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AN ELEMENTARY PROOF OF THE POSITIVITY OF THE INTERTWINING OPERATOR IN ONE–DIMENSIONAL TRIGONOMETRIC DUNKL THEORY

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ABSTRACT. This note is devoted to the intertwining operator in the one–dimensional trigonometric Dunkl setting. We obtain a simple integral expression of this operator and deduce its positivity.

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1. INTRODUCTION

We use the lecture notes [6] as a general reference about trigonometric Dunkl theory. In dimension 1, this special function theory is a deformation of Fourier analysis on $\mathbb{R}$, depending on two complex parameters k_1 and k_2, where the classical derivative is replaced by the Cherednik operator

$$\begin{aligned} Df(x) &= \left(\tfrac{d}{dx}\right) f(x) + \left\{\tfrac{k_1}{1-e^{-x}} + \tfrac{2\,k_2}{1-e^{-2x}}\right\}\left\{f(x)-f(-x)\right\} - \left(\tfrac{k_1}{2}+k_2\right) f(x) \\ &= \left(\tfrac{d}{dx}\right) f(x) + \left\{\tfrac{k_1+k_2}{2}\coth\tfrac{x}{2} + \tfrac{k_2}{2}\tanh\tfrac{x}{2}\right\}\left\{f(x)-f(-x)\right\} - \left(\tfrac{k_1}{2}+k_2\right) f(-x)\,, \end{aligned}$$

the Lebesgue measure by $A(x)\,dx$, where

$$A(x) = |2\sinh\tfrac{x}{2}|^{2k_1}\,|2\sinh x|^{2k_2}\,,$$

and the exponential function $e^{i\lambda x}$ by the Opdam hypergeometric function

$$G_{i\lambda}(x) = \underbrace{\varphi_{2\lambda}^{k_1+k_2-\frac12,\,k_2-\frac12}\left(\tfrac{x}{2}\right)}_{{}_2\mathrm{F}_1(\frac{k_1}{2}+k_2+i\lambda,\frac{k_1}{2}+k_2-i\lambda;k_1+k_2+\frac12;-\sinh^2\frac{x}{2})} + \frac{\frac{k_1}{2}+k_2+i\lambda}{2k_1+2k_2+1}\,(\sinh x)\,\underbrace{\varphi_{2\lambda}^{k_1+k_2+\frac12,\,k_2+\frac12}\left(\tfrac{x}{2}\right)}_{{}_2\mathrm{F}_1(\frac{k_1}{2}+k_2+1+i\lambda,\frac{k_1}{2}+k_2+1-i\lambda;k_1+k_2+\frac32;-\sinh^2\frac{x}{2})}\,.$$

Here $\varphi_\lambda^{\alpha,\beta}(x)$ denotes the Jacobi function and ${}_2\mathrm{F}_1(a,b;c;Z)$ the classical hypergeometric function.

In a series of papers ([2], [5], [7], [3], [8], [9], [10], [11], …), Trimèche and his collaborators have studied an intertwining operator $\mathcal{V}\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$, which is characterized by

$$\mathcal{V}\circ\left(\tfrac{d}{dx}\right) = D\circ\mathcal{V} \quad\text{and}\quad \delta_0\circ\mathcal{V} = \delta_0\,,$$

and the dual operator $\mathcal{V}^t\colon C_c^\infty(\mathbb{R}) \longrightarrow C_c^\infty(\mathbb{R})$, which satisfies

$$\int_{-\infty}^{+\infty} \mathcal{V}f(x)\,g(x)\,A(x)\,dx = \int_{-\infty}^{+\infty} f(y)\,\mathcal{V}^t g(y)\,dy\,.$$

Let us mention in particular the following facts.

- *Eigenfunctions*. For every $\lambda\in\mathbb{C}$,

$$\mathcal{V}(x\longmapsto e^{i\lambda x}) = G_{i\lambda}\,.$$

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- *Explicit expression.* An integral representation of $\mathcal{V}$ was computed in [2] (and independently in [1]), under the assumption that $k_1 \geqslant 0$, $k_2 \geqslant 0$ with $k_1 + k_2 > 0$.
- *Analytic continuation.* It was shown in [3] that the intertwining operator $\mathcal{V}$ extends meromorphically with respect to $k \in \mathbb{C}^2$, with singularities in $\{k \in \mathbb{C}^2 \mid k_1 + k_2 + \frac{1}{2} \in -\mathbb{N}\}$.
- *Positivity.* On the one hand, the positivity of $\mathcal{V}$ was disproved in [2], by using the above–mentioned expression of $\mathcal{V}$ in the case $k_1 > 0$, $k_2 > 0$. On the other hand, the positivity of $\mathcal{V}$ was investigated in [8], [9], [10], [11] by using the positivity of a heat type kernel in the case $k_1 \geqslant 0$, $k_2 \geqslant 0$.

In Section 2, we obtain an integral representation of $\mathcal{V}$ and $\mathcal{V}^t$ when $\operatorname{Re} k_1 > 0$ and $\operatorname{Re} k_2 > 0$. The expression is simpler and the proof is quicker than the previous ones in [2] or [1]. In Section 3, we deduce the positivity of $\mathcal{V}$ and $\mathcal{V}^t$ when $k_1 > 0$, $k_2 > 0$, and comment on the positivity issue.

2. Integral representation of the intertwining operator

In this section, we resume the computations in [2, Section 2] and prove the following result.

Theorem 2.1. *Let* $k = (k_1, k_2) \in \mathbb{C}^2$ *with* $\operatorname{Re} k_1 > 0$ *and* $\operatorname{Re} k_2 > 0$. *Then*

$$\mathcal{V} f(x) = \int_{|y|<|x|} \mathcal{K}(x,y)\, f(y)\, dy \qquad \forall\, x \in \mathbb{R}^*$$

and

$$\mathcal{V}^t g(y) = \int_{|x|>|y|} \mathcal{K}(x,y)\, g(x)\, A(x)\, dx\,,$$

where

$$\begin{aligned} \mathcal{K}(x,y) = \tfrac{c}{4}\, A(x)^{-1} \int_{|y|}^{|x|} & \sigma(x,y,z)\, (\cosh \tfrac{z}{2} - \cosh \tfrac{y}{2})^{k_1-1} (\cosh x - \cosh z)^{k_2-1} \\ & \times (\sinh \tfrac{z}{2})\, dz\,, \end{aligned} \tag{1}$$

with

$$c = 2^{3k_1+3k_2}\, \frac{\Gamma(k_1+k_2+\frac{1}{2})}{\sqrt{\pi}\, \Gamma(k_1)\, \Gamma(k_2)} \tag{2}$$

and

$$\sigma(x,y,z) = (\operatorname{sign} x) \left\{ e^{\frac{x}{2}} (2 \cosh \tfrac{x}{2}) - e^{-\frac{y}{2}} (2 \cosh \tfrac{z}{2}) \right\}. \tag{3}$$

Proof. As observed in [2] and [5],

$$\mathcal{V} f(x) = \int_{-|x|}^{+|x|} \mathcal{K}(x,y)\, f(y)\, dy$$

is an integral operator, whose kernel

$$\mathcal{K}(x,y) = \tfrac{1}{4}\, K(\tfrac{x}{2}, \tfrac{y}{2}) + (\operatorname{sign} x) \left(\tfrac{k_1}{4} + \tfrac{k_2}{2}\right) A(x)^{-1}\, \widetilde{K}(\tfrac{x}{2}, \tfrac{y}{2}) - (\operatorname{sign} x)\, \tfrac{1}{2}\, A(x)^{-1} \tfrac{\partial}{\partial y}\, \widetilde{K}(\tfrac{x}{2}, \tfrac{y}{2}) \tag{4}$$

can be expressed in terms of the kernel

$$\begin{aligned} K(x,y) = {} & 2\, c\, A(2x)^{-1} |\sinh 2x| \\ & \times \int_{|y|}^{|x|} (\cosh z - \cosh y)^{k_1-1} (\cosh 2x - \cosh 2z)^{k_2-1} (\sinh z)\, dz \end{aligned} \tag{5}$$

of the intertwining operator in the Jacobi setting (see [4, Subsection 5.3]) and of its integral

$$\begin{aligned}\widetilde{K}(x,y) &= \int_{|y|}^{|x|} K(w,y)\,A(2w)\,dw \\ &= \tfrac{c}{k_2}\int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_1-1}\,(\cosh 2x-\cosh 2z)^{k_2}\,(\sinh z)\,dz\,.\end{aligned} \tag{6}$$

Let us integrate by parts (6) and differentiate the resulting expression with respect to y. This way, we obtain

$$\widetilde{K}(x,y) = \tfrac{4\,c}{k_1}\int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_1}\,(\cosh 2x-\cosh 2z)^{k_2-1}\,(\cosh z)\,(\sinh z)\,dz\,, \tag{7}$$

and

$$\begin{aligned}\tfrac{\partial}{\partial y}\,\widetilde{K}(x,y) = -\,4\,c\,(\sinh y)\int_{|y|}^{|x|}&(\cosh z-\cosh y)^{k_1-1}\,(\cosh 2x-\cosh 2z)^{k_2-1} \\ &\times\,(\cosh z)\,(\sinh z)\,dz\,.\end{aligned} \tag{8}$$

We conclude by substituting (5), (6), (7), (8) in (4) and more precisely (6), respectively (7) in

$$(\operatorname{sign} x)\,\tfrac{k_2}{2}\,A(x)^{-1}\,\widetilde{K}(\tfrac{x}{2},\tfrac{y}{2})\,,\quad\text{respectively}\quad(\operatorname{sign} x)\,\tfrac{k_1}{4}\,A(x)^{-1}\,\widetilde{K}(\tfrac{x}{2},\tfrac{y}{2})\,.$$

□

Remark 2.2. *Let $x,y\in\mathbb{R}$ such that $|x|>|y|$. The expression* (1) *extends meromorphically with respect to $k\in\mathbb{C}^2$, with singularities in $\{k\in\mathbb{C}^2\,|\,k_1+k_2+\frac{1}{2}\in-\mathbb{N}\}$, produced by the factor $\Gamma(k_1+k_2+\frac{1}{2})$ in* (2). *In the limit cases where either k_1 or k_2 vanishes,* (1) *reduces to the following expressions, already obtained in* [2] *and* [1]*:*

- *Assume that $k_1=0$ and $\operatorname{Re}k_2>0$. Then*

$$\mathcal{K}(x,y) = 2^{k_2-1}\,\tfrac{\Gamma(k_2+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k_2)}\,|\sinh x|^{-2k_2}\,(\cosh x-\cosh y)^{k_2-1}\,(\operatorname{sign} x)\,(e^x-e^{-y})\,. \tag{9}$$

- *Assume that $k_2=0$ and $\operatorname{Re}k_1>0$. Then*

$$\mathcal{K}(x,y) = 2^{k_1-2}\,\tfrac{\Gamma(k_1+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k_1)}\,|\sinh\tfrac{x}{2}|^{-2k_1}\,(\cosh\tfrac{x}{2}-\cosh\tfrac{y}{2})^{k_1-1}\,(\operatorname{sign} x)\,(e^{\frac{x}{2}}-e^{-\frac{y}{2}})\,. \tag{10}$$

3. Positivity of the intertwining operator

Corollary 3.1. *Assume that $k_1>0$ and $k_2>0$. Then the kernel* (1) *is strictly positive, for every $x,y\in\mathbb{R}$ such that $|x|>|y|$. Hence the intertwining operator $\mathcal{V}$ and its dual $\mathcal{V}^t$ are positive.*

Proof. Let us check the positivity of (3) when $x,y,z\in\mathbb{R}$ satisfy $|x|>z>|y|$. On the one hand, if $x>0$, then

$$\begin{aligned}\sigma(x,y,z) &= e^{\frac{x}{2}}\,(2\cosh\tfrac{x}{2})-e^{-\frac{y}{2}}\,(2\cosh\tfrac{z}{2}) \\ &> (e^{\frac{x}{2}}-e^{-\frac{y}{2}})\,(2\cosh\tfrac{x}{2})>0\,.\end{aligned}$$

On the other hand, if $x<0$, then

$$\begin{aligned}\sigma(x,y,z) &= e^{-\frac{y}{2}}\,(2\cosh\tfrac{z}{2})-e^{\frac{x}{2}}\,(2\cosh\tfrac{x}{2}) \\ &> e^{-\frac{y}{2}}\,(2\cosh\tfrac{y}{2})-e^{\frac{x}{2}}\,(2\cosh\tfrac{x}{2}) = e^{-y}-e^{x}>0\,.\end{aligned}$$

□

Remark 3.2. *As already observed in* [2], *the positivity of* (9), *respectively* (10) *is immediate in the limit case where* $k_1\!=\!0$ *and* $k_2\!>\!0$, *respectively* $k_2\!=\!0$ *and* $k_1\!>\!0$.

Remark 3.3. *The positivity of* $\mathcal{V}$ *was mistakenly disproved in* [2, Theorem 2.11] *when* $k_1>0$ *and* $k_2>0$. *More precisely, by using a more complicated formula than* (1), *the density* $\mathcal{K}(x,y)$ *was shown to be negative, when* $x>0$ *and* $y\searrow -x$. *The error in the proof lies in the expression* A_1, *which is equal to* $\frac{k}{k'}\frac{\sinh(2x)-\sinh(2|y|)}{E}$ *and which tends to* $+2\,\frac{k}{k'}\,\frac{\cosh(2x)}{\sinh(2x)}>0$.

Remark 3.4. *A different approach, based on the positivity of a heat type kernel, was used in* [8], [9], [10] *and* [11] *in order to tackle the positivity of* $\mathcal{V}$. *While* [8] *may be right, the same flaw occurs in* [9], [10], [11], *namely the cut–off* $\mathbf{1}_{Y_\ell}$ *breaks down the differential–difference equations, which are not local.*

In conclusion, the present note settles in a simple way the positivity issue in dimension 1 and hence in the product case. Otherwise, the positivity of the interwining operator $\mathcal{V}$ and its dual $\mathcal{V}^t$, when the multiplicity function k is $\geqslant 0$, remains an open problem in higher dimensions.

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