Infinite sized magic square

Antoine Salomon

Infinite sized magic square

Antoine Salomon*

Abstract. We investigate a generalisation of the definition of magic squares, which allows for infinite sized square grids. If the grid is $[0,1]^2$, a magic square will be a bijection C from $[0,1]^2$ to $[0,1]$ such that integrals $\int_0^1 C(x,y)dx$ (resp. $\int_0^1 C(x,y)dx$) are constant w.r.t. y (resp. x). We show that such objects exist by means of an explicit construction, and discuss some of their properties. We also propose explicit constructions of magic squares when the grid is $\mathbb{R}^2$ or some countably infinite set.

Introduction Magic square is one of the most famous recreational mathematical object. For a given size $n \geq 1$, it consists in an arrangement of integers 1, 2,..., n^2 in a $n \times n$ grid such that the sums of numbers of each row, column and diagonal are all equal. It can be dated back to ancient China [**7**], and still spawns some publications, for example in computer science [**2**]. Many variations have been imagined such as magical rectangles [**4**], magic squares labelled with integers or real numbers which are not 1,...,n^2 [**5**], etc... Yet, as far we know, no generalisation to arbitrary, possibly infinite, square grids has ever been studied. Let us formalize such a generalisation.

Definition. Let $(A, \mathcal{A}, \mu)$ be a measured space and let B be a subset of $\mathbb{R}$. A magic square from A to B is a bijection C from A^2 to B such that $x \mapsto C(x,y)$ (resp. $y \mapsto C(x,y)$) is integrable for all y (resp. x) and such that there exist $\alpha, \beta \in \mathbb{R}$ with:

- $\forall x \in A,\ \int_A C(x,y)d\mu(y) = \alpha,$
- $\forall y \in A,\ \int_A C(x,y)d\mu(x) = \beta.$

Note that we do not consider diagonal sums. Indeed, we could require the integral $\int_A C(x,x)d\mu(x)$ to equal a certain constant, but there may be no obvious definition for the "second" diagonal (e.g. if $A = \mathbb{N}$ or $\mathbb{R}_+$). Moreover, even if it is not difficult to define the second diagonal sum if $A = [0,1]$ or $\mathbb{R}$, we will see that such considerations are not really interesting.

This definition underlies all the objects we will study in this article. Thus, what will be called a magic square of order n is a magic square from $\{0,\dots,n-1\}$ to $\{0,\dots,n^2-1\}$, and the corresponding measure is the counting measure. Up to a shift in the numbering (the reason of which will appear later), this is equivalent to a "classical" magic square (which is rather numbered from 1 to n^2). A magic square on $[0,1]$ (resp. $\mathbb{R}$) will be a magic square from $[0,1]$ to $[0,1]$ (resp. $\mathbb{R}$ to $\mathbb{R}$), and the corresponding measure is the Lebesgue measure. Also, a magic square on a countably infinite set $\mathcal{D} \subset \mathbb{R}$ is a magic square from $\mathcal{D}$ to $\mathcal{D}$, and the corresponding measure is the counting measure. Yet, as we do not want to be overly abstract and formal, each result will be formulated elementarily without directly referring to the former definition, and in some cases we will also give particular and simpler definitions.

Constant α will be referred to as the column sum and β as the row sum. When $\int_A \int_A |C(x,y)|d\mu(x)d\mu(y)$ and $\mu(A)$ are finite, Fubini's theorem implies that $\alpha = \beta = \frac{1}{\mu(A)} \int_A \int_A C(x,y)d\mu(x)d\mu(y)$. This always holds for magic squares of order n and for magic squares on $[0,1]$, but α may be different from β in other cases.

*LEDa, Université Paris-Dauphine. E-mail: antoine.salomon@dauphine.fr

Let us give the idea behind our main result, i.e. the construction of a magic square on $[0,1]$. This is based on decomposition of real numbers under a base $N \geq 2$: it is well known that every real of $x \in [0,1]$ can be written as $0.a_1a_2\ldots a_k\ldots$, where $a_k \in \{0,\ldots,N-1\}$ for all k, and this means that $x = \sum_{k\geq 1} a_k N^{-k}$. A magic square on $[0,1]$ can be described as follows: the value at a given point $(x_0,y_0) \in [0,1]^2$ is $0.a_1a_2\ldots a_k\ldots$ in a base of the form $N = n^2$, where a_k is the integer of $\{0,\ldots,n^2-1\}$ spotted by (x_0,y_0) when $[0,1]^2$ is regularly split into $n^{2(k-1)}$ pieces so that a copy of a given magic square of order n (numbered from 0 to n^2-1) is "printed" on each piece. For example, in Figure 1 the value at (x_0,y_0) of the resulting magic square on $[0,1]$ is $0.128\ldots$ in base nine, i.e. $1.9^{-1} + 2.9^{-2} + 8.9^{-3} + \ldots$

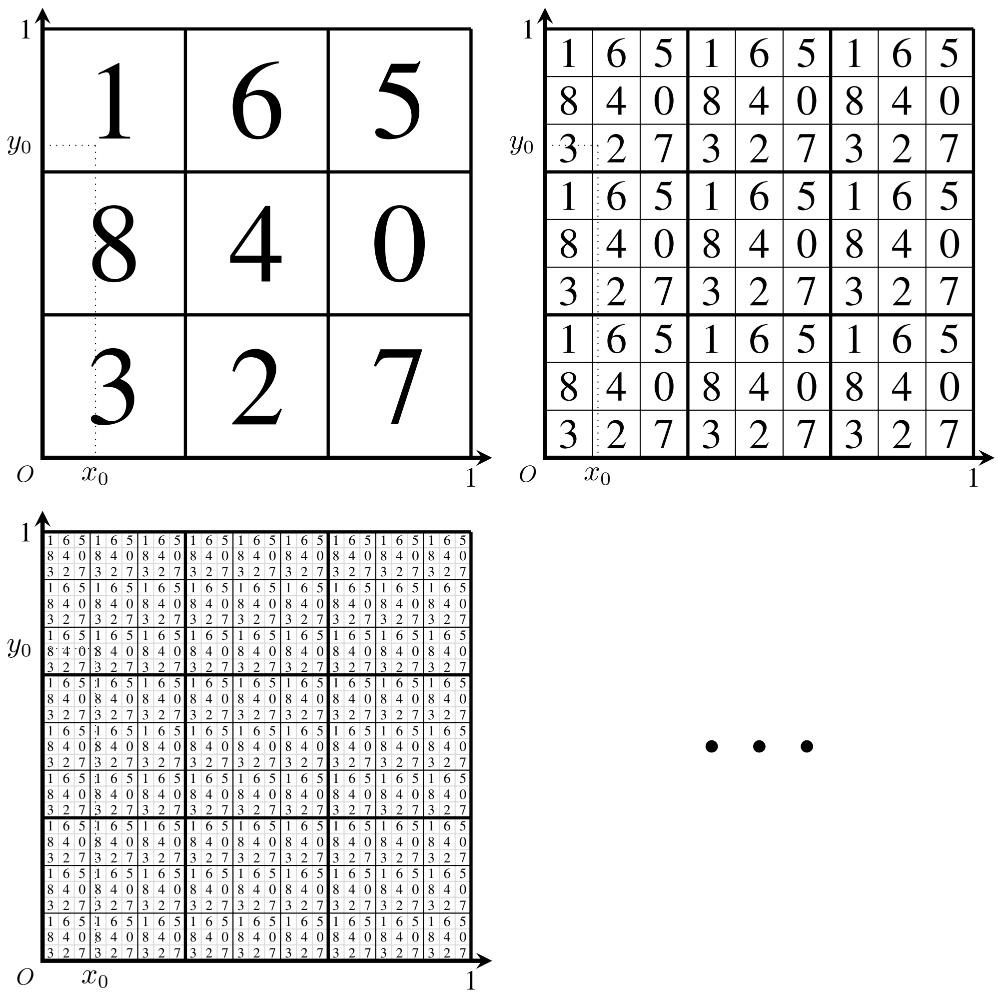

Figure 1. Illustration of main idea for $n = 3$.

The constance of column and row sums results from the corresponding property of the magic square of order n. It is a surjection because every sequence $a_1, a_2, \ldots, a_k, \ldots$ can be reached, and an injection because a_1 reveals in which of the first n^2 parts of $[0,1]^2$ (x_0,y_0) lies, then a_2 reveals in which of the n^2 parts of the previous part

(x_0, y_0) lies, etc... Actually, this is not entirely correct because some real numbers have two possible decompositions in a given base (e.g. in base ten 1=0.999...). A significant part of the proof will be devoted to patching this difficulty. Note also that the "printed" magic square of order n need not be the same at every stage (and it will not in our formal construction), and we could even choose every "printed" magic square at any stage arbitrarily.
We will show that this construction ensures that the values of the corresponding magic square on $[0, 1]$ are uniformly distributed on $[0, 1]$, though this property does not hold for every magic squares on $[0, 1]$. Also, our construction of magic squares on $\mathbb{R}$ basically consists in a tessellation of $\mathbb{R}^2$ with modified versions of the previously constructed a magic square on $[0, 1]$. However, the existence of magic squares on countably infinite sets is an independent result, and the corresponding construction is mostly based on manipulations of numerical series.

The outline of the article is the following. First section is devoted to preliminary basic results about decomposition of real numbers in a base n and about "classical" magic squares. In a second section, we show the existence of magic squares on $[0, 1]$ with an explicit method, and discuss their properties. Then, we construct magic squares on $\mathbb{R}$ and on any countably infinite set dense in $[0, 1]$.
Throughout the paper, the notation $[x]$ stands for the integer part of the real x and $n \; mod \; m$ is the remainder of the euclidean division of the integer n by the integer m.

1. NOTATIONS AND PRELIMINARY RESULTS

Decomposition of a real number in an arbitrary base. Let us fix an integer $n \geq 2$. It is well known that every natural number m may be written as a sum $m = \sum_{\ell \in \mathbb{N}} x_\ell n^\ell$ with $x_l \in \{0, 1, \ldots, n-1\}$, and that this decomposition is unique. Let us recall the form of this decomposition for real numbers $x \in [0, 1)$.
Every real $x \in (0, 1)$ can be decomposed into a sum $x = \sum_{\ell \geq 1} \frac{x_\ell}{n^\ell}$ with $x_\ell \in \{0, 1, \ldots, n-1\}$: the equality holds with $x_\ell = [n^\ell x] \; mod \; n$. Yet, the decomposition is not unique iff there exists an integer $\ell \geq 1$ such that $n^\ell x \in \mathbb{N}$. Precisely, if such an ℓ exists there are exactly two possibilities:

- either $x_\ell = [n^\ell x] \; mod \; n$ for every $\ell \geq 1$,
- or $x_\ell = [n^\ell x] \; mod \; n$ for all $\ell \in \{0, 1, \ldots, \ell_0 - 1\}$, $x_{\ell_0} = ([n^{\ell_0} x] \; mod \; n) - 1$, and $x_\ell = n - 1$ for all $\ell \geq \ell_0 + 1$, where ℓ_0 is the smallest integer $\ell \geq 1$ such that $n^\ell x \in \mathbb{N}$.

Thus, in this case, the sequence $(x_\ell)_{\ell \geq 1}$ is stationary and the corresponding limit is either 0 or $n - 1$.
Let us denote N_n the set of the real numbers whose decomposition in base n is not unique.

Definition. $N_n := \{x \in [0, 1) | \, \exists \ell \geq 1, \; n^\ell x \in \mathbb{N}\}$.

As an example, N_{10} is the set of decimal numbers of the interval $[0, 1)$.
Note also that, if $x = 0$, we have $x = \sum_{\ell=1}^{+\infty} \frac{0}{n^\ell}$ and this decomposition is unique.

Conversely, every sequence valued in $\{0, 1, \ldots, n-1\}$ can be written as $([n^\ell x] \; mod \; n)_{\ell \geq 1}$ for a given $x \in [0, 1)$, except for stationary sequences whose limit value is $n - 1$.
Let us formalize all these classical results in the following definitions and lemmas.

Definition. Let k be an integer of the set $\{0, 1, \ldots, n-1\}$. Let us denote by $D_{k,n}$ the set of stationary sequences valued in $\{0, 1, \ldots, n-1\}$ and whose limit is k:

$$D_{k,n} := \left\{(x_\ell)_{\ell \geq 1} \in \{0, 1, \ldots, n-1\}^{\mathbb{N}^*} | \exists m \geq 1,\ \forall \ell \geq m,\ x_\ell = k\right\}.$$

Also, the set of set of sequences valued in $\{0, 1, \ldots, n-1\}$ which are not stationary with limit $n-1$ is denoted by E_n:

$$E_n := \{0, 1, \ldots, n-1\}^{\mathbb{N}^*} \setminus D_{n-1,n}.$$

Definition. Let us define the following functions:

$$d_n : \begin{array}{ccc} [0,1) & \to & \{0, 1, \ldots, n-1\}^{\mathbb{N}^*} \\ x & \mapsto & ([n^\ell x] \, mod \, n)_{\ell \geq 1} \end{array},$$

$$\mathcal{S}_n : \begin{array}{ccc} \{0, 1, \ldots, n-1\}^{\mathbb{N}^*} & \to & [0,1] \\ (x_\ell)_{\ell \geq 1} & \mapsto & \sum_{\ell \geq 1} \frac{x_\ell}{n^\ell} \end{array}.$$

Proposition 1. The function d_n is a bijection from $[0, 1)$ to E_n such that

- $d_n(N_n) = D_{0,n}$,
- $d_n([0, 1[\setminus N_n) = \{0, 1, \ldots, n-1\}^{\mathbb{N}^*} \setminus (D_{0,n} \cup D_{n-1,n})$,

and the corresponding inverse function (from E_n to $[0, 1)$) coincides with $\mathcal{S}_n$.

Let us finish this section with an elementary property.

Proposition 2. Let $x, y \in [0, 1)$ and $n \geq 1$. One has:

$x < y \Leftrightarrow \exists \ell \geq 1,\ \forall m \in \{1, \ldots, \ell-1\},\ d_n(x)_m = d_n(y)_m$ and $d_n(x)_\ell < d_n(y)_\ell$.

Magic squares Let us first recall the usual definition of (finite) magic squares.

Definition. A magic square of order n is a bijection c from $\{0, 1, \ldots, n-1\}^2$ to $\{0, 1, \ldots, n^2-1\}$ such that:

- $\forall i, i' \in \{0, 1, \ldots, n-1\},\ \sum_{j=0}^{n-1} c(i, j) = \sum_{j=0}^{n-1} c(i', j)$,
- $\forall j, j' \in \{0, 1, \ldots, n-1\},\ \sum_{i=0}^{n-1} c(i, j) = \sum_{i=0}^{n-1} c(i, j')$.

As already mentioned in the introduction, magic squares are here numbered from 0 to $n^2 - 1$ for convenience, whereas the classical convention dictates to number from 1 to n^2. Also, row sums and column sums are always equal. The corresponding constant does not depend on the magic square of order n considered, and is often called the magic constant.

Proposition 3. For all magic square c of order n, one has:

$$\forall i_0, j_0 \in \{0, 1, \ldots, n-1\},\ \sum_{j=0}^{n-1} c(i_0, j) = \sum_{i=0}^{n-1} c(i, j_0) = \frac{n(n^2-1)}{2}.$$

To give an idea of how we will prove the existence of magic squares on $[0,1]$ in the next section, let us describe a simple construction of high order magic squares.
In the lemma below, we describe a classical way to construct a magic square of order mn from two magic squares of respective orders n and m (see Figure 2 for an example).

Lemma 1. Let c_1 be a magic square of order n and c_2 a magic square of order m. The function c defined by

$$c(i,j) := c_1(q_i, q_j)m^2 + c_2(r_i, r_j),$$

where $i = q_i m + r_i$ are $j = q_j m + r_j$ are respectively the euclidean division of i and j by m, is a magic square of order mn.

This elementary result gives a simple method to show that there exist magic squares of arbitrarily high orders. Yet, one has to use more sophisticated methods to show that there exist magic squares of any order $n \geq 3$, such as De la Loubère method [**3**] (for odd orders), W.S. Andrews method [**1**] (for orders of the form $4n$), or Strachey method [**6**] (for order of the form $4n+2$).
Moreover, when this lemma is iterated, a link with the decomposition of integers appears.

Corollary 1. Let $c_0, c_1, \ldots, c_{k-1}$ be k magic squares of order n. The function $\mathfrak{C}_k(c_0, c_1, \ldots, c_{k-1})$ defined by

$$\mathfrak{C}_k(c_0, c_1, \ldots, c_{k-1})(i,j) := \sum_{\ell=0}^{k-1} c_l(i_\ell, j_\ell) n^{2\ell},$$

where $i, j \in \{0, 1, \ldots, n^k - 1\}$, and where $i_0, j_0, i_1, j_1, \ldots, i_{k-1}, j_{k-1}$ are the unique integers of $\{0, 1, \ldots, n-1\}$ such that $i = \sum_{\ell=0}^{k-1} i_\ell n^\ell$ and $j = \sum_{\ell=0}^{k-1} j_\ell n^\ell$, is a magic square of order n^k.

The values taken by the magic squares which result from this method are expressed in base n^2, and the corresponding algorithm has to decompose i and j in base n. In the next section, we will adapt this method to the decomposition of real numbers of $[0,1)$ in a certain base n.

2. MAGIC SQUARES ON $[0,1]$ In this section, n is an integer greater than 3 and $(c_k)_{k\geq 1}$ is an arbitrary sequence of magic squares of order n.

Formal definition For those who have in mind the definition in the introduction, the object we will study in this section is a magic square from $[0,1]$ to $[0,1]$ (w.r.t Lebesgue measure). Let us recall what it formally means.

Definition. A magic square on $[0,1]$ is a measurable bijection C from $[0,1]^2$ to $[0,1]$ such that:

- $\forall x, x' \in [0,1],\ \int_0^1 C(x,y)dy = \int_0^1 C(x',y)dy,$
- $\forall y, y' \in [0,1],\ \int_0^1 C(x,y)dx = \int_0^1 C(x,y')dy.$

As mentioned in the introduction, it easily follows from Fubini's theorem that row sums and column sums are equal, i.e. $\int_0^1 C(x,y_0)dx = \int_0^1 C(x_0,y)dy$ for all magic

3	**8**	**1**
2	**4**	**6**
7	**0**	**5**

1	6	5
8	4	0
3	2	7

3·9+1	**3**·9+6	**3**·9+5	**8**·9+1	**8**·9+6	**8**·9+5	**1**·9+1	**1**·9+6	**1**·9+5
3·9+8	**3**·9+4	**3**·9+0	**8**·9+8	**8**·9+4	**8**·9+0	**1**·9+8	**1**·9+4	**1**·9+0
3·9+3	**3**·9+2	**3**·9+7	**8**·9+3	**8**·9+2	**8**·9+7	**1**·9+3	**1**·9+2	**1**·9+7
2·9+1	**2**·9+6	**2**·9+5	**4**·9+1	**4**·9+6	**4**·9+5	**6**·9+1	**6**·9+6	**6**·9+5
2·9+8	**2**·9+4	**2**·9+0	**4**·9+8	**4**·9+4	**4**·9+0	**6**·9+8	**6**·9+4	**6**·9+0
2·9+3	**2**·9+2	**2**·9+7	**4**·9+3	**4**·9+2	**4**·9+7	**6**·9+3	**6**·9+2	**6**·9+7
7·9+1	**7**·9+6	**7**·9+5	**0**·9+1	**0**·9+6	**0**·9+5	**5**·9+1	**5**·9+6	**5**·9+5
7·9+8	**7**·9+4	**7**·9+0	**0**·9+8	**0**·9+4	**0**·9+0	**5**·9+8	**5**·9+4	**5**·9+0
7·9+3	**7**·9+2	**7**·9+7	**0**·9+3	**0**·9+2	**0**·9+7	**5**·9+3	**5**·9+2	**5**·9+7

28	33	32	73	78	77	10	15	14
35	31	27	80	76	72	17	13	9
30	29	34	75	74	79	12	11	16
19	24	23	37	42	41	55	60	59
26	22	18	44	40	36	62	58	54
21	20	25	39	38	43	57	56	61
64	69	68	1	6	5	46	51	50
71	67	63	8	4	0	53	49	45
66	65	70	3	2	7	48	47	52

Figure 2. An example of Lemma 1, for $n = m = 3$ (or of Corollary 1, for $k = 2$ and $n = 3$).

square on $[0, 1]$ and all $x_0, y_0 \in [0, 1]$. Yet, we will see that there is no "magic constant" as in the finite case, which means that the value of the former integrals depends on C (cf. Proposition 5).

As announced in the former section, we are going to construct a magic square on $[0, 1]$ thanks to an adaptation of Corollary 1. The main idea is to consider the function[1]

$$\begin{array}{ccc} [0,1)^2 & \to & [0,1] \\ (x,y) & \mapsto & \sum_{\ell\geq 1} c_\ell(d_n(x)_\ell, d_n(y)_\ell) n^{-2\ell}. \end{array}$$

Yet its domain of definition is not the entire set $[0, 1]^2$, and it is not a bijection. So, our goal will be to construct a bijection from $[0, 1]^2$ to $[0, 1]$ that coincides with it almost everywhere.

Construction of a magic square on $[0, 1]$ Let us first modify d_n into a bijection from $[0, 1]$ to $\{0, 1, \ldots, n-1\}^{\mathbb{N}^*}$.

Lemma 2. There exists a measurable bijection d'_n from $[0, 1]$ to $\{0, 1, \ldots, n-1\}^{\mathbb{N}^*}$ which coincides almost everywhere with d_n.

Proof. The set N_n is countably infinite, and so are the sets $D_{k,n}$. Thus, sets $N_n \cup \{1\}$ and $D_{0,n} \cup D_{n-1,n}$ are both countably infinite, and there exists a bijection σ_n from

[1]As in the example in the introduction, the construction could be simpler and only use one magic square of order n, rather than a sequence $(c_k)_{k\geq 1}$. On the contrary, it could be more general and be based on a sequence $(c_{k,\ell,m})_{k\geq 1, 0\leq \ell\leq n^{k-1}-1, 0\leq m\leq n^{k-1}-1}$, so that the corresponding expression would be $\sum_{\ell\geq 1} c_{\ell,[n^{\ell-1}x],[n^{\ell-1}y]}(d_n(x)_\ell, d_n(y)_\ell)n^{-2\ell}$. None of these changes significantly affect our proofs.

$N_n \cup \{1\}$ to $D_{0,n} \cup D_{n-1,n}$.[2]
Let us then define the function d'_n as follows:

$$d'_n : \begin{array}{ccc} [0,1] & \to & \{0,1,\ldots,n-1\}^{\mathbb{N}^*} \\ x & \mapsto & \left\{ \begin{array}{ll} d_n(x) & \text{if } x \in [0,1) \setminus N_n \\ \sigma_n(x) & \text{if } x \in N_n \cup \{1\} \end{array} \right. \end{array} .$$

Thanks to Proposition 1, d'_n is bijective, and it is also measurable by construction. It coincides with d_n on $[0,1) \setminus N_n$, hence almost everywhere because $N_n \cup \{1\}$ is countable.

■

Lemma 3. Let $\mathfrak{C}_0$ denote the function

$$\begin{array}{ccc} [0,1]^2 & \to & [0,1] \\ (x,y) & \mapsto & \sum_{\ell \geq 1} c_\ell(d'_n(x)_\ell, d'_n(y)_\ell) n^{-2\ell}. \end{array}$$

There exists a measurable bijection $\mathfrak{C}$ from $[0,1]^2$ to $[0,1]$ such that, for all $y \in [0,1]$ (resp. $x \in [0,1]$), $x \mapsto \mathfrak{C}(x,y)$ coincides with $x \mapsto \mathfrak{C}_0(x,y)$ (resp. $y \mapsto \mathfrak{C}_0(x,y)$) almost everywhere.

Proof. Thanks to Lemma 2, it is easy to construct a bijection r_n from $[0,1]^2$ to $\{0,1,\ldots,n-1\}^{\mathbb{N}^*} \times \{0,1,\ldots,n-1\}^{\mathbb{N}^*}$, by simply setting:

$$r_n(x,y) := (d'_n(x), d'_n(y)).$$

Let us also define the following function:

$$F_n : \begin{array}{ccc} \{0,1,\ldots,n-1\}^{\mathbb{N}^*} \times \{0,1,\ldots,n-1\}^{\mathbb{N}^*} & \to & \{0,1,\ldots,n^2-1\}^{\mathbb{N}^*} \\ ((x_\ell)_{\ell\geq 1}, (y_\ell)_{\ell \geq 1}) & \mapsto & (c_\ell(x_\ell, y_\ell))_{\ell \geq 1} \end{array} .$$

As functions c_ℓ are bijective, F_n is also bijective.
The composed function $\mathcal{S}_{n^2} \circ F_n \circ r_n$ then coincides with $\mathfrak{C}_0$ (everywhere), but is not bijective because the image of F_n is not restricted to E_{n^2} (cf. Proposition 1). Moreover, the image of $\mathfrak{C}$ has to be the entire set $[0,1]$ (and not $[0,1)$, as in Proposition 1). Consequently, we are going to define $\mathfrak{C}$ as a slight modification of $\mathcal{S}_{n^2} \circ F_n \circ r_n$, as follows.
First, let us remark that the constant sequence $(n-1)_{\ell \geq 1}$ is the unique sequence of $\{0,1,\ldots,n-1\}^{\mathbb{N}^*}$ whose image by $\mathcal{S}_n$ is 1. Consequently, $\mathcal{S}_n$ can be seen as bijection from $E_n \cup \{(n-1)_{\ell \geq 1}\}$ to $[0,1]$ (cf. Proposition 1). Then, let us consider a bijection γ_n from $D_{0,n} \cup D_{n-1,n}$ to $D_{0,n} \cup \{(n-1)_{\ell\geq 1}\}$, which is made possible by the fact that we are dealing with countably infinite sets. The function b_n defined by

$$b_n : \begin{array}{ccc} \{0,1,\ldots,n-1\}^{\mathbb{N}^*} & \to & E_n \cup \{(n-1)_{\ell \geq 1}\} \\ (x_\ell)_{\ell \geq 1} & \mapsto & \left\{ \begin{array}{ll} \gamma_n((x_\ell)_{\ell\geq 1}) & \text{if } (x_\ell)_{\ell \geq 1} \in D_{0,n} \cup D_{n-1,n} \\ (x_\ell)_{\ell \geq 1} & \text{otherwise} \end{array} \right. \end{array} .$$

is then bijective, and we set $\mathfrak{C} := \mathcal{S}_{n^2} \circ b_{n^2} \circ F_n \circ r_n$. Indeed, with this definition, $\mathfrak{C}$ is measurable by construction, and is also bijective as a composition of bijective functions. Moreover, b_{n^2} and the identity function coincides everywhere but a countable set. Hence the result. ■

[2] It is not difficult to describe explicitly such a bijection. This applies to the whole article: all results can be made entirely explicit, and in particular there is no need for the axiom of choice.

Theorem 1. *There exists a magic square on* $[0,1]$, *which coincides almost everywhere with* $(x,y) \mapsto \sum_{\ell \geq 1} c_\ell(d_n(x)_\ell, d_n(y)_\ell) n^{-2\ell}$.

Proof. Let us show that the function $\mathfrak{C}$ given by Lemma 3 is indeed the magic square on $[0,1]$ we are looking for. Thanks to Lemmas 2 and 3, we only need to prove the constance of row and column sums. One has, for all $x \in [0,1]$:

$$\begin{aligned}
\int_0^1 \mathfrak{C}(x,y)dy &= \int_0^1 \sum_{\ell \geq 1} c_\ell(d'_n(x)_\ell, d'_n(y)_\ell) n^{-2\ell} dy \\
&= \int_0^1 \sum_{\ell \geq 1} c_\ell(d'_n(x)_\ell, d_n(y)_\ell) n^{-2\ell} dy \\
&= \sum_{\ell \geq 1} n^{-2\ell} \int_0^1 c_\ell(d'_n(x)_\ell, d_n(y)_\ell) dy \\
&= \sum_{\ell \geq 1} n^{-2\ell} \int_0^1 c_\ell(d'_n(x)_\ell, [n^\ell y] \ mod \ n) dy \\
&= \sum_{\ell \geq 1} n^{-2\ell} \sum_{j=0}^{n^\ell - 1} \int_{\frac{j}{n^\ell}}^{\frac{j+1}{n^\ell}} c_\ell(d'_n(x)_\ell, [n^\ell y] \ mod \ n) dy \\
&= \sum_{\ell \geq 1} n^{-2\ell} \sum_{j=0}^{n^\ell - 1} \int_{\frac{j}{n^\ell}}^{\frac{j+1}{n^\ell}} c_\ell(d'_n(x)_\ell, j \ mod \ n) dy \\
&= \sum_{\ell \geq 1} n^{-2\ell} \sum_{j=0}^{n^\ell - 1} \frac{1}{n^\ell} c_\ell(d'_n(x)_\ell, j \ mod \ n) \\
&= \sum_{\ell \geq 1} n^{-2\ell} \frac{1}{n^\ell} \sum_{j=0}^{n^\ell - 1} c_\ell(d'_n(x)_\ell, j \ mod \ n) \\
&= \sum_{\ell \geq 1} n^{-2\ell} \frac{1}{n^\ell} n^{\ell - 1} \sum_{j=0}^{n-1} c_\ell(d'_n(x)_\ell, j).
\end{aligned}$$

By Proposition 3, $\sum_{j=0}^{n-1} c_\ell(d'_n(x)_\ell, j) = \frac{n(n^2-1)}{2}$, so that one has:

$$\begin{aligned}
\int_0^1 \mathfrak{C}(x,y)dy &= \sum_{\ell \geq 1} n^{-2\ell} \frac{1}{n^\ell} n^{\ell-1} \frac{n(n^2-1)}{2} = \frac{n^2-1}{2} \sum_{\ell \geq 1} n^{-2\ell} \\
&= \frac{n^2-1}{2} \frac{1}{n^2} \frac{1}{1 - \frac{1}{n^2}} = \frac{1}{2}.
\end{aligned}$$

Similarly, one can show that $\int_0^1 \mathfrak{C}(x,y)dx = \frac{1}{2}$ for all $y \in [0,1]$. ■

Properties

Proposition 4. Let $\mathfrak{C}$ be the magic square given by Theorem 1. The law of $\mathfrak{C}$, viewed as a random variable from $[0,1]^2$ to $[0,1]$ equipped with their respective Lebesgue σ-algebras and measures, is the uniform distribution on $[0,1]$.

Proof. For convenience, we are going to prove the following equivalent formulation of Proposition 4: if the law of some random variable (X,Y) is the uniform distribution on $[0,1]^2$, then the law of $\mathfrak{C}(X,Y)$ is the uniform distribution on $[0,1]$.

First, as $\mathfrak{C}(X,Y) = \sum_{\ell\geq 1} c_\ell(d_n(X)_\ell, d_n(Y)_\ell)n^{-2\ell}$ *a.e.*, one has:

$$d_{n^2}(\mathfrak{C}(X,Y))_\ell = c_\ell(d_n(X)_\ell, d_n(Y)_\ell)\ a.e.$$

Then, let us fix $t \in [0,1[$. By Proposition 2, the event $\{\mathfrak{C}(X,Y) < t\}$ is then (up to a null event) the disjoint union of events (indexed by $\ell \geq 1$)

$$\left(\bigcap_{m\in\{1,\dots,\ell-1\}} \{d_{n^2}(\mathfrak{C}(X,Y))_m = d_{n^2}(t)_m\}\right) \cap \{d_{n^2}(\mathfrak{C}(X,Y))_\ell < d_{n^2}(t)_\ell\}$$

$$= \left(\bigcap_{m\in\{1,\dots,\ell-1\}} \{c_m(d_n(X)_m, d_n(Y)_m) = d_{n^2}(t)_m\}\right) \cap \{c_\ell(d_n(X)_\ell, d_n(Y)_\ell) < d_{n^2}(t)_\ell\},$$

and the latter are themselves the disjoint union of the following events (indexed by $k = 0, 1, \dots, d_{n^2}(t)_\ell - 1$):

$$\left(\bigcap_{m\in\{1,\dots,\ell-1\}} \{c_m(d_n(X)_m, d_n(Y)_m) = d_{n^2}(t)_m\}\right) \cap \{c_\ell(d_n(X)_\ell, d_n(Y)_\ell) = k\}.$$

Functions $c_1, \dots, c_\ell$ being bijective, the former events correspond to fixing the ℓ first values of $d_n(X)$ and $d_n(Y)$. Moreover, one has

$$(X,Y) = \sum_{m\geq 1} (d_n(X)_m, d_n(Y)_m)n^{-m}\ p.p.,$$

so that each of the former event is equivalent to (X,Y) being in a square whose down left corner is

$$(x_0, y_0) = \sum_{m=1}^{\ell-1} c_m^{-1}(d_{n^2}(t)_m)n^{-m} + c_\ell^{-1}(k)n^{-\ell}$$

and whose sides have length $n^{-\ell}$. Consequently, one has:

$$\begin{aligned}\mathbb{P}(\mathfrak{C}(X,Y) < t) &= \sum_{\ell\geq 1} \big|\{0, 1, \dots, d_{n^2}(t)_\ell - 1\}\big|(n^{-\ell})^2 = \sum_{\ell\geq 1} d_{n^2}(t)_\ell n^{-2\ell} \\ &= \mathcal{S}_{n^2}(d_{n^2}(t)) = t.\end{aligned}$$

Hence the result.

∎

We could have proposed an other generalisation of the definition of magic squares, based on this property of uniformity. Thus, a classical magic square would have been viewed as a r.v. that uniformly distributes integers $1, \dots, n^2$ on a $n \times n$ grid rather than as a bijection. Yet, a definition of magic squares on $[0,1]$ that would only require uniformity and constance of row and column sums (and not bijectivity) would have trivial examples, such as

$$C_0 : \begin{array}{ccc} [0,1]^2 & \to & [0,1] \\ (x,y) & \mapsto & \left\{ \begin{array}{cc} x & \text{si } y < 1/2 \\ 1-x & \text{si } y \geq 1/2 \end{array} \right. \end{array}.$$

Moreover, our definition of magic squares on $[0,1]$ does not imply that they distribute their values uniformly on $[0,1]$. This results from the following proposition.

Proposition 5. There exists a magic square $\tilde{\mathfrak{C}}$ on $[0,1]$ such that

$$\int_0^1 \int_0^1 \tilde{\mathfrak{C}}(x,y)dxdy \neq \frac{1}{2}.$$

Note that this proposition also implies that row and column sums do not necessarily equal $\frac{1}{2}$.

Proof. Let us first consider the function $\mathfrak{C}_1$ defined by $\mathfrak{C}_1 := \frac{1}{2}\mathfrak{C}$, where $\mathfrak{C}$ is the magic square given by Theorem 1. It is a bijection from $[0,1]^2$ to $[0,\frac{1}{2}]$ such that $\int_0^1 \mathfrak{C}_1(x_0,y)dy = \frac{1}{4}$ for all $x_0 \in [0,1]$ et $\int_0^1 \mathfrak{C}_1(x,y_0)dx = \frac{1}{4}$ for all $y_0 \in [0,1]$. We can modify $\mathfrak{C}_1$ into a bijection $\tilde{\mathfrak{C}}$ from $[0,1]^2$ to $[0,1]$ without changing the values of the former integrals. Indeed, this can be done by modifying $\mathfrak{C}_1$ on a diagonal of $[0,1]^2$, as follows:

$$\tilde{\mathfrak{C}} : \begin{array}{ccc} [0,1]^2 & \to & [0,1] \\ (x,y) & \mapsto & \left\{ \begin{array}{cc} \mathfrak{C}_1(x,y) & \text{if } x \neq y \\ \mathfrak{C}_1(2x,2y) & \text{if } x = y \text{ and } x \leq \frac{1}{2} \\ x & \text{if } x = y \text{ et } x > \frac{1}{2} \end{array} \right. \end{array}.$$

■

As the function $\tilde{\mathfrak{C}}$ of the former proof equals $\frac{1}{2}\mathfrak{C}$ almost everywhere, $\tilde{\mathfrak{C}}$ distributes its value uniformly over $[0,\frac{1}{2}]$, and one could argue that that there may still be a link between magic squares on $[0,1]$ and uniform distributions. Yet, a more sophisticated construction shows that it is not the case. The idea is the same as the sketch in the introduction, except that the "printed" magic square is a non-uniform one. Figure 3 illustrates this: the values of the corresponding magic square on $[0,1]$ are written as $0.a_1a_2\ldots a_k\ldots$ in base thirteen, the first table gives the values of a_1, the second gives the value of a_2, and the process can be iterated for a_3, a_4, etc... The value of this magic square lies in $[6.13^{-1}, 7.13^{-1})$ iff $a_1 = 6$, so that the corresponding probability equals the area of $[0,\frac{1}{2}] \times [\frac{1}{2},1]$, which is $\frac{1}{4}$. This differs from $7.13^{-1} - 6.13^{-1} = \frac{1}{13}$, therefore the distribution is not uniform.

To finish this section, let us make a comment on diagonal sums, i.e. $\int_0^1 C(x,x)dx$ and $\int_0^1 C(x,1-x)dx$. We did not require these sums to equal a given constant, as usually required with "classical" magic squares. Indeed it would not be interesting: as in the former proof, it is not difficult to manipulate the value of diagonal sums, because changing the values of the magic square on its diagonals does not affect its row and column sums. Yet, one can show that if all magic squares of order n of the sequence $(c_k)_{k\geq 1}$ with which $\mathfrak{C}$ is constructed in Theorem 1 have this diagonal property (i.e. $\sum_{i=0}^{n-1} c_k(i,i) = \sum_{i=0}^{n-1} c_k(i,n-1-i) = \frac{n(n^2-1)}{2}$ for all $k \geq$

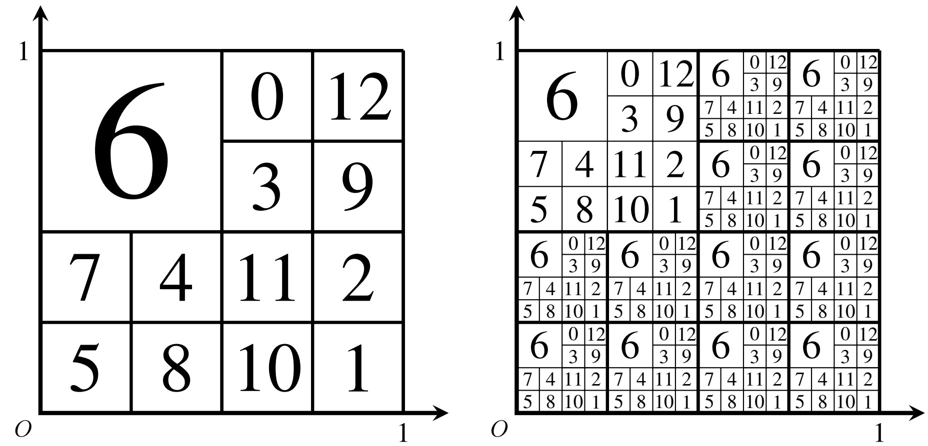

Figure 3. Construction of a non-uniform magic square on $[0,1]$.

0), then $\int_0^1 \mathfrak{C}(x,x)dx = \int_0^1 \mathfrak{C}(x,1-x)dx = \frac{\sqrt{2}}{2}$. Factor $\sqrt{2}$ is consistent with the Pythagorean theorem.

3. MAGIC SQUARES ON $\mathbb{R}$ In this section, we will show the existence of magic squares on $\mathbb{R}$, i.e. magic square from $\mathbb{R}$ to $\mathbb{R}$ according to the definition in the introduction. As the integral of such magic squares on $\mathbb{R}^2$ may not be finite, Fubini's theorem does not enable to conclude that row and column sums are equal. Actually, any two arbitrary values can be met, as shown below.

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}$. There exists a measurable bijection C from $\mathbb{R}^2$ to $\mathbb{R}$ such that:*

- $\forall x \in \mathbb{R}$, $\int_{\mathbb{R}} \mathsf{C}(x,y)dy = \alpha$,
- $\forall y \in \mathbb{R}$, $\int_{\mathbb{R}} \mathsf{C}(x,y)dx = \beta$.

Proof.
Note: in this proof, we will call a magic square from $A \times B$ to C ($A, B, C \subset \mathbb{R}$) a bijection that respects the usual conditions on row and column sums.

Construction of magic squares with images of the form $[-a,-b) \cup [a,b)$.
It is not difficult to go back over the construction behind Theorem 1 to get a magic square C from $[0,1)^2$ to $[0,1)$, and whose row and column sums equal $\frac{1}{2}$.[3] Now, let us consider functions $c_{(p,q),m}$ ($p, q \in \mathbb{Z}$, $m \in \mathbb{R}$), defined by

$$c_{(p,q),m} : \begin{array}{ccc} R_{(p,q)} & \to & [m - \frac{1}{2}, m + \frac{1}{2}) \\ (x,y) & \mapsto & m - \frac{1}{2} + C(x-p, y-q) \end{array},$$

where $R_{(p,q)} := [p, p+1) \times [q, q+1)$. These are magic squares whose row and column sums equal m. We can then arrange 16 of these magic squares on $[0,4) \times [0,4)$,

[3]Precisely, to do that one has to construct d'_n only on $[0,1)$ (and not $[0,1]$), and to restrict the image of b_n to E_n (and not $E_n \cup \{(n-1)_{\ell \geq 1}\}$).

as follows (the integers represent the value of m on each set $R_{(p,q)}, p, q \in \{0, 1, 2, 3\}$):

-8	-7	8	7
6	3	-6	-3
4	5	-4	-5
-2	-1	2	1

The resulting function is a magic square C' from $[0,4)^2$ to $[-\frac{17}{2}, -\frac{1}{2}) \cup [\frac{1}{2}, \frac{17}{2})$, whose sum is 0. Finally, let us set ($p, q \in \mathbb{Z}$, $k \in \mathbb{N}$)

$$\tilde{c}_{(p,q),k} : \begin{array}{ccc} R_{(p,q)} & \to & [-\frac{1}{2.17^k}, -\frac{1}{2.17^{k+1}}) \cup [\frac{1}{2.17^{k+1}}, \frac{1}{2.17^k}) \\ (x,y) & \mapsto & \frac{1}{17^{k+1}} C'(\frac{x-p}{4}, \frac{y-q}{4}) \end{array} .$$

These are magic squares whose column and row sums equal zero, and their image sets $[-\frac{1}{2.17^k}, -\frac{1}{2.17^{k+1}}) \cup [\frac{1}{2.17^{k+1}}, \frac{1}{2.17^k})$ are a partition of $[-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$.

Case $\alpha = \beta = 0$.
Let σ be a bijection from $\mathbb{N}$ to $\mathbb{Z}^2$. The function C'' defined as a juxtaposition of functions $\tilde{c}_{\sigma(k),k}$ ($k \in \mathbb{N}$), i.e. $C''(x,y) = c_{\sigma(k),k}(x,y)$ where k is the unique integer such that $(x,y) \in R_{\sigma(k)}$, is then a magic square from $\mathbb{R}^2$ to $[-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$, and its row and column sums equal zero. Note that the corresponding integrals are convergent because the values of $\tilde{c}_{(p,q),k}$ geometrically decrease with k. One then just need to extend to the image of C'' to $\mathbb{R}$. This can be done by modifying C'' on a diagonal, for example as follows:

$$\mathsf{C} : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & \begin{cases} C''(x,y) & \text{si } x \neq y \\ 0 & \text{si } x = y = 0 \\ \frac{1}{2x} & \text{si } x = y \text{ et } x \in (-1,1] \setminus \{0\} \\ C''(x-1, y-1) & \text{si } x = y \text{ et } x > 1 \\ C''(x+1, y+1) & \text{si } x = y \text{ et } x \leq -1 \end{cases} \end{array} .$$

Case $\alpha\beta > 0$.
Let us first solve the case $\alpha = \beta = 1$. The method is similar to the former case. It consists in covering $\mathbb{R}^2$ with some functions $m + \tilde{c}_{(p,q),k}$, the values of m being arranged as follows:

$$\begin{array}{c|c|c|c|c|c|c} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \hline \dots & 0 & 0 & 0 & 0 & 1 & \dots \\ \hline \dots & 0 & 0 & 0 & 1 & 0 & \dots \\ \hline \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \hline \dots & 0 & 1 & 0 & 0 & 0 & \dots \\ \hline \dots & 1 & 0 & 0 & 0 & 0 & \dots \\ \hline \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

More precisely, we consider a bijection σ_1 from $\mathbb{N}$ to $\mathbb{Z}$ and a bijection σ_2 from $\mathbb{N}$ to $\mathbb{Z}^2 \setminus \{(\ell, \ell), \ell \in \mathbb{Z}\}$. The function which results from the juxtaposition of functions $1 + \tilde{c}_{(\sigma_1(k), \sigma_1(k)),k}$ and functions $\tilde{c}_{\sigma_2(k),k}$ is a magic square from $\mathbb{R}^2$ to $[-\frac{1}{2}, \frac{3}{2}) \setminus \{0, 1\}$, and row (resp. column) sums equal 1, i.e. $\alpha = \beta = 1$. As before, we can then modify this magic square on a diagonal to get a magic square C from $\mathbb{R}^2$

to $\mathbb{R}$.
Any other case left is solved by setting $(x, y) \mapsto \mathsf{C}(\frac{x}{\alpha}, \frac{y}{\beta})$ if $\alpha, \beta > 0$ and $(x, y) \mapsto -\mathsf{C}(-\frac{x}{\alpha}, -\frac{y}{\beta})$ if $\alpha, \beta < 0$.

Other cases.
Case $\alpha = 1, \beta = -1$ can be solved as before, with the following arrangement:

⋱	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
…	0	0	0	0	0	0	0	0	**1**	0	0	0	…
…	0	0	0	0	0	0	0	0	**1**	0	0	0	…
…	0	0	0	0	0	0	0	**1**	0	0	0	0	…
…	**-1**	**-1**	0	0	0	0	0	**1**	0	0	0	0	…
…	0	0	**-1**	**-1**	0	0	**1**	0	0	0	0	0	…
…	0	0	0	0	**-1**	**-1**	**1**	0	0	0	0	0	…
…	0	0	0	0	0	**1**	**-1**	**-1**	0	0	0	0	…
…	0	0	0	0	0	**1**	0	0	**-1**	**-1**	0	0	…
…	0	0	0	0	**1**	0	0	0	0	0	**-1**	**-1**	…
…	0	0	0	0	**1**	0	0	0	0	0	0	0	…
…	0	0	0	**1**	0	0	0	0	0	0	0	0	…
…	0	0	0	**1**	0	0	0	0	0	0	0	0	…
⋰	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Case $\alpha = 1, \beta = 0$ is solved as follows:

⋱	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
…	**-1**	0	0	0	0	0	0	0	**1**	0	0	0	…
…	0	**-1**	0	0	0	0	0	0	**1**	0	0	0	…
…	0	0	**-1**	0	0	0	0	**1**	0	0	0	0	…
…	0	0	0	**-1**	0	0	0	**1**	0	0	0	0	…
…	0	0	0	0	**-1**	0	**1**	0	0	0	0	0	…
…	0	0	0	0	0	**-1**	**1**	0	0	0	0	0	…
…	0	0	0	0	0	**1**	**-1**	0	0	0	0	0	…
…	0	0	0	0	0	**1**	0	**-1**	0	0	0	0	…
…	0	0	0	0	**1**	0	0	0	**-1**	0	0	0	…
…	0	0	0	0	**1**	0	0	0	0	**-1**	0	0	…
…	0	0	0	**1**	0	0	0	0	0	0	**-1**	0	…
…	0	0	0	**1**	0	0	0	0	0	0	0	**-1**	…
⋰	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Cases left are again solved be rescaling variables, and/or by transposition (i.e. by considering $(x, y) \mapsto \mathsf{C}(y, x)$). ∎

4. MAGIC SQUARES ON COUNTABLY INFINITE SETS In this section, we will show the existence of magic squares on some countably infinite set $\mathcal{D} \subset \mathbb{R}$. Formally, such a magic square is a bijection from $\mathcal{D}^2$ to $\mathcal{D}$, with the corresponding conditions on row and column sums. Note that this existence is equivalent to the existence of similar magic squares where the domain of definition $\mathcal{D}^2$ is replaced by a set of the form $\mathcal{N}^2$, where $\mathcal{N}$ is an arbitrary infinitely countable set. Indeed, that results from a simple use of a bijection between $\mathcal{N}$ and $\mathcal{D}$, which does not affect row and column sums conditions.
Yet, $\mathcal{D}$, as the image set, will have the following properties.

- To ensure that sums are well-defined, $\mathcal{D}$ will only contain positive elements.
- Then, if a magic square on $\mathcal{D}$ exists, $\mathcal{D}$ has to be bounded. Indeed, the values of row and column sums are necessarily greater than any arbitrary element of $\mathcal{D}$. Without loss of generality, we will assume that $\mathcal{D}$ is bounded by 1.
- Also, 0 has to be a limit point of $\mathcal{D}$, as $\mathcal{D}$ has to contain the terms of convergent numerical series.
- Finally, we need a density condition on $\mathcal{D}$. Indeed, former conditions are still not strong enough: for example there can not exist magic square on $\mathcal{D} = \{\frac{1}{2^n}, n \geq 0\} \cup \{3\}$, as any bijection from $\mathcal{D}^2$ to $\mathcal{D}$ would have one, and only one, row (resp. column) sum greater than 3. Thus our construction, which is based on gradual adjustments of row and column sums, will require $\mathcal{D}$ to contain elements in some small open sets. That is why we will assume that $\mathcal{D}$ is dense in $[0, 1]$.

These requirements being set, any two possible values of the row and column sums can be met, as in the former section.

Theorem 3. *Let $\mathcal{D} \subseteq [0, 1]$ be an countably infinite set dense in $[0, 1]$, and let $\alpha > 1$ and $\beta > 1$. There exists a bijection $\mathscr{C}$ from $\mathcal{D}^2$ to $\mathcal{D}$ such that:*

- $\forall i \in \mathcal{D}, \sum_{j\in\mathcal{D}} \mathscr{C}(i, j) = \alpha,$
- $\forall j \in \mathcal{D}, \sum_{i\in\mathcal{D}} \mathscr{C}(i, j) = \beta.$

Proof. Let $\mathcal{D}$ be described be a sequence $(a_n)_{n\geq1}$ of pairwise distinct elements. Let us denote A_n the set defined by:

$$A_n := \{(a_k, a_\ell) | k, \ell \in \{1, \ldots, n\}\}.$$

We also fix an arbitrary real $M > \max\{0, \alpha - 3, \beta - 3\}$ and define

$$\forall n, k \geq 1,\ H_{k,n} := \begin{cases} \sum_{\ell=k+1}^{n} \frac{1}{M+\ell} & \text{if } n \geq k+1 \\ 0 & \text{otherwise} \end{cases} .$$

We will construct $\mathscr{C}$ inductively.
Let us first show that if there exists a function $\mathscr{C}_n$ such that

- $\mathscr{C}_n$ is an injection from A_n to $\mathcal{D}$,
- $a_1, \ldots, a_n \in \mathscr{C}_n(A_n)$,
- $\forall k \in \{1, \ldots, n\}$, $\min(\frac{\beta-1}{2} H_{k,n}, \alpha - \frac{\beta-1}{M+n}) \leq \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) < \alpha$ and $\min(\frac{\alpha-1}{2} H_{k,n}, \beta - \frac{\alpha-1}{M+n}) \leq \sum_{\ell=1}^{n} \mathscr{C}_n(a_\ell, a_k) < \beta$,

then there exists a function $\mathscr{C}_{n+1}$ satisfying the same three conditions for $n + 1$, and also such that $\mathscr{C}_n(a, a') = \mathscr{C}_{n+1}(a, a')$ for all $(a, a') \in A_n$.
So, let us assume that such a function $\mathscr{C}_n$ exists. To construct $\mathscr{C}_{n+1}$, we just need to choose appropriate values of $\mathscr{C}_{n+1}(a_{n+1}, a_{n+1})$, $\mathscr{C}_{n+1}(a_k, a_{n+1})$ and $\mathscr{C}_{n+1}(a_{n+1}, a_k)$ for k in $\{1, \ldots, n\}$.
First, to ensure that $a_{n+1} \in \mathscr{C}_{n+1}(A_{n+1})$, we set $\mathscr{C}_{n+1}(a_{n+1}, a_{n+1}) = a_{n+1}$ if $a_{n+1} \notin \mathscr{C}_n(A_n)$ and $\mathscr{C}_{n+1}(a_{n+1}, a_{n+1}) = a$ otherwise, where a is an arbitrary element of $\mathcal{D} \setminus \mathscr{C}_n(A_n)$.
Then, let us assign values to $\mathscr{C}_{n+1}(a_k, a_{n+1})$, $k \in \{1, \ldots, n\}$. For each k, this choice depends on whether $\sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) < \alpha - \frac{\beta-1}{M+n}$ or not.
If $\sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) < \alpha - \frac{\beta-1}{M+n}$, we necessarily have $\sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) \geq \frac{\beta-1}{2} H_{k,n}$.

In this case, $\mathscr{C}_{n+1}(a_k, a_{n+1})$ is chosen among the elements of $\mathscr{D}$ which are in $[\frac{\beta-1}{2(M+n+1)}, \frac{\beta-1}{M+n+1}]$ and which have not previously been chosen as a value of $\mathscr{C}_{n+1}$, i.e. which are not $\mathscr{C}_{n+1}(a_{n+1}, a_{n+1})$, not in the finite set $\mathscr{C}_n(A_n)$ and, assuming choices has been made in that order, not in $\{\mathscr{C}_{n+1}(a_1, a_{n+1}), \mathscr{C}_{n+1}(a_2, a_{n+1}), \cdots, \mathscr{C}_{n+1}(a_{k-1}, a_{n+1})\}$. This is made possible by density of $\mathscr{D}$ in $[0, 1]$ and by the fact that $\frac{\beta-1}{M+n+1} \leq 1$, the latter being a simple consequence of the definition of M. Note also that this choice is designed to ensure that $\mathscr{C}_{n+1}$ will be injective. Thus, we have on the one hand

$$\begin{aligned}
\sum_{\ell=1}^{n+1} \mathscr{C}_{n+1}(a_k, a_\ell) &= \sum_{\ell=1}^{n} \mathscr{C}_{n+1}(a_k, a_\ell) + \mathscr{C}_{n+1}(a_k, a_{n+1}) \\
&= \sum_{\ell=1}^{n} \mathscr{C}_{n}(a_k, a_\ell) + \mathscr{C}_{n+1}(a_k, a_{n+1}) \\
&\geq \frac{\beta-1}{2} H_{k,n} + \frac{\beta-1}{2(M+n+1)} = \frac{\beta-1}{2} H_{k,n+1} \\
&\geq \min\left(\frac{\beta-1}{2} H_{k,n+1}, \alpha - \frac{\beta-1}{M+n+1}\right),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\sum_{\ell=1}^{n+1} \mathscr{C}_{n+1}(a_k, a_\ell) &= \sum_{\ell=1}^{n} \mathscr{C}_{n}(a_k, a_\ell) + \mathscr{C}_{n+1}(a_k, a_{n+1}) \\
&\leq \alpha - \frac{\beta-1}{M+n} + \frac{\beta-1}{M+n+1} < \alpha.
\end{aligned}$$

If, on the contrary, $\sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) \geq \alpha - \frac{\beta-1}{M+n}$, then $\mathscr{C}_{n+1}(a_k, a_{n+1})$ is chosen as an element of $\mathscr{D}$ which has not been previously assigned, lying in the interval

$$\left[\alpha - \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) - \frac{\beta-1}{M+n+1}, \min\left(\alpha - \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell), \frac{\beta-1}{M+n+1}\right)\right).$$

Note that this interval has a non-empty interior, because:

$$\begin{aligned}
\alpha - \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) - \frac{\beta-1}{M+n+1} &\leq \alpha - (\alpha - \frac{\beta-1}{M+n}) - \frac{\beta-1}{M+n+1} \\
&= \frac{\beta-1}{(M+n)(M+n+1)} \\
&< \frac{\beta-1}{M+n+1},
\end{aligned}$$

and

$$\alpha - \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) - \frac{\beta-1}{M+n+1} < \alpha - \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell).$$

Thus, we have on the one hand

$$\begin{aligned}\sum_{\ell=1}^{n+1} \mathscr{C}_{n+1}(a_k, a_\ell) &= \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) + \mathscr{C}_{n+1}(a_k, a_{n+1}) \\ &\geq \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) + \alpha - \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) - \frac{\beta - 1}{M + n + 1} \\ &\geq \alpha - \frac{\beta - 1}{M + n + 1} \geq \min\left(\frac{\beta - 1}{2} H_{k,n}, \alpha - \frac{\beta - 1}{M + n + 1}\right),\end{aligned}$$

and on the other hand

$$\begin{aligned}\sum_{\ell=1}^{n+1} \mathscr{C}_{n+1}(a_k, a_\ell) &= \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) + \mathscr{C}_{n+1}(a_k, a_{n+1}) \\ &< \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) + \alpha - \sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) = \alpha.\end{aligned}$$

We proceed symmetrically to choose the values of $\mathscr{C}_{n+1}(a_{n+1}, a_k)$ for k in $\{1, \ldots, n\}$, with an exchange between α and β.
Thus, it has been shown that our construction ensures that

$$\min\left(\frac{\beta - 1}{2} H_{k,n+1}, \alpha - \frac{\beta - 1}{M + n + 1}\right) \leq \sum_{\ell=1}^{n+1} \mathscr{C}_{n+1}(a_k, a_\ell) < \alpha$$

and

$$\min\left(\frac{\alpha - 1}{2} H_{k,n+1}, \beta - \frac{\alpha - 1}{M + n + 1}\right) \leq \sum_{\ell=1}^{n+1} \mathscr{C}_{n+1}(a_\ell, a_k) < \beta$$

for k in $\{1, \ldots, n\}$, but we still have to prove these inequalities for $k = n + 1$. The latter result from the fact that $H_{n+1,n+1} = 0$ and that $\mathscr{C}_{n+1}(a_k, a_{n+1}) \leq \frac{\beta - 1}{M+n+1}$ for all $k \in \{1, \ldots, n\}$, so that

$$\begin{aligned}\sum_{\ell=1}^{n+1} \mathscr{C}_{n+1}(a_\ell, a_{n+1}) &= \sum_{\ell=1}^{n} \mathscr{C}_n(a_\ell, a_{n+1}) + \mathscr{C}_n(a_{n+1}, a_{n+1}) \\ &\leq \sum_{\ell=1}^{n} \frac{\beta - 1}{M + n + 1} + 1 = \frac{n}{M + n + 1}(\beta - 1) + 1 < \beta.\end{aligned}$$

Inequality $\sum_{\ell=1}^{n+1} \mathscr{C}_{n+1}(a_{n+1}, a_\ell) < \alpha$ is proven similarly, with a simple exchange of α and β.

So, starting from $\mathscr{C}_1$ defined by $\mathscr{C}_1(a_1, a_1) := a_1$, we can construct a sequence $(\mathscr{C}_n)_{n \geq 1}$ of functions which all satisfy the three conditions of our induction. We then define $\mathscr{C} : \mathscr{D}^2 \to \mathscr{D}$ by:

$$\mathscr{C}(a_k, a_\ell) := \mathscr{C}_{\max(k,\ell)}(a_k, a_\ell).$$

Because of the first two conditions, $\mathscr{C}$ is a bijection.
Moreover, as $\sum_{\ell=1}^{n} \mathscr{C}_n(a_k, a_\ell) = \sum_{\ell=1}^{n} \mathscr{C}(a_k, a_\ell)$, we have:

$$\forall n, k \geq 1,\ \min\left(\frac{\beta-1}{2} H_{k,n}, \alpha - \frac{\beta-1}{M+n}\right) \leq \sum_{\ell=1}^{n} \mathscr{C}(a_k, a_\ell) < \alpha.$$

Because $H_{k,n} \xrightarrow[n\to+\infty]{} +\infty$, we can conclude that $\sum_{\ell\geq 1} \mathscr{C}(a_k, a_\ell) = \alpha$ for all $k \geq 1$, and this can be reformulated as:

$$\forall i \in \mathcal{D},\ \sum_{j\in\mathcal{D}} \mathscr{C}(i,j) = \alpha.$$

Similarly:

$$\forall j \in \mathcal{D},\ \sum_{i\in\mathcal{D}} \mathscr{C}(i,j) = \beta.$$

■

In this theorem, the density of $\mathcal{D}$ in $[0,1]$ is maybe too strong an assumption. A possible extension of this problem of magic squares with countably infinite grids is to find weaker conditions on $\mathcal{D}$ and/or α and β that ensure that such a magic square exists.

REFERENCES

1. William Symes Andrews (1917), “Magic Squares and Cubes 2nd edition”, *Open Court Publishing Company* p. 122.
2. Duan, Z., Liu, J., Li, J., & Tian, C. (2015), “Improved even order magic square construction algorithms and their applications in multi-user shared electronic accounts ”, *Theoretical Computer Science*, 607, 391-410.
3. Simon de la Loubère (1693), “A New Historical Relation of the Kingdom of Siam”, reprint: *Oxford University Press*, Kuala Lumpur (1969).
4. Phillips, J. P. N. (1968), “A simple method of constructing certain magic rectangles of even order”, *The Mathematical Gazette*, 52(379), 9-12.
5. Ruberg, S. (1981), “An infinite number of 4×4 magic squares”, *Pi Mu Epsilon Journal*, 7(5), 299-302.
6. Walter William Rouse Ball (1911),“Mathematical Recreations and Essays”, *Macmillan*, London.
7. Swetz, F. J., & Luo, R. (2008),“Legacy of the Luoshu”, *The Australian Mathematical Society*, 184.