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Adiabatic elimination for open quantum systems with effective Lindblad master equations *

R. Azouit1, A. Sarlette2, P. Rouchon1

Abstract—We consider an open quantum system described by a Lindblad-type master equation with two time scales. The fast time scale is strongly dissipative and drives the system towards a low-dimensional decoherence-free space. To perform the adiabatic elimination of this fast relaxation, we propose a geometric asymptotic expansion based on the small positive parameter describing the time scale separation. This expansion exploits geometric singular perturbation theory and center-manifold techniques. We conjecture that, at any order, it provides an effective slow Lindblad master equation and a completely positive parameterization of the slow invariant sub-manifold associated to the low-dimensional decoherence-free space. By preserving complete positivity and trace, two important structural properties attached to open quantum dynamics, we obtain a reduced-order model that directly conveys a physical interpretation since it relies on effective Lindbladian descriptions of the slow evolution. At the first order, we derive simple formulae for the effective Lindblad master equation. For a specific type of fast dissipation, we show how any Hamiltonian perturbation yields Lindbladian second-order corrections to the first-order slow evolution governed by the Zeno-Hamiltonian. These results are illustrated on a composite system made of a strongly dissipative harmonic oscillator, the ancilla, weakly coupled to another quantum system.

I. INTRODUCTION

Solving the equation of evolution for an open quantum system - the Lindblad master equation [5] - is generally tedious. To gain better physical insight and/or for numerical simulations, it is of wide interest to compute rigorous reduced models of quantum dynamical systems. In a typical case, a system of interest is coupled to an ancillary system expressing a measurement device or a perturbing environment [12]. The quantum dynamics describes the joint evolution of both systems and we want to determine a dynamical equation for the system of interest only, from which we have “eliminated” the ancillary system.

A standard tool for model reduction is to use the different time scales of the complete system to separate the quantum dynamics into fast and slow variables and then eliminate the fast ones. This technique is known as adiabatic elimination. In quantum Hamiltonian systems, regular perturbation theory can be easily applied as the propagator remains unitary, and the construction of the reduced model to various orders of approximation is standard [19]. In contrast, for open quantum systems, described by a Lindblad master equation [5], the case is much more complicated and involves singular perturbation theory. Several particular examples have been treated separately. In [6] different methods are proposed to perform an adiabatic elimination up to second order on a lambda system. In [16] and [18] the problem of excited states decaying towards n ground states is treated. A specific atom optics dynamics is investigated in [1]. In the presence of continuous measurement, [9] presents a method of adiabatic elimination for systems with Gaussian dynamics.

Perturbation theory for open quantum systems, with e.g. Lindblad dynamics, has attracted much less attention than their Hamiltonian counterpart. Treating the Lindblad master equation as a usual linear system, or applying the Schrieffer-Wolff formalism which is generalized in [13] to Lindblad dynamics, requires the inversion of super-operators which can be troublesome both numerically and towards physical interpretation. In [2] we made a first attempt to circumvent this inversion, using invariants of the dynamics. This provides a first-order expansion only, and in general linear form — i.e. not necessarily with the structure of a Lindblad equation. A more physical expression for the slow dynamics has been derived for the formalism of quantum stochastic models introduced by Hudson and Parthasarathy, first in [3] then generalized for unbounded operators in [4]. The original dynamics, of the full system, is proven to converge to the reduced slow dynamics as the speed of the fast dynamics tends to infinity. In the present paper, we make this picture more precise (for finite-dimensional systems). First, we characterize the order of convergence via an asymptotic expansion, in Lindblad form, up to second order. Second, we also give the quantum geometric expansion for the modified center manifold, i.e. the invariant subspace on which the slow dynamics evolves; the latter was not considered in [3], [4].

Compared to standard perturbation theory for dynamical systems, the key feature of a geometric method to perform an adiabatic elimination for open quantum systems is that the resulting reduced model is explicitly described with the structure an effective Lindblad equation; and the reduced state is parameterized by a reduced density operator, whose mapping to the initial system state space is expressed in terms of Kraus operators, ensuring a trace-preserving completely positive map. By preserving these structural properties of open quantum dynamics, we obtain a reduced model that directly conveys a physical interpretation. As far as we know, combining asymptotic expansion with both completely positive map and Lindbladian formulation has never been

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addressed before. This work is a first attempt to investigate
the interest of such combination with lemmas 1, 2 and 3
underlying the conjecture illustrated on figure 1.

Our method applies to general open quantum systems
with two time scales, described by two general Lindbladian
super-operators (1), and where the fast Lindbladian makes
the system converge to a decoherence-free subspace of the
overall Hilbert space. We then use a geometric approach
based on center manifold techniques [8] and geometric sin-
gular perturbation theory [11] to obtain an expansion of the
effect of the perturbation introduced by the slow Lindbladian
on this decoherence-free subspace. For general Lindbladians
satisfying this setting, we get explicit formulas for the
Lindblad operators describing the first order expansion. In
the particular case of a Hamiltonian perturbation, we retrieve
the well known Zeno effect. Furthermore, for a fast Lindbladian
described by a single decoherence operator and subject to a
Hamiltonian perturbation, we derive explicit formulas for the
first-order effect on the location of the center manifold and
for Lindblad operators describing the second order expansion
of the dynamics. This allows to highlight how a first-order
Zeno effect is associated to second-order decoherence.

We apply our method to a quantum system coupled to
a highly dissipative quantum harmonic oscillator (ancilla).
Our general formulas directly provide an effective Lindblad
master equation of the reduced model where this ancilla is
eliminated. The result for this example is well known, which
allows us to emphasize how the correct results are obtained
also on infinite-dimensional systems, and to appreciate the
computational simplicity of applying our formulas in com-
parison with specific computations like [7].

The paper is organized as follows. Section II presents
a class of perturbed master equations

II. A class of perturbed master equations

Denote by $\mathcal{H}$ a Hilbert space of finite dimension, by
$\mathcal{D}$ the compact convex set of density operators $\rho$ on $\mathcal{H}$
($\rho$ is Hermitian, nonnegative and trace one). We consider
a two-time scales dynamics on $\mathcal{D}$ described by the master
differential equation

$$
\frac{d}{dt}\rho = \mathcal{L}_0(\rho) + \epsilon \mathcal{L}_1(\rho)
$$

where $\epsilon$ is a small positive parameter and the linear super-
operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are of Lindbladian forms [5]. That is,
there exist two finite families of operators on $\mathcal{H}$, denoted
by $(L_{0,r})$ and $(L_{1,r})$, and two Hermitian operators $H_0$ and $H_1$
called (Hamiltonians) such that, for $r = 0, 1$, we have

$$
\mathcal{L}_r(\rho) = -i[H_r, \rho] + \sum_r L_{r,x} \rho L_{r,y}^\dagger - \frac{1}{2} L_{r,x}^\dagger L_{r,y} - \frac{1}{2} \rho L_{r,y}^\dagger L_{r,x}.
$$

It is well known that if the initial condition $\rho(0)$ belongs to $\mathcal{D}$,
then the solution $\rho(t)$ of (1) remains in $\mathcal{D}$ and is defined
for all $t \geq 0$. It is also known that the flow of (1) is a
contraction for many distances, such as the one derived from
the nuclear norm $||\cdot||$; for any trajectories $\rho_1$ and $\rho_2$ of (1),
we have $||\rho_1(t) - \rho_2(t)|| \leq ||\rho_1(t') - \rho_2(t')||$ for all $t \geq t'$; see e.g. [17, Th.9.2].

We assume that, for $\epsilon = 0$, the unperturbed master equation
$\frac{d}{dt}\rho = \mathcal{L}_0(\rho)$ converges to a stationary regime. More precisely,
we assume that the unperturbed master equation admits a
sub-manifold of stationary operators coinciding with the $\Omega$
limit set of its trajectories. Denote by

$$
\mathcal{D}_0 = \{ \rho \in \mathcal{D} \mid \mathcal{L}_0(\rho) = 0 \}
$$

this stationary manifold: it is compact and convex. We thus
assume that for all $\rho_0 \in \mathcal{D}$, the solution of $\frac{d}{dt}\rho = \mathcal{L}_0(\rho)$
with $\rho(0) = \rho_0$ converges for $t \to +\infty$ towards an element of $\mathcal{D}_0$
denoted by $R(\rho_0) = \lim_{t \to +\infty} \rho(t)$. Since for any $t \geq 0$
the propagator $e^{t\mathcal{L}_0}$ is a completely positive linear map [17, Chap.8], $R$ is also a completely positive map. By
Choi’s theorem [10] there exists a finite set of operators on
$\mathcal{H}$ denoted by $(M_\mu)$ such that

$$
R(\rho_0) = \sum_\mu M_\mu \rho_0 M_\mu^\dagger
$$

with $\sum_\mu M_\mu^\dagger M_\mu = I$, the identity operator on $\mathcal{H}$. The form
(3) is called a Kraus map. We thus assume that

$$
\mathcal{D}_0 = \{ R(\rho) \mid \rho \in \mathcal{D}_0 \} \quad \text{and} \quad \forall \rho \in \mathcal{D}_0, \ R(\rho) = \rho.
$$

An invariant operator attached to the dynamics $\frac{d}{dt}\rho = \mathcal{L}_0(\rho)$
is a Hermitian operator $J$ such that for any time $t \geq 0$ and
any initial state $\rho_0 = \rho(0)$, we have $\Tr(J \rho(t)) = \Tr(J \rho_0)$.
Such invariant operators $J$ are characterized by the fact that
$\mathcal{L}_0(J) = 0$ where the adjoint map to $\mathcal{L}_0$ is given by

$$
\mathcal{L}_0^*(A) = i[H_0, A] + \sum_r L_{0,r}^\dagger A L_{0,r} - \frac{1}{2} L_{0,r}^\dagger L_{0,r} A - \frac{1}{2} A L_{0,r}^\dagger L_{0,r}
$$

for any Hermitian operator $A$.

Thus by taking the limit for $t$ tending to $+\infty$ in
$\Tr(J \rho(t)) = \Tr(J \rho_0)$, we have, for all Hermitian operators
$\rho_0$, $\Tr(J R(\rho_0)) = \Tr(J \rho_0)$. Denote by $R^*$ the adjoint map
associated to $R$:

$$
R^*(A) = \sum_\mu M_\mu^\dagger A M_\mu
$$

for any Hermitian operator $A$ on $\mathcal{H}$. Then, $\Tr(R^*(J) \rho_0) = \Tr(J_{\rho_0})$ for all $\rho_0$, implying $R^*(J) = J$. I.e. invariant operators
$J$ are characterized by $\mathcal{L}_0^*(J) = 0$ and satisfy $R^*(J) = J$. 

We assume additionally that $D_0$ coincides with the set of density operators with support in $H_0$, a subspace of $\mathcal{H}$. In other words, the unperturbed master equation features a decoherence-free space $\mathcal{H}_0$. Denote by $P_0$ the operator on $\mathcal{H}$ corresponding to orthogonal projection onto $\mathcal{H}_0$. Consequently, for any Hermitian operator $\rho$, we have $P_0\rho P_0 = R(\rho)P_0 = R(\rho)$. Thus $\text{Tr} \left( R^* (P_0) \rho \right) = \text{Tr} \left( R(\rho) \right) = \text{Tr} \left( \rho \right)$ for all $\rho$ which implies:

$$R^* (P_0) = I. \quad (5)$$

Moreover, for any vector $|c\rangle$ in $\mathcal{H}_0$, $R(|c\rangle\langle c|) = |c\rangle\langle c|$. This implies that, for the Kraus map (3), there exists a family of complex numbers $A_\mu$ such that $\sum_\mu |A_\mu|^2 = 1$ and

$$\forall |c\rangle \in \mathcal{H}_0, \quad M_\mu |c\rangle = A_\mu |c\rangle. \quad (6)$$

### III. First order expansion for arbitrary perturbations

We consider here the perturbed master equation (1) whose unperturbed part $\frac{d}{dt}\rho = \mathcal{L}_0(\rho)$ satisfies the assumptions of Section II: any trajectory converges to a steady-state, the set of steady-states $D_0$ coincides with the set of density operators with support on a subspace $\mathcal{H}_0$ of $\mathcal{H}$. This section develops a first-order expansion versus $\epsilon$ of (1) around $D_0$.

Denote by $\mathcal{H}_s$ (subscript $s$ for slow), an abstract Hilbert space with the same dimension as $\mathcal{H}_0$. Denote by $D_s$ the set of density operators on $\mathcal{H}_s$. Denote by $|\nu\rangle$ (resp. $|v_s\rangle$) a Hilbert basis of $\mathcal{H}_s$ (resp. $\mathcal{H}_0$). Consider the Kraus map $K_0$ defined by

$$\forall \rho_0 \in D_0, \quad K_0(\rho_0) = S_0 \rho_0 S_0^\dagger \in D \quad (7)$$

where $S_0 = \sum_s |v_s\rangle \langle v_s|$. We have $S_0 S_0^\dagger = P_0$, the orthogonal projector onto $\mathcal{H}_0$ and $S_0^\dagger S_0 = I_s$, the identity operator on $\mathcal{H}_s$.

As illustrated on figure 1, we are looking for an expansion based on linear super-operators $\{K_m\}_{m \geq 0}$ between $\mathcal{D}_s$ and $\mathcal{D}$ and on Lindblad dynamics $\{\mathcal{L}_s\}_{m \geq 0}$ on $\mathcal{D}_s$ such that any solution $\rho_s(t) \in \mathcal{D}_s$ of the reduced Lindblad master equation

$$\frac{d}{dt} \rho_s = \mathcal{L}_s(\rho_s) = \sum_{m \geq 0} e^{\epsilon m} K_m(\rho_s)$$

yields, via the map

$$\rho(t) = K(\rho_s(t)) = \sum_{m \geq 0} e^{\epsilon m} K_m(\rho_s(t)) \quad (8)$$

a trajectory of the perturbed system (1). We combine here geometric singular perturbation theory [11] with center manifold techniques based on Carr asymptotic expansion lemma [8] to derive recurrence relationships for $K_m$ and $\mathcal{L}_s$. These recurrences are obtained by identifying the terms of the same order in the formal invariance condition

$$\mathcal{L}_0(\rho_s) + \epsilon \mathcal{L}_1(\rho_s) = \frac{d}{dt} \rho = \mathcal{L}(\rho_s) = K(\rho_s).$$

This means that, for any $\rho_s \in \mathcal{D}_s$, we have

$$\mathcal{L}_0 \left( \sum_{m \geq 0} e^{\epsilon m} K_m(\rho_s) \right) + \epsilon \mathcal{L}_1 \left( \sum_{m \geq 0} e^{\epsilon m} K_m(\rho_s) \right) = \sum_{m \geq 0} e^{\epsilon m} K_m \left( \sum_{n \geq 0} e^{\epsilon n} \mathcal{L}_s(\rho_s) \right). \quad (10)$$

The zeroth order terms in epsilon yield

$$\mathcal{L}_0 \left( K_0(\rho_s) \right) = K_0(\mathcal{L}_s(\rho_s)). \quad (11)$$

With $K_0$ defined in (7), we have $\mathcal{L}_0(\rho_s) = 0$ and thus $\mathcal{L}_s(\rho_s) = 0$. Consequently, for $m \geq 1$, we have

$$\mathcal{L}_0 \left( K_m(\rho_s) \right) + \mathcal{L}_1 \left( K_{m-1}(\rho_s) \right) = \sum_{m \geq 1} e^{\epsilon m} \mathcal{L}_s(\rho_s). \quad (12)$$

The first order terms in epsilon define $K_1$ and $\mathcal{L}_s$ by

$$\mathcal{L}_0 \left( K_1(\rho_s) \right) + \mathcal{L}_1 \left( K_0(\rho_s) \right) = K_0(\mathcal{L}_s(\rho_s)). \quad (13)$$

The following lemma proves that the super-operator $\mathcal{L}_s(\rho_s)$ defined by this equation is always of Lindblad form.

**Lemma 1:** Assume that $\mathcal{L}_s(\rho) = -i[H_s, \rho]$ for some Hermitian operator $H_s$ on $\mathcal{H}_s$. Then, if $\mathcal{L}_s(\rho_s)$ satisfies (13), we have
\( \mathcal{L}_{s,1}(\rho_s) = -i[H_{s,1}, \rho_s] \) where \( H_{s,1} = S^\dagger_0 H_1 S_0 \) is a Hermitian operator on \( \mathcal{H}_s \).

Assume that \( L_1(\rho) = L_1 \rho L_1^\dagger \) for some operator \( L_1 \) on \( \mathcal{H} \). Then, if \( \mathcal{L}_{s,1} \) satisfies (13), we have

\[
\mathcal{L}_{s,1}(\rho_s) = \sum_{\mu} A_\mu \rho_s A_\mu^\dagger - \frac{1}{2} (A_\mu^\dagger A_\mu - \rho_s A_\mu A_\mu^\dagger) \tag{14}
\]

with \( A_\mu = S^\dagger_0 M_\mu L_1 S_0 \) and the Kraus operators \( M_\mu \) defined by (3).

The result for a general Lindbladian dynamics (2) for \( r = 1 \) follows by linearity.

Proof: By definition we have \( R \circ K_0 = K_0 \) and \( R \circ L_0 = L_0 \). Then \( R(\mathcal{L}_{s,1}(K_0(\rho_s))) = K_0(\mathcal{L}_{s,1}(\rho_s)) \). Left multiplication by \( S_0^\dagger \) and right multiplication by \( S_0 \) yields

\[
\mathcal{L}_{s,1}(\rho_s) = S_0^\dagger R \left( \mathcal{L}_{s,1}(S_0 \rho S_0^\dagger) \right) S_0
\]

since \( S_0^\dagger S_0 = I_s \) is the identity operator on \( \mathcal{H}_s \).

For \( \mathcal{L}_{s,1}(\rho) = -i[H_{s,1}, \rho] \) we have, exploiting the fact that \( M_\mu S_0 = A_\mu S_0 \) and \( S_0^\dagger S_0 = I_s \):

\[
S_0^\dagger R \left( \mathcal{L}_{s,1}(S_0 \rho S_0^\dagger) \right) S_0
= -i \sum_{\mu} S_0^\dagger M_\mu (H_1 S_0 \rho S_0^\dagger - S_0 \rho S_0^\dagger H_1) M_\mu^\dagger S_0
= -i \sum_{\mu} \left( \left( S_0^\dagger M_\mu S_0 \right) \left( \left( H_1 S_0 \rho S_0^\dagger - S_0 \rho S_0^\dagger H_1 \right) \right) M_\mu^\dagger \right) S_0
= -i \sum_{\mu} \left( S_0^\dagger M_\mu S_0 \right) \left( \left( H_1 S_0 \rho S_0^\dagger - S_0 \rho S_0^\dagger H_1 \right) \right) M_\mu^\dagger S_0
= -i \left( S_0^\dagger H_1 S_0 \right) \left( \left( H_1 S_0 \rho S_0^\dagger - S_0 \rho S_0^\dagger H_1 \right) \right) M_\mu^\dagger S_0
\]

since \( \sum_{\mu} M_\mu^\dagger M_\mu = I_s \). We get the Zeno Hamiltonian \( H_{s,1} = S_0^\dagger H_1 S_0 \).

For \( \mathcal{L}_0(\rho) = L_1 \rho L_1^\dagger - \frac{1}{2} (L_1^\dagger L_1 \rho + \rho L_1 L_1^\dagger) \), similar computations yield

\[
S_0^\dagger R \left( \mathcal{L}_0(S_0 \rho S_0^\dagger) \right) S_0
= \left( \sum_{\mu} S_0^\dagger M_\mu L_1 S_0 \rho S_0^\dagger L_1^\dagger M_\mu^\dagger S_0 \right) S_0
= \left( \sum_{\mu} A_\mu \rho_s A_\mu^\dagger - \frac{1}{2} (A_\mu^\dagger A_\mu - \rho_s A_\mu A_\mu^\dagger) \right) S_0
\]

with \( A_\mu = S_0^\dagger M_\mu L_1 S_0 \). It remains to prove that \( \sum_{\mu} A_\mu^\dagger A_\mu = S_0^\dagger L_1^\dagger L_1 S_0 \) for showing that we indeed have a Lindblad formulation. This results from the following computations:

\[
\sum_{\mu} A_\mu^\dagger A_\mu = \sum_{\mu} S_0^\dagger L_1^\dagger M_\mu^\dagger S_0^\dagger M_\mu L_1 S_0
= S_0^\dagger \left( L_1^\dagger R^\dagger (S_0 S_0^\dagger) L_1 S_0 \right) = S_0^\dagger L_1^\dagger L_1 S_0,
\]

where we use that \( S_0 S_0^\dagger = P_0 \) and \( R^\dagger (P_0) = I \).
Using, once again, $R \circ L_0 \equiv 0$ and $R \circ K_0 = K_0$, we get
\begin{equation}
\mathcal{L}_{s,2}(\rho_s) = S^\dagger_0 R \left( K_1(\rho_s) \right) - K_1 \left( \mathcal{L}_{s,1}(\rho_s) \right) S_0. \tag{16}
\end{equation}

The following lemma shows that $\mathcal{L}_{s,2}(\rho_s)$ admits a Lindbladian form.

**Lemma 3:** The super-operator $\mathcal{L}_{s,2}$ defined by (16) admits the following Lindbladian formulation
\begin{equation}
\mathcal{L}_{s,2}(\rho_s) = \sum_{\mu} B_{\mu} \rho_s B_{\mu}^\dagger - \frac{1}{2} \left( B_{\mu}^\dagger B_{\mu} \rho_s + \rho_s B_{\mu} B_{\mu}^\dagger \right)
\end{equation}
with $B_{\mu} = 2S^\dagger_0 M_{\mu} L_o (L_o^{-1})^{-1} H_1 S_0$. $M_{\mu}$ defined by (3) and $(L_o^{-1})^{-1}$ standing for the Moore-Penrose pseudo-inverse of $L_o^{-1}$.

**Proof:** We have $R \left( K_1(\mathcal{L}_{s,1}(\rho_s)) \right) = 0$. This results from $(K_0^s)$ stands for $S_0 \rho_s S_0^\dagger = K_0(\rho_s)$.

Repeating computations similar to (15), we see that for any operator $A$ on $\mathcal{H}$, $R(\mathcal{A}P_0) = R(P_0 \mathcal{A}) = P_0 \mathcal{A} P_0$. Since $P_0 K_0^s = K_0^0 P_0 = K_0^s$ we moreover have $R(\mathcal{A} K_0^s) = P_0 \mathcal{A} K_0^s$ and $R(K_0^s A) = K_0^s \mathcal{A} P_0$. This gives the result of applying $R$ on all the terms in (17), and since $P_0 (L_o^{-1})^{-1} P_0 = 0$, we conclude that $R(K_1(\mathcal{L}_{s,1})) = 0$.

Thus $\mathcal{L}_{s,2}(\rho_s) = S^\dagger_0 R \left( K_1(\rho_s) \right) S_0$. Exploiting similar simplifications, we have
\begin{align}
\mathcal{L}_1(\rho_s) &= -H_1(\rho_s - C_1 \rho_s C_1^\dagger) + (C_1 K_0^s - K_0^s C_1) H_1 \\
&= H_1 K_0^s C_1 + C_1 K_0^s H_1 - (H_1 C_1 K_0^s + K_0^s C_1 H_1) \\
&= 2H_1 K_0^s H_1 (L_o^{-1})^{-1} + 2(L_o^{-1})^{-1} H_1 K_0^s H_1 \\
&\quad - 2H_1 (L_o^{-1})^{-1} H_1 K_0^s - 2K_0^s H_1 (L_o^{-1})^{-1} H_1
\end{align}
and, using $S^\dagger_0 R(\mathcal{A} K_0^s) = S^\dagger_0 P_0 \mathcal{A} K_0^s = S^\dagger_0 \mathcal{A} K_0^s$ and the definition $K_0^s = S_0 \rho_s S_0^\dagger$, we get
\begin{equation}
\mathcal{L}_{s,2}(\rho_s) = 2S^\dagger_0 R \left( H_1 K_0^s H_1 (L_o^{-1})^{-1} + (L_o^{-1})^{-1} H_1 K_0^s H_1 \right) S_0 \\
- 2S^\dagger_0 H_1 (L_o^{-1})^{-1} H_1 S_0 \rho_s - 2\rho_s S^\dagger_0 H_1 (L_o^{-1})^{-1} H_1 S_0.
\end{equation}

Since for all $A$, $R(\mathcal{A}(A)) = 0$, we have the identity
\begin{equation}
R(L_0 A L_0^\dagger) = R \left( \frac{1}{2} (L_0^\dagger L_0 A + A L_0^\dagger) \right).
\end{equation}

With $A = (L_o^{-1})^{-1} H_1 K_0^s H_1 (L_o^{-1})^{-1}$ we get
\begin{equation}
R \left( (I - P_0) H_1 K_0^s H_1 (L_o^{-1})^{-1} + (L_o^{-1})^{-1} H_1 K_0^s H_1 (I - P_0) \right) = R \left( H_1 K_0^s H_1 (L_o^{-1})^{-1} + (L_o^{-1})^{-1} H_1 K_0^s H_1 \right)
\end{equation}
\begin{equation}
\text{since } R \left( P_0 H_1 K_0^s H_1 (L_o^{-1})^{-1} \right) = P_0 H_1 K_0^s H_1 (L_o^{-1})^{-1} P_0 \text{ and } \left( L_o^{-1} H_1 \right)^{-1} P_0 = 0. \text{ Thus }
\mathcal{L}_{s,2}(\rho_s) =
\begin{align}
4S^\dagger_0 R \left( L_o (L_o^{-1})^{-1} H_1 K_0^s H_1 (L_o^{-1})^{-1} L_o \right) S_0 \\
- 2S^\dagger_0 H_1 (L_o^{-1})^{-1} H_1 S_0 \rho_s - 2\rho_s S^\dagger_0 H_1 (L_o^{-1})^{-1} H_1 S_0.
\end{align}
\end{equation}

Using the decomposition (3) of $R$ we have
\begin{equation}
4S^\dagger_0 R \left( L_o (L_o^{-1})^{-1} H_1 K_0^s H_1 (L_o^{-1})^{-1} L_o \right) S_0 = \sum_{\mu} B_{\mu} \rho_s B_{\mu}.
\end{equation}

We conclude by the following computations:
\begin{equation}
\frac{1}{2} \sum_{\mu} B_{\mu} B_{\mu}^\dagger =
\begin{align}
2 \sum_{\mu} &S^\dagger_0 H_1 (L_o^{-1})^{-1} L_o^\dagger M^\dagger_{\mu} S_0 S_0^\dagger M_{\mu} L_o (L_o^{-1})^{-1} H_1 S_0 \\
&= 2S^\dagger_0 H_1 (L_o^{-1})^{-1} L_o^\dagger R(\mathcal{A} P_0) L_o (L_o^{-1})^{-1} H_1 S_0 \\
&= 2S^\dagger_0 H_1 (L_o^{-1})^{-1} L_o^\dagger L_o (L_o^{-1})^{-1} H_1 S_0 \\
&= 2S^\dagger_0 H_1 (L_o^{-1})^{-1} H_1 S_0.
\end{align}
\end{equation}

V. ILLUSTRATIVE EXAMPLE: LOW-Q CYCIFICATION TO ANOTHER QUANTUM SYSTEM

The developments above are rigorous in finite dimension, but they can be formally applied also on infinite-dimensional systems, as illustrated in the following example.

We consider a strongly dissipative driven harmonic oscillator (low-Q cavity) coupled to another, undamped quantum system with the same transition frequency ("target" system). Denote $\mathcal{H}_A$ (resp. $\mathcal{H}_B$) the infinite-dimensional Hilbert space of the strongly dissipative harmonic oscillator (resp. the target system), spanned by the Fock states $|n_A\rangle_{\text{Fock}}$ (resp. a possibly infinite basis $|n_B\rangle_{\text{Fock}}$); $\rho$ is the density operator of the composite system, on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

In the frame rotating at the common frequency of the two systems, their coupled evolution is described by the standard master differential equation:
\begin{equation}
\frac{d}{dt} \rho = \left[ a^\dagger a - a^\dagger a^\dagger, \rho \right] + \kappa \left[ a^\dagger a^\dagger \rho + \rho a^\dagger a \right] - ig \left[ a^\dagger b + ab^\dagger, \rho \right].
\end{equation}

Here $a = a \otimes I_B$ and $b = I_A \otimes b$ are the annihilation operators respectively for the harmonic oscillator $A$ and for the quantum system $B$ (possibly generalized if $B$ is not a harmonic oscillator; e.g. if $B$ is a qubit, we have $b = |g\rangle\langle e|$ the transition operator from excited to ground state). The first line describes the driven and damped evolution of harmonic oscillator $A$, while the second line describes the exchange of energy quanta between the two quantum systems. The constants $(\kappa, g) \in \mathbb{R}^2$ govern the speed of these dynamics. We here consider $\kappa \gg g$, with the goal to adiabatically eliminate the fast dynamics of the low-Q cavity and compute its effect.
on the other quantum system. The dynamics (18) is then
equivalent to
d
dt ρ= L0(ρ) + εL1(ρ) (19)
with L0 = √κ(˜a − α) and εL1(ρ) = −ig [a ˆb + ˆb a†, ρ]. For this typical example, the results of
L0,1 and L0,2 are well known (see e.g. [7, chap.12]). Our
results allow to readily retrieve their expression and thus
completely circumvent the trouble of the usual calculation.

In the absence of coupling between the two subsystems
(ε = 0), the overall system trivially converges towards
R(ρ0) = |α⟩⟨α| ⊗ TrA(ρ0). Here TrA is the partial trace
over HA and |α⟩ denotes the coherent state of amplitude
α ∈ C, towards which a classically driven and damped
harmonic oscillator is known to converge. Therefore we have
HA = |α⟩|α⟩ ⊗ HB, PO = |α⟩⟨α| ⊗ IB, and Mμ = |μ⟩⟨μ| ⊗ IB
with μ spanning N. We will naturally describe ρS on
the Hilbert space HA = HB and with basis {|ni⟩|α⟩}, so S0 =
∑ |α⟩⟨n|β⟨α⟩. For the first-order perturbation, using the property
\( \hat{a}\alpha = a|\alpha⟩ \), Lemma 1 readily yields
\[
H_{1,1} = a\hat{b} + α^∗ \hat{b} + a^∗ \hat{b} ,
\]
denoting by \( q_s \) the operator on HA equivalent to \( q \) on HB.
This standard result shows that the oscillator A can be
approximated as a classical field of amplitude α. Indeed,
H1,1 describes e.g. Rabi oscillations for a qubit driven by a
classical field \( (HA = span|α⟩, |\bar{\alpha}⟩) \); or, when HB describes
another harmonic oscillator, H1,1 is the same Hamiltonian in
fact as in the first line of (18), with classical drive amplitude
\( iu \) replaced by α.

Next, using \( \Theta_\alpha \) the unitary displacement operator on HA,
which satisfies \( \Theta_\alpha a^\dagger \Theta_\alpha = a = α I \), we compute \( L_0 L_0^\dagger =
\sum_k N_k^{-1} \Theta_\alpha^\dagger k \), where \( N_k = a^\dagger a = \sum_{n \geq 1} n |\alpha⟩⟨n| \) and
the Moore-Penrose pseudo-inverse of \( N_k \) is just \( N_k^{-1} =
\sum_{n \geq 1} \frac{1}{n} |n⟩⟨α| \). We then compute
\[
C_1 = \frac{2}{κ} \sum_k N_k^{-1} (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) |α⟩⟨α| ⊗ IB + h.c.
\]
\[
= \frac{2}{κ} \sum_k N_k^{-1} ((\hat{a}^\dagger + α^∗ I)b + (\hat{b}^\dagger + α I)\hat{b}) |α⟩⟨α| ⊗ IB + h.c.
\]
\[
= \frac{2}{κ} \sum_k N_k^{-1} ((\hat{a}^\dagger + α^∗ I)b + (\hat{b}^\dagger + α I)\hat{b}) |0⟩⟨α| ⊗ IB + h.c.
\]
\[
= \frac{2}{κ} \sum_k N_k^{-1} ((α b + α^∗ b^\dagger) |0⟩ + \hat{b}|1⟩)|α⟩ + h.c.
\]
\[
= \frac{2}{κ} \sum_k N_k^{-1} |α⟩⟨α| ⊗ \hat{b} + h.c.
\]
From Lemma 2, we see that a pure state \( |\psi_S⟩ \in HA \) gets
mapped at order zero to \( |α⟩ ⊗ |\varphi_S⟩ \) with \( |\varphi_S⟩ \equiv |\bar{\alpha}⟩ \), but at
order one to a slightly rotated state \( |α⟩ ⊗ |\varphi_S⟩ - \frac{2i}{κ} \sum_k |1⟩⟨1| ⊗
(⟨b| |\varphi_S⟩)⟩). This expresses that the coupled low-Q cavity A
contains slightly more energy than a coherent state, to the
detriment of system B.

For the second order perturbation, from Lemma 3 we must compute
\[ B_\mu = 2S_0^\dagger M_\mu L_0(I_0^\dagger L_0^{-1})^\dagger H_1 S_0. \]
The computations
made for \( C_1 \) above can be used, writing:
\[ B_\mu = S_0^\dagger M_\mu L_0 \left( \frac{2}{κ} \sum_k |α⟩⟨1| \otimes b \right) S_0 \]
\[ = \frac{2}{κ} \sum_k |α⟩⟨n|⟨b| |μ⟩ |α⟩ ⟨a| ⊗ b |m⟩ ⟨m|⟩ S_0 \]
\[ = \frac{2}{κ} \sum |n⟩⟨m| ⟨μ⟩ |α⟩ ⟨a| ⊗ b |m⟩ ⟨n|⟩ S_0 \]
\[ = \frac{2}{κ} \sum |α⟩⟨n| μ |α⟩ ⟨a| ⊗ b |m⟩ ⟨n|⟩ S_0 \]
All the obtained \( B_\mu \) are in fact identical up to a scalar factor, so they may be combined into a single operator:
\[ e^2 \sum_s (n_s)(\rho_s) = g^2 \sum_\mu B_\mu \rho_s B_\mu^\dagger - \frac{1}{2} \left( b_\mu^\dagger b_\mu \rho_s + \rho_\mu, b_\mu^\dagger b_\mu \right) \]
\[ = \frac{4g^2}{κ} \sum_\mu |⟨μ|α⟩|^2 \left( b_\mu^\dagger b_\mu ρ_s - \frac{1}{2} \left( b_\mu^\dagger b_\mu ρ_s + ρ_\mu b_\mu^\dagger b_\mu \right) \right) \]
(Noting that \( |⟨μ|α⟩|^2 \) just corresponds to the expansion of the coherent state \(|α⟩\) of unit norm, in the Fock basis.) We thus get the expected reduced dynamics:
\[ \frac{d}{dt} \rho_s = -ig \left( a b_\mu^\dagger + α^∗ b_\mu^\dagger , \rho_s \right) \]
\[ + \frac{4g^2}{κ} \left( b_\mu^\dagger b_\mu ρ_s - \frac{1}{2} \left( b_\mu^\dagger b_\mu ρ_s + ρ_\mu b_\mu^\dagger b_\mu \right) \right) \]
which expresses that the B system is subject to slow damping
due to the presence of the low-Q cavity.

Remark: Note that if the slow dynamics includes a
Hamiltonian that acts only on the B system, i.e. of the form
\( HB = I_A \otimes H_B \) (acting only on B), then \( C_1 \) features an
additional term
\[ \frac{2}{κ} \sum_k N_k^{-1} \Theta_\alpha^\dagger (I_A \otimes H_B) |α⟩⟨α| ⊗ IB + h.c. \]
\[ = \frac{2}{κ} \left( \Theta_\alpha N_k^{-1} |α⟩⟨α| \right) H_B \]
\[ = \frac{2}{κ} \left( \Theta_\alpha N_k^{-1} |0⟩⟨α| \right) H_B = 0. \]
Thus the second-order correction vanishes and the Zeno
dynamics is the only addition up to second order:
\[ \frac{d}{dt} \rho_s = -ig \left( a b_\mu^\dagger + α^∗ b_\mu^\dagger , H_B, ρ_s \right) \]
\[ + \frac{4g^2}{κ} \left( b_\mu^\dagger b_\mu ρ_s - \frac{1}{2} \left( b_\mu^\dagger b_\mu ρ_s + ρ_\mu b_\mu^\dagger b_\mu \right) \right) \]

VI. CONCLUSION
We have shown how to eliminate the fast dynamics in
an open quantum system (Lindblad equation) with two time
scales, and obtain the resulting reduced dynamics explicitly
as an expansion in Lindblad form, where the slow system
is hence parameterized explicitly with a quantum state on
a lower-dimensional Hilbert space, and mapped to the
complete Hilbert space by a completely positive trace preserving
map (Kraus map). This is important to guarantee that the approximate models at various orders preserve the structure of quantum states (positivity and trace). The Kraus map and characterization of convergence orders is new with respect to previous work [3], [4]. We have illustrated on a benchmark system (highly dissipating quantum oscillator resonantly coupled to another quantum system) how our explicit formulae directly retrieve the results previously obtained with ad hoc computations.

We have obtained explicit formulae for the second-order corrections only in the particular case of a fast Lindbladian with single-channel damping \( L_0 \), and a slow “perturbation” in Hamiltonian form. Conceptually there should be no obstacle to extending this theory to any Lindbladrians, the key point being an appropriate way to generalize the pseudo-inversion \((L_0^\dagger L_0)^{-1}\). However, the special case completed here will already allow to answer currently open questions about the second-order influence of small Hamiltonian perturbations on stable open quantum systems built e.g. with engineered reservoirs [15], [14].

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References