)}^{1-(p+1)-p}$ of $1-(p+1)-(p+1)$ motion, bare and real grazing

Fig. 4. Saddle-node, grazing bifurcations and transition phenomena of $1-p-p$ motions: (a) grazing bifurcation boundaries $L: R_{1-p-p} \cap R_{1-(p+1)-(p+1)}$; (b) hysteresis region HR_p existing between grazing boundary G_{1-p-p} and saddle-node bifurcation boundary $SN_{1-(p+1)-(p+1)}$.

boundaries G_{1-p-p} and $G_{1-(p+1)-(p+1)}^b$ of $1-p-p$ motion alternately and mutually cross themselves at the points X_i of intersection. It needs to be emphasized that the singular points of intersection of several bifurcation boundaries have a special significance and create inevitably two types of transition regions (hysteresis and tongue-shaped regions). A special example is inserted to expound the above-mentioned analysis. A singular point X , the intersection of boundaries SN_{1-2-2}^{1-2-1} , SN_{1-2-2}^{1-1-1} , G_{1-1-1} and G_{1-1-1}^b , is described and illustrated in Fig. 4(b). Crossing the grazing boundary G_{1-1-1} , the number of impacts increases by two and $1-1-1$ motion transits to $1-2-2$ motion. This means that the grazing boundary G_{1-1-1} is real. It is important to note that the grazing boundary G_{1-1-1}^b is bare. Consequently, along G_{1-1-1}^b the mass block M_1 of the system begins to touch the stops with almost zero before-impact velocity and $1-1-1$ motion develops into a $1-2-2$ motion which is unstable in the whole tongue-shaped region and the system transits to other type of impact motion by the unstable $1-2-2$ motion. A tongue-shaped region is surrounded by the bare-grazing bifurcation boundary G_{1-1-1}^b of $1-1-1$ motion and saddle-node bifurcation boundary SN_{1-2-2}^{1-2-1} of $1-2-2$ motion, and corresponding hysteresis region is limited by real-grazing boundary G_{1-1-1} and saddle-node bifurcation boundary SN_{1-2-2}^{1-1-1} . The grazing and saddle-node boundaries between the particular regions of $1-p-p$ and $1-(p+1)-(p+1)$ motions alternately intersect at singular points and so create two kinds of transient zones. There are partly zones of hysteresis and partly zones of tongue-shaped regions including fundamental $1-(p+1)-p$, subharmonic $3-(3p+1)-(3p+1)$ and $2-(2p+1)-2p$ motions. The analysis for singular points X can be extended to ascertain the limit of hysteresis and tongue-shaped regions. The hysteresis regions are limited at the top by the saddle-node bifurcation boundaries $SN_{1-(p+1)-(p+1)}$ of $1-(p+1)-(p+1)$ motion and at the bottom by the real-grazing boundaries G_{1-p-p} of $1-p-p$ motion. The tongue-shaped regions are limited at the top by the bare-grazing bifurcation boundary G_{1-p-p}^b of $1-p-p$ motion and at the bottom by the saddle-node bifurcation boundary $SN_{1-(p+1)-(p+1)}^{1-(p+1)-p}$ of $1-(p+1)-(p+1)$ motion. A series of zones of fundamental and subharmonic impact motions, such as $1-(p+1)-p$, $2-(2p+1)-2p$ and $3-(3p+1)-(3p+1)$ impact orbits, etc., are found to occur in the tongue-shaped regions. Existence region of a periodic or subharmonic impact motion, in the parameter plane, is called parameter island of the corresponding motion. The phenomena can be observed in Fig. 3(c). Details of Fig. 3(c), associated with the tongue-shaped regions and subharmonic impact characters, are shown, respectively, in Fig. 3(b) and (d). The positions of the

corresponding parameter islands of $1-(p+1)-p$, $3-(3p+1)-(3p+1)$ and $2-(2p+1)-2p$ motions arrange from the bottom up (or from left to right) in this order in the tongue-shaped region, and the area of the parameter islands becomes smaller and smaller in this order. It is important to note that no motion with the exciting force period and symmetric impacts (i.e., symmetric $1-p-p$ motion) is stable or can exist in the tongue-shaped regions. Consequently, $1-(p+1)-p$, $2-(2p+1)-2p$ and $3-(3p+1)-(3p+1)$ motions and chaotic ones dominate in the regions. The continuous transition between neighboring $1-p-p$ and $1-(p+1)-(p+1)$ motions can exist only at the singular points X of the intersections of grazing and saddle-node boundaries G_{1-p-p} , G_{1-p-p}^b and $SN_{1-(p+1)-(p+1)}$. The motions which the system exhibits at the singular points X_i are identical. The $1-p-p$ motion, associated with the new-appeared periodic touching the stops of the clearance, can be regarded as that the certain two periodic impacts of the $1-(p+1)-(p+1)$ motion degenerates into the touching. Consequently, $1-p-p$ motion will end on its grazing boundary and $1-(p+1)-(p+1)$ motion will emerge. In contrast, the adverse transition from one neighboring fundamental motion to the other, at the singular points X_i , is also similar to that above-mentioned.

A tongue-shaped region nearby the critical line $L_0: R_{1-0-0} \cap R_{1-1-1}$ between adjacent regions of $1-0-0$ and $1-1-1$ motions are shown in Fig. 3(a). The impact motion is invariably found below the grazing boundary G_{1-0-0} . The regions of the kinds of impact motions ascertained are marked by different colors and corresponding symbol $n-p-q$. The tongue-shaped region is limited at the bottom by the pitchfork bifurcation boundary PF_{1-1-1} of symmetric $1-1-1$ motion and at the top by the bare-grazing bifurcation boundary G_{1-0-0}^b of the impactless motion. Increasing the exciting frequency ω (or decreasing the clearance δ) up to the boundary $L_C: R_{1-0-0} \cap R_{1-1-1}$, the mass block M_1 of the system, associated with $1-0-0$ impactless forced vibration, begins to touch the stops with almost zero impact velocities, i.e., the bare touching. On the boundary $L_C: R_{1-0-0} \cap R_{1-1-1}$, the impactless motion develops into a $1-1-1$ motion which is unstable in the whole tongue-shaped region and the system transits to other type of impact motion by the unstable $1-1-1$ motion. In the tongue-shaped region R_{1-0-0} no motion with the exciting force period and symmetric impacts is stable or can exist so that $1-1-0$ (or $1-0-1$), $2-1-0$ (or $2-0-1$) and $3-1-1$ motions, $2-2-1$ (or $2-1-2$), $3-3-1$ (or $3-1-3$), $3-2-1$ (or $3-1-2$), $3-2-2$ motions and chaotic motions dominate. By means of a theoretical or numerical analysis it is possible to find this zone of the existence and stability

subregions of 1-1-0, 2-1-0 and symmetric 3-1-1 motions. The zones of 1-1-0, 2-1-0 and symmetric 3-1-1 motions are limited from above by stability boundaries PD_{1-1-0} , SN_{2-1-0} and PF_{3-1-1} and from down by grazing boundaries G_{1-1-0} , G_{2-1-0} and G_{3-1-1} which probably correspond to stable or unstable grazing bifurcations, respectively. Partial subregions are drawn in Fig. 3(a) and surrounded by the stability and grazing boundaries. With increasing the exciting frequency ω up to crossing the boundary L_{PF} : $R_{1-1-1} \cap R_{T1-0-0}$, symmetric 1-1-1 motion stabilizes. And then the symmetric 1-1-1 motion transits by jump to the impactless motion on the stability boundary L_{SN} : $R_{1-1-1} \cap R_{1-0-0}$, where the differences $\Delta X^{(i)} = (\Delta \tau, \Delta x_2^{(i)}, \Delta \dot{x}_2^{(i)}, \Delta \dot{x}_1^{(i)})^T$ of the fixed point $X^* = (\tau_0, x_{20}, \dot{x}_{1+}, \dot{x}_{20})^T$ of symmetric 1-1-1 motion in Poincaré section σ_p grow with the same sign and the impact finally vanishes and 1-0-0 motion stabilizes. As a result, beyond the boundary L_{SN} : $R_{1-1-1} \cap R_{1-0-0}$ no solution of symmetric 1-1-1 motion is real; the saddle-node bifurcation of the motion corresponds to the state along the stability boundary.

With ω increasing and crossing a tongue-shaped zone between adjacent regions of 1-0-0 and 1-1-1 motions, a most complex transition, associated with a horizontal scan for certain fixed value of clearance (or a vertical scan for certain fixed value of exciting frequency), is summarized as follows:

$$\begin{aligned} \omega \uparrow \text{ or } \delta \downarrow : 1-0-0 \rightarrow \text{Unstable G Bif} \rightarrow \dots \rightarrow 2-1-0 \rightarrow \text{G Bif} \rightarrow \dots \rightarrow 3-1-1_{AS} \rightarrow \\ \text{Inverse Pitchfork Bif} \rightarrow 3-1-1_S \rightarrow \text{G Bif} \rightarrow \dots \rightarrow 2-2-0 \rightarrow 1-1-0 \rightarrow \text{G Bif} \rightarrow \\ 1-1-1_{AS} \rightarrow \text{Inverse Pitchfork Bif} \rightarrow 1-1-1_S, \end{aligned} \quad (10)$$

where G Bif is probably stable or unstable grazing bifurcation. The unstable grazing bifurcation causes long-periodic multi-impact or chaotic motion due to the bare touching of the mass block M_1 with the stops. Motion amplitude $x_{1mp}(\omega)$ and impact velocity $\dot{x}_{1A-}(\omega)$ associated with a transition crossing a tongue-shaped zone between adjacent regions of 1-0-0 and 1-1-1 motions, along a horizontal scan for $\delta=5.0$, are shown in Fig. 5. During a horizontal scan of change of the value of exciting frequency through regions of different types of impact motions, the diversity and evolution of the system motion can be observed by the bifurcation diagrams shown in Fig. 5(a) and (b). It is important to note that the bifurcation diagram $x_{1mp}(\omega)$ corresponds to the Poincaré section $\sigma_n = \{(x_1, \dot{x}_1, x_2, \dot{x}_2, t) \in \mathbf{R}^4 \times \mathbf{S}, x_1 = x_{1 \min} \bmod(t = 2\pi/\omega)\}$, which represents the minimum displacements of mass M_1 appearing in every excitation period $T_1 = 2\pi/\omega$. The bifurcation diagram $\dot{x}_{1A-}(\omega)$ is related to the impact Poincaré section $\sigma_p = \{(x_1, \dot{x}_1, x_2, \dot{x}_2, t) \in \mathbf{R}^4 \times \mathbf{S}, x_1 = \delta, \dot{x}_1 = \dot{x}_{1A-}\}$, which corresponds to the before-impact velocity of every impact. As a result, it is convenient to ascertain $n-p-q$ motion according to the number p of branches

of $\dot{x}_{1A-}(\omega)$ and the number n of branches of $x_{1mp}(\omega)$ of a certain impact motion. The windows of 2-1-0 (or 2-0-1), 3-1-1 and 1-1-0 (or 1-0-1) motions are easily observed by Fig. 5. Period-doubling bifurcation of 1-1-0 (or 1-0-1) motion occurs with decreasing the exciting frequency ω . However we can note that existence region of the period-doubling orbit is too small to be observed; see Fig. 6 (the detail of Fig. 5(b)). Long-periodic multi-impact or irregular impact motions arise successively via bare-grazing bifurcations of 2-1-0 (2-1-0) and asymmetric 3-1-1 motions with increasing the exciting frequency ω . A representative 3-1-1 motion is shown for $\omega=0.51$ in Fig. 7(a). A bare-grazing contact instability of 3-1-1 motion occurs at $\omega = 0.525659$. The transition to chaos via the bare-grazing bifurcation of the motion is depicted by the phase plane portraits shown in Fig. 7(a)-(c).

The zones of tongue-shaped regions between adjacent regions of $1-p-p$ and $1-(p+1)-(p+1)$ motions are shown, respectively, in Fig. 3(b) and (d). Only the zones of the existence and stability subregions of $1-(p+1)-p$, $3-(3p+1)-(3p+1)$ and $2-(2p+1)-2p$ motions are provided in the tongue-shaped regions. The zones of some impact motions caused by correlative bifurcations of the three motions are too small to be observed clearly, so these motions and other motions not ascertained in the tongue-shaped regions are all marked by gray color. Part zones of $1-(p+1)-p$, $3-(3p+1)-(3p+1)$ and $2-(2p+1)-2p$ motions are drawn in Fig. 3 (c), the details of which are shown by the next enlarged subregions in Fig. 3(b) and (d). The zones of $1-(p+1)-p$, $2-(2p+1)-2p$ and $3-(3p+1)-(3p+1)$ motions are limited from above in turn by period doubling bifurcation boundaries $PD_{1-(p+1)-p}$, $PD_{2-(2p+1)-2p}$ and $PD_{3-(3p+1)-(3p+1)}$ and from down in turn by saddle-node bifurcation boundary $SN_{1-(p+1)-p}$, grazing bifurcation boundaries $G_{2-(2p+1)-2p}$ and $G_{3-(3p+1)-(3p+1)}$, respectively. With ω or δ decreasing and crossing a tongue-shaped zone between adjacent regions of $1-p-p$ and $1-(p+1)-(p+1)$ motions ($p \geq 1$), a most complex transition, associated with a horizontal scan for certain fixed value of clearance (or a vertical scan for certain fixed value of exciting frequency), is summarized as follows:

$$\begin{aligned} \omega \downarrow \text{ or } \delta \downarrow : 1-(p+1)-(p+1)_S \leftarrow \text{SN Bif} \leftarrow 1-(p+1)-p \leftarrow 2-(2p+2)-2p \\ \leftarrow \dots \leftarrow \text{G Bif} \leftarrow 3-(3p+1)-(3p+1) \leftarrow 6-(6p+2)-(6p+2) \\ \leftarrow \dots \leftarrow \text{G Bif} \leftarrow 2-(2p+1)-2p \leftarrow 4-(4p+2)-4p \leftarrow \dots \leftarrow \text{unstable G Bif} \leftarrow 1-p-p_S, \end{aligned} \quad (11)$$

where G Bif is probably real or bare unstable grazing bifurcation. Motion amplitude $x_{1mp}(\delta)$ and impact velocity $\dot{x}_{1A-}(\delta)$ associated with a transition crossing a tongue-shaped zone between adjacent regions of 1-1-1 and 1-2-2 motions, along a vertical scan for $\omega=0.26$, are shown in Fig. 8. The windows of 1-2-1, 3-4-4 and 2-3-2 motions are easily found by analyzing Fig. 8(a) and (b).

Fig. 5. A transition crossing a tongue-shaped zone between adjacent regions of 1-0-0 and 1-1-1 motions along a horizontal scan for $\delta=5.0$: (a) bifurcation diagram for the lowest displacement $x_{1mp}(\omega)$ of the motion appearing in every period T ; (b) bifurcation diagram for impact velocity $\dot{x}_{1A-}(\omega)$.

Fig. 6. Details of Fig. 5(b).

Fig. 7. Phase plane portraits of the mass block M_1 : (a) 3-1-1 motion, $\omega=0.51$, $\delta=5.0$; (b) 3-1-1 motion with grazing contact, $\omega=0.525659$, $\delta=5.0$; (c) chaotic impact motion, $\omega=0.5256589$, $\delta=5.0$.

Fig. 8. A transition crossing a tongue-shaped zone between adjacent regions of 1-1-1 and 1-2-2 motions along a vertical scan for $\omega=0.26$: (a) bifurcation diagram for $x_{1mp}(\delta)$; (b) bifurcation diagram for impact velocity $\dot{x}_{1A-}(\delta)$.

Period-doubling bifurcations of 1-2-1, 3-4-4 and 2-3-2 motions occur successively with increasing the clearance δ . Existence regions of 2-4-2, 6-8-8 and 4-6-4 motions are too narrow to be observed in Fig. 8. However they can be observed by Fig. 9 (details of Fig. 8(b)). Long-periodic multi-impact or chaotic motions arise successively due to the grazing bifurcation of 3-4-4 and 2-3-2 motions with decreasing the clearance δ ; see Fig. 8. A bare-grazing contact instability of 3-4-4 motion occurs at $\delta=2.047421$. Consequently, the unstable grazing bifurcation of the motion leads to irregular impact motion immediately, which is depicted in Fig. 10.

For very small clearance, the transition from 1- p - p to 1-($p+1$)-($p+1$) motion ($p \geq 2$) is far more complicated than that

associated with large clearance, which basically goes through three processes. First, pitchfork bifurcation of symmetric 1- p - p motion occurs with decreasing ω . Consequently, asymmetric 1- p - p motion stabilizes, two antisymmetric forms of which can exist in the dependence on the initial conditions of the system. Secondly, period-doubling bifurcation of asymmetric 1- p - p motion occurs and it undergoes a succession of period-doubling bifurcations with decrease in ω , which eventually result in apparently non-periodic or chaotic motions. Finally, symmetric 1-($p+1$)-($p+1$) motion is generated by a degeneration of chaos with decreasing ω . Moreover, there exists the zone of 3-($p+1$)-($p+1$) motion in the transition region from 1- p - p to 1-($p+1$)-($p+1$) motion, which are observed in Fig. 3(e) and (f). For small

Fig. 9. Details of Fig. 8(b).

Fig. 10. Phase plane portraits of the mass block M_1 : (a) 3-4-4 motion, $\omega=0.26$, $\delta=2.09$; (b) 3-4-4 motion with grazing contact, $\omega=0.26$, $\delta=2.047421$; (c) chaotic impact motion, $\omega=0.26$, $\delta=2.047420$.

clearance, another transition from asymmetric 1-1-1 to symmetric 1-2-2 motion may also occur. First, Neimark-Sacker bifurcation of asymmetric 1-1-1 motion occurs with decreasing ω so that quasi-periodic impact motions arise; secondly, the quasi-periodic impact motion transits to chaos via phase locking; finally, symmetric 1-2-2 motion is generated by a degeneration of chaos with decrease in ω . The transition from $1-p-p$ to $1-(p+1)-(p+1)$ motion ($p \geq 2$) can be observed in Fig. 3(e) and Fig. 11.

For very small clearance, symmetric 1-1-1 motion first goes through pitchfork bifurcation with decreasing ω . Secondly, period-doubling bifurcation of asymmetric 1-1-1 motion occurs; perhaps a succession of period-doubling bifurcations occur and eventually result in apparently non-periodic or chaotic motions with decreasing ω . Finally, symmetric 1-2-2 motion is generated by a degeneration of chaos. Moreover, there exist the zones of 1-2-1, 2-3-2, 3-4-4 and 2-4-2 motions in the transition region from asymmetric 1-1-1 to symmetric 1-2-2 motion. Based on the bottom of Fig. 3(f), the whole transition can be observed.

Period doubling bifurcation of asymmetric 1-1-1 motion basically corresponds to very small clearance. However, it is possible to occur Neimark-Sacker bifurcation of asymmetric 1-1-1 motion in certain interval associated with very small clearances, as seen in Fig. 3(f). For slightly large clearance, Neimark-Sacker bifurcation of symmetric 1-1-1 motion occurs in high exciting frequency range and quasi-periodic impact motions transit to chaotic impact motions via phase locking or torus doubling, and impactless 1-0-0 motion is finally generated by a degeneration of chaos with decrease in ω ; Neimark-Sacker bifurcation of asymmetric 1-1-1 motion generally corresponds to slightly low exciting frequencies, as seen in Fig. 2(a). Representative bifurcation diagrams of before-impact velocity as a function of the exciting frequency are shown for $\delta=0.1, 0.3, 0.545, 0.6, 0.8$ and

1.1 in Fig. 11(a)-(f), which illustrate bifurcation characteristics of symmetric 1-1-1 motion.

4. The correlative relationship between dynamic performance and system parameters

It is important to analyze the detailed influence of each parameter on dynamics of the system to reveal the correlative relationship and matching law between dynamic performance and system parameters. In the following text, we will discuss the influence of each parameter on dynamics of the system by comparing with dynamics associated with the criterion parameters. In Fig. 12 are plotted the existence regions of different types of periodic-impact motions for the non-smooth vibratory system with variation of the mass distribution μ_m . The selected values of parameter μ_m are given in graphical interpretation of Fig. 12, the other parameters are the same as the remainder of the criterion parameters. For the following analyses associated with the selected parameters $\mu_k, \mu_c, \zeta, f_{20}$, and R , the graphical interpretations are similar to that of Fig. 12. The effects of changes in μ_m are emphatically analyzed by selecting its different values and scanning the exciting frequency ω and clearance δ . As the mass distribution μ_m becomes small enough, the case corresponds to high mass concentration on the impacting mass block, the system is actually equivalent to the mechanical model that the mass block M_1 is connected to the supporting base by the linear springs K_1 and K_2 in series and linear viscous dashpots C_1 and C_2 in series and behaves as a single-degree-of-freedom impact oscillator. The increase of mass parameter μ_m brings great changes in the distribution and existence region of symmetric 1-1-1 motion of the system. With increasing μ_m in the interval $\mu_m \in [0.05, 0.5]$, one can find that

Fig. 15. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, δ) -parameter plane, associated with the selected values of ζ and the remainder of the criterion parameters: (a) $\zeta = 0.02$; (b) $\zeta = 0.05$; (c) $\zeta = 0.3$; (d) $\zeta = 0.5$.

Fig. 16. Existence regions of different types of periodic-impact motions for small ζ : (a) local detail of Fig. 15(a), $\zeta = 0.02$; (b) local detail of Fig. 15(b), $\zeta = 0.05$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

or chaotic motions, marked by blue color, are much larger than that of symmetric 1-1-1 motion in high exciting frequency range; see Fig. 15(a). The reasons that some periodic-impact motions can be not marked are mainly that their existence regions are too small to be observed clearly or the number n of excitation period or the number p of impacts in the whole motion period is too big

for the corresponding periodic-impact motions to be ascertained effectively.

Based on the criterion parameters, the correlative relationship between dynamic performance and the exciting forces acting on two mass blocks is studied. The results from simulation, associated with the influence of the exciting forces on main motions of

Fig. 17. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, δ) -parameter plane, associated with the selected values of f_{20} and the remainder of the criterion parameters: (a) $f_{20} = 0.25$; (b) $f_{20} = 0.5$; (c) $f_{20} = 0.75$; (d) $f_{20} = 1.0$.

the system, are shown for $f_{20} \in [0, 1]$ in Figs. 2, 17 and 25. Variation of the exciting force f_{20} has an effect on the impact velocities and existence regions of $1-p-p$ and complete chattering-impact motions, but it hardly changes general forms of a series of periodic-impact motions shown in Figs. 2 and 17. No anomaly type of periodic-impact motions occurs even for two extreme cases of $f_{20} = 0$ and 1. As the parameter f_{20} is small enough, the exciting force acting on the mass block M_1 is much larger than that on the other mass, the existence regions of $1-p-p$ ($p \geq 2$) and complete chattering-impact motions enlarge towards the interval of large clearances, see Fig. 2 and Fig. 17(a). The exciting force acting on the mass block M_1 gradually decreases whereas that on the other mass correspondingly increases with gradual increase of f_{20} . The result causes a gradual reduction of the peak of the existence region of symmetric $1-1-1$ motion in the (ω, δ) -parameter plane so that the existence regions of $1-p-p$ ($p \geq 2$) and complete chattering-impact motions become correspondingly small. Because the largest tongue-shaped region nearby the critical line L_0 : $R_{1-0-0} \cap R_{1-1-1}$ always exists near the peak of the existence region of symmetric $1-1-1$ motion, a following example is provided to demonstrate that increase of f_{20} leads to a gradual reduction of the peak of the existence region of symmetric $1-1-1$ motion. A tongue-shaped region nearby the critical line L_0 : $R_{1-0-0} \cap R_{1-1-1}$ between adjacent regions of $1-0-0$ and $1-1-1$ motions becomes larger and more obvious due to a significant reduction of the peak of existence region of symmetric $1-1-1$ motion, in which $1-1-0$ (or $1-1-0$), $3-1-1$, $2-1-0$ (or $2-0-1$), ..., etc., fundamental and subharmonic impact motions are found. The positions of these tongue-shaped zones gradually lower and the shape and internal structure of one of them

become more and more observable with increase of f_{20} , as seen by comparing Figs. 2 and 17. It is interesting to note that change in f_{20} hardly affect the distribution and existence regions of asymmetric $1-1-1$ motions in the (ω, δ) -parameter plane except very small clearance. In high exciting frequency range, the gradual increase of f_{20} leads to smaller and smaller existence region of symmetric $1-1-1$ motion.

The influence of restitution coefficient R of impact on dynamics of the system is epitomized in Fig. 18. Based on the existing simulation results for dynamic analysis, the correlative relationship between dynamic performance and the restitution coefficient of impact are analyzed by comparing Figs. 2 and 18. The change in restitution coefficient of impact mainly affects the impact velocities and existence regions of $1-p-p$ and complete chattering-impact motions. In the interval $R \in [0.05, 0.8]$, the total area of existence regions of $1-p-p$ motions ($p \geq 1$) and complete chattering-impact motions, especially in low exciting frequency range, is basically equal. As the restitution coefficient R of impact is small enough, the existence region of complete chattering-impact motions is very large so that the regions of $1-p-p$ ($p \geq 1$) motions are badly extruded and substantially reduced. Especially the existence regions of some $1-p-p$ motions ($p \geq 4$) are too small to be shown in Fig. 18(a) and (b). It is possible that there exist no asymmetric $1-1-1$ motion for extreme small R . For enough small R , the system exhibits subharmonic $2-1-1$, $3-1-1$, $3-2-2$ motions and so on in high exciting frequency range; see Fig. 18(a). With increase in R , the existence region of complete chattering-impact motion reduces gradually. Consequently, the regions of $1-p-p$ motions correspondingly extends by degrees to the ranges

Fig. 18. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, δ) -parameter plane, associated with the selected values of R and the remainder of the criterion parameters: (a) $R = 0.1$; (b) $R = 0.2$; (c) $R = 0.4$; (d) $R = 0.6$.

Fig. 19. Time series of complete chattering-impact motion ($\omega = 0.5$, $\delta = 0.1$ and the criterion parameters (except for R)): (a) $R = 0.2$; (b) $R = 0.5$; (c) $R = 0.8$; (a1) detail of Fig. 19(a); (b1) detail of Fig. 19(b); (c1) detail of Fig. 19(c).

Fig. 20. Before-impact velocities and existence regions of different types of impact motions for the periodically-forced system with a clearance in the (ω, μ_m) -parameter plane related to $\delta=0.2$ and the remainder of the criterion parameters.

Fig. 21. Before-impact velocities and existence regions of different types of impact motions for the periodically-forced system with a clearance in the (ω, μ_k) -parameter plane related to $\delta=0.2$ and the remainder of the criterion parameters.

Fig. 22. Before-impact velocities and existence regions of different types of impact motions for the periodically-forced system with a clearance in the (ω, μ_c) -parameter plane related to $\delta=0.2$ and the remainder of the criterion parameters.

Fig. 23. Before-impact velocities and existence regions of different types of impact motions for the periodically-forced system with a clearance in the (ω, ζ) -parameter plane related to $\delta=0.2$ and the remainder of the criterion parameters.

Fig. 24. Before-impact velocities and existence regions of different types of impact motions for the periodically-forced system with a clearance in the (ω, f_{20}) -parameter plane related to $\delta=0.2$ and the remainder of the criterion parameters.

Fig. 25. Before-impact velocities and existence regions of different types of impact motions for the periodically-forced system with a clearance in the (ω, R) -parameter plane related to $\delta=0.2$ and the remainder of the criterion parameters.

Fig. 26. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, μ_m) -parameter plane: (a) $\delta=0.5$; (b) $\delta=1.0$.

observed obviously. Correspondingly, the region of symmetric 1-1-1 motion becomes wider in the exciting frequency interval and the existence region of asymmetric 1-1-1 motion extends slightly. On the other hand, the restitution coefficient of impact brings also great influences on the impact sequences and the sticking times of $1-\bar{p}-\bar{p}$ motion in a motion period. For large restitution coefficient R , the impact sequence of $1-\bar{p}-\bar{p}$ motion is also more obvious in the motion period and the sticking time correspondingly shorter. Conversely, as R is small enough, the

impact strength is markedly attenuated, the impact sequence of $1-\bar{p}-\bar{p}$ motion decays quickly in the motion period and becomes not obvious in visual sensation and quantity (but the chatter sequence is still infinite), the sticking time is correspondingly longer. A simulation result, associated with $\omega=0.5$, $\delta=0.1$ and the remainder of the criterion parameters, is provided to observe the influence of restitution coefficient R on the impact number and the sticking time of $1-\bar{p}-\bar{p}$ motion in a motion period, which is shown for $R=0.2, 0.5$ and 0.8 in Fig. 19.

Fig. 27. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, μ_k) -parameter plane: (a) $\delta=0.5$; (b) $\delta=1.0$.

Fig. 28. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, μ_c) -parameter plane: (a) $\delta=0.5$; (b) $\delta=1.0$.

5. The influence of system parameters to before-impact velocity

The clearances or constraints among mechanical components exist inevitably in mechanical systems due to the restriction on technological level in the mechanical machining process, the demand for mechanical design and assembly errors, the need for the heat-expansion and cold-contraction of some part of mechanical components or other reason. The existence of the clearances or constraints is bound to bring about the vibro-impact effects in service of the mechanisms. The vibro-impact effects have a great influence on performance of mechanical equipments, increase noise levels and intensify fatigue and abrasion of mechanical parts. An important scientific research subject in the field of mechanical engineering is urgent to be solved, i.e., how to reduce wear of mechanical components and improve use efficiency, service life and operation reliability of non-smooth mechanical systems. Therefore, it is a primary purpose of dynamic research on mechanical vibrating systems with clearances or constraints how both vibration and impact are effectively utilized or furthest decreased to prevent from chattering-impact or chaotic impact motions so as to restrain noise levels and reduce the abrasion between components and constraints.

The influence of clearance nonlinearity on before-impact velocity of the mass block M_1 is epitomized in Fig. 2(b). The diagram is

useful when the distribution of impact velocity is of interest over the whole (ω, δ) plane. Based on the simulation results of the system with the criterion parameters, the correlative relationship between before-impact velocity and parameters (ω, δ) is analyzed to find their reasonable matching law. Impact velocities of symmetric and asymmetric 1-1-1 motions are specially marked in Fig. 2(b) by red color, the existence regions of which are shown in Fig. 2(a). With increase in the value of clearance, the impact velocity of 1-1-1 motion, associated with small δ , raises gradually. The resonance peak of function curve $\dot{x}_{1A}(\omega)$ of impact velocity of symmetric 1-1-1 motion, corresponding to large δ , rises gradually with increasing δ . The regularities can be observed by Fig. 2(b). In general case, small clearances lead to low impact velocity of symmetric 1-1-1 motion. The function curve $\dot{x}_{1A}(\omega)$ of impact velocity of symmetric 1-1-1 motion, associated with $\delta=0.2$, is lower on the whole than that corresponding to $\delta>0.2$. As the clearance is located at the interval of approximately [0.6, 2.0], 1- p - p motions ($p \geq 2$), complete chattering-impact motions, 1-($p+1$)- p , 2-($2p+1$)- $2p$ and 3-($3p+1$)-($3p+1$) motions ($p \geq 1$) in tongue-shaped zones play major roles in low exciting frequency range, respectively. Consequently, these motions bring about the increase of average impact number in an exciting period, which increases noise levels and intensifies wear of the impact mass and stops. The choice of small clearance and the determination of reasonable parameter matching and effective exciting

Fig. 29. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, ζ) -parameter plane: (a) $\delta=0.5$; (b) $\delta=1.0$.

Fig. 30. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, f_{20}) -parameter plane: (a) $\delta=0.5$; (b) $\delta=1.0$.

frequency range are conducive to obtain symmetric 1-1-1 motion, reduce its impact strength, abrasion and noise levels.

A conclusion has been obtained from Fig. 2 that the function curve $\dot{x}_{1A-}(\omega)$ of before-impact velocity of 1-1-1 motion, associated with the criterion parameters and $\delta=0.2$, is lower on the whole than those corresponding to $\delta > 0.2$. Based on the conclusion above-mentioned, the influence of parameter variation on before-impact velocity of the mass block M_1 is epitomized in Figs. 20–25. The correlative relationship between impact velocity $\dot{x}_{1A-}$ and parameters (ω, μ_m) are shown for $\delta=0.2$ in Fig. 20. The change in μ_m hardly affects the impact velocity of the mass block M_1 for the same exciting frequency, and the range of exciting frequency of symmetric 1-1-1 motion is basically constant for small clearance and large mass distribution. As for the same mass distribution, impact velocity of symmetric 1-1-1 motion gradually raises with increase in ω . As for the mass distribution interval $\mu_m \in [0.05, 0.95]$, 1- p - p fundamental motions ($p \geq 2$), complete chattering-impact motions, 1-2-1, 2-3-2 and 2-4-2 motions and chaos, etc., dominate in the exciting frequency range of $\omega < 2$, respectively. These motions bring about the increase of average impact number in a motion period. Whether large or small μ_m , these motions have the negative effects on the vibrating system associated with small clearances and low exciting frequencies. With increase in δ , large μ_m causes that the existence region of symmetric 1-1-1 motion becomes relatively small in the exciting

frequency range. Two examples associated with $\delta=0.5$ and 1 are provided for analyzing the influence of parameters δ and μ_m on dynamic performance of the system. As for a little larger clearance, the existence region of symmetric 1-1-1 motion in the (ω, μ_m) -parameter plane is divided into two parts by that of 1-0-0 motion, and subharmonic 2-1-1, 3-2-2, 5-3-3, 6-4-4 and chaotic motions, etc., are found to appear in the interval of large μ_m ; see Fig. 26.

The correlative relationship between before-impact velocity $\dot{x}_{1A-}$ and parameters (ω, μ_k) are shown for $\delta=0.2$ in Fig. 21. It is interesting to note that the change in μ_k hardly affects the exciting frequency range of existence regions of 1- p - p fundamental motions ($p > 3$) and complete chattering-impact motions associated with small δ . The existence region of symmetric 1-1-1 motion is almost invariant in a certain range of exciting frequency as μ_k is not large enough, and the impact velocity of the motion is basically constant whether μ_k increases or decreases for the same exciting frequency. The range of exciting frequency of asymmetric 1-1-1 motion slightly increases as μ_k is large enough. As for a little larger clearance δ , small μ_k causes that the existence region of symmetric 1-1-1 motion becomes narrow in the exciting frequency range, and subharmonic 2-1-1 motion is found to exist in the range of $\mu_k \in (0.05, 0.2377)$; the existence region of symmetric 1-1-1 motion, associated with large μ_k , is still wide in the exciting frequency range, as seen in Fig. 27. Dynamic performance of the

Fig. 31. Existence regions of different types of periodic-impact motions for the periodically-forced system with a clearance in the (ω, R) -parameter plane: (a) $\delta=0.5$; (b) $\delta=1.0$.

system can be improved significantly by selecting slightly larger stiffness distribution μ_k .

The correlative relationship between before-impact velocity $\dot{x}_{1A-}$ and parameters (ω, μ_c) are shown for $\delta=0.2$ in Fig. 22. As the damping distribution μ_c is not large enough, the existence regions of $1-p-p$ motions ($p \geq 3$) and complete chattering-impact motions are almost invariant in ω interval. The range of exciting frequency of symmetric $1-1-1$ motion is almost invariant for μ_c , and impact velocity of the motion is basically constant for the same exciting frequency with change in μ_c . The range of exciting frequency of asymmetric $1-1-1$ motion slightly increases as μ_c is large enough. As for small clearance, small damping distribution μ_c hardly affects dynamics of the system, as seen in Fig. 22. As for a little larger clearance δ , small μ_c causes that the existence region of symmetric $1-1-1$ motion becomes narrow in the exciting frequency range and the system exhibits subharmonic $3-2-2$ and $5-3-3$ motions in a little higher exciting frequency interval; see Fig. 28.

The correlative relationship between impact velocity $\dot{x}_{1A-}$ and parameters (ω, ζ) is shown for $\delta=0.2$ in Fig. 23. With increase in ζ , impact velocity of $1-1-1$ motion, corresponding to the same and relatively small exciting frequency, slightly attenuates at the outset to the lower bound and after that begins to slightly increase. As the exciting frequency keeps invariant and is higher than some value, impact velocity of $1-1-1$ motion attenuates gradually with increasing ζ . It is important to note that the function curve $\dot{x}_{1A-}(\omega)$ of symmetric $1-1-1$ motion is divided into two sections in the parameter range associated with large ζ and higher ω . The section of the function curve $\dot{x}_{1A-}(\omega)$ of symmetric $1-1-1$ motion, in low exciting frequency range, is higher than the other in high exciting frequency interval, both are mainly separated by the windows of quasi-periodic, chaotic and impactless motions. There exists Neimark-Sacker bifurcation of symmetric $1-1-1$ motion as ζ is very small or large enough. As for large ζ , impact velocity of symmetric $1-1-1$ motion is very small in certain a range of high exciting frequency; see Fig. 23. The correlative relationship between dynamic performance and parameters (ω, ζ) are shown for $\delta=0.5$ and 1 in Fig. 29. Small ζ badly affects dynamics of the system, slightly large ζ leads to more energy consumption of the system to reduce its use efficiency. Based on comprehensive consideration of large existence region and low impact velocity of $1-1-1$ motion, the damping ratio range $\zeta \in [0.1, 0.2]$ is feasible and effective to the operation reliability of the system.

The correlative relationship between before-impact velocity $\dot{x}_{1A-}$ and parameters (ω, f_{20}) are shown for $\delta=0.2$ in Fig. 24. The change in f_{20} mainly affects the existence regions of $1-1-1$

motions associated with small δ in high exciting frequency range. As f_{20} is large enough, the existence region of $1-1-1$ motion is very narrow for small δ in the (ω, f_{20}) -parameter plane, as seen in Figs. 17 and 24 and Fig. 30(a). With increase in parameter f_{20} , impact velocity of symmetric $1-1-1$ motion, corresponding to the same exciting frequency, attenuates gradually. As for invariant exciting frequency, the impact strength of complete chattering-impact motions can enhance gradually with increasing f_{20} . As for small clearances, large f_{20} causes that the existence region of symmetric $1-1-1$ motion becomes narrower in the exciting frequency range; see Fig. 17 and Fig. 30(a).

The correlative relationship between before-impact velocity $\dot{x}_{1A-}$ and parameters (ω, R) is shown for $\delta=0.2$ in Fig. 25. As the restitution coefficient R of impact is not large enough, the system exhibits complete chattering-impact motions $(1-\bar{p}-\bar{p})$ in a large exciting frequency interval due to fast attenuation in impact velocities, though the first impact velocities of these motions in an exciting period are very high. Furthermore, the existence regions of $1-p-p$ motions ($p \geq 3$), associated with small R , are too narrow in ω interval to be observed. It is important to note that the highest branches of function curves $\dot{x}_{1A-}(\omega)$ of $1-\bar{p}-\bar{p}$ motions, in certain an interval of ω , are possibly higher than the whole curve $\dot{x}_{1A-}(\omega)$ of $1-1-1$ motion for small R . With increase in the restitution coefficient R , the existence region of complete chattering-impact motions is gradually diminished in the ω interval and those of $1-p-p$ motions accordingly expand, and impact velocity of $1-1-1$ motion, corresponding to the same and relatively small exciting frequency, slightly increases at the outset to the upper bound and after that begins to slightly attenuate. However, as the exciting frequency is higher than some value and keeps invariant, impact velocity of $1-1-1$ motion increases gradually with increasing R . As for small R , the function curves $\dot{x}_{1A-}(\omega)$ of impact velocity of symmetric $1-1-1$ motion attenuate gradually with increasing the exciting frequency ω . The case is contrary after R is larger than some value, the function curves $\dot{x}_{1A-}(\omega)$ of symmetric $1-1-1$ motion increases progressively with increasing ω . As R is not large enough, some narrow windows of $1-1-1$ motions exist in high exciting frequency range, which are isolated by other periodic-impact or chaotic motions. As for a little larger clearance δ , small R leads to narrower existence region of symmetric $1-1-1$ motion in the exciting frequency range; see Fig. 31. Some characteristics of complete chattering-impact motions must be emphasized. As R is small enough, the impact strength is markedly attenuated, the impact sequence of $1-\bar{p}-\bar{p}$ motion decays quickly in the motion period and becomes not obvious in visual sensation and

quantity (but the chatter sequence is still infinite), and the sticking time is correspondingly longer. Conversely, for large restitution coefficient R , the impact sequence of $1-\bar{p}-\bar{p}$ motion is also more obvious in the motion period and the sticking time becomes correspondingly shorter due to slow attenuation in impacts. The effect, i.e., large impact velocities and increase of average impact number in a motion period, badly intensifies abrasion and pitting fatigue between the restrained mass and clearances and increase noise levels. It is valuable to avoid the existence regions of complete and incomplete chattering-impact motions and $1-p-p$ motions ($p \geq 2$) in parameter design as much as possible to improve service life and operation reliability of vibrating systems with clearances or constraints.

6. Conclusions

In this paper we consider a two-degree-of-freedom periodically-forced system with a clearance represented by two symmetric rigid stops. The correlative relationship between dynamic performance and system parameters is analyzed to find their reasonable matching law. Two key parameters of the system, the dimensionless excitation frequency ω and clearance δ , are emphasized to analyze the influence of the main factors on dynamics of the system.

The system usually exhibits $1-p-p$ motions ($p \geq 1$) and complete chattering-impact motions in low exciting frequency or small clearance range. As the clearance δ is small or small enough, the transition between adjacent fundamental motions basically goes through three processes with decrease in ω , which are epitomized as follows: pitchfork bifurcation of symmetric $1-p-p$ motion; period-doubling bifurcation of asymmetric $1-p-p$ motion; non-periodic or chaotic motions caused by a succession of period-doubling bifurcations; symmetric $1-(p+1)-(p+1)$ motion generated by a degeneration of chaos. Both $1-p-p$ and $1-(p+1)-(p+1)$ motions are separated by a region including other periodic-impact and chaotic motions, in which only the zones of $3-(p+1)-(p+1)$ and $1-(p+1)-p$ motions are obvious enough to be observed.

For slightly large clearance δ , the transition from $1-p-p$ to $1-(p+1)-(p+1)$ motion arises with decreasing the exciting frequency ω up to the grazing bifurcation boundary G_{1-p-p} of $1-p-p$ motion. A series of grazing bifurcations occur with decreasing ω so that the number p of impacts of the fundamental group of motions increases two by two. As p becomes big enough, $1-p-p$ motion will exhibit incomplete chattering-impact characteristics, which is the so-called an incomplete chattering sequence in an excitation period followed by a finite sequence of impacts with successively reduced velocity. Complete chattering-impact motion with sticking will occur with decreasing ω up to the sliding bifurcation boundary. Transition from $1-(p+1)-(p+1)$ to $1-p-p$ motion occurs with increasing ω up to the saddle-node bifurcation boundary $SN_{1-(p+1)-(p+1)}$ of $1-(p+1)-(p+1)$ motion. The transition between neighbor fundamental motions $1-p-p$ and $1-(p+1)-(p+1)$ is generally non-reversible. There exist a series of singular points on the boundaries between the existence regions of $1-p-p$ and $1-(p+1)-(p+1)$ motions, at which different saddle-node bifurcation boundaries of $1-(p+1)-(p+1)$ motion, real-grazing and bare-grazing boundaries of $1-p-p$ motion alternately and mutually intersect. Consequently, two types of transition regions (hysteresis and tongue-shaped regions) are created, by which neighboring fundamental impact motions are separated. The hysteresis regions are limited at the upper bound by the saddle-node bifurcation boundaries $SN_{1-(p+1)-(p+1)}^{1-p-p}$ of $1-(p+1)-(p+1)$ motion and at the lower bound by the real-grazing boundaries G_{1-p-p} of $1-p-p$ motion. The tongue-shaped regions are limited at the upper bound by the bare-grazing boundary G_{1-p-p}^b of $1-p-p$ motion and at the lower bound by the saddle-node bifurcation boundary $SN_{1-(p+1)-(p+1)}^{1-(p+1)-p}$ of $1-(p+1)-(p+1)$ motion. A series of

zones of fundamental $1-(p+1)-p$, subharmonic $3-(3p+1)-(3p+1)$ and $2-(2p+1)-2p$ impact orbits ($p \geq 0$) dominate in the tongue-shaped regions between adjacent regions of $1-p-p$ and $1-(p+1)-(p+1)$ motions.

As for large clearance δ , the system exhibits mainly $1-0-0$ or symmetric $1-1-1$ motion in most regions of the (ω, δ) -parameter plane except small or narrow tongue-shaped zones nearby the critical line $L_0: R_{1-0-0} \cap R_{1-1-1}$, in which there exist narrower zones of fundamental $1-1-0$ (or $1-0-1$), subharmonic $3-1-1$ and $2-1-0$ (or $2-0-1$) motions.

The system exhibits more complex dynamical characteristics as μ_m becomes large enough. Consequently, more types of periodic-impact motions appear and the existence region of complete chattering-impact motions is extended to larger clearance interval. It is possible for more complex sequences of subharmonic impact motions to occur in large μ_m case, such as $2-1-1$, $3-1-1$, $3-2-2$ and $5-3-3$, etc. As for small μ_k , the existence regions of $1-p-p$ motions ($p \geq 2$) are extended to larger clearance interval and the existence regions of symmetric $1-1-1$ motion are divided into two parts. Many complex sequences of subharmonic impact motions are found to exist in small μ_k case. The increase of stiffness parameter μ_k brings significant expansion in the existence region of symmetric $1-1-1$ motion and obvious shrinkage in the existence regions of $1-p-p$ ($p \geq 2$) and complete chattering-impact motions. As the damping distribution μ_c is not large enough, its variation hardly affects the distribution and existence regions of $1-p-p$ and complete chattering-impact motions. As for large μ_c , the exciting frequency range of symmetric $1-1-1$ motion, associated with small clearance interval, becomes correspondingly large. The serious deformation of regions of $1-p-p$ motions generally occurs in enough small ζ case. As a result, the existence regions of these motions mutually interweave to cause block segmentation which makes the regions of fundamental $1-p-p$ motions ($p \geq 2$) present irregular distribution characteristics. Small ζ badly affects dynamical performance of the system, slightly large ζ leads to more energy consumption of the system to reduce its use efficiency. Based on comprehensive consideration of large existence region and low impact velocity of $1-1-1$ motion, the damping ratio range $\zeta \in [0.1, 0.2]$ is feasible and effective to the operation reliability of the system. Variation of the exciting force f_{20} has an effect on the impact velocities and existence regions of $1-p-p$ motions and complete chattering-impact ones, but it hardly changes general forms of a series of periodic-impact motions. No anomaly type of periodic-impact motions occurs even for the extreme case of $f_{20}=0$ or 1. As the restitution coefficient R of impact is small enough, the existence region of complete chattering-impact motions is very large so that the regions of $1-1-1$ and $1-p-p$ ($p \geq 2$) motions are badly extruded and substantially reduced. The restitution coefficient of impact brings also great influences on the number of impacts and the sticking time of chattering motion in a motion period. As for large R , the impact sequence of chattering motion becomes more obvious in the motion period and the sticking time is correspondingly shorter due to slow attenuation in impacts. Conversely, as R is relatively small, the impact strength is markedly attenuated so that the impact sequence of chattering motion decays quickly in the motion period and the sticking time becomes correspondingly longer.

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