The Mann-Whitney U-statistic for α-dependent sequences

Jérôme Dedecker, Guillaume Saulière

The Mann-Whitney U-statistic for α-dependent sequences

Jérôme Dedecker[a] and Guillaume Saulière[b]

November 21, 2016

[a] Université Paris Descartes, Sorbonne Paris Cité, Laboratoire MAP5 (UMR 8145).
Email: jerome.dedecker@parisdescartes.fr

[b] Université Paris Sud, Institut de Recherche bioMédicale et d'Epidémiologie du Sport (IRMES), Institut National du Sport de l'Expertise et de la Performance (INSEP).
Email: Guillaume.SAULIERE@insep.fr

Abstract

We give the asymptotic behavior of the Mann-Whitney U-statistic for two independent stationary sequences. The result applies to a large class of short-range dependent sequences, including many non-mixing processes in the sense of Rosenblatt [15]. We also give some partial results in the long-range dependent case, and we investigate other related questions. Based on the theoretical results, we propose some simple corrections of the usual tests for stochastic domination; next we simulate different (non-mixing) stationary processes to see that the corrected tests perform well.

1 Introduction and main result

Let $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ be two independent stationary sequences of real-valued random variables. Let $m = m_n$ be a sequence of positive integers such that $m_n \to \infty$ as $n \to \infty$. The Mann-Whitney U-statistic may be defined as

$$U_n = U_{n,m} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{1}_{X_i<Y_j}\,.$$

We shall also use the notations $\pi = \mathbb{P}(X_1 < Y_1)$, $H_Y(t) = \mathbb{P}(Y_1 > t)$ and $G_X(t) = \mathbb{P}(X_1 < t)$.

It is well known that, if $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ are two independent sequences of independent and identically (iid) random variables, then U_n may be used to test $\pi = 1/2$ against $\pi \neq 1/2$ (or $\pi > 1/2$, $\pi < 1/2$). When $\pi > 1/2$ we shall say that Y stochastically dominates X in a weak sense. This property of weak domination is true for instance if Y stochastically dominates X, that is if $H_Y(t) \geq H_X(t)$ for any real t, with a strict inequality for some t_0. But it also holds in many other situations, for instance if X and Y are two Gaussian random variables with $\mathbb{E}(X) < \mathbb{E}(Y)$, whatever the variances of X and Y.

In this paper, we study the asymptotic behavior of U_n under some conditions on the α-dependence coefficients of the sequences $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$. Let us recall the definition of these coefficients.

Definition 1.1. Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence, and let $\mathcal{F}_0 = \sigma(X_k, k \leq 0)$. For any non-negative integer n, let

$$\alpha_{1,\mathbf{X}}(n) = \sup_{x\in\mathbb{R}} \left\|\mathbb{E}\left(\mathbf{1}_{X_n\leq x}|\mathcal{F}_0\right) - F(x)\right\|_1 ,$$

where F is the distribution function of X_0.

We shall also always require that the sequences are 2-ergodic in the following sense.

Definition 1.2. A strictly stationary sequence $(X_i)_{i\in\mathbb{Z}}$ is 2-ergodic if, for any bounded mesurable function f from $\mathbb{R}^2$ to $\mathbb{R}$, and any non-negative integer k

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i, X_{i+k}) = \mathbb{E}(f(X_0, X_k)) \quad \text{almost surely.} \tag{1.1}$$

Note that the almost sure limit in (1.1) always exists, by the ergodic theorem. The property of 2-ergodicity means only that this limit is constant. This property is weaker than usual ergodicity, which would implies the same property for any bounded function f from $\mathbb{R}^\ell$ to $\mathbb{R}$, $\ell \in \mathbb{N} - \{0\}$.

Our main result is the following theorem.

Theorem 1.1. *Let $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ be two independent stationary 2-ergodic sequences of real-valued random variables. Assume that*

$$\sum_{k=1}^{\infty} \alpha_{1,\mathbf{X}}(k) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha_{1,\mathbf{Y}}(k) < \infty\,. \tag{1.2}$$

Let (m_n) be a sequence of positive integers such that

$$\lim_{n\to\infty} \frac{n}{m_n} = c\,, \quad \text{for some } c \in [0, \infty). \tag{1.3}$$

Then the random variables $\sqrt{n}(U_n - \pi)$ converge in distribution to $\mathcal{N}(0, V)$, with

$$\begin{aligned} V = \mathrm{Var}(H_Y(X_0)) + 2\sum_{k=1}^{\infty} \mathrm{Cov}(H_Y(X_0), H_Y(X_k)) \\ + c\left(\mathrm{Var}(G_X(Y_0)) + 2\sum_{k=1}^{\infty}\mathrm{Cov}(G_X(Y_0), G_X(Y_k))\right)\,. \end{aligned} \tag{1.4}$$

Note that the result of Theorem 1.1 has been obtained by Serfling [16], provided that the more restrictive condition

$$\sum_{k\geq 0} \alpha(k) < \infty \quad \text{and} \quad \alpha(k) = O\left(\frac{1}{k\log k}\right) \tag{1.5}$$

is satisfied by the two sequences $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$, where the $\alpha(k)$'s are the usual α-mixing coefficients of Rosenblatt [15].

The notion of α-dependence that we consider in the present paper is much weaker than α-mixing in the sense of Rosenblatt [15]. Some properties of these α-dependence coefficients are stated in the book by Rio [14], and many examples are given in the two papers [7], [8]. Let us recall one of these examples. Let

$$X_i = \sum_{k\geq 0} a_k \varepsilon_{i-k} \tag{1.6}$$

where $(a_k)_{k\geq 0} \in \ell^2$, and the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ is a sequence of independent and identically distributed real-valued random variables, with mean zero and finite variance. If the cumulative distribution function F of X_0 is γ-Hölder for some $\gamma \in (0,1]$, then

$$\alpha_{1,\mathbf{X}}(n) \leq C\left(\sum_{k\geq n} a_k^2\right)^{\frac{\gamma}{\gamma+2}}$$

for some positive constant C (this follows from the computations in Section 6.1 of [8]). In particular, if a_k is geometrically decreasing, then so is $\alpha_{1,\mathbf{X}}(n)$ (whatever the index γ).

Note that, without extra assumptions on the distribution of ε_0, such linear processes have no reasons to be mixing in the sense of Rosenblatt. For instance, it is well known that the linear process

$$X_i = \sum_{k\geq 0} \frac{\varepsilon_{i-k}}{2^{k+1}}, \quad \text{where } \mathbb{P}(\varepsilon_1 = -1/2) = \mathbb{P}(\varepsilon_1 = 1/2) = 1/2,$$

is not α-mixing (see for instance [1]). By contrast, for this particular example, $\alpha_{1,\mathbf{X}}(n)$ is geometrically decreasing.

To conclude, let us also mention the paper [5] where it is proved that the coefficients $\alpha_{1,\mathbf{X}}(n)$ can be computed for a large class of dynamical systems on the unit interval (see Subsection 6.2 of the present paper for more details).

Remark 1.1. Other types of dependence are considered in the papers by Dewan and Prakasa Rao [11] and Dehling and Fried [9]. The paper [11] deals with the U-statistics U_n for two independent samples of associated random variables. The second paper [9] deals with general U-statistics of functions of β-mixing sequences, for either two independent samples, or for two adjacent samples from the same time series (see Subsection 5.2 for an extension of our results to this more difficult context). The class of unctions of β-mixing sequences contains also many examples, with a large intersection with the class of α-dependent sequences. The advantage of our approach is that it leads to conditions that are optimal in a precise sense (see Section 3). On another hand, it seems more difficult to deal with general U-statistics in our context (some kind of regularity is needed, for instance we could certainly deal with U-statistics based on functions which have bounded variation in each coordinates).

2 Testing the stochastic domination from two independent time series

The statistic $\sqrt{n}(U_n - 1/2)$ cannot be used to test $H_0 : \pi = 1/2$ versus one of the possible alternatives, because under H_0 the asymptotic distribution of $\sqrt{n}(U_n - 1/2)$ depends of the unknow quantity V defined in (1.4). In the next Proposition, we propose an estimator of V under some slightly stronger conditions on the dependence coefficients.

Definition 2.1. Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables, and let $\mathcal{F}_0 = \sigma(X_k, k \le 0)$. Let $f_x(z) = \mathbf{1}_{z\le x} - F(x)$, where F is the cumulative distribution function of X_0. For any non-negative integer n, let

$$\alpha_{2,\mathbf{X}}(n) = \max\left\{\alpha_{1,\mathbf{X}}(n), \sup_{x,y\in\mathbb{R}, j\ge i\ge n} \left\|\mathbb{E}\left(f_x(X_i)f_y(X_j)|\mathcal{F}_0\right) - \mathbb{E}\left(f_x(X_i)f_y(X_j)\right)\right\|_1\right\} .$$

Proposition 2.1. *Let $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ be two independent stationary sequences, and let (m_n) be a sequence of positive inetegers satisfying* (1.3). *Assume that*

$$\sum_{k=1}^{\infty} \alpha_{2,\mathbf{X}}(k) < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} \alpha_{2,\mathbf{Y}}(k) < \infty . \tag{2.1}$$

Let

$$H_m(t) = \frac{1}{m}\sum_{j=1}^{m} \mathbf{1}_{Y_j>t} \quad \textit{and} \quad G_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{X_i<t} ,$$

and

$$\hat\gamma_X(k) = \frac{1}{n}\sum_{i=1}^{n-k}(H_m(X_i)-\bar H_m)(H_m(X_{i+k})-\bar H_m) , \quad \hat\gamma_Y(\ell) = \frac{1}{m}\sum_{j=1}^{m-\ell}(G_n(Y_j)-\bar G_n)(G_n(Y_{j+\ell})-\bar G_n) ,$$

where $n\bar H_m = \sum_{i=1}^n H_m(X_i)$ and $m\bar G_n = \sum_{j=1}^m G_n(Y_j)$. Let also (a_n) and (b_n) be two sequences of positive integers tending to infinity as n tends to infinity, such that $a_n = o(\sqrt{n})$ and $b_n =$

$o(\sqrt{m_n})$. *Then*

$$V_n = \hat{\gamma}_X(0) + 2\sum_{k=1}^{a_n} \hat{\gamma}_X(k) + \frac{n}{m_n}\left(\hat{\gamma}_Y(0) + 2\sum_{\ell=1}^{b_n} \hat{\gamma}_Y(\ell)\right)$$

converges in $\mathbb{L}^2$ *to the quantity* V *defined in* (1.4).

Combining Theorem 1.1 and Proposition 2.1, we obtain that, under $H_0 : \pi = 1/2$, if $V > 0$, the random variables

$$T_n = \frac{\sqrt{n}(U_n - 1/2)}{\sqrt{\max\{V_n, 0\}}} \tag{2.2}$$

converge in distribution to $\mathcal{N}(0, 1)$.

Remark 2.1. The condition (2.1) is not restrictive: in all the natural examples we know, the coefficients $\alpha_{2,\mathbf{X}}(k)$ behave as $\alpha_{1,\mathbf{X}}(k)$ (this is the case for instance for the linear process (1.6)). Note that, if we assume (2.1) instead of (1.2), the ergodicity is no longer required in Theorem 1.1. Finally, following the proof of Theorem 4.1 page 89 in [2], one can also build a consistent estimator of V under the condition

$$\sum_{k=1}^{\infty} \sqrt{\frac{\alpha_{1,\mathbf{X}}(k)}{k}} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \sqrt{\frac{\alpha_{1,\mathbf{Y}}(k)}{k}} < \infty\,, \tag{2.3}$$

which is not directly comparable to (2.1) (for instance, the first part of (2.1) is satisfied if $\alpha_{2,\mathbf{X}}(k) = O(k^{-1}(\log k)^{-a})$ for some $a > 1$, while the first part of (2.3) requires $\alpha_{1,\mathbf{X}}(k) = O(k^{-1}(\log k)^{-a})$ for some $a > 2$).

Remark 2.2. The choice of the two sequences (a_n) and (b_n) is a delicate matter. If the coefficients $\alpha_{2,\mathbf{X}}(k)$ decrease very quickly, then a_n should increase very slowly (it suffices to take $a_n \equiv 0$ in the iid setting). On the contrary, if $\alpha_{2,\mathbf{X}}(k) = O(k^{-1}(\log k)^{-a})$ for some $a > 1$, then the terms in the covariance series have no reason to be small, and one should take a_n close to $\sqrt{n}$ to estimate many of these covariance terms. A data-driven criterion for choosing a_n and b_n is an interesting (but probably difficult) question, which is far beyond the scope of the present paper.

However, from a practical point of view, there is an easy way to proceed: one can plot the estimated covariances $\hat{\gamma}_X(k)$'s and choose a_n (not too large) in such a way that

$$\hat{\gamma}_X(0) + 2\sum_{k=1}^{a_n} \hat{\gamma}_X(k)$$

should represent an important part of the covariance series

$$\mathrm{Var}(H_Y(X_0)) + 2\sum_{k=1}^{\infty} \mathrm{Cov}(H_Y(X_0), H_Y(X_k))\,.$$

Similarly, one may choose b_n from the graph of the $\hat{\gamma}_Y(\ell)$'s. As we shall see in the simulations (Section 6), if the decays of the true covariances $\gamma_X(k)$ and $\gamma_Y(\ell)$ are not too slow, this provides an easy and reasonable choice for a_n and b_n.

3 Long range dependence

In this section, we shall see that the condition (1.2) cannot be weakened for the validity of Theorem 1.1. More precisely, we shall give some examples where (1.2) fails, and the asymptotic distribution of $a_n(U_n - \pi)$ (if any) can be non Gaussian.

Let us begin with the boundary case, where

$$\alpha_{2,\mathbf{X}}(k) = O((k+1)^{-1}) \quad \text{and} \quad \alpha_{2,\mathbf{Y}}(k) = O((k+1)^{-1}) \,. \tag{3.1}$$

In that case, one can prove the following result.

Proposition 3.1. *Let $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ be two independent stationary sequences. Assume that* (3.1) *is satisfied, and let (m_n) be a sequence a positive integers satisfying* (1.3). *Then:*

1. *For any $r \in (2,4)$, there exists a positive constant C_r such that, for any $x > 0$,*

$$\limsup_{n\to\infty} \mathbb{P}\left(\sqrt{n/\log n}(U_n - \pi) > x\right) \leq \frac{C_r}{x^r} \,.$$

2. *There are some examples of stationary Markov chains $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ for which the sequence $\sqrt{n/\log n}(U_n - \pi)$ converges in distribution to a non-degenerate normal distribution.*

We consider now the case where

$$\sum_{k=1}^{\infty} \alpha_{1,\mathbf{X}}(k) < \infty \quad \text{and} \quad \alpha_{1,\mathbf{Y}}(k) = O((k+1)^{-(p-1)}) \,, \text{ for some } p \in (1,2). \tag{3.2}$$

This means that $(X_i)_{i\in\mathbb{Z}}$ is short-range dependent, while $(Y_i)_{i\in\mathbb{Z}}$ is possibly long-range dependent. In that case, we shall prove the following result.

Proposition 3.2. *Let $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ be two independent stationary sequences. Assume that* (3.2) *is satisfied for some $p \in (1,2)$, and that $(X_i)_{i\in\mathbb{Z}}$ is 2-ergodic. Let (m_n) be a sequence a positive integers such that*

$$\lim_{n\to\infty} \frac{n}{m_n^{2(p-1)/p}} = c \,, \quad \textit{for some } c \in [0,\infty). \tag{3.3}$$

Then:

1. *There exists a positive constant C_p such that, for any $x > 0$,*

$$\limsup_{n\to\infty} \mathbb{P}\left(\sqrt{n}(U_n - \pi) > x\right) \leq \frac{C_p}{x^p}\,.$$

2. *There are some examples of stationary Markov chains $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ for which the sequence $\sqrt{n}(U_n - \pi)$ converges in distribution to $Z + \sqrt{c}S$, where Z is independent of S, Z is a Gaussian random variable, and S is a stable random variable of order p.*

Remark 3.1. Note that we do not deal with the case where the two samples are long-range dependent; in that case the degenerate U-statistic from the Hoeffding decomposition (see (4.2)) is no longer negligible, which gives rise to other possible limits. See the paper [10] for a treatment of this difficult question in the case of functions of Gaussian processes.

4 Proofs

4.1 Proof of Theorem 1.1

The main ingredient of the proof is the following proposition:

Proposition 4.1. *Let $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ be two independent stationary sequences. Let also*

$$f(x,y) = \mathbf{1}_{x<y} - H_Y(x) - G_X(y) + \pi\,. \tag{4.1}$$

Then

$$\mathbb{E}\left(\left(\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(X_i,Y_j)\right)^2\right) \leq \frac{16}{mn}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\min\{\alpha_{1,\mathbf{X}}(i), \alpha_{1,\mathbf{Y}}(j)\}\,.$$

Let us admit this proposition for a while, and see how it can be used to prove the theorem. Starting from (4.1), we easily see that

$$\sqrt{n}(U_n - \pi) = \frac{\sqrt{n}}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(X_i,Y_j) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(H_Y(X_i) - \pi) + \frac{\sqrt{n}}{m}\sum_{j=1}^{m}(G_X(Y_j) - \pi)\,. \tag{4.2}$$

This is the well known Hoeffding decomposition of U-statistics; the first term on the right-hand side of (4.2) is called a *degenerate* U-statistic, and is handled *via* Proposition 4.1. Indeed, it follows easily from this proposition that

$$n\mathbb{E}\left(\left(\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(X_i,Y_j)\right)^2\right) \leq \frac{16}{m}\sum_{i=0}^{n-1}(i+1)\alpha_{1,\mathbf{X}}(i) + \frac{16}{m}\sum_{j=0}^{m-1}(j+1)\alpha_{1,\mathbf{Y}}(j)\,. \tag{4.3}$$

Since (1.2) holds, then $\alpha_{1,\mathbf{X}}(n) = o(n^{-1})$ and $\alpha_{1,\mathbf{Y}}(n) = o(m^{-1})$ (because these coefficients decrease) and by Cesaro's lemma

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n}(i+1)\alpha_{1,\mathbf{X}}(i) = 0 \quad \text{and} \quad \lim_{m\to\infty} \frac{1}{m}\sum_{j=0}^{m}(j+1)\alpha_{1,\mathbf{Y}}(j) = 0\,.$$

From (4.3) and Condition (1.3), we infer that

$$\frac{\sqrt{n}}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(X_i, Y_j) \quad \text{converges to 0 in } \mathbb{L}^2 \text{ as } n\to\infty. \tag{4.4}$$

It remains to control the two last terms on the right-hand side of (4.2). We first note that these two terms are independent, so it is enough to prove the convergence in distribution for both terms. The functions H_Y and G_X being monotonic, the central limit theorem for stationary and 2-ergodic α-dependent sequences (see for instance [5]) implies that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(H_Y(X_i)-\pi) \quad \text{converges in distribution to } \mathcal{N}(0, v_1), \text{ where}$$

$$v_1 = \mathrm{Var}(H_Y(X_0)) + 2\sum_{k=1}^{\infty}\mathrm{Cov}(H_Y(X_0), H_Y(X_k))\,, \tag{4.5}$$

and

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}(G_X(Y_j)-\pi) \quad \text{converges in distribution to } \mathcal{N}(0, v_2), \text{ where}$$

$$v_2 = \mathrm{Var}(G_X(Y_0)) + 2\sum_{k=1}^{\infty}\mathrm{Cov}(G_X(Y_0), G_X(Y_k))\,. \tag{4.6}$$

Combining (4.2), (4.4), (4.5) and (4.6), the proof of Theorem 1.1 is complete.

It remains to prove Proposition 4.1. We start from the elementary inequality

$$\mathbb{E}\left(\left(\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(X_i, Y_j)\right)^2\right) \leq \frac{4}{n^2m^2}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=i}^{n}\sum_{\ell=j}^{m}\mathbb{E}(f(X_i,Y_j)f(X_k,Y_\ell))\,. \tag{4.7}$$

To control the term $\mathbb{E}(f(X_i,Y_j)f(X_k,Y_\ell))$, we first work conditionally on $\mathbf{Y} = (Y_i)_{i\in\mathbb{Z}}$. Note first that, since $\mathbf{Y}$ is independent of $\mathbf{X}$, $\mathbb{E}(f(X_i,Y_j)|\mathbf{Y}) = 0$. Since $|f(x,y)| \leq 2$,

$$|\mathbb{E}(f(X_i,Y_j)f(X_k,Y_l)|\mathbf{Y}=\mathbf{y})| \leq 2\|\mathbb{E}(f(X_k,y_\ell)|\mathcal{F}_i)\|_1\,, \tag{4.8}$$

where $\mathcal{F}_i = \sigma(X_k, k\leq i)$. Now, by definition of f,

$$\|\mathbb{E}(f(X_k,y_\ell)|\mathcal{F}_i)\|_1 \leq \|\mathbb{E}(\mathbf{1}_{X_k<y_\ell}|\mathcal{F}_i) - G_X(y_\ell)\|_1 + \|\mathbb{E}(H_Y(X_k)|\mathcal{F}_i)-\pi\|_1 \leq 2\alpha_{1,\mathbf{X}}(k-i)\,. \tag{4.9}$$

From (4.8) and (4.9), it follows that

$$|\mathbb{E}(f(X_i,Y_j)f(X_k,Y_\ell))| \le \|\mathbb{E}(f(X_i,Y_j)f(X_k,Y_\ell)|\mathbf{Y})\|_1 \le 4\alpha_{1,\mathbf{X}}(k-i)\,. \tag{4.10}$$

In the same way

$$|\mathbb{E}(f(X_i,Y_j)f(X_k,Y_\ell))| \le 4\alpha_{1,\mathbf{Y}}(\ell-j)\,. \tag{4.11}$$

Proposition 4.1 follows from (4.7), (4.10) and (4.11).

4.2 Proof of Proposition 2.1

We proceed in two steps.

Step 1. Let

$$\gamma_X^*(k) = \frac{1}{n}\sum_{i=1}^{n-k}(H_Y(X_i)-\pi)(H_Y(X_{i+k})-\pi)\,, \quad \gamma_Y^*(\ell) = \frac{1}{m}\sum_{j=1}^{m-\ell}(G_X(Y_j)-\pi)(G_X(Y_{i+k})-\pi)\,,$$

and note that

$$\mathbb{E}(\gamma_X^*(k)) = \frac{n-k}{n}\mathrm{Cov}(H_Y(X_0),H_Y(X_k)) \quad \text{and} \quad \mathbb{E}(\gamma_Y^*(\ell)) = \frac{n-k}{n}\mathrm{Cov}(G_X(Y_0),G_X(Y_\ell))\,. \tag{4.12}$$

In this first step, we shall prove that

$$V_n^* = \gamma_X^*(0) + 2\sum_{k=1}^{a_n}\gamma_X^*(k) + \frac{n}{m_n}\left(\gamma_Y^*(0) + 2\sum_{\ell=1}^{b_n}\gamma_Y^*(\ell)\right)\,,$$

converges in $\mathbb{L}^2$ to V. Clearly, by (4.12), it suffices to show that $\mathrm{Var}(V_n^*)$ converges to 0 as $n\to\infty$. We only deal with the second term in the expression of V_n^*, the other ones may be handled in the same way. Letting $Z_i = H_Y(X_i)-\pi$, and $\gamma_k = \mathrm{Cov}(H_Y(X_0),H_Y(X_k))$ we have

$$\mathrm{Var}\left(\sum_{k=1}^{a_n}\gamma_X^*(k)\right) = \frac{1}{n^2}\sum_{k=1}^{a_n}\sum_{\ell=1}^{a_n}\sum_{i=1}^{n-k}\sum_{j=1}^{n-\ell}\mathbb{E}((Z_iZ_{i+k}-\gamma_k)(Z_jZ_{j+\ell}-\gamma_\ell))\,. \tag{4.13}$$

We must examine several cases.

On $\Gamma_1 = \{j \ge i+k\}$,

$$|\mathbb{E}((Z_iZ_{i+k}-\gamma_k)(Z_jZ_{j+\ell}-\gamma_\ell))| \le 2\alpha_{2,\mathbf{X}}(j-i-k).$$

Consequently

$$\frac{1}{n^2}\sum_{k=1}^{a_n}\sum_{\ell=1}^{a_n}\sum_{i=1}^{n-k}\sum_{j=1}^{n-\ell}|\mathbb{E}((Z_iZ_{i+k}-\gamma_k)(Z_jZ_{j+\ell}-\gamma_\ell))|\mathbf{1}_{\Gamma_1} \le \frac{2}{n^2}\sum_{k=1}^{a_n}\sum_{\ell=1}^{a_n}\sum_{i=1}^{n}\left(\sum_{j\ge 0}\alpha_{2,\mathbf{X}}(j)\right) \le \frac{Ca_n^2}{n} \tag{4.14}$$

for some positive constant C.

On $\Gamma_2 = \{i \le j < i+k\} \cap \{i+k \le j+\ell\}$,

$$|\mathbb{E}((Z_i Z_{i+k} - \gamma_k)(Z_j Z_{j+\ell} - \gamma_\ell))| = |\mathbb{E}((Z_i Z_{i+k} - \gamma_k) Z_j Z_{j+\ell})| \le 2\alpha_{1,\mathbf{X}}(j+\ell-i-k).$$

Consequently

$$\frac{1}{n^2} \sum_{k=1}^{a_n} \sum_{\ell=1}^{a_n} \sum_{i=1}^{n-k} \sum_{j=1}^{n-\ell} |\mathbb{E}((Z_i Z_{i+k} - \gamma_k)(Z_j Z_{j+\ell} - \gamma_\ell))| \mathbf{1}_{\Gamma_2} \le \frac{2}{n^2} \sum_{k=1}^{a_n} \sum_{\ell=1}^{a_n} \sum_{i=1}^{n} \left(\sum_{j \ge 0} \alpha_{1,\mathbf{X}}(j) \right) \le \frac{C a_n^2}{n} . \tag{4.15}$$

On $\Gamma_3 = \{i \le j\} \cap \{i+k > j+\ell\}$,

$$|\mathbb{E}((Z_i Z_{i+k} - \gamma_k)(Z_j Z_{j+\ell} - \gamma_\ell))| = |\mathbb{E}((Z_j Z_{j+\ell} - \gamma_\ell) Z_i Z_{i+k})| \le 2\alpha_{1,\mathbf{X}}(i+k-j-\ell).$$

Consequently

$$\frac{1}{n^2} \sum_{k=1}^{a_n} \sum_{\ell=1}^{a_n} \sum_{i=1}^{n-k} \sum_{j=1}^{n-\ell} |\mathbb{E}((Z_i Z_{i+k} - \gamma_k)(Z_j Z_{j+\ell} - \gamma_\ell))| \mathbf{1}_{\Gamma_3} \le \frac{2}{n^2} \sum_{k=1}^{a_n} \sum_{\ell=1}^{a_n} \sum_{j=1}^{n} \left(\sum_{i \ge 0} \alpha_{1,\mathbf{X}}(i) \right) \le \frac{C a_n^2}{n} . \tag{4.16}$$

From (4.14), (4.15) and (4.16), setting $\Gamma = \{i \le j\} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, one gets

$$\frac{1}{n^2} \sum_{k=1}^{a_n} \sum_{\ell=1}^{a_n} \sum_{i=1}^{n-k} \sum_{j=1}^{n-\ell} |\mathbb{E}((Z_i Z_{i+k} - \gamma_k)(Z_j Z_{j+\ell} - \gamma_\ell))| \mathbf{1}_{\Gamma} \le \frac{3C a_n^2}{n}$$

Of course, interchanging i and j, the same is true on $\Gamma^c = \{i > j\}$, and finally

$$\frac{1}{n^2} \sum_{k=1}^{a_n} \sum_{\ell=1}^{a_n} \sum_{i=1}^{n-k} \sum_{j=1}^{n-\ell} |\mathbb{E}((Z_i Z_{i+k} - \gamma_k)(Z_j Z_{j+\ell} - \gamma_\ell))| \le \frac{6C a_n^2}{n} \tag{4.17}$$

From (4.13), (4.17) and the fact that $a_n = o(\sqrt{n})$, we infer that $\sum_{k=1}^{a_n} \gamma_X^*(k)$ converges in $\mathbb{L}^2$ to $\sum_{k>0} \mathrm{Cov}(H_Y(X_0), H_Y(X_k))$. Since the other terms in the definition of V_n^* can be handled in the same way, it follows that V_n^* converges in $\mathbb{L}^2$ to V.

Step 2. In this second step, in the expression of $\gamma_X^*(k)$, we replace $H_Y(X_i)$ by $H_m(X_i)$, $H_Y(X_{i+k})$ by $H_m(X_{i+k})$, and π by $\bar{H}_m$. Similarly, in the expression of $\gamma_Y^*(\ell)$, we replace $G_X(Y_j)$ by $G_n(Y_j)$, $G_X(Y_{j+\ell})$ by $G_n(Y_{j+\ell})$, and π by $\bar{G}_n$. If we can prove that these replacements in the expression of V_n^* are negligible in $\mathbb{L}^2$, the conclusion of Proposition 2.1 will follow.

Let

$$\gamma_{1,X}(k) = \frac{1}{n} \sum_{i=1}^{n-k} (H_m(X_i) - \pi)(H_Y(X_{i+k}) - \pi) .$$

Then

$$\|\gamma_X^*(k) - \gamma_{1,X}(k)\|_2 \le \|H_m(X_i) - H_Y(X_i)\|_2 = \left(\int \|H_m(x) - H_Y(x)\|_2^2 \mathbb{P}_X(dx)\right)^{1/2}, \qquad (4.18)$$

the last equality being true because $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ are independent. Now,

$$\|H_m(x) - H_Y(x)\|_2^2 \le \frac{1}{m}\left(\mathrm{Var}(\mathbf{1}_{Y_0>x}) + 2\sum_{k=1}^{n-1} |\mathrm{Cov}(\mathbf{1}_{Y_0>x}, \mathbf{1}_{Y_k>x})|\right) \le \frac{2\sum_{k=0}^{n-1}\alpha_{1,\mathbf{Y}}(k)}{m}. \quad (4.19)$$

From (4.18) and (4.19), we infer that there exist a positive constant C_1 such that

$$\left\|\gamma_X^*(0) - \gamma_{1,X}(0) + 2\sum_{k=1}^{a_n}(\gamma_X^*(k) - \gamma_{1,X}(k))\right\|_2 \le \frac{C_1 a_n}{\sqrt{m}}.$$

By condition (1.3) and the fact that $a_n = o(\sqrt{n})$, this last quantity tends to 0 as $n \to \infty$.

Let now

$$\gamma_{2,X}(k) = \frac{1}{n}\sum_{i=1}^{n-k}(H_m(X_i) - \bar{H}_m)(H_Y(X_{i+k}) - \pi).$$

Then

$$\|\gamma_{1,X}(k) - \gamma_{2,X}(k)\|_2 \le \left\|\bar{H}_m - \frac{1}{n}\sum_{i=1}^{n} H_Y(X_i)\right\|_2 + \left\|\frac{1}{n}\sum_{i=1}^{n} H_Y(X_i) - \pi\right\|_2. \qquad (4.20)$$

Proceeding as in (4.19) for the first term on the right-hand side of (4.20), and noting that

$$\left\|\frac{1}{n}\sum_{i=1}^{n} H_Y(X_i) - \pi\right\|_2^2 \le \frac{2\sum_{k=0}^{n-1}\alpha_{1,\mathbf{X}}(k)}{n},$$

we get that there exist a positive constant C_2 such that

$$\left\|\gamma_{1,X}(0) - \gamma_{2,X}(0) + 2\sum_{k=1}^{a_n}(\gamma_{1,X}(k) - \gamma_{2,X}(k))\right\|_2 \le \frac{C_2 a_n}{\sqrt{m}} + \frac{C_2 a_n}{\sqrt{n}}.$$

Again, this last quantity tends to 0 as $n \to \infty$.

Finally, all the other replacements may be done in the same way, and lead to negligible contributions in $\mathbb{L}^2$. Hence

$$\lim_{n\to\infty} \|V_n - V_n^*\|_2 = 0$$

and the result follows from Step 1.

4.3 Proof of Proposition 3.1

We start from (4.2) again, but with the normalization $\sqrt{n/\log n}$ instead of $\sqrt{n}$. Since (3.1) holds,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n}(i+1)\alpha_{1,\mathbf{X}}(i) < \infty \quad \text{and} \quad \limsup_{m\to\infty} \frac{1}{m}\sum_{j=0}^{m}(j+1)\alpha_{1,\mathbf{Y}}(j) < \infty\,.$$

From Inequality (4.3) and Condition (1.3), we infer that

$$\frac{\sqrt{n}}{nm\sqrt{\log n}}\sum_{i=1}^{n}\sum_{j=1}^{m} f(X_i, Y_j) \quad \text{converges to 0 in } \mathbb{L}^2 \text{ as } n \to \infty. \tag{4.21}$$

Let us prove Item 1. From (4.2) and (4.21),

$$\limsup_{n\to\infty} \mathbb{P}\left(\sqrt{n/\log n}(U_n - \pi) > x\right) \leq \limsup_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^{n}(H_Y(X_i) - \pi) > x\sqrt{n\log n}/2\right)$$
$$+ \limsup_{n\to\infty} \mathbb{P}\left(\frac{n}{m}\sum_{j=1}^{m}(G_X(Y_i) - \pi) > x\sqrt{n\log n}/2\right)\,. \tag{4.22}$$

Since (3.1) holds, it follows from Proposition 5.1 in [6] that: for any $r \in (2,4)$, there exists a positive constant $c_{1,r}$ such that, for any $x > 0$

$$\limsup_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^{n}(H_Y(X_i) - \pi) > x\sqrt{n\log n}/2\right) \leq \frac{c_{1,r}}{x^r}\,.$$

Since (1.3) holds, the same is true for the second term on the right-hand side of (4.22) (for a positive constant $c_{2,r}$), and Item 1 follows.

Let us prove Item 2. We use a result by Gouëzel [12] about intermittent maps. Let $\theta_{1/2}$ be the map described in Subsection 6.2 of the present paper. As explained in Subsection 6.2, there exists an unique absolutely continuous $\theta_{1/2}$-invariant measure $\nu_{1/2}$ on $[0,1]$, and (see the comments on the case $\alpha = 1/2$ in Section 1.3 of Gouëzel's paper), on the probability space $([0,1], \nu_{1/2})$,

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left(H_Y(\theta_{1/2}^k) - \nu_{1/2}(H_Y)\right)$$

converges in distribution to a non-degenerate normal distribution provided $H_Y(0) \neq \nu(H_Y)$ (which is true for instance if Y has distribution $\nu_{1/2}$, since in that case $H_Y(0) = 1$ and $\nu(H_Y) = 1/2$). As fully explained in Subsection 6.2, there exists a stationary Markov chain $(X_i)_{i\in\mathbb{Z}}$ such that the distribution of $(X_1, \dots, X_n)$ is the same as $(\theta_{1/2}^n, \dots, \theta_{1/2})$ on the probability space $([0,1], \nu_{1/2})$. Take $(Y_i)_{i\in\mathbb{Z}}$ an independent copy of the chain $(X_i)_{i\in\mathbb{Z}}$. For such chains, both

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(H_Y(X_i)-\pi) \quad \text{and} \quad \frac{1}{\sqrt{m}}\sum_{j=1}^{m}(G_X(Y_j)-\pi)$$

converge to the same non-degenerate normal distribution (note that $\pi = 1/2$ in that case). Moreover, as recalled in Subsection 6.2, there exist two positive constants A, B such that

$$\frac{A}{k+1} \le \alpha_{2,\mathbf{X}}(k) = \alpha_{2,\mathbf{Y}}(k) \le \frac{B}{k+1}\,.$$

This completes the proof of Item 2.

4.4 Proof of Proposition 3.2

We start from (4.2) again. At the end of the proof, we shall prove that if (3.2) and (3.3) are satisfied, then (4.4) holds.

Let us now prove Item 1. From (4.2) and (4.4),

$$\limsup_{n\to\infty}\mathbb{P}\left(\sqrt{n}(U_n-\pi)>x\right) \le \limsup_{n\to\infty}\mathbb{P}\left(\sum_{i=1}^{n}(H_Y(X_i)-\pi)>x\sqrt{n}/2\right) + \limsup_{n\to\infty}\mathbb{P}\left(\frac{n}{m}\sum_{j=1}^{m}(G_X(Y_i)-\pi)>x\sqrt{n}/2\right)\,. \quad (4.23)$$

Since the first part of (3.2) is satisfied, it follows from the central limit theorem for stationary and 2-ergodic α-dependent sequences that

$$\lim_{n\to\infty}\mathbb{P}\left(\sum_{i=1}^{n}(H_Y(X_i)-\pi)>x\sqrt{n}/2\right) = \mathbb{P}(Z>x/2)\,, \quad (4.24)$$

where Z is a Gaussian random variable. From Condition (3.2) and Remark 3.3 in [6], we infer that if (3.3) holds, then there exists a positive constant κ_p such that, for any $x>0$,

$$\limsup_{n\to\infty}\mathbb{P}\left(\frac{n}{m}\sum_{j=1}^{m}(G_X(Y_i)-\pi)>x\sqrt{n}/2\right) \le \frac{\kappa_p}{x^p}\,. \quad (4.25)$$

Item 1 follows from (4.23), (4.24) and (4.25).

Let us prove Item 2. For the sequence $(X_i)_{i\in\mathbb{Z}}$, take a sequence of iid random variables uniformly distributed over $[0,1]$. To find the sequence $(Y_i)_{i\in\mathbb{Z}}$ we use again a result by Gouëzel [12] about intermittent maps. Let $\theta_{1/p}$ be the map described in Subsection 6.2 of the present paper. As explained is Subsection 6.2, there exists an unique absolutely continuous $\theta_{1/p}$-invariant measure $\nu_{1/p}$ on $[0,1]$, and (see Theorem 1.3 in [12]), on the probability space $([0,1],\nu_{1/p})$,

$$\frac{1}{m^{1/p}}\sum_{k=1}^{m}\big(G_X(\theta_{1/p}^k)-\nu_{1/p}(G_X)\big)$$

converges in distribution to a non-degenerate stable distribution of order p provided $G_X(0)\neq \nu(G_X)$ (this condition is satisfied here, because $G_X(0)=0$ and $\nu_{1/p}(G_X)>0$). As fully explained in Subsection 6.2, there exists a stationary Markov chain $(Y_i)_{i\in\mathbb{Z}}$ such that the distribution of $(Y_1,\ldots,Y_n)$ is the same as $(\theta_{1/p}^n,\ldots,\theta_{1/p})$ on the probability space $([0,1],\nu_{1/p})$. For such a chain,

$$\frac{1}{m^{1/p}}\sum_{j=1}^{m}(G_X(Y_j)-\pi)$$

converges in distribution to a non-degenerate stable distribution of order p. Moreover, as recalled in Subsection 6.2, there exist two positive constants A,B such that

$$\frac{A}{(k+1)^{p-1}}\leq \alpha_{1,\mathbf{Y}}(k)\leq \frac{B}{(k+1)^{p-1}}\,.$$

This completes the proof of Item 2.

It remains to prove (4.4). From Proposition 4.1 we get that

$$n\mathbb{E}\left(\left(\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}f(X_i,Y_j)\right)^2\right)\leq \sum_{i=0}^{\infty}\left(\frac{16}{m}\sum_{j=0}^{m-1}\min\{\alpha_{1,\mathbf{X}}(i),\alpha_{1,\mathbf{Y}}(j)\}\right)\,. \tag{4.26}$$

Note that

$$\frac{16}{m}\sum_{j=0}^{m-1}\min\{\alpha_{1,\mathbf{X}}(i),\alpha_{1,\mathbf{Y}}(j)\}\leq 16\alpha_{1,\mathbf{X}}(i)\,, \tag{4.27}$$

and that, for any positive integer i, since $\alpha_{1,\mathbf{Y}}(j)\to 0$ as $j\to\infty$,

$$\lim_{m\to\infty}\frac{16}{m}\sum_{j=0}^{m-1}\min\{\alpha_{1,\mathbf{X}}(i),\alpha_{1,\mathbf{Y}}(j)\}=0\,, \tag{4.28}$$

by Cesaro's lemma. Since $\sum_{i>0}\alpha_{1,\mathbf{X}}(i)<\infty$, it follows from (4.26), (4.27), (4.28), and the dominated convergence theorem that

$$\lim_{n\to\infty}n\mathbb{E}\left(\left(\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}f(X_i,Y_j)\right)^2\right)=0\,,$$

proving (4.4).

5 Other related results

5.1 Testing the stochastic domination by a known distribution

We briefly describes here a procedure for testing the (weak) domination by a known distribution μ. Let $H(t) = \mu((t, \infty))$, and let $(X_i)_{i \in \mathbb{Z}}$ be a stationary 2-ergodic sequence of real-valued random variables. Let $\pi = \mathbb{E}(H(X_0))$, and define

$$\bar{H}_n = \frac{1}{n} \sum_{i=1}^{n} H(X_i) \,.$$

If

$$\sum_{k=1}^{\infty} \alpha_{1,\mathbf{X}}(k) < \infty \,, \tag{5.1}$$

then the random variables $\sqrt{n}(\bar{H}_n - \pi)$ converge in distribution to $\mathcal{N}(0, V_1)$, where

$$V_1 = \mathrm{Var}(H(X_0)) + 2 \sum_{k=1}^{\infty} \mathrm{Cov}(H(X_0), H(X_k)) \,. \tag{5.2}$$

Now, we want to test $H_0 : \pi = 1/2$ (no relation of weak domination between μ and $\mathbb{P}_X$) against one of the possible alternatives: $H_1 : \pi \neq 1/2$ (one of the two distributions weakly dominates the other one), $H_1 : \pi > 1/2$ (μ weakly dominates $\mathbb{P}_X$), $H_1 : \pi < 1/2$ ($\mathbb{P}_X$ weakly dominates μ).

Again, one cannot use directly the statistics $\sqrt{n}(\bar{H}_n - 1/2)$ to test $H_0 : \pi = 1/2$ versus one of the possibles alternatives, because under H_0 the asymptotic distribution of $\sqrt{n}(\bar{H}_n - 1/2)$ depends of the unknown quantity V_1 defined in (5.2).

To estimate V_1, we take

$$V_{1,n} = \hat{\gamma}(0) + 2 \sum_{k=1}^{a_n} \hat{\gamma}(k)$$

for some sequence $a_n = o(\sqrt{n})$, where

$$\hat{\gamma}(k) = \frac{1}{n} \sum_{i=1}^{n-k} (H(X_i) - \bar{H}_n)(H(X_{i+k}) - \bar{H}_n) \,.$$

Now, as in Proposition 2.1, if (5.1) holds for $\alpha_{2,\mathbf{X}}(k)$ instead of $\alpha_{1,\mathbf{X}}(k)$, then v_n converges in $\mathbb{L}^2$ to v. It follows that, under $H_0 : \pi = 1/2$, if $v > 0$, the random variables

$$T'_n = \frac{\sqrt{n}(\bar{H}_n - 1/2)}{\sqrt{\max\{V_{1,n}, 0\}}}$$

converges in distribution to $\mathcal{N}(0, 1)$.

5.2 The case of two adjacent samples from the same time series

Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of real-valued random variables. Let $m = m_n$ be a sequence of positive integers such that $m_n \to \infty$ as $n \to \infty$. We consider here the U-statistic

$$U_n = U_{n,m} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=n+1}^{n+m} \mathbf{1}_{X_i<Y_j}\,,$$

where $Y_j = f(X_j)$, f being a monotonic function. This statistic can be used to test if there is a change in the time series after the time n, and more precisely if Y_{n+1} weakly dominates X_1. Let $\pi = \mathbb{P}(X_1 < f(X_1^*))$, where X_1^* is an independent copy of X_1. Let also $H_Y(t) = \mathbb{P}(f(X_1) > t)$ and $G_X(t) = \mathbb{P}(X_1 < t)$.

Finding the asymptotic behavior of U_n is a slightly more difficult problem than for two independent samples, because we cannot use the independence to control the degenerate U-statistic as in Proposition 4.1. However, it can be done by using a more restrictive coefficient than $\alpha_{2,\mathbf{X}}$.

Definition 5.1. Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables, and let $\mathcal{F}_0 = \sigma(X_k, k \le 0)$. Let P be the law of X_0 and $P_{(X_i,X_j)}$ be the law of (X_i, X_j). Let $P_{X_k|\mathcal{F}_0}$ be the conditional distribution of X_k given $\mathcal{F}_0$, and let $P_{(X_i,X_j)|\mathcal{F}_0}$ be the conditional distribution of (X_i, X_j) given $\mathcal{F}_0$. Define the random variables

$$\begin{aligned} b(k) &= \sup_{x\in\mathbb{R}} \left|P_{X_k|\mathcal{F}_0}(f_x)\right| , \\ b(i,j) &= \sup_{(x,y)\in\mathbb{R}^2} \left|P_{(X_i,X_j)|\mathcal{F}_0}\left(f_x \otimes f_y\right) - P_{(X_i,X_j)}\left(f_x \otimes f_y\right)\right| , \end{aligned}$$

where the function f_x has been defined in Definition 2.1. Define now the coefficients

$$\beta_{1,\mathbf{X}}(k) = \mathbb{E}(b(k)) \text{ and } \beta_{2,\mathbf{X}}(k) = \max\left\{\tilde\beta_{1,\mathbf{X}}(k), \sup_{i>j\ge k}\mathbb{E}((b(i,j)))\right\} .$$

Of course, by definition of the coefficients, we always have that $\alpha_{2,\mathbf{X}}(k) \le \beta_{2,\mathbf{X}}(k)$. The coefficients $\beta_{2,\mathbf{X}}$ are weaker than the usual β-mixing coefficients of $(X_i)_{i\in\mathbb{Z}}$. Many examples of non-mixing process for which $\beta_{2,\mathbf{X}}$ can be computed are given in [8]. For all the examples that we mention in the present paper, the coefficient $\beta_{2,\mathbf{X}}(k)$ is of the same order than the coefficient $\alpha_{2,\mathbf{X}}(k)$ (see the paper [4] for the example of the intermittent maps described in Subsection 6.2).

Theorem 5.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of real-valued random variables. Assume that*

$$\sum_{k=1}^{\infty} \beta_{2,\mathbf{X}}(k) < \infty\,. \tag{5.3}$$

Let (m_n) be a sequence of positive integers satisfying (1.3)*. Then $\sqrt{n}(U_n - \pi)$ converge in distribution to $\mathcal{N}(0, V)$, with*

$$V = \mathrm{Var}(H_Y(X_0)) + 2\sum_{k=1}^{\infty} \mathrm{Cov}(H_Y(X_0), H_Y(X_k)) \\ + c\left(\mathrm{Var}(G_X(f(X_0))) + 2\sum_{k=1}^{\infty} \mathrm{Cov}(G_X(f(X_0)), G_X(f(X_k)))\right). \quad (5.4)$$

For the estimation of V, one can prove the following analogue of Proposition 2.1.

Proposition 5.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be an independent stationary sequence, and let $Y_j = f(X_j)$ where f is a monotonic function. Let (m_n) be a sequence of positive integers satisfying* (1.3)*. Assume that* (5.3) *is satisfied. Let*

$$H_m(t) = \frac{1}{m}\sum_{j=n+1}^{n+m} \mathbf{1}_{Y_j>t} \quad \textit{and} \quad G_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{X_i<t}\,,$$

and

$$\hat{\gamma}_X(k) = \frac{1}{n}\sum_{i=1}^{n-k}(H_m(X_i)-\bar{H}_m)(H_m(X_{i+k})-\bar{H}_m)\,,\ \hat{\gamma}_Y(\ell) = \frac{1}{m}\sum_{j=n+1}^{n+m-\ell}(G_n(Y_j)-\bar{G}_n)(G_n(Y_{j+\ell})-\bar{G}_n)\,,$$

where $n\bar{H}_m = \sum_{i=1}^{n} H_m(X_i)$ and $m\bar{G}_n = \sum_{j=n+1}^{n+m} G_n(Y_j)$. Let also (a_n) and (b_n) be two sequences of positive integers tending to infinity as n tends to infinity, such that $a_n = o(\sqrt{n}/\log(n))$ and $b_n = o(\sqrt{m_n}/\log(m_n))$. Then

$$V_n = \hat{\gamma}_X(0) + 2\sum_{k=1}^{a_n} \hat{\gamma}_X(k) + \frac{n}{m_n}\left(\hat{\gamma}_Y(0) + 2\sum_{\ell=1}^{b_n} \hat{\gamma}_Y(\ell)\right)$$

converges in $\mathbb{L}^2$ to the quantity V defined in (5.4)*.*

Combining Theorem 5.1 and Proposition 5.2, we obtain that, under $H_0 : \pi = 1/2$, if $V > 0$, the random variables

$$T_n = \frac{\sqrt{n}(U_n - 1/2)}{\sqrt{\max\{V_n, 0\}}} \quad (5.5)$$

converge in distribution to $\mathcal{N}(0, 1)$.

The proof of Proposition 5.2 follows the same lines as that of Proposition 2.1, so we do not give the details here. The only main change concerns Step 2, Inequalities (4.18) and (4.19), where we used the independence between the samples. Here instead, one can use the fact that

$$\left\|\sup_{x\in\mathbb{R}} |H_n(x) - H(x)|\right\|_2^2 \leq C\,(\log(m))^2\, \frac{\sum_{k=0}^{m-1} \beta_{1,\mathbf{X}}(k)}{m}\,.$$

This can be easily proved by following the proof of Proposition 7.1 in Rio [14] up to (7.8) and by using the computations in the proof of Theorem 2.1 in [3].

The proof of Theorem 5.1 follows also the line of that of Theorem 1.1; the only non-trivial change concerns Proposition 4.1. Here, we have to deal with

$$L_n = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=n+1}^{n+m} f(X_i, Y_j)$$

where f is defined in (4.1). We want to prove that $n\mathbb{E}(L_n^2)$ tends to 0 as n tends to infinity. As in the proof of Proposition 4.1, we start from the elementary inequality

$$\mathbb{E}(L_n^2) \leq \frac{4}{n^2m^2} \sum_{i=1}^{n} \sum_{j=n+1}^{n+m} \sum_{k=i}^{n} \sum_{\ell=j}^{n+m} \mathbb{E}(f(X_i, Y_j) f(X_k, Y_\ell)) \,. \tag{5.6}$$

Let $(X_i^*)_{i\in\mathbb{Z}}$ be an independent copy of $(X_i^*)_{i\in\mathbb{Z}}$ and $Y_i^* = f(X_i^*)$, and write

$$\begin{aligned} \mathbb{E}(f(X_i, Y_j) f(X_k, Y_\ell)) = \mathbb{E}(f(X_i, Y_j) f(X_k, Y_\ell)) - \mathbb{E}(f(X_i, Y_j^*) f(X_k, Y_\ell^*)) \\ + \mathbb{E}(f(X_i, Y_j^*) f(X_k, Y_\ell^*)) \,. \end{aligned} \tag{5.7}$$

Now, using the independence, the second term on the right-hand side of (5.7) can be handled exactly as in Proposition 4.1. It remains to deal with the first term in the right-hand side of (5.7). Note that

$$\begin{aligned} &|\mathbb{E}(f(X_i, Y_j) f(X_k, Y_\ell)) - \mathbb{E}(f(X_i, Y_j^*) f(X_k, Y_\ell^*))| \\ &\quad \leq \int |\mathbb{E}(f(x_i, Y_j) f(x_k, Y_\ell) | X_i = x_i, X_k = x_k) - \mathbb{E}(f(x_i, Y_j) f(x_k, Y_\ell))| P_{(X_i, X_k)}(dx_i, dx_k) \end{aligned} \tag{5.8}$$

For any fixed x, the function $y \to h_x(y) = f(x, y)$ is a bounded variation function whose variation is bounded by 2, and $\mathbb{E}(h_x(X_i)) = 0$. Proceeding as in Lemma 1 of [7], one can then prove that

$$\begin{aligned} \int |\mathbb{E}(f(x_i, Y_j) f(x_k, Y_\ell) | X_i = x_i, X_k = x_k) - \mathbb{E}(f(x_i, Y_j) f(x_k, Y_\ell))| P_{(X_i, X_k)}(dx_i, dx_k) \\ \leq 4\beta_{2,\mathbf{X}}(j-k) \,. \end{aligned} \tag{5.9}$$

Considering separately the two cases $j - k > n^{1/4}$ or $j - k \leq n^{1/4}$, we get the upper bound

$$\begin{aligned} \frac{1}{n^2m^2} \sum_{i=1}^{n} \sum_{j=n+1}^{n+m} \sum_{k=i}^{n} \sum_{\ell=j}^{n+m} |\mathbb{E}(f(X_i, Y_j) f(X_k, Y_\ell) - \mathbb{E}(f(X_i, Y_j^*) f(X_k, Y_\ell^*))| \\ \leq \frac{4}{\sqrt{n}m} + \frac{4}{m} \sum_{k > n^{1/4}} \beta_{2,\mathbf{X}}(k) \end{aligned} \tag{5.10}$$

from which we easily derive that $n\mathbb{E}(L_n^2)$ tends to 0 as n tends to infinity (since (1.3) and (5.3) are satisfied).

6 Simulations

6.1 Example 1: functions of an auto-regressive process

In this section, we first simulate $(Z_1, \ldots, Z_n)$, according to the simple AR(1) equation

$$Z_{k+1} = \frac{1}{2}\left(Z_k + \varepsilon_{k+1}\right) ,$$

where Z_1 is uniformly distributed over $[0, 1]$, and $(\varepsilon_i)_{i\geq 2}$ is a sequence of iid random variables with distribution $\mathcal{B}(1/2)$, independent of Z_1.

One can check that the transition Kernel of the chain $(Z_i)_{i\geq 1}$ is

$$K(f)(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right) ,$$

and that the uniform distribution on $[0, 1]$ is the unique invariant distribution by K. Hence, the chain $(Z_i)_{i\geq 1}$ is strictly stationary.

It is well known that the chain $(Z_i)_{i\geq 1}$ is not α-mixing in the sense of Rosenblatt [15] (see for instance [1]). However, one can prove that the coefficients $\alpha_{2,\mathbf{X}}$ of the chain $(Z_i)_{i\geq 1}$ are such that

$$\alpha_{2,\mathbf{Z}}(k) \leq 2^{-k}$$

(and the same upper bound is true for $\beta_{2,\mathbf{Z}}(k)$, see for instance Section 6.1 in [8]).

Let now Q_{μ,σ^2} be the inverse of the cumulative distribution function of the law $\mathcal{N}(\mu, \sigma^2)$. Let then

$$X_i = Q_{\mu_1,\sigma_1^2}(Z_i) .$$

The sequence $(X_i)_{i\leq 1}$ is also a stationary Markov chain (as an invertible function of a stationary Markov chain), and one can easily check that $\alpha_{2,\mathbf{X}}(k) = \alpha_{2,\mathbf{Z}}(k)$. By construction, X_i is $\mathcal{N}(\mu, \sigma^2)$-distributed, but the sequence $(X_i)_{i\geq 1}$ is not a Gaussian process (otherwise it would be mixing in the sense of Rosenblatt).

Next, we simulate iid random variables $(Y_1, \ldots, Y_m)$ with common distribution $\mathcal{N}(\mu_2, \sigma_2^2)$.

Note that, if $\mu_1 = \mu_2$, the hypothesis $H_0 : \pi = 1/2$ is satisfied and the random variables T_n defined in (2.2) converge in distribution to $\mathcal{N}(0, 1)$.

We shall study the behavior of T_n for $\sigma_1^2 = 4$, $\sigma_2^2 = 1$ and different choices of μ_1, μ_2, a_n and b_n. As explained in Section 2, the quantities a_n and b_n may be chosen by analyzing the graph of the estimated auto-covariances $\hat{\gamma}_X(k)$ and $\hat{\gamma}_Y(\ell)$. For this examples, these graphs suggest a choice of $a_n = 3$ or 4 and $b_n = 0$ (see Figure 1).

We shall compute T_n for different choices of (n, m) (from $(150, 100)$ to $(750, 500)$ with $n/m = 1.5$). We estimate the three quantities $\mathrm{Var}(T_n)$, $\mathbb{P}(T_n > 1.645)$ and $\mathbb{P}(|T_n| > 1.96)$ via a classical Monte-Carlo procedure, by averaging over $N = 2000$ independent trials.

If $\mu_1 = \mu_2$, we say that $\mathbb{P}(T_n > 1.645)$ is level 1 (the level of the one-sided test) and $\mathbb{P}(|T_n| > 1.96)$ is level 2 (the level of the two-sided test). If a_n, b_n are well chosen, the estimated variance should be close to 1 and the estimated levels 1 and 2 should be close to 0.05.

If $\mu_1 \neq \mu_2$, we say that $\mathbb{P}(T_n > 1.645)$ is power 1 (the power of the one-sided test) and $\mathbb{P}(|T_n| > 1.96)$ is power 2 (the power of the two-sided test).

- Case $\mu_1 = \mu_2 = 0$ and $a_n = 0, b_n = 0$ (no correction):

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	2.13	2.199	2.118	2.112	2.273
Estimated level 1	0.129	0.145	0.121	0.132	0.14
Estimated level 2	0.177	0.194	0.18	0.18	0.191

Here, since $a_n = 0$, we do not estimate any of the covariance terms $\gamma_X(\ell)$; hence, we compute T_n as if both series $(X_i)_{i\geq 1}$ and $(Y_j)_{j\geq 1}$ were uncorrelated. The result is that the estimated variance is too large (around 2.1 whatever (n, m)) and so are the estimated levels 1 and 2 (around 0.13 for level 1 and 0.18 for level 2, whatever (n, m)). This means that the-one sided or two-sided tests build with T_n will reject the null hypothesis too often.

- Case $\mu_1 = \mu_2 = 0$ and $a_n = 3, b_n = 0$:

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	1.227	1.148	1.172	1.119	1.018
Estimated level 1	0.071	0.061	0.065	0.06	0.046
Estimated level 2	0.074	0.069	0.071	0.066	0.054

As suggested by the graphs of the estimated auto-covariances (see Figure 1), the choice $a_n = 3, b_n = 0$ should give a reasonable estimation of V. This is indeed the case: the estimated variance is now around 1.1, the estimated level 1 is around 0.06 and the estimated level 2 around 0.07.

- Case $\mu_1 = \mu_2 = 0$ and $a_n = 4, b_n = 0$:

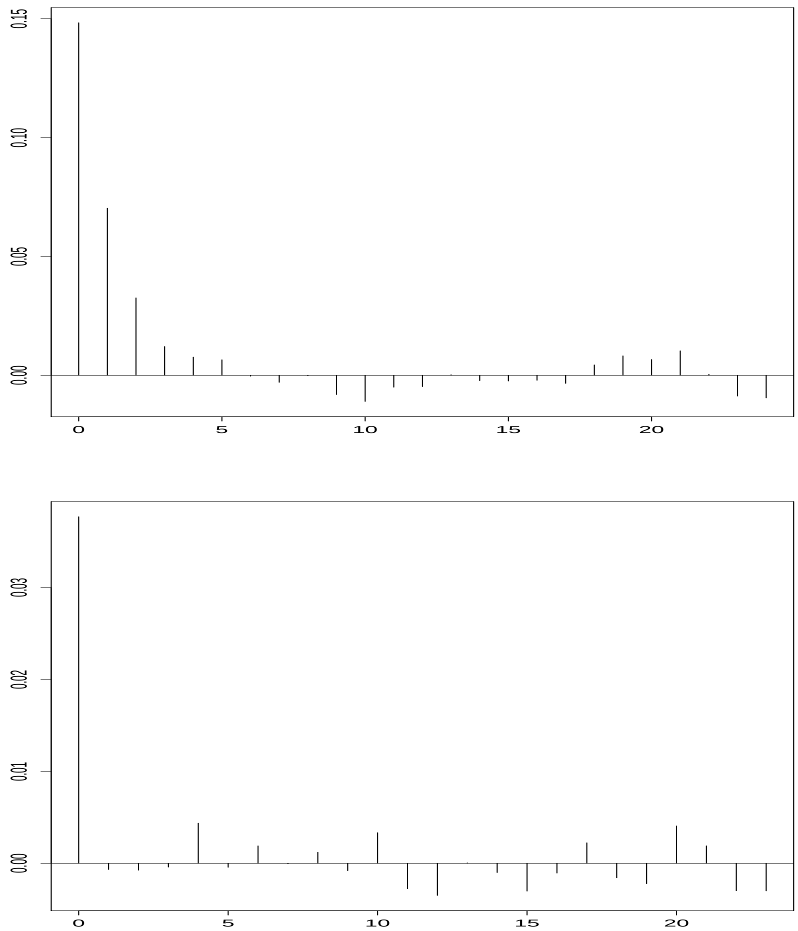


Figure 1: Estimation of the $\gamma_X(k)$'s (top) and $\gamma_Y(\ell)$'s (bottom) for Example 1, with $\mu_1 = \mu_2 = 0$, $n = 300$ and $m = 200$.

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	1.151	1.11	1.121	1.03	0.987
Estimated level 1	0.06	0.05	0.058	0.049	0.042
Estimated level 2	0.062	0.067	0.06	0.057	0.046

Here, we see that the choice $a_n = 4, b_n = 0$ works well also, and seems even slightly better than the choice $a_n = 3, b_n = 0$.

• Case $\mu_1 = 0$, $\mu_2 = 0.2$ and $a_n = 4, b_n = 0$:

In this example, H_0 is not satisfied, and $\pi > 1/2$. However, μ_2 is close to μ_1, and since $\sigma_1 = 2$ and $\sigma_2 = 1$, Y_1 does not stochastically dominate X_1 (there is only stochastic domination in a weak sense). We shall see how the two tests are able to see this weak domination, by giving some estimation of the powers.

$n;m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	1.229	1.136	1.107	1.066	1.059
Estimated power 1	0.185	0.276	0.352	0.4	0.476
Estimated power 2	0.13	0.187	0.25	0.291	0.344

As one can see, the powers of the one sided and two-sided tests are always greater than 0.05, as expected. Still as expected, these powers increase with the size of the samples. For $n = 750, m = 500$, the power of the one-sided test is around 0.47 and that of the two-sided test is around 0.34.

6.2 Example 2: Intermittent maps

For γ in $]0, 1[$, we consider the intermittent map θ_γ from $[0, 1]$ to $[0, 1]$, introduced by Liverani, Saussol and Vaienti [13]:

$$\theta_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases} \tag{6.1}$$

It follows from [13] that there exists a unique absolutely continuous θ_γ-invariant probability measure ν_γ, with density h_γ.

Let us briefly describe the Markov chain associated with θ_γ, and its properties. Let first K_γ be the Perron-Frobenius operator of θ_γ with respect to ν_γ, defined as follows: for any functions u, v in $\mathbb{L}^2([0, 1], \nu_\gamma)$

$$\nu_\gamma(u \cdot v \circ \theta_\gamma) = \nu_\gamma(K_\gamma(u) \cdot v)\,. \tag{6.2}$$

The relation (6.2) states that K_γ is the adjoint operator of the isometry $U : u \mapsto u \circ \theta_\gamma$ acting on $\mathbb{L}^2([0, 1], \nu_\gamma)$. It is easy to see that the operator K_γ is a transition kernel, and that ν_γ is invariant by K_γ. Let now $(\xi_i)_{i\geq 1}$ be a stationary Markov chain with invariant measure ν_γ and transition kernel K_γ. It is well known that on the probability space $([0, 1], \nu_\gamma)$, the random vector $(\theta_\gamma, \theta_\gamma^2, \ldots, \theta_\gamma^n)$ is distributed as $(\xi_n, \xi_{n-1}, \ldots, \xi_1)$. Now it is proved in [5] that there exist two positive constants A, B such that

$$\frac{A}{(n+1)^{(1-\gamma)/\gamma}} \leq \alpha_{2,\xi}(n) \leq \frac{B}{(n+1)^{(1-\gamma)/\gamma}}$$

(this is also true for the coefficient $\beta_{2,\xi}(n)$, see [4]). Moreover, the chain $(\xi_i)_{i\geq 1}$ is not α-mixing in the sense of Rosenblatt [15].

Let $X_i(x) = \theta_{\gamma_1}^i(x)$ and $Y_j(y) = \theta_{\gamma_2}^j(y)$. On the probability space $([0,1] \times [0,1], \nu_{\gamma_1} \otimes \nu_{\gamma_2})$ the two sequences $(X_i)_{i \geq 1}$ and $(Y_j)_{j \geq 1}$ are independent.

Note that, if $\gamma_1 = \gamma_2$, the hypothesis $H_0 : \pi = 1/2$ is satisfied and the random variables T_n defined in (2.2) converge in distribution to $\mathcal{N}(0,1)$.

We shall study the behavior of T_n for different choices of γ_1, γ_2, a_n and b_n. As explained in Section 2, the quantities a_n and b_n may be chosen by analyzing the graph of the estimated auto-covariances $\hat{\gamma}_X(k)$ and $\hat{\gamma}_Y(\ell)$.

As in Subsection 6.1, we shall compute T_n for different choices of (n,m) (from $(150,100)$ to $(750,500)$ with $n/m = 1.5$). We estimate the three quantities $\mathrm{Var}(T_n)$, $\mathbb{P}(T_n > 1.645)$ and $\mathbb{P}(|T_n| > 1.96)$ via a classical Monte-Carlo procedure, by averaging over $N = 2000$ independent trials.

If $\gamma_1 = \gamma_2$, we say that $\mathbb{P}(T_n > 1.645)$ is level 1 (the level of the one-sided test) and $\mathbb{P}(|T_n| > 1.96)$ is level 2 (the level of the two-sided test). If a_n, b_n are well chosen, the estimated variance should be close to 1 and the estimated levels 1 and 2 should be close to 0.05.

If $\gamma_1 \neq \gamma_2$, we say that $\mathbb{P}(T_n > 1.645)$ is power 1 (the power of the one-sided test) and $\mathbb{P}(|T_n| > 1.96)$ is power 2 (the power of the two-sided test).

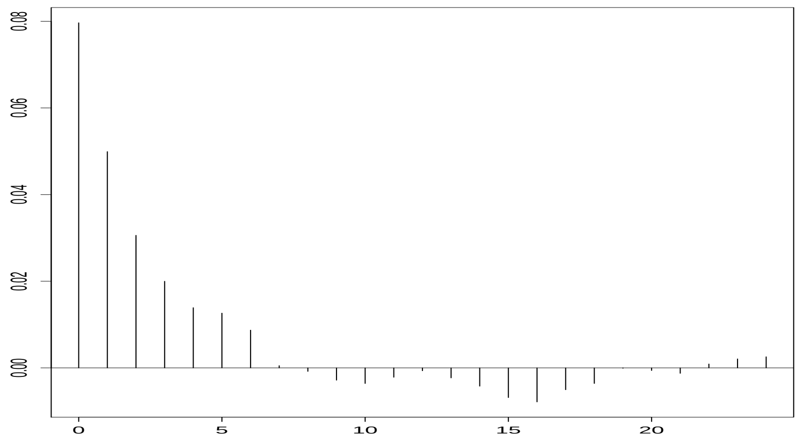
Note that we cannot simulate exactly $(X_i)_{i \geq 1}$ and $(Y_j)_{j \geq 1}$, since the distribution $\nu_{1/4}$ is unknown. We start by picking X_1 and Y_1 independently over $[0, 0.05]$ (in order to start near the neutral fixed point 0) and then generate $X_i = \theta_{\gamma_1}^{i-1}(X_1)$ and $Y_j = \theta_{\gamma_2}^{j-1}(Y_1)$ for $i > 1, j > 1$. We shall see that the tests are robust to this (slight) lack of stationarity.

- Case $\gamma_1 = \gamma_2 = 1/4$ and $a_n = 0, b_n = 0$ (no correction):

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	5.172	4.658	4.858	4.764	4.751
Estimated level 1	0.204	0.196	0.19	0.206	0.218
Estimated level 2	0.358	0.359	0.361	0.361	0.37

Here again, since $a_n = b_n = 0$, we compute T_n as if both series $(X_i)_{i \geq 1}$ and $(Y_j)_{j \geq 1}$ were uncorrelated. It leads to a disastrous result, with an estimated variance around 4.8, an estimated level 1 around 0.2, and an estimated level 2 around 0.36. This means that the-one sided or two-sided tests build with T_n will reject the null hypothesis much too often.

- Case $\gamma_1 = \gamma_2 = 1/4$ and $a_n = 4, b_n = 4$:



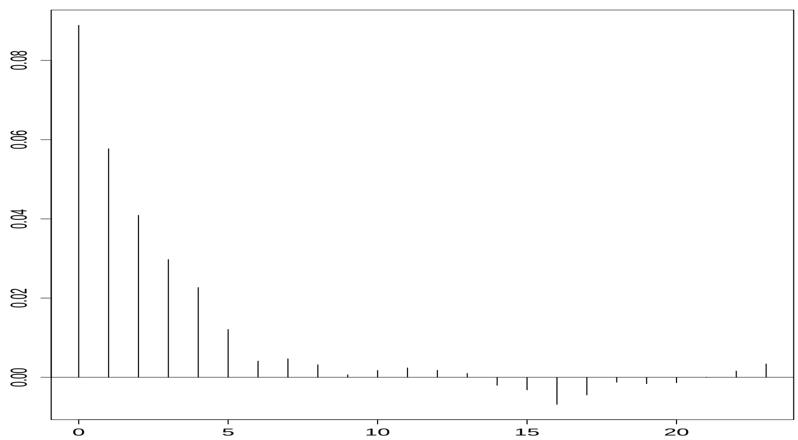


Figure 2: Estimation of the $\gamma_X(k)$'s (top) and $\gamma_Y(\ell)$'s (bottom) for Example 2, with $\gamma_1 = \gamma_2 = 1/4$, $n = 300$ and $m = 200$.

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	1.352	1.308	1.329	1.324	1.28
Estimated level 1	0.057	0.0645	0.062	0.069	0.067
Estimated level 2	0.089	0.082	0.087	0.087	0.075

As suggested by the graphs of the estimated auto-covariances (see Figure 2), the choice $a_n = 4, b_n = 4$ should give a reasonable estimation of V. We see that the procedure is much better than if no correction is made: the estimated variance is now around 1.3, the estimated level 1 is around 0.065 and the estimated level 2 is around 0.085.

- Case $\gamma_1 = \gamma_2 = 1/4$ and $a_n = 5, b_n = 5$:

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	1.349	1.249	1.164	1.181	1.185
Estimated level 1	0.057	0.062	0.055	0.055	0.063
Estimated level 2	0.084	0.077	0.068	0.071	0.07

For $a_n = 5, b_n = 5$, the results seem slightly better than for $a_n = 4, b_n = 4$. Even for large samples, the estimated level 2 is still larger than the expected level: around 0.07 instead of 0.05 (but 0.07 has to be compared to 0.36 which is the estimated level 2 without correction). Recall that the coefficients $\alpha_{2,\mathbf{X}}(n)$ and $\alpha_{2,\mathbf{Y}}(n)$ are exactly of order $n^{-1/3}$, which is quite slow. The procedure can certainly be improved for larger n, m by estimating more covariance terms (in accordance with Proposition 2.1).

- Case $\gamma_1 = 0.25, \gamma_2 = 0.1$ and $a_n = 5, b_n = 4$.

We first estimate the quantity $\pi = \mathbb{P}(X_1 < Y_1)$ via a realization of

$$\frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{1}_{X_i < Y_j} .$$

For $n = m = 30000$, this gives an estimation around 0.529 for π, meaning that Y weakly dominates X. Let us see how the tests based on T_n can see this departure from $H_0 : \pi = 1/2$.

Based on the estimated covariances, a reasonable choice is $a_n = 5, b_n = 4$. This choice leads to the following estimated powers (based on $N = 2000$ independent trials).

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	1.24	1.2	1.2	1.23	1.117
Estimated power 1	0.11	0.165	0.185	0.226	0.241
Estimated power 2	0.091	0.12	0.13	0.158	0.17

As one can see, the powers of the one sided and two-sided tests are always greater than 0.05, as expected. Still as expected, these powers increase with the size of the samples. For $n = 750, m = 500$, the power of the one-sided test is around 0.24 and that of the two-sided test is around 0.17. This has to be compared with the estimated levels when T_n is centered on 0.529 instead of 0.5 (recall that 0.529 is the previous estimation of π on very large samples): in that case, for $n = 750, m = 500$ and $a_n = 5, b_n = 4$, one gets an estimated level 1 = 0.059, and an estimated level 2 = 0.068, which are both close to 0.05 as expected.

6.3 Intermittents maps: the case of two adjacent samples

Here, we shall illustrate the result of Subsection 5.2. As in Subsection 6.2, we first simulate X_1 according to the uniform distribution over $[0, 0.05]$, and then generate $X_i = \theta_\gamma^{i-1}(Z_1)$, but now for the indexes $i \in \{2, \ldots, n+m\}$. For $j \in \{n+1, \ldots, n+m\}$, we take $Y_j = f(X_j)$ for a given monotonic function f.

In our first experiment, we simply take $f = \mathrm{Id}$, so that $Y_j = X_j$. Hence the assumption $H_0 : \pi = 1/2$ is satisfied.

For the second experiment, we take $f(x) = x^{4/5}$, so that $f(X_1) > X_1$ almost surely. Hence $\pi > 1/2$.

Let T_n be defined in (5.5). We follow the same scheme as in Subsection 6.2, and we estimate the three quantities $\mathrm{Var}(T_n)$, $\mathbb{P}(T_n > 1.645)$ and $\mathbb{P}(|T_n| > 1.96)$ via a classical Monte-Carlo procedure, by averaging over $N = 2000$ independent trials.

- Case $\gamma = 1/4$, $f = \mathrm{Id}$ and $a_n = 0, b_n = 0$ (no correction):

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	4.851	4.908	4.551	4.926	4.85
Estimated level 1	0.294	0.276	0.252	0.256	0.251
Estimated level 2	0.368	0.373	0.354	0.366	0.362

Here, as in the corresponding two-sample case (see the table for $\gamma_1 = \gamma_2 = 1/4$), the uncorrected tests lead to a disastrous result, with an estimated level 1 around 0.25 and an estimated level 2 around 0.36.

- Case $\gamma = 1/4$, $f = \mathrm{Id}$ and $a_n = 5, b_n = 5$:

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	1.369	1.302	1.272	1.205	1.171
Estimated level 1	0.114	0.097	0.086	0.081	0.075
Estimated level 2	0.109	0.096	0.079	0.072	0.066

The corrected tests with the choice $a_n = b_n = 5$ give a much better result than if no correction is made. One can observe, however, that the estimated levels are still too high ($\geq 8\%$) for moderately large samples (up to $n = 450, m = 300$). The results seem less good than in the corresponding two-sample case (see the table for $\gamma_1 = \gamma_2 = 1/4$), which is of course not surprising.

- Case $\gamma = 1/4$, $f(x) = x^{4/5}$ and $a_n = 5, b_n = 5$:

$n; m$	150 ; 100	300 ; 200	450 ; 300	600 ; 400	750 ; 500
Estimated variance	1.596	1.448	1.334	1.311	1.451
Estimated power 1	0.321	0.385	0.451	0.49	0.548
Estimated power 2	0.242	0.293	0.34	0.383	0.442

As one can see, the powers of the one sided and two-sided tests are always greater than 0.05, as expected. Still as expected, these powers increase with the size of the samples. For $n = 750, m = 500$, the power of the one-sided test is around 0.55 and that of the two-sided test is around 0.44.

Acknowledgments. We thank Herold Dehling for helpful comments on a preliminary version of the paper.

References

[1] R. C. Bradley (1986), Basic properties of strong mixing conditions, *Dependence in probability and statistics. A survey of recent results. Oberwolfach, 1985.* E. Eberlein and M. S. Taqqu editors, Birkäuser, 165-192.

[2] S. Dédé, Théorèmes limites fonctionnels et estimation de la densité spectrale pour des suites stationnaires. Phd thesis, Université Pierre et Marie Curie, 2009. https://tel.archives-ouvertes.fr/tel-00440850

[3] J. Dedecker, An empirical central limit theorem for intermittent maps, *Probab. Theory Related Fields* **148** (2010) 177-195.

[4] J. Dedecker, H. Dehling, and M. S. Taqqu, Weak convergence of the empirical process of intermittent maps in $\mathbb{L}^2$ under long-range dependence, *Stochastics and Dynamics* **15** (2015) 29 pages.

[5] J. Dedecker, S. Gouëzel and F. Merlevède, Some almost sure results for unbounded functions of intermittent maps and their associated Markov chains, *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010) 796-821.

[6] J. Dedecker and F. Merlevède, A deviation bound for α-dependent sequences with applications to intermittent maps, to appear in Stochastics and Dynamics (2016).

[7] J. Dedecker and C. Prieur, New dependence coefficients. Examples and applications to statistics, *Probab. Theory Related Fields* **132** (2005) 203-236.

[8] J. Dedecker and C. Prieur, An empirical central limit theorem for dependent sequences, *Stochastic Process. Appl.* **117** (2007) 121-142.

[9] H. Dehling and R. Fried, Asymptotic distribution of two-sample empirical U-quantiles with applications to robust tests for shifts in location. *J. Multivariate Anal.* **105** (2012), 124170.

[10] H. Dehling, A. Rooch and M. S. Taqqu, Non-parametric change-point tests for long-range dependent data. *Scand. J. Stat.* **40** (2013) 153173.

[11] I. Dewan and B. L. S. Prakasa Rao, Mann-Whitney test for associated sequences, *Ann. Inst. Statist. Math.* **55** (2003) 111-119.

[12] S. Gouëzel, Central limit theorems and stable laws for intermittent maps, *Probab. Theory Related Fields* **128** (2004) 82-122.

[13] C. Liverani, B. Saussol and S. Vaienti (1999), A probabilistic approach to intermittency, *Ergodic Theory Dynam. Systems* **19** 671-685.

[14] E. Rio, *Théorie asymptotique des processus aléatoires faiblement dépendants*, Mathématiques et Applications **31** (Springer-Verlag, 2000).

[15] M. Rosenblatt, A central limit theorem and a strong mixing condition, *Proc. Nat. Acad. Sci. U. S. A.* **42** (1956) 43-47.

[16] R. J. Serfling, The Wilcoxon two-sample statistic on strongly mixing processes, *Ann. Math. Statist.* **39** (1968) 1202-1209.