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Generators of the pro-$p$ Iwahori and Galois representations

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Abstract

For an odd prime $p$, we determine a minimal set of topological generators of the pro-$p$ Iwahori subgroup of a split reductive group $G$ over $\mathbb{Z}_p$. In the simple adjoint case and for any sufficiently large regular prime $p$, we also construct Galois extensions of $\mathbb{Q}$ with Galois group between the pro-$p$ and the standard Iwahori subgroups of $G$.

1 Introduction

Let $p$ be an odd prime, let $G$ be a split reductive group over $\mathbb{Z}_p$, fix a Borel subgroup $B = U \rtimes T$ of $G$ with unipotent radical $U \triangleleft B$ and maximal split torus $T \subset B$. The Iwahori subgroup $I$ and pro-$p$-Iwahori subgroup $I(1)$ of $G(\mathbb{Z}_p)$ are defined \cite{13, 3.7} by

$$I = \{ g \in G(\mathbb{Z}_p) : \text{red}(g) \in B(\mathbb{F}_p) \},$$
$$I(1) = \{ g \in G(\mathbb{Z}_p) : \text{red}(g) \in U(\mathbb{F}_p) \}.$$ 

where ‘red’ is the reduction map $\text{red}: G(\mathbb{Z}_p) \to G(\mathbb{F}_p)$. The subgroups $I$ and $I(1)$ are both open subgroups of $G(\mathbb{Z}_p)$. Thus $I = I(1) \rtimes T_{\text{tors}}$ and $T(\mathbb{Z}_p) = T(1) \rtimes T_{\text{tors}}$ where $T(1)$ and $T_{\text{tors}}$ are respectively the pro-$p$ and torsion subgroups of $T(\mathbb{Z}_p)$. Following \cite{3} (who works with $G = GL_n$), we construct in section 2 a minimal set of topological generators for $I(1)$.

More precisely, let $M = X^*(T)$ be the group of characters of $T$, $R \subset M$ the set of roots of $T$ in $\mathfrak{g} = \text{Lie}(G)$, $\Delta \subset R$ the set of simple roots with respect to $B$, $R = \bigsqcup_{c \in \mathcal{C}} R_c$ the decomposition of $R$ into irreducible components, $\Delta_c = \Delta \cap R_c$ the simple roots in $R_c$, $\alpha_{c, \text{max}}$ the highest positive root in $R_c$. We let $\mathcal{D} \subset \mathcal{C}$ be the set of irreducible components of type $G_2$ and for $d \in \mathcal{D}$, we denote by $\delta_d \in R_{d,+}$ the sum of the two simple roots in $\Delta_d$. We denote by $M^\vee = X_*(T)$ the group of cocharacters of $T$, by $ZR^\vee$ the subgroup spanned by the coroots $R^\vee \subset M^\vee$ and we fix a set of representatives $\mathcal{S} \subset M^\vee$ for an $\mathbb{F}_p$-basis of

$$(M^\vee/ZR^\vee) \otimes \mathbb{F}_p = \bigoplus_{s \in \mathcal{S}} \mathbb{F}_p \cdot s \otimes 1.$$
We show (see theorem 2.4.1):

**Theorem.** The following elements form a minimal set of topological generators of the pro-$p$-Iwahori subgroup $I(1)$ of $G = G(\mathbb{Q}_p)$:

1. The semi-simple elements $\{s(1 + p) : s \in S\}$ of $T(1)$,
2. For each $c \in C$, the unipotent elements $\{x_\alpha(1) : \alpha \in \Delta_c\}$,
3. For each $c \in C$, the unipotent element $x_{-\alpha_{c, \text{max}}}(p)$,
4. (If $p = 3$) For each $d \in D$, the unipotent element $x_{\delta_d}(1)$.

This result generalizes Greenberg [3] proposition 5.3, see also Schneider and Ollivier ([9], proposition 3.64, part i) for $G = SL_2$.

Let $T^{ad}$ be the image of $T$ in the adjoint group $G^{ad}$ of $G$. The action of $G^{ad}$ on $G$ induces an action of $T^{ad}(\mathbb{Z}_p)$ on $I$ and $I(1)$ and the latter equips the Frattini quotient $\tilde{I}(1)$ of $I(1)$ with a structure of $\mathbb{F}_p[T^{ad}_{\text{tors}}]$-module, where $T^{ad}_{\text{tors}}$ is the torsion subgroup of $T^{ad}(\mathbb{Z}_p)$ (cf. section 2.12). Any element $\beta$ in $\mathbb{Z}R = M^{ad} = X^*(T^{ad})$ induces a character $\beta : T^{ad}_{\text{tors}} \to \mathbb{F}_p^\times$ and we denote by $\mathbb{F}_p(\beta)$ the corresponding simple (1-dimensional) $\mathbb{F}_p[T^{ad}_{\text{tors}}]$-module. With these notations, the theorem implies that

**Corollary.** The $\mathbb{F}_p[T^{ad}_{\text{tors}}]$-module $\tilde{I}(1)$ is isomorphic to

$$
\mathbb{F}_p^\aleph S \oplus \left( \oplus_{\alpha \in \Delta} \mathbb{F}_p(\alpha) \right) \oplus \left( \oplus_{c \in C} \mathbb{F}_p(-\alpha_{c, \text{max}}) \right) \left( \oplus_{d \in D} \mathbb{F}_p(\delta_d) \right) \text{ if } p = 3.
$$

Here $\aleph S$ is the cardinality of $S$. Suppose from now on in this introduction that $G$ is simple and of adjoint type. Then:

**Corollary** The $\mathbb{F}_p[T^{ad}_{\text{tors}}]$-module $\tilde{I}(1)$ is multiplicity free unless $p = 3$ and $G$ is of type $A_1$, $B_\ell$ or $C_\ell$ ($\ell \geq 2$), $F_4$ or $G_2$.

Let now $K$ be a Galois extension of $\mathbb{Q}$, $\Sigma_p$ the set of primes of $K$ lying above $p$. Let $M$ be the compositum of all finite $p$-extensions of $K$ which are unramified outside $\Sigma_p$, a Galois extension over $\mathbb{Q}$. Set $\Gamma = \text{Gal}(M/K)$, $\Omega = \text{Gal}(K/\mathbb{Q})$ and $\Pi = \text{Gal}(M/\mathbb{Q})$. We say that $K$ is $p$-rational if $\Gamma$ is a free pro-$p$ group, see [6]. The simplest example is $K = \mathbb{Q}$, where $\Gamma = \Pi$ is also abelian and $M$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Other examples of $p$-rational fields are $\mathbb{Q}(\mu_p)$ where $p$ is a regular prime.

Assume $K$ is a $p$-rational, totally complex, abelian extension of $\mathbb{Q}$ and $(p - 1) \cdot \Omega = 0$. Then Greenberg in [3] constructs a continuous homomorphism

$$
\rho_0 : \text{Gal}(M/\mathbb{Q}) \to GL_n(\mathbb{Z}_p)
$$

such that $\rho_0(\Gamma)$ is the pro-$p$ Iwahori subgroup of $SL_n(\mathbb{Z}_p)$, assuming that there exists $n$ distinct characters of $\Omega$, trivial or odd, whose product is the trivial character.

In section 3, we are proving results which show the existence of $p$-adic Lie extensions of $\mathbb{Q}$ where the Galois group corresponds to a certain specific $p$-adic Lie algebra. More precisely, for $p$-rational fields, we construct continuous morphisms with open image $\rho : \Pi \to I$ such that $\rho(\Gamma) = I(1)$. We
show in corollary 3.3.1 that

**Corollary** Suppose that $K$ is a $p$-rational totally complex, abelian extension of $\mathbb{Q}$ and $(p-1)\cdot \Omega = 0$. Assume also that if $p = 3$, our split simple adjoint group $G$ is not of type $A_1$, $B_\ell$ or $C_\ell$ $(\ell \geq 2)$, $F_4$ or $G_2$. Then there is a morphism $\rho : \Pi \to I$ such that $\rho(\Gamma) = I(1)$ if and only if there is a morphism $\overline{\rho} : \Omega \to T_{\text{tors}}$ such that the characters $\alpha \circ \overline{\rho} : \Omega \to \mathbb{F}_p^\times$ for $\alpha \in \{\Delta \cup -\alpha_{\text{max}}\}$ are all distinct and belong to $\tilde{\Omega}_{\text{odd}}^T$.

Here $\tilde{\Omega}_{\text{odd}}^S$ is a subset of the characters of $\Omega$ with values in $\mathbb{F}_p^\times$ (it is defined after proposition 3.2.1). Furthermore assuming $K = \mathbb{Q}(\mu_p)$ we show the existence of such a morphism $\overline{\rho} : \Omega \to T_{\text{tors}}$ provided that $p$ is a sufficiently large regular prime (cf. section 3.2):

**Corollary** There is a constant $c$ depending only upon the type of $G$ such that if $p > c$ is a regular prime, then for $K = \mathbb{Q}(\mu_p)$, $M$, $\Pi$ and $\Gamma$ as above, there is a continuous morphism $\rho : \Pi \to I$ with $\rho(\Gamma) = I(1)$. The constant $c$ can be determined from lemmas 3.4.1, 3.4.2 and remark 3.4.3.

In section 2, we find a minimal set of topological generators of $I(1)$ and study the structure of $\tilde{I}(1)$ as an $\mathbb{F}_p[T_{\text{tors}}^{\text{ad}}]$-module. In section 3, assuming our group $G$ to be simple and adjoint, we discuss the notion of $p$-rational fields and construct continuous morphisms $\rho : \Pi \to I$ with open image.

We would like to thank Marie-France Vignéras for useful discussions and for giving us the reference [9]. We are also deeply grateful to Ralph Greenberg for numerous conversations on this topic.

## 2 Topological Generators of the pro-$p$ Iwahori

This section is organized as follows. In sections (2.1 – 2.3) we introduce the notations, then section 2.4 states our main result concerning the minimal set of topological generators of $I(1)$ (see theorem 2.4.1) with a discussion of the Iwahori factorisation in section 2.5. Its proof for $G$ simple and simply connected is given in sections (2.6 – 2.10), where section 2.10 deals with the case of a group of type $G_2$. The proof for an arbitrary split reductive group over $\mathbb{Z}_p$ is discussed in sections (2.11 – 2.14). In particular, section 2.14 establishes the minimality of our set of topological generators. Finally, in section 2.15 we study the structure of the Frattini quotient $\tilde{I}(1)$ of $I(1)$ as an $\mathbb{F}_p[T_{\text{tors}}^{\text{ad}}]$-module and determine the cases when it is multiplicity free.

### 2.1 Let $p$ be an odd prime, $G$ be a split reductive group over $\mathbb{Z}_p$. Fix a pinning of $G$ [11, XXIII 1]

$$(T, M, R, \Delta, (X_\alpha)_{\alpha \in \Delta}) .$$

Thus $T$ is a split maximal torus in $G$, $M = X^*(T)$ is its group of characters,

$$g = g_0 \oplus \oplus_{\alpha \in R} g_\alpha$$

is the weight decomposition for the adjoint action of $T$ on $g = \text{Lie}(G)$, $\Delta \subset R$ is a basis of the root system $R \subset M$ and for each $\alpha \in \Delta$, $X_\alpha$ is a $\mathbb{Z}_p$-basis of $g_\alpha$. 
2.2 We denote by \( M'^{\vee} = X_*(T) \) the group of cocharacters of \( T \), by \( \alpha' \) the coroot associated to \( \alpha \in \mathfrak{R} \) and by \( R' \subseteq M'^{\vee} \) the set of all such coroots. We expand \((X_\alpha)_{\alpha \in \Delta}\) to a Chevalley system \((X_\alpha)_{\alpha \in \mathfrak{R}} \) of \( G \) [11, XXIII 6.2]. For \( \alpha \in \mathfrak{R} \), we denote by \( U_\alpha \subseteq \mathfrak{G} \) the corresponding unipotent group, by \( x_\alpha : G_{a,Z_p} \to U_\alpha \) the isomorphism given by \( x_\alpha(t) = \exp(tX_\alpha) \). The height \( h(\alpha) \in \mathbb{Z} \) of \( \alpha \in \mathfrak{R} \) is the sum of the coefficients of \( \alpha \) in the basis \( \Delta \) of \( \mathfrak{R} \). Thus \( R_+ = h^{-1}(\mathbb{Z}_{>0}) \) is the set of positive roots in \( \mathfrak{R} \), corresponding to a Borel subgroup \( B = U \times T \) of \( G \) with unipotent radical \( U \). We let \( C \) be the set of irreducible components of \( R \), so that

\[
R = \prod_{c \in C} R_c, \quad \Delta = \prod_{c \in C} \Delta_c, \quad R_+ = \prod_{c \in C} R_{c,+}
\]

with \( R_c \) irreducible, \( \Delta_c = \Delta \cap R_c \) is a basis of \( R_c \) and \( R_{c,+} = R_+ \cap R_c \) is the corresponding set of positive roots in \( R_c \). We denote by \( \alpha_{c,\text{max}} \in R_{c,+} \) the highest root of \( R_c \). We let \( D \subseteq C \) be the set of irreducible components of type \( G_2 \) and for \( d \in D \), we denote by \( \delta_d \in R_{d,+} \) the sum of the two simple roots in \( \Delta_d \).

2.3 Since \( G \) is smooth over \( \mathbb{Z}_p \), the reduction map

\[
\text{red} : G(\mathbb{Z}_p) \to G(\mathbb{F}_p)
\]

is surjective and its kernel \( G(1) \) is a normal pro-\( p \)-subgroup of \( G(\mathbb{Z}_p) \). The Iwahori subgroup \( I \) and pro-\( p \)-Iwahori subgroup \( I(1) \subseteq I \) of \( G(\mathbb{Z}_p) \) are defined [13, 3.7] by

\[
I = \{ g \in G(\mathbb{Z}_p) : \text{red}(g) \in B(\mathbb{F}_p) \},
I(1) = \{ g \in G(\mathbb{Z}_p) : \text{red}(g) \in U(\mathbb{F}_p) \}.
\]

Thus \( I(1) \) is a normal pro-\( p \)-sylow subgroup of \( I \) which contains \( U(\mathbb{Z}_p) \) and

\[
I/I(1) \simeq B(\mathbb{F}_p)/U(\mathbb{F}_p) \simeq T(\mathbb{F}_p).
\]

Since \( T(\mathbb{Z}_p) \to T(\mathbb{F}_p) \) is split by the torsion subgroup \( T_{\text{tors}} \simeq T(\mathbb{F}_p) \) of \( T(\mathbb{Z}_p) \),

\[
T(\mathbb{Z}_p) = T(1) \times T_{\text{tors}} \quad \text{and} \quad I = I(1) \rtimes T_{\text{tors}}
\]

where

\[
T(1) = T(\mathbb{Z}_p) \cap I(1) = \ker (T(\mathbb{Z}_p) \to T(\mathbb{F}_p))
\]

is the pro-\( p \)-sylow subgroup of \( T(\mathbb{Z}_p) \). Note that

\[
T(1) = \text{Hom}(M, 1 + p\mathbb{Z}_p) = M'^{\vee} \otimes (1 + p\mathbb{Z}_p),
T_{\text{tors}} = \text{Hom}(M, \mu_{p-1}) = M'^{\vee} \otimes \mathbb{F}_p^\times.
\]

2.4 Let \( S \subseteq M'^{\vee} \) be a set of representatives for an \( \mathbb{F}_p \)-basis of

\[
(M'^{\vee}/\mathbb{Z}R'^{\vee}) \otimes \mathbb{F}_p = \bigoplus_{s \in \mathcal{S}} \mathbb{F}_p : s \otimes 1.
\]

**Theorem 2.4.1.** The following elements form a minimal set of topological generators of the pro-\( p \)-Iwahori subgroup \( I(1) \) of \( G = G(\mathbb{Q}_p) \):

1. The semi-simple elements \( \{ s(1 + p) : s \in \mathcal{S} \} \) of \( T(1) \).
2. For each \( c \in C \), the unipotent elements \( \{ x_\alpha(1) : \alpha \in \Delta_c \} \).
3. For each \( c \in C \), the unipotent element \( x_{-\alpha_{c,\text{max}}}(p) \).
4. (If \( p = 3 \)) For each \( d \in D \), the unipotent element \( x_{\delta_d}(1) \).
2.5 By [11, XXII 5.9.5] and its proof, there is a canonical filtration
\[ U = U_1 \supset U_2 \supset \cdots \supset U_h \supset U_{h+1} = 1 \]
of \( U \) by normal subgroups such that for \( 1 \leq i \leq h \), the product map (in any order)
\[ \prod_{h(\alpha)=i} U_\alpha \to U \]
factors through \( U_i \) and yields an isomorphism of group schemes
\[ \prod_{h(\alpha)=i} U_\alpha \cong U_i, \quad U_i = U_i/U_{i+1}. \]

By [11, XXII 5.9.6] and its proof,
\[ U_i(R) = U_i(R)/U_{i+1}(R) \]
for every \( \mathbb{Z}_p \)-algebra \( R \). It follows that the product map
\[ \prod_{h(\alpha)=i} U_\alpha \times U_{i+1} \to U_i \]
is an isomorphism of \( \mathbb{Z}_p \)-schemes and by induction, the product map
\[ \prod_{h(\alpha)=1} U_\alpha \times \prod_{h(\alpha)=2} U_\alpha \times \cdots \times \prod_{h(\alpha)=h} U_\alpha \to U \]
is an isomorphism of \( \mathbb{Z}_p \)-schemes. Similarly, the product map
\[ \prod_{h(\alpha)=h} U_\alpha \times \prod_{h(\alpha)=h+1} U_\alpha \times \cdots \times \prod_{h(\alpha)=1} U_\alpha \to U^- \]
is an isomorphism of \( \mathbb{Z}_p \)-schemes, where \( U^- \) is the unipotent radical of the Borel subgroup \( B^- = U^- \rtimes T \) opposed to \( B \) with respect to \( T \). Then by [11, XXII 4.1.2], there is an open subscheme \( \Omega \) of \( G \) (the “big cell”) such that the product map
\[ U^- \times T \times U \to G \]
is an open immersion with image \( \Omega \). Plainly, \( B = U \rtimes T \) is a closed subscheme of \( \Omega \). Thus by definition of \( I \), \( I \subset \Omega(\mathbb{Z}_p) \) and therefore any element of \( I \) (resp. \( I(1) \)) can be written uniquely as a product
\[ \prod_{h(\alpha)=h} x_\alpha(a_\alpha) \times \cdots \times \prod_{h(\alpha)=1} x_\alpha(a_\alpha) \times t \times \prod_{h(\alpha)=-1} x_\alpha(a_\alpha) \times \cdots \times \prod_{h(\alpha)=-h} x_\alpha(a_\alpha) \]
where \( a_\alpha \in \mathbb{Z}_p \) for \( \alpha \in R_+ \), \( a_\alpha \in p\mathbb{Z}_p \) for \( \alpha \in R_- = -R_+ \) and \( t \in T(\mathbb{Z}_p) \) (resp. \( T(1) \)). This is the Iwahori decomposition of \( I \) (resp. \( I(1) \)). If \( I^+ \) is the group spanned by \( \{ x_\alpha(\mathbb{Z}_p) : \alpha \in R_+ \} \) and \( I^- \) is the group spanned by \( \{ x_\alpha(p\mathbb{Z}_p) : \alpha \in R_- \} \), then \( I^+ = U(\mathbb{Z}_p), I^- \subset U^-(\mathbb{Z}_p) \) and every \( x \in I \) (resp. \( I(1) \)) has a unique decomposition \( x = u^-tu^+ \) with \( u^\pm \in I^\pm \) and \( t \in T(\mathbb{Z}_p) \) (resp. \( t \in T(1) \)).
2.6 Suppose first that $G$ is semi-simple and simply connected. Then $M^\vee = \mathbb{Z}R^\vee$, thus $S = \emptyset$. Moreover, everything splits according to the decomposition $R = \bigsqcup R_c$:

$$G = \prod G_c, \quad T = \prod T_c, \quad B = \prod B_c, \quad I = \prod I_c \quad \text{and} \quad I(1) = \prod I_c(1).$$

To establish the theorem in this case, we may thus furthermore assume that $G$ is simple. From now on until section 2.11, we therefore assume that $G$ is (split) simple and simply connected.

2.7 As a first step, we show that

Lemma 2.7.1. The group generated by $I^+$ and $I^-$ contains $T(1)$.

Proof. Since $G$ is simply connected,

$$\prod_{\alpha \in \Delta} \alpha^\vee : \prod_{\alpha \in \Delta} G_{m, \mathbb{Z}_p} \to T$$

is an isomorphism, thus

$$T_c(1) = \prod_{\alpha \in \Delta} \alpha^\vee (1 + p\mathbb{Z}_p).$$

Now for any $\alpha \in \Delta$, there is a unique morphism $[11, XX 5.8]$

$$f_\alpha : SL(2)_{\mathbb{Z}_p} \to G$$

such that for every $u, v \in \mathbb{Z}_p$ and $x \in \mathbb{Z}_p^\times$,

$$f_\alpha \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = x_\alpha(u), \quad f_\alpha \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = x_{-\alpha}(v) \quad \text{and} \quad f_\alpha \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \alpha^\vee(x).$$

Since for every $x \in 1 + p\mathbb{Z}_p$ $[11, XX 2.7],$

$$\begin{pmatrix} 1 & 0 \\ x^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

in $SL(2)(\mathbb{Z}_p)$, it follows that $\alpha^\vee(1 + p\mathbb{Z}_p)$ is already contained in the subgroup of $G(\mathbb{Z}_p)$ generated by $x_\alpha(\mathbb{Z}_p^\times)$ and $x_{-\alpha}(p\mathbb{Z}_p)$. This proves the lemma. $\square$

2.8 Recall from $[11, XXI 2.3.5]$ that for any pair of non-proportional roots $\alpha \neq \pm \beta$ in $R$, the set of integers $k \in \mathbb{Z}$ such that $\beta + k\alpha \in R$ is an interval of length at most 3, i.e. there are integers $r \geq 1$ and $s \geq 0$ with $r + s \leq 4$ such that

$$R \cap \{\beta + \mathbb{Z}\alpha\} = \{\beta - (r - 1)\alpha, \cdots, \beta + s\alpha\}.$$

The above set is called the $\alpha$-chain through $\beta$ and any such set is called a root chain in $R$. Let $\| - \| : R \to \mathbb{R}_+$ be the length function on $R$. 

Proposition 2.8.1. Suppose $||\alpha|| \leq ||\beta||$. Then for any $u, \in G_a$ the commutator

$$[x_\beta(v) : x_\alpha(u)] = x_\beta(v)x_\alpha(u)x_\beta(-v)x_\alpha(-u)$$

is given by the following table, with $(r, s)$ as above:

<table>
<thead>
<tr>
<th>$(r, s)$</th>
<th>$[x_\beta(v) : x_\alpha(u)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 0)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(1, 1)$</td>
<td>$x_{\alpha+\beta}(\pm uv)$</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>$x_{\alpha+\beta}(\pm uv) \cdot x_{2\alpha+\beta}(\pm u^2 v)$</td>
</tr>
<tr>
<td>$(1, 3)$</td>
<td>$x_{\alpha+\beta}(\pm uv) \cdot x_{2\alpha+\beta}(\pm u^2 v) \cdot x_{3\alpha+\beta}(\pm u^3 v) \cdot x_{3\alpha+2\beta}(\pm u^3 v^2)$</td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$x_{\alpha+\beta}(\pm 2 uv)$</td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$x_{\alpha+\beta}(\pm 2 uv) \cdot x_{2\alpha+\beta}(\pm 3 u^2 v) \cdot x_{\alpha+2\beta}(\pm 3 uv^2)$</td>
</tr>
<tr>
<td>$(3, 1)$</td>
<td>$x_{\alpha+\beta}(\pm 3 uv)$</td>
</tr>
</tbody>
</table>

The signs are unspecified, but only depend upon $\alpha$ and $\beta$.

Proof. This is [11, XXIII 6.4].

Corollary 2.8.2. If $r + s \leq 3$ and $\alpha + \beta \in R$ (i.e. $s \geq 1$), then for any $a, b \in \mathbb{Z}$, the subgroup of $G$ generated by $x_\alpha(p^a \mathbb{Z}_p)$ and $x_\beta(p^b \mathbb{Z}_p)$ contains $x_{\alpha+\beta}(p^{a+b} \mathbb{Z}_p)$.

Proof. This is obvious if $(r, s) = (1, 1)$ or $(2, 1)$ (using $p \neq 2$ in the latter case). For the only remaining case where $(r, s) = (1, 3)$, note that

$$[x_\beta(v) : x_\alpha(u)] [x_\beta(w^2 v) : x_\alpha(u w^{-1})]^{-1} = x_{\alpha+\beta}(\pm uv(1-w)).$$

Since $p \neq 2$, we may find $w \in \mathbb{Z}_p^\times$ with $(1-w) \in \mathbb{Z}_p^\times$. Our claim easily follows.

Lemma 2.8.3. If $R$ contains any root chain of length 3, then $G$ is of type $G_2$.

Proof. Suppose that the $\alpha$-chain through $\beta$ has length 3. By [11, XXI 3.5.4], there is a basis $\Delta'$ of $R$ such that $\alpha \in \Delta'$ and $\beta = a\alpha + b\alpha'$ with $\alpha' \in \Delta'$, $a, b \in \mathbb{N}$. The root system $R'$ spanned by $\Delta' = \{\alpha, \alpha'\}$ [11, XXI 3.4.6] then also contains an $\alpha$-chain of length 3. By inspection of the root systems of rank 2, for instance in [11, XXIII 3], we find that $R'$ is of type $G_2$. In particular, the Dynkin diagram of $R$ contains a triple edge (linking the vertices corresponding to $\alpha$ and $\alpha'$), which implies that actually $R = R'$ is of type $G_2$.

2.9 We now establish our theorem 2.4.1 for a group $G$ which is simple and simply connected, but not of type $G_2$.

Lemma 2.9.1. The group $I^+$ is generated by $\{x_\alpha(\mathbb{Z}_p) : \alpha \in \Delta\}$.

Proof. Let $H \subset I^+$ be the group spanned by $\{x_\alpha(\mathbb{Z}_p) : \alpha \in \Delta\}$. We show by induction on $h(\gamma) \geq 1$ that $x_\gamma(\mathbb{Z}_p) \subset H$ for every $\gamma \in R_+$. If $h(\gamma) = 1$, $\gamma$ already belongs to $\Delta$ and there is nothing to prove. If $h(\gamma) > 1$, then by [1, VI.1.6 Proposition 19], there is a simple root $\alpha \in \Delta$ such that $\beta = \gamma - \alpha \in R_+$. Then $h(\beta) = h(\gamma) - 1$, thus by induction $x_\beta(\mathbb{Z}_p) \subset H$. Since also $x_\alpha(\mathbb{Z}_p) \subset H$, $x_\gamma(\mathbb{Z}_p) \subset H$ by Corollary 2.8.2.

Lemma 2.9.2. The group generated by $I^+$ and $x_{-\alpha_{\text{max}}}(p \mathbb{Z}_p)$ contains $I^-$. 


Proof. Let $H \subset I$ be the group spanned by $I^+$ and $x_{-\alpha_{\text{max}}}(p\mathbb{Z}_p)$. We show by descending induction on $h(\gamma) \geq 1$ that $x_{-\gamma}(p\mathbb{Z}_p) \subset H$ for every $\gamma \in R_+$. If $h(\gamma) = h(\alpha_{\text{max}})$, then $\gamma = \alpha_{\text{max}}$ and there is nothing to prove. If $h(\gamma) < h(\alpha_{\text{max}})$, then by [1, VI.1.6 Proposition 19], there is a pair of positive roots $\alpha, \beta$ such that $\beta = \gamma + \alpha$. Then $h(\beta) = h(\gamma) + h(\alpha) > h(\gamma)$, thus by induction $x_{-\beta}(p\mathbb{Z}_p) \subset H$. Since also $x_\alpha(\mathbb{Z}_p) \subset H$, $x_{-\gamma}(p\mathbb{Z}_p) \subset H$ by Corollary 2.8.2.

Remark 2.9.3. From the Hasse diagrams in [10], it seems that in the previous proof, we may always require $\alpha$ to be a simple root.

Proof. (Of theorem 2.4.1 for $G$ simple, simply connected, not of type $G_2$) By Lemma 2.10.1, the group generated by $x_{\alpha}(\mathbb{Z}_p)$, $\alpha \in \Delta$, thus topologically generated by $x_\alpha(1)$, $\alpha \in \Delta$, and $x_{-\alpha_{\text{max}}}(p)$. None of these topological generators can be removed: the first ones are contained in $I^+ \subset I(1)$, and all of them are needed to span the image of

$$I(1) \twoheadrightarrow U(F_p) \twoheadrightarrow U_1(F_p) \simeq \prod_{\alpha \in \Delta} U_\alpha(F_p),$$

a surjective morphism that kills $x_{-\alpha_{\text{max}}}(p)$.

2.10 Let now $G$ be simple of type $G_2$, thus $\Delta = \{\alpha, \beta\}$ with $\|\alpha\| < \|\beta\|$ and

$$R_+ = \{\alpha, \beta, \beta + \alpha, \beta + 2\alpha, \beta + 3\alpha, 2\beta + 3\alpha\}.$$  

The whole root system looks like this:

![Root System Diagram](attachment:image.png)

Lemma 2.10.1. The group generated by $I^+$ and $x_{-2\beta-3\alpha}(p\mathbb{Z}_p)$ contains $I^-$.

Proof. Let $H \subset I(1)$ be the group generated by $I^+$ and $x_{-2\beta-3\alpha}(p\mathbb{Z}_p)$. Then, for every $u, v \in \mathbb{Z}_p$, $H$ contains

$$[x_{-\beta-3\alpha}(pv) : x_\beta(u)] = x_{-\beta-3\alpha}(\pm pv)$$

$$[x_{-\beta-3\alpha}(pv) : x_{\beta+3\alpha}(u)] = x_\beta(\pm pv)$$

$$[x_{-\beta-3\alpha}(pv) : x_{\beta+2\alpha}(u)] = x_{-\beta-\alpha}(\pm pv) \cdot x_\alpha(\pm pu^2v) \cdot x_{\beta+3\alpha}(\pm pu^3v) \cdot x_{-\beta}(\pm p^2u^3v^2)$$
It thus contains $x_{-\beta-3\alpha}(p\mathbb{Z}_p)$, $x_{-\beta}(p\mathbb{Z}_p)$ and $x_{-\beta-\alpha}(p\mathbb{Z}_p)$, along with

$$
[x_{-\beta-3\alpha}(pu) : x_{\alpha}(u)] = \pm x_{-\beta-2\alpha}(\pm pu^2) \cdot x_{-\beta}(\pm pu^3) \cdot x_{-\beta-3\alpha}(\pm pu^2v^2)
$$

$$
[x_{-\beta-3\alpha}(pu) : x_{\beta+2\alpha}(u)] = x_{\alpha}(\pm pu^2) \cdot x_{\beta+\alpha}(\pm pu^3) \cdot x_{2\beta+3\alpha}(\pm pu^3) \cdot x_{\beta}(\pm pu^3v^2)
$$

It therefore also contains $x_{-\beta-2\alpha}(p\mathbb{Z}_p)$ and $x_{-\beta}(p\mathbb{Z}_p)$.

The filtration $(U_i)_{i\geq 1}$ of $U$ in section 2.5 induces a filtration

$$I^+ = I_1^+ \supset \cdots \supset I_5^+ \supset I_6^+ = 1$$

of $I^+ = U(\mathbb{Z}_p)$ by normal subgroups $I_i^+ = U_i(\mathbb{Z}_p)$ whose graded pieces

$$T_i^+ = U_i(\mathbb{Z}_p) = I_i^+/I_{i+1}^+$$

are free $\mathbb{Z}_p$-modules, namely

$$
T_1^+ = \mathbb{Z}_p \cdot \bar{x}_\alpha \oplus \mathbb{Z}_p \cdot \bar{x}_\beta,
$$

$$T_2^+ = \mathbb{Z}_p \cdot \bar{x}_{\alpha+\beta},
$$

$$T_3^+ = \mathbb{Z}_p \cdot \bar{x}_{2\alpha+\beta},
$$

$$T_4^+ = \mathbb{Z}_p \cdot \bar{x}_{3\alpha+\beta},
$$

$$T_5^+ = \mathbb{Z}_p \cdot \bar{x}_{3\alpha+2\beta}
$$

where $\bar{x}_\gamma$ is the image of $x_\gamma(1)$. The commutator defines $\mathbb{Z}_p$-linear pairings

$$[-,-]_{i,j} : T_i^+ \times T_j^+ \to T_{i+j}^+
$$

with $[y,x]_{i,j} = -[x,y]_{i,j}$, $[x,x]_{i,i} = 0$ and, by Proposition 2.8.1,

$$
[\bar{x}_\beta, \bar{x}_\alpha] = \pm 2\bar{x}_{2\alpha+\beta},
$$

$$
[\bar{x}_{\alpha+\beta}, \bar{x}_{\alpha}] = \pm 2\bar{x}_{2\alpha+\beta},
$$

$$
[\bar{x}_{\alpha+\beta}, \bar{x}_{2\alpha+\beta}] = \pm x_{3\alpha+2\beta}.
$$

Let $H$ be the subgroup of $I^+$ generated by $x_\alpha(\mathbb{Z}_p)$ and $x_\beta(\mathbb{Z}_p)$ and denote by $H_i$ its image in $I_i^+/I_{i+1}^+ = G_i$. Then $H_1 = G_1$, $H_2$ contains $[\bar{x}_\beta, \bar{x}_\alpha] = \pm \bar{x}_{\alpha+\beta}$ thus $H_2 = G_2$, $H_3$ contains $[\bar{x}_{\alpha+\beta}, \bar{x}_\alpha] = \pm 2\bar{x}_{2\alpha+\beta}$ thus $H_3 = G_3$ since $p \neq 2$, $H_4$ contains $[\bar{x}_{2\alpha+\beta}, \bar{x}_\alpha] = \pm 3\bar{x}_{3\alpha+\beta}$ thus $H_4 = G_4$ if $p \neq 3$, in which case actually $H = H_5 = G_5 = I^+$ since $H$ always contains $[\bar{x}_{\alpha+\beta}, \bar{x}_{2\alpha+\beta}] = \pm x_{3\alpha+2\beta}$.

If $p = 3$, let us also consider the exact sequence

$$0 \to J_4 \to G_4 \to \bar{T}_1^+ \to 0
$$

The group $J_4 = I_2^+/I_5^+$ is commutative, and in fact again a free $\mathbb{Z}_3$-module:

$$J_4 = (U_2/U_5)(\mathbb{Z}_p) = \mathbb{Z}_3 \bar{x}_{\alpha+\beta} \oplus \mathbb{Z}_3 \bar{x}_{2\alpha+\beta} \oplus \mathbb{Z}_3 \bar{x}_{3\alpha+\beta}
$$

where $\bar{x}_\gamma$ is the image of $x_\gamma(1)$. The action by conjugation of $\bar{T}_1^+$ on $J_4$ is given by

$$
\bar{x}_\alpha \mapsto \begin{pmatrix} 1 & \pm 2 & 1 \\ \pm 3 & \pm 3 & 1 \end{pmatrix},
$$

$$\bar{x}_\beta \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
$$

in the indicated basis of $J_4$. The $\mathbb{Z}_3$-submodule $H'_4 = H_4 \cap J_4$ of $J_4$ satisfies

$$H'_4 + \mathbb{Z}_3 \bar{x}_{3\alpha+\beta} = J_4 \text{ and } 3\bar{x}_{3\alpha+\beta} \in H'_4.$$
Naming signs $\epsilon_i \in \{\pm 1\}$ in formula (1, 3) of proposition 2.8.1, we find that $H'_4$ contains

$$\epsilon_1 uv \cdot \bar{x}_{\alpha+\beta} + \epsilon_2 u^2 v \cdot \bar{x}_{2\alpha+\beta} + \epsilon_3 u^3 v \cdot \bar{x}_{3\alpha+\beta}$$

for every $u, v \in \mathbb{Z}_3$. Adding these for $v = 1$ and $u = \pm 1$, we obtain

$$\bar{x}_{2\alpha+\beta} \in H'_4.$$ 

It follows that $H'_4$ actually contains the following $\mathbb{Z}_3$-submodule of $J'_4$:

$$J'_4 = \{a \cdot \bar{x}_{\alpha+\beta} + b \cdot \bar{x}_{2\alpha+\beta} + c \cdot \bar{x}_{3\alpha+\beta} : a, b, c \in \mathbb{Z}_3, \epsilon_1 a \equiv \epsilon_3 c \mod 3\}.$$ 

Now observe that $J'_4$ is a normal subgroup of $G_4$, and the induced exact sequence

$$0 \to J'_4 / J'_4 \to G_4 / J'_4 \to T_1^+ \to 0$$

is an abelian extension of $T_1^+ \simeq \mathbb{Z}_3^2$ by $J'_4 / J'_4 \simeq \mathbb{F}_3$. Since $H_4 / J'_4$ is topologically generated by two elements and surjects onto $T_1^+$, it actually defines a splitting:


Thus $H'_4 = J'_4$, $H_4$ is a normal subgroup of $G_4$, $H$ is a normal subgroup of $I^+$ and

$$I^+ / H \simeq G_4 / H_4 \simeq J'_4 / J'_4 \simeq \mathbb{F}_3$$

is generated by the class of $x_{\alpha+\beta}(1)$ or $x_{3\alpha+\beta}(1)$. We have shown:

**Lemma 2.10.2.** The group $I^+$ is spanned by $x_{\alpha}(\mathbb{Z}_p)$ and $x_{\beta}(\mathbb{Z}_p)$ plus $x_{\alpha+\beta}(1)$ if $p = 3$.

**Proof.** (Of theorem 2.4.1 for $G$ simple of type $G_2$) By lemma 2.7.1, 2.10.1, 2.10.2 and the Iwahori decomposition of section 2.5, the pro-$p$-Iwahori $I(1)$ is generated by $x_{\alpha}(\mathbb{Z}_p)$, $x_{\beta}(\mathbb{Z}_p)$, $x_{-2\beta-3\alpha}(p\mathbb{Z}_p)$, along with $x_{\alpha+\beta}(1)$ if $p = 3$. It is therefore topologically generated by $x_{\alpha}(1)$, $x_{\beta}(1)$, $x_{-2\beta-3\alpha}(p)$, along with $x_{\alpha+\beta}(1)$ if $p = 3$. The surjective reduction morphism $I(1) \to U(\mathbb{F}_p) \to \overline{U}_1(\mathbb{F}_p)$ shows that the first two generators can not be removed. The third one also can not, since all the others belong to the closed subgroup $I_+ \subseteq I(1)$. Finally, suppose that $p = 3$ and consider the extension

$$1 \to U_2 / U_5 \to U / U_5 \to U / U_1 \to 1$$

With notations as above, the reduction of

$$J'_4 \subset J_4 = U_2(\mathbb{Z}_3) / U_5(\mathbb{Z}_3) = (U_2 / U_5)(\mathbb{Z}_3)$$

is a normal subgroup $Y$ of $X = (U / U_5)(\mathbb{F}_3)$ with quotient $X / Y \simeq \mathbb{F}_3^3$. The surjective reduction morphism

$$I(1) \to U(\mathbb{F}_3) \to U(\mathbb{F}_3) / U_5(\mathbb{F}_3) = X \to X / Y$$

then kills $x_{-2\beta-3\alpha}(p)$. The fourth topological generator $x_{\alpha+\beta}(1)$ of $I(1)$ thus also can not be removed, since the first two certainly do not span $X / Y \simeq \mathbb{F}_3^3$. \qed
2.11 We now return to an arbitrary split reductive group $G$ over $\mathbb{Z}_p$. Let

$$G^{sc} \rightarrow G^{der} \hookrightarrow G \rightarrow G^{ad}$$

be the simply connected cover $G^{sc}$ of the derived group $G^{der}$ of $G$, and the adjoint group $\pi : G \rightarrow G^{ad}$ of $G$. Then

$$(T^{ad}, M^{ad}, R^{ad}, \Delta^{ad}, (X^{ad}_\alpha)_{\alpha \in \Delta^{ad}}) = (\pi(T), \mathbb{Z}R, R, \Delta, (\pi(X_\alpha))_{\alpha \in \Delta})$$

is a pinning of $G^{ad}$ and this construction yields a bijection between pinnings of $G$ and pinnings of $G^{ad}$. Applying this to $G^{sc}$ or $G^{der}$, we obtain pinnings

$$(T^{sc}, M^{sc}, R^{sc}, \Delta^{sc}, (X^{sc}_\alpha)_{\alpha \in \Delta^{sc}}) \quad \text{and} \quad \left(T^{der}, M^{der}, R^{der}, \Delta^{der}, (X^{der}_\alpha)_{\alpha \in \Delta^{der}}\right)$$

for $G^{sc}$ and $G^{der}$; all of the above constructions then apply to $G^{ad}$, $G^{sc}$ or $G^{der}$, and we will denote with a subscript $ad$, $sc$ or $der$ for the corresponding objects. For instance, we have a sequence of Iwahori (resp. pro-$p$-Iwahori) subgroups

$$I^{sc} \rightarrow I^{der} \hookrightarrow I \rightarrow I^{ad} \quad \text{and} \quad I^{sc}(1) \rightarrow I^{der}(1) \hookrightarrow I(1) \rightarrow I^{ad}(1).$$

2.12 The action of $G$ on itself by conjugation factors through a morphism

$$\text{Ad} : G^{ad} \rightarrow \text{Aut}(G).$$

For $b \in B^{ad}(\mathbb{F}_p)$, $\text{Ad}(b)(B_{\mathbb{F}_p}) = B_{\mathbb{F}_p}$ and $\text{Ad}(b)(U_{\mathbb{F}_p}) = U_{\mathbb{F}_p}$. We thus obtain an action of the Iwahori subgroup $I^{ad}$ of $G^{ad} = G^{ad}(\mathbb{Q}_p)$ on $I$ or $I(1)$. Similar consideration of course apply to $G^{sc}$ and $G^{der}$, and the sequence

$$I^{sc}(1) \rightarrow I^{der}(1) \hookrightarrow I(1) \rightarrow I^{ad}(1)$$

is equivariant for these actions of $I^{ad} = I^{ad}(1) \rtimes T^{ad}_{\text{tors}}$.

2.13 Let $J$ be the image of $I^{sc}(1) \rightarrow I(1)$, so that $J$ is a normal subgroup of $I$. From the compatible Iwahori decompositions for $I(1)$ and $I^{sc}(1)$ in section 2.5, we see that $T(1) \hookrightarrow I(1)$ induces a $T^{ad}$-equivariant isomorphism

$$T(1)/T(1) \cap J \rightarrow I(1)/J.$$

Since the inverse image of $T(\mathbb{Z}_p)$ in $G^{sc}(\mathbb{Z}_p)$ equals $T^{sc}(\mathbb{Z}_p)$ and since also

$$T^{sc}(1) = T^{sc}(\mathbb{Z}_p) \cap I^{sc}(1),$$

we see that $T(1) \cap J$ is the image of $T^{sc}(1) \rightarrow T(1)$. Also, the kernel of $I^{sc}(1) \rightarrow I(1)$ equals $Z \cap I^{sc}(1)$ where

$$Z = \ker(G^{sc} \rightarrow G)(\mathbb{Z}_p) = \ker(T^{sc} \rightarrow T)(\mathbb{Z}_p).$$

Therefore $Z \cap I^{sc}(1)$ is the kernel of $T^{sc}(1) \rightarrow T(1)$, which is trivial since $Z$ is finite and $T^{sc}(1) \simeq \text{Hom}(M^{sc}, 1 + p\mathbb{Z}_p)$ has no torsion. We thus obtain exact sequences

$$1 \rightarrow T^{sc}(1) \rightarrow T(1) \rightarrow Q \rightarrow 0$$

$$1 \rightarrow I^{sc}(1) \rightarrow I(1) \rightarrow Q \rightarrow 0$$

where the cokernel $Q$ is the finitely generated $\mathbb{Z}_p$-module

$$Q = (M^\vee / ZR^\vee) \otimes (1 + p\mathbb{Z}_p).$$
Remark 2.13.1. If $G$ is simple, then $M'/ZR'$ is a finite group of order $c$, with $c \mid \ell + 1$ if $G$ is of type $A_\ell$, $c \mid 3$ if $G$ is of type $E_6$ and $c \mid 4$ in all other cases. Thus $Q = 0$ and $I^{sc}(1) = I(1)$ unless $G$ is of type $A_\ell$ with $p \mid c \mid \ell + 1$ or $p = 3$ and $G$ is adjoint of type $E_6$. In these exceptional cases, $M'/ZR'$ is cyclic, thus $Q \simeq \mathbb{F}_p$.

2.14 It follows that $I(1)$ is generated by $I^{sc}(1)$ and $s(1 + p\mathbb{Z}_p)$ for $s \in S$, thus topologically generated by $I^{sc}(1)$ and $s(1 + p)$ for $s \in S$. In view of the results already established in the simply connected case, this shows that the elements listed in $(1 - 4)$ of Theorem 2.4.1 indeed form a set of topological generators for $I(1)$.

None of the semi-simple elements in (1) can be removed: they are all needed to generate the above abelian quotient $Q$ of $I(1)$ which indeed kills the unipotent generators in $(2 - 4)$. Likewise, none of the unipotent elements in (2) can be removed: they are all needed to generate the abelian quotient

$$I(1) \twoheadrightarrow U(\mathbb{F}_p) \twoheadrightarrow U_1(\mathbb{F}_p) \simeq \prod_{\alpha \in \Delta} U_\alpha(\mathbb{F}_p)$$

which kills the other generators in (1), (3) and (4). One checks easily using the Iwahori decomposition of $I(1)$ and the product decomposition $U^- = \prod_{c \in C} U_c^-$ that none of the unipotent elements in (3) can be removed. Finally if $p = 3$ and $d \in D$, the central isogeny $G^{sc} \to G^{ad}$ induces an isomorphism $G_d^{sc} \to G_d^{ad}$ between the simple (simply connected and adjoint) components corresponding to $d$, thus also an isomorphism between the corresponding pro-$p$-Iwahori’s $I_d^{sc}(1) \to I_d^{ad}(1)$. In particular, the projection $I(1) \to I^{ad}(1) \to I_d^{ad}(1)$ is surjective. Composing it with the projection $I_d^{ad}(1) \to \mathbb{F}_3^3$ constructed in section 2.10, we obtain an abelian quotient $I(1) \to \mathbb{F}_3^3$ that kills all of our generators except $x_\alpha(1), x_\beta(1)$ and $x_{\alpha + \beta}(1)$ where $\Delta_3 = \{\alpha, \beta\}$. In particular, the generator $x_{\alpha + \beta}(1)$ from (4) is also necessary. This finishes the proof of Theorem 2.4.1.

2.15 The action of $I^{ad} = I^{ad}(1) \rtimes T^{ad}_{\text{tors}}$ on $I(1)$ induces an $\mathbb{F}_p$-linear action of $T^{ad}_{\text{tors}} = \text{Hom} \left( M^{ad}, \mu_{p - 1} \right) = \text{Hom} \left( \mathbb{Z}R, \mathbb{F}_p^{\times} \right)$ on the Frattini quotient $\bar{I}(1)$ of $I(1)$. Our minimal set of topological generators of $I(1)$ reduces to an eigenbasis of $\bar{I}(1)$, i.e. an $\mathbb{F}_p$-basis of $\bar{I}(1)$ made of eigenvectors for the action of $T^{ad}_{\text{tors}}$. We denote by $\mathbb{F}_p(\alpha)$ the 1-dimensional representation of $T^{ad}_{\text{tors}}$ on $\mathbb{F}_p$ defined by $\alpha \in \mathbb{Z}R$. We thus obtain:

**Corollary 2.15.1.** The $\mathbb{F}_p[T^{ad}_{\text{tors}}]$-module $\bar{I}(1)$ is isomorphic to

$$\mathbb{F}_p^S \oplus \left( \oplus_{\alpha \in \Delta} \mathbb{F}_p(\alpha) \right) \oplus \left( \oplus_{c \in C} \mathbb{F}_p(-\alpha_{c,\text{max}}) \right) \left( \oplus_{d \in D} \mathbb{F}_p(\delta_c) \right) \text{ if } p = 3.$$

Here $\sharp S$ denotes the cardinality of the set $S$. The map $\alpha \mapsto \mathbb{F}_p(\alpha)$ yields a bijection between $\mathbb{Z}R/(p - 1)\mathbb{Z}R$ and the isomorphism classes of simple $\mathbb{F}_p[T^{ad}_{\text{tors}}]$-modules. In particular some of the simple modules in the previous corollary may happen to be isomorphic. For instance if $G$ is simple of type $B_\ell$ and $p = 3$, then $-\alpha_{\text{max}} \equiv \alpha \mod 2$ where $\alpha \in \Delta$ is a long simple root. An inspection of the tables in [1] yields the following:

**Corollary 2.15.2.** If $G$ is simple, the $\mathbb{F}_p[T^{ad}_{\text{tors}}]$-module $\bar{I}(1)$ is multiplicity free unless $p = 3$ and $G$ is of type $A_1$, $B_\ell$ or $C_\ell$ ($\ell \geq 2$), $F_4$ or $G_2$.

In the next section we use this result to construct Galois representations landing in $I^{ad}$ with image containing $I^{ad}(1)$. 
3 The Construction of Galois Representations

Let $G$ be a split simple adjoint group over $\mathbb{Z}_p$ and let $I(1)$ and $I = I(1) \rtimes T_{\text{tors}}$ be the corresponding Iwahori groups, as defined in the previous section. We want here to construct Galois representations of a certain type with values in $I$ with image containing $I(1)$. After a short review of $p$-rational fields in section 3.1, we establish a criterion for the existence of our representations in sections 3.2 and 3.3 and finally give some examples in section 3.4.

3.1 Let $K$ be a number field, $r_2(K)$ the number of complex primes of $K$, $\Sigma_p$ the set of primes of $K$ lying above $p$, $M$ the compositum of all finite $p$-extensions of $K$ which are unramified outside $\Sigma_p$, $M^{ab}$ the maximal abelian extension of $K$ contained in $M$, and $L$ the compositum of all cyclic extensions of $K$ of degree $p$ which are contained in $M$ or $M^{ab}$. If we let $\Gamma$ denote $\text{Gal}(M/K)$, then $\Gamma$ is a pro-$p$ group, $\Gamma^{ab} \cong \text{Gal}(M^{ab}/K)$ is the maximal abelian quotient of $\Gamma$, and $\hat{\Gamma} \cong \Gamma^{ab}/p\Gamma^{ab} \cong \text{Gal}(L/K)$ is the Frattini quotient of $\Gamma$.

**Definition** A number field $K$ is $p$-rational if the following equivalent conditions are satisfied:

1. $\text{rank}_{\mathbb{Z}_p}(\Gamma^{ab}) = r_2(K) + 1$ and $\Gamma^{ab}$ is torsion-free as a $\mathbb{Z}_p$-module,
2. $\Gamma$ is a free pro-$p$ group with $r_2(K) + 1$ generators,
3. $\Gamma$ is a free pro-$p$ group.

The equivalence of (1), (2) and (3) follows from [6], see also proposition 3.1 and the discussion before remark 3.2 of [3]. There is a considerable literature concerning $p$-rational fields, including [8], [4].

**Examples:**

1. Suppose that $K$ is a quadratic field and that either $p \geq 5$ or $p = 3$ and is unramified in $K/\mathbb{Q}$. If $K$ is real, then $K$ is $p$-rational if and only if $p$ does not divide the class number of $K$ and the fundamental unit of $K$ is not a $p$-th power in the completions $K_v$ of $K$ at the places $v$ above $p$. On the other hand, if $K$ is complex and $p$ does not divide the class number of $K$, then $K$ is a $p$-rational field (cf. proposition 4.1 of [3]). However, there are $p$-rational complex $K$’s for which $p$ divides the class number (cf. chapter 2, section 1, p. 25 of [7]). For similar results, see also [2] and [5] if $K$ is complex.

2. Let $K = \mathbb{Q}(\mu_p)$. If $p$ is a regular prime, then $K$ is a $p$-rational field (cf. [12], see also [3], proposition 4.9 for a shorter proof).

3.2 Suppose that $K$ is Galois over $\mathbb{Q}$ and $p$-rational with $p \nmid [K : \mathbb{Q}]$.

Since $K$ is Galois over $\mathbb{Q}$, so is $M$ and we have an exact sequence

$$1 \to \Gamma \to \Pi \to \Omega \to 1$$  \hspace{1cm} (3.2.1)

where $\Omega = \text{Gal}(K/\mathbb{Q})$ and $\Pi = \text{Gal}(M/\mathbb{Q})$. Conjugation in $\Pi$ then induces an action of $\Omega$ on the Frattini quotient $\hat{\Gamma} = \text{Gal}(L/K)$ of $\Gamma$. Any continuous morphism $\rho : \Pi \to I$ maps $\Gamma$ to $I(1)$ and induces a morphism $\overline{\rho} : \Omega \to I/I(1) = T_{\text{tors}}$ and a $\overline{\rho}$-equivariant morphism $\tilde{\rho} : \hat{\Gamma} \to \tilde{I}(1)$. If $\rho(\Gamma) = I(1)$, then $\tilde{\rho}$ is also surjective. Suppose conversely that we are given the finite data

$$\overline{\rho} : \Omega \to T_{\text{tors}} \quad \text{and} \quad \tilde{\rho} : \hat{\Gamma} \to \tilde{I}(1).$$
Then as $\Omega$ has order prime to $p$, the Schur-Zassenhaus theorem ([14], proposition 2.3.3) implies that the exact sequence 3.2.1 splits. The choice of a splitting $\Pi \cong \Gamma \rtimes \Omega$ yields a non-canonical action of $\Omega$ on $\Gamma$ which lifts the canonical action of $\Omega$ on the Frattini quotient $\hat{\Gamma}$. By [3], proposition 2.3, $\hat{\rho}$ lifts to a continuous $\Omega$-equivariant surjective morphism $\rho : \Gamma \to I(1)$, which plainly gives a continuous morphism

$$\rho = (\rho', \overline{\rho}) : \Pi \cong \Gamma \rtimes \Omega \to I = I(1) \rtimes T_{\text{tors}}$$

inducing $\overline{\rho} : \Omega \to T_{\text{tors}}$ and $\hat{\rho} : \hat{\Gamma} \to \hat{I}(1)$. Thus:

**Proposition 3.2.1.** Under the above assumptions on $K$, there is a continuous morphism $\rho : \Pi \to I$ such that $\rho(\Gamma) = I(1)$ if and only if there is a morphism $\overline{\rho} : \Omega \to T_{\text{tors}}$ such that the induced $\mathbb{F}_p[\Omega]$-module $\overline{\rho}^* \hat{I}(1)$ is a quotient of $\hat{\Gamma}$.

The Frattini quotient $\hat{I}(1)$ is an $\mathbb{F}_p[T_{\text{tors}}]$-module and by the map $\overline{\rho}$, we can consider $\hat{I}(1)$ as an $\mathbb{F}_p[\Omega]$-module which we denote by $\overline{\rho}^* \hat{I}(1)$.

### 3.3 Suppose now that

**A($K$):** $K$ is a totally complex abelian (thus CM) Galois extension of $\mathbb{Q}$ which is $p$-rational of degree $[K : \mathbb{Q}] \mid p - 1$.

Let $\hat{\Omega}$ be the group of characters of $\Omega$ with values in $\mathbb{F}_p^\times$, $\hat{\Omega}_{\text{odd}} \subset \hat{\Omega}$ the subset of odd characters (those taking the value $-1$ on complex conjugation), and $\chi_0 \in \hat{\Omega}$ the trivial character. Then by [3] proposition 3.3,

$$\hat{\Gamma} = \bigoplus_{\chi \in \Omega_{\text{odd}} \cup \{\chi_0\}} \mathbb{F}_p(\chi)$$

as an $\mathbb{F}_p[\Omega]$-module. In particular, $\hat{\Gamma}$ is multiplicity free. Suppose therefore also that the $\mathbb{F}_p[T_{\text{tors}}]$-module $I(1)$ is multiplicity free, i.e. by corollary 2.15.2,

**B($G$):** If $p = 3$, then $G$ is not of type $A_1$, $B_\ell$ or $C_\ell$ ($\ell \geq 2$), $F_4$ or $G_2$.

For $S$ as in section 2.4, we define

$$\hat{\Omega}_S^{\text{odd}} = \begin{cases} 
\hat{\Omega}_{\text{odd}} \cup \chi_0, & \text{if } S = \emptyset \\
\hat{\Omega}_{\text{odd}}, & \text{if } S \neq \emptyset.
\end{cases}$$

Note that $S = \emptyset$ unless $G$ of type $A_\ell$ with $p \mid \ell + 1$ or $G$ is of type $E_6$ with $p = 3$, in which both cases $S$ is a singleton. We thus obtain:

**Corollary 3.3.1.** Under the assumptions **A($K$)** on $K$ and **B($G$)** on $G$, there is a morphism $\rho : \Pi \to I$ such that $\rho(\Gamma) = I(1)$ if and only if there is morphism $\overline{\rho} : \Omega \to T_{\text{tors}}$ such that the characters $\alpha \circ \overline{\rho} : \Omega \to \mathbb{F}_p^\times$ for $\alpha \in \Delta \cup \{-\alpha_{\text{max}}\}$ are all distinct and belong to $\hat{\Omega}_S^{\text{odd}}$. 

3.4 Some examples. Write \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \) and \( \alpha_{\text{max}} = n_1 \alpha_1 + \cdots + n_\ell \alpha_\ell \) using the conventions of the tables in [1]. In this part we suppose that \( p \) is a regular (odd) prime and take \( K = \mathbb{Q}(\mu_p) \), so that \( K \) is \( p \)-rational and \( \Omega = \mathbb{Z}/(p-1)\mathbb{Z} \).

Lemma 3.4.1. Suppose \( G \) is of type \( A_\ell, B_\ell, C_\ell \) or \( D_\ell \) and \( p \geq 2l + 3 \) (resp. \( p \geq 2l + 5 \)) if \( p \equiv 1 \) mod 4 (resp. \( p \equiv 3 \) mod 4). Then we can find distinct characters \( \phi_1, \ldots, \phi_{\ell+1} \in \Omega_{\text{odd}} \cup \chi_0 \) such that \( \phi_1^{n_1} \phi_2^{n_2} \cdots \phi_{\ell}^{n_\ell} \phi_{\ell+1} = \chi_0 \). Furthermore, if \( G \) is of type \( A_\ell \) and \( \ell \) is odd, then one can even choose the characters \( \phi_1, \ldots, \phi_{\ell+1} \) to be inside \( \Omega_{\text{odd}} \).

Proof. Since \( \Omega \) is (canonically) isomorphic to \( \mathbb{Z}/(p-1)\mathbb{Z} \), \( \# \Omega_{\text{odd}} = \frac{p-1}{2} \) and there are exactly \( \left\lfloor \frac{p-1}{4} \right\rfloor \) pairs of characters \( \{ \chi, \chi^{-1} \} \) with \( \chi \neq \chi^{-1} \) in \( \Omega_{\text{odd}} \). The condition on \( p \) is equivalent to \( \ell \leq 2 \frac{(p-1)}{4} - 1 \).

If \( G \) is of type \( A_\ell \), then \( \alpha_{\text{max}} = \alpha_1 + \cdots + \alpha_\ell \). If \( \ell \) is even and \( \frac{\ell}{2} \leq \left\lfloor \frac{p-1}{4} \right\rfloor \), then we can pick \( \frac{\ell}{2} \) distinct pairs of odd characters \( \{ \chi, \chi^{-1} \} \) as above for \( \{ \phi_1, \ldots, \phi_\ell \} \) and set \( \phi_{\ell+1} = \chi_0 \). If \( \ell \) is odd and \( \frac{\ell+1}{2} \leq \left\lfloor \frac{p-1}{4} \right\rfloor \), then we can choose \( \frac{\ell+1}{2} \) distinct such pairs for the whole set \( \{ \phi_1, \ldots, \phi_{\ell+1} \} \).

If \( G \) is of type \( D_\ell \) (with \( \ell \geq 4 \)), then \( \alpha_{\text{max}} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell+1} - \alpha_\ell \). Now if \( \ell \) is odd we can pick \( \frac{\ell+1}{2} \) such pairs \( \{ \chi, \chi^{-1} \} \), one for \( \{ \phi_1, \phi_\ell \} \), another pair for \( \{ \phi_1, \phi_{\ell+1} \} \) and \( \frac{\ell-3}{2} \) such pairs for \( \{ \phi_2, \ldots, \phi_{\ell-2} \} \). If \( \ell \) is even we let \( \phi_2 \) be the trivial character, and we can choose \( \frac{\ell}{2} \) such pairs of characters \( \{ \chi, \chi^{-1} \} \), one pair for \( \{ \phi_1, \phi_{\ell-1} \} \), another pair for \( \{ \phi_{\ell+1}, \phi_1 \} \) and \( \frac{\ell-2}{2} \) such pairs for \( \{ \phi_3, \ldots, \phi_{\ell-2} \} \). So the inequality that we will need is \( 4 \leq \ell \leq 2 \left\lfloor \frac{p-1}{4} \right\rfloor - 1 \).

If \( G \) is of type \( B_\ell \) (with \( \ell \geq 2 \)), then \( \alpha_{\text{max}} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell \). If \( \ell \) is odd then we pick \( \frac{\ell+1}{2} \) pairs of characters \( \{ \chi, \chi^{-1} \} \); one pair for \( \{ \phi_1, \phi_{\ell+1} \} \) and \( \frac{\ell-1}{2} \) such pairs for \( \{ \phi_2, \ldots, \phi_{\ell} \} \). If \( \ell \) is even then we need \( \frac{\ell}{2} \) pairs of \( \{ \chi, \chi^{-1} \} \); one pair for \( \{ \phi_1, \phi_{\ell+1} \} \) and \( \frac{\ell-2}{2} \) such pairs for \( \{ \phi_3, \ldots, \phi_{\ell} \} \) and we let \( \phi_2 \) be the trivial character. So in this case we need \( 3 \leq \ell \leq 2 \left\lfloor \frac{p-1}{4} \right\rfloor - 1 \).

The remaining \( C_\ell \) case is analogous. \( \square \)

Lemma 3.4.2. Suppose \( G \) is of type \( E_6, E_7, E_8, F_4 \) or \( G_2 \) and \( p \geq \sum_{i=1}^{\ell} (2i-1) n_i + 2 \ell \). Then we can find distinct characters \( \phi_1, \ldots, \phi_{\ell+1} \in \Omega_{\text{odd}} \) such that \( \phi_1^{n_1} \phi_2^{n_2} \cdots \phi_{\ell}^{n_\ell} \phi_{\ell+1} = \chi_0 \).

Proof. The choice of a generator \( \xi \) of \( \mathbb{F}_p^\times \) yields an isomorphism \( \mathbb{Z}/(p-1)\mathbb{Z} \simeq \hat{\Omega} \), mapping \( i \) to \( \chi_i \) and \( 1 + 2\mathbb{Z}/(p-1)\mathbb{Z} \) to \( \hat{\Omega}_{\text{odd}} \). Set \( \phi_i = \chi_{2i-1} \in \hat{\Omega}_{\text{odd}} \) for \( i = 1, \ldots, \ell \) and \( \phi_{\ell+1} = \chi_{-r} \) where \( r = \sum_{i=1}^{\ell} n_i \cdot (2i-1) \). The tables in [1] show that \( h = \sum_{i=1}^{\ell} n_i \) is odd, thus also \( \phi_{\ell+1} \in \Omega_{\text{odd}} \) and plainly \( \phi_1^{n_1} \cdots \phi_{\ell}^{n_\ell} \phi_{\ell+1} = 1 \). If \( p \geq \sum_{i=1}^{\ell} (2i-1) n_i + 2 \ell \), the elements \( \{ 2i-1, -\sum_{i=1}^{\ell} n_i \cdot (2i-1); i \in [1, \ell] \} \) are all distinct modulo \( p \) and hence proves the lemma. \( \square \)

Remark 3.4.3. For \( G \) of type \( E_6, E_7, E_8, F_4 \) or \( G_2 \), the tables in [1] show that the constant \( \sum_{i=1}^{\ell}(2i-1) n_i + 2 \ell \) of lemma 3.4.2 is 79, 127, 247, 53, 13 respectively.

Corollary 3.4.4. There is a constant \( c \) depending only upon the type of \( G \) such that if \( p > c \) is a regular prime, then for \( K = \mathbb{Q}(\mu_p) \), \( M, \Pi \) and \( \Gamma \) as above, there is a continuous morphism \( \rho : \Pi \to I \) with \( \rho(\Gamma) = I(1) \).

In conclusion, we have determined a minimal set of topological generators of the pro-p Iwahori subgroup of a split reductive groups over \( \mathbb{Z}_p \) (theorem 2.4.1) and used it to study the structure of the Frattini quotient \( \overline{I}(1) \) as an \( \mathbb{F}_p[T_{\text{tors}}^d] \)-module (corollary 2.15.1). Then we have used corollary 2.15.1 to determine when \( \overline{I}(1) \) is multiplicity free (see corollary 2.15.2). Furthermore in proposition 3.2.1 and corollary 3.3.1, assuming \( p \)-rationality, we have shown that we can construct Galois representations if and only if we can find a suitable list of distinct characters in \( \Omega \), the existence of which is discussed in section 3.4 under the assumption \( K = \mathbb{Q}(\mu_p) \), for any sufficiently large regular prime \( p \) (see corollary 3.4.4).
References


