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On locally irregular decompositions of subcubic graphs*

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Abstract

A graph G is locally irregular if every two adjacent vertices of G have different degrees. A locally irregular decomposition of G is a partition $E_1, \ldots, E_k$ of $E(G)$ such that each $G[E_i]$ is locally irregular. Not all graphs admit locally irregular decompositions, but for those who are decomposable, in that sense, it was conjectured by Baudon, Bensmail, Przybyło and Woźniak that they decompose into at most 3 locally irregular graphs. Towards that conjecture, it was recently proved by Bensmail, Merker and Thomassen that every decomposable graph decomposes into at most 328 locally irregular graphs.

We here focus on locally irregular decompositions of subcubic graphs, which form an important family of graphs in this context, as all non-decomposable graphs are subcubic. As a main result, we prove that decomposable subcubic graphs decompose into at most 5 locally irregular graphs, and only at most 4 when the maximum average degree is less than $\frac{12}{5}$. We then consider weaker decompositions, where subgraphs can also include regular connected components, and prove the relaxations of the conjecture above for subcubic graphs.

1 Introduction

Throughout this paper, we deal with so-called *locally irregular decompositions*, which are defined as follows. We consider undirected simple graphs only. A graph G is said *locally irregular* if, for every edge uv of G, we have $d(u) \neq d(v)$. The concept of locally irregular graph arose in the context of neighbour-distinguishing edge-weightings, where one aims at weighting the edges of a given graph so that a particular aggregate, computed from the weighting, yields a proper vertex-colouring. The well-known **1-2-3 Conjecture**, raised by Karoński, Łuczak and Thomason [6], and its variants (see the survey [9] by Seamone), are perhaps the most representative examples where locally irregular graphs arise naturally, as the "best graphs" for these problems are precisely the locally irregular ones.

Still in the context of those weighting problems related to locally irregular graphs, there are situations where, though a given graph G is not locally irregular, knowing that G *decomposes* into a certain number of locally irregular graphs may have some consequences. Here, by a *decomposition* of G, we mean an edge-partition $E_1, \ldots, E_k$ of $E(G)$. Alternatively, a decomposition of G may be regarded as an edge-colouring of G. A decomposition of G is said *locally irregular* when all parts or colour classes induce locally irregular graphs. Locally

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irregular decompositions were formally introduced in [2] by Baudon, Bensmail, Przybyło and Woźniak, who noted that, in particular contexts, a graph admitting a particular locally irregular decomposition agrees with the 1-2-3 Conjecture, or variants of it.

As a more general perspective, we are interested in determining, given a graph G, the smallest number of locally irregular subgraphs that decompose G. Following the edge-colouring point of view, we denote by $\chi'_{\text{irr}}(G)$ that chromatic parameter, which we call the *irregular chromatic index* (of G). Note that the irregular chromatic index is not defined for all graphs, consider for instance any odd-length path or odd-length cycle, which cannot be decomposed at all. From that point of view, we say that G is *decomposable* when $\chi'_{\text{irr}}(G)$ is defined. Otherwise, we call G *exceptional*.

One first important result in the study of locally irregular decompositions is the full characterization of exceptional graphs, due to Baudon, Bensmail, Przybyło and Woźniak [2]. So that we can state this characterization, we first need to formally define the following family $\mathfrak{T}$ of graphs. The definition is recursive:

1. The triangle K_3 belongs to $\mathfrak{T}$.
2. Every other graph in $\mathfrak{T}$ can be constructed by 1) taking an auxiliary graph H being either an even-length path or an odd-length path with a triangle glued to one of its ends, then 2) choosing a graph $G \in \mathfrak{T}$ containing a triangle with at least one vertex, say v, of degree 2 in G, and finally 3) identifying v with a vertex of degree 1 of H.

The full characterization of exceptional graphs is then the following:

Theorem 1.1 (Baudon, Bensmail, Przybyło, Woźniak [2])**.** *A connected graph G is exceptional if and only if G is either 1) an odd-length path, 2) an odd-length cycle, or 3) a member of $\mathfrak{T}$.*

Let us emphasize that all exceptional graphs are subcubic (*i.e.*, have maximum degree at most 3), and are of odd size (number of edges). For these main reasons, we believe that understanding locally irregular decompositions of subcubic graphs is of prime importance.

Concerning decomposable graphs, the main conjecture is that they should admit decompositions into at most three locally irregular graphs.

Conjecture 1.2 (Baudon, Bensmail, Przybyło, Woźniak [2])**.** *For every decomposable graph G, we have $\chi'_{\text{irr}}(G) \leq 3$.*

Conjecture 1.2 was verified for several classes of graphs, including decomposable trees (*i.e.*, trees not being an odd-length path), decomposable complete graphs (*i.e.*, K_n with $n \geq 4$), and some classes of decomposable bipartite graphs and Cartesian products [2]. Using probabilistic methods, Conjecture 1.2 has also been verified for regular graphs with degree at least 10^7 [2], and for graphs with minimum degree at least 10^{10} [8] (by Przybyło). Let us further point out that the bound in Conjecture 1.2, if true, would be best possible, as some decomposable graphs, just as *e.g.* decomposable complete graphs or cycles with length congruent to 2 modulo 4, cannot be decomposed into two locally irregular graphs only. In general, Baudon, Bensmail and Sopena [3] showed that determining the irregular chromatic index of a given graph is an NP-complete problem.

At that moment, though, it was not known whether χ'_{irr} is, in general, bounded above by a constant. This was also not known in the particular case of decomposable bipartite graphs, for which we still do not know whether Conjecture 1.2 holds. These two questions were later considered by Bensmail, Merker and Thomassen [4], who proved the following:

Theorem 1.3 (Bensmail, Merker, Thomassen [4]). *For every decomposable graph G, we have $\chi'_{\rm irr}(G) \leq 328$. Furthermore, if G is bipartite, then we have $\chi'_{\rm irr}(G) \leq 10$.*

In this paper, we consider Conjecture 1.2 in the context of bounded-degree graphs, giving a special focus on subcubic graphs. One first point for that is that it is still not known whether decomposable subcubic graphs verify Conjecture 1.2. Another important motivation is that subcubic graphs are intimately related to exceptional graphs, as all exceptional graphs are subcubic. For these two reasons, it is interesting to understand how locally irregular decompositions behave in subcubic graphs.

Our work is organized as follows. In Section 2, we start by recalling some arguments and results from [4] that are used in our proofs, and which we also use to deduce a first upper bound on the irregular chromatic index of decomposable bounded-degree graphs. In the case of decomposable subcubic graphs G, this yields that $\chi'_{\rm irr}(G) \leq 7$ always holds. Through a more involved proof, we decrease, in Section 3, this bound down to 5. In Section 4, we further decrease this bound down to 4 for decomposable subcubic graphs with maximum average degree less than $\frac{12}{5}$. We then consider, in Sections 5 and 6, two relaxed versions of Conjecture 1.2 that were considered by Bensmail and Stevens [5], where one allows locally irregular decompositions to also induce subgraphs with regular connected components. We show that, in this context, the two relaxations of Conjecture 1.2 are true for subcubic graphs. We end up this paper in Section 7, where we gather some possible directions for future work.

Remark: Right before the submission of the current paper, the authors have been notified of the appearance, on arXiv [7], of a new paper by Lužar, Przybyło and Soták. In that paper, the bounds in Theorem 1.3 have been reduced to 220 and 7, respectively. It was also proved that $\chi'_{\rm irr}(G) \leq 4$ holds for every decomposable subcubic graph G, which improves our main results in Section 3, and partially those in Section 4. However, the results in the current paper and [7] were obtained independently, and the proof arguments we use are different from those from [7], and may thus be of interest for future works on locally irregular decompositions. Furthermore, to the best of our knowledge, the questions we consider in Sections 5 and 6 have not been considered by other authors.

2 Locally irregular decompositions of bounded-degree graphs

One first ingredient in the proof of Theorem 1.3 is a general reduction of Conjecture 1.2 to graphs with even size. We generalize it in the following way, where, by a *hereditary family* of graphs, we mean a family of graphs that is closed under taking subgraphs.

Theorem 2.1 (Bensmail, Merker, Thomassen [4]). *Let $\mathcal{G}$ be a hereditary family of graphs. Then, we have*

$$\max\{\chi'_{\rm irr}(G) : G \in \mathcal{G} \text{ is decomposable}\} \leq \max\{\chi'_{\rm irr}(G) : G \in \mathcal{G} \text{ has even size}\} + 1.$$

Hence, in order to exhibit constant upper bounds on the irregular chromatic index of decomposable graphs among a class $\mathcal{G}$, one may focus on the even-size graphs of $\mathcal{G}$ only. One convenient point for focusing on even-size graphs is that they are all decomposable. In particular, when considering an even-size subgraph of a graph, we do not have to wonder about whether it is exceptional or not.

The proof of Theorem 2.1 relies on the following two lemmas, which we use in the next section.

Lemma 2.2 (Bensmail, Merker, Thomassen [4]). *Let G be a connected graph with even size. Then, for every vertex v of G, there is a path P of length 2 in G, such that P contains v, and all connected components of $G - E(P)$ have even size.*

Recall that, when referring to a *claw*, we mean the star $K_{1,3}$ on 4 vertices.

Lemma 2.3 (Bensmail, Merker, Thomassen [4]). *Let G be a decomposable connected graph with odd size. Then, there is, in G, a claw H with 0 or 2 of its edges subdivided, such that all connected components of $G - E(H)$ have even size.*

Clearly, the graph property "being of maximum degree at most k" is a hereditary property. Thus, using Theorem 2.1 and Lemma 2.2, we can already state a general upper bound on the irregular chromatic index of a decomposable graph with given maximum degree. Throughout this paper, by a *k-vertex* (resp. *k^--vertex*, *k^+-vertex*), we refer to a vertex with degree k (resp. at most k, at least k).

Observation 2.4. *For every connected graph G with even size, we have $\chi'_{\text{irr}}(G) \leq 3\Delta(G) - 3$.*

Proof. We prove the claim by induction on $|V(G)| + |E(G)|$. As it can easily be verified whenever G is small, we proceed with the inductive step. Let v be a $\Delta(G)$-vertex of G. According to Lemma 2.2, we can find, in G, a path P of length 2 such that P contains v, and all connected components of $G' := G - E(P)$ have even size. Since G' is smaller than G, all its connected components have even size, and $\Delta(G') \leq \Delta(G)$, there exists a locally irregular $(3\Delta(G) - 3)$-edge-colouring of G'. By that edge-colouring, there is necessarily, in G, at least one of the $3\Delta(G)-3$ colours, say α, which is not assigned to any edge incident to the vertices of P. Hence, by assigning colour α to the edges of P, we get a locally irregular $(3\Delta(G) - 3)$-edge-colouring of G, since a path of length 2 is locally irregular. □

Corollary 2.5. *For every decomposable graph G, we have $\chi'_{\text{irr}}(G) \leq 3\Delta(G) - 2$.*

3 Locally irregular decompositions of subcubic graphs

Concerning lower bounds on the maximum irregular chromatic index of a decomposable subcubic graph, let us first mention that there are infinitely many subcubic graphs G verifying $\chi'_{\text{irr}}(G) = 3$. This is, in particular, the case for cycles with length congruent to 2 modulo 4 (see [2]). It is actually NP-complete to decide whether a given cubic graph G verifies $\chi'_{\text{irr}}(G) \leq 2$, implying that much more subcubic graphs, with possibly a more complex structure, can have irregular chromatic index 3. This follows from a result of Dehghan, Sadeghi and Ahadi [1], who proved, in the context of the 1-2-3 Conjecture, that deciding whether a cubic graph has a neighbour-sum-distinguishing 2-edge-weighting is an NP-complete problem. This result implies exactly the claim above, as neighbour-sum-distinguishing 2-edge-weightings and locally irregular 2-edge-colourings are equivalent notions in regular graphs (see [2]).

We now turn our attention towards upper bounds on the irregular chromatic index of decomposable subcubic graphs. According to Observation 2.4, we know that connected subcubic graphs with even size have irregular chromatic index at most 6. From that, we

get, according to Corollary 2.5, that decomposable subcubic graphs have irregular chromatic index at most 7. In this section, we decrease these two bounds to 4 and 5, respectively. We actually focus on connected subcubic graphs with even size that are *strictly subcubic*, meaning that they are not cubic. By proving that they have irregular chromatic index at most 4, we are then able to prove the upper bound 5 on the irregular chromatic index of both cubic graphs with even size, and decomposable (not necessarily strictly) subcubic graphs with odd size.

Theorem 3.1. *For every connected strictly subcubic graph G with even size, we have $\chi'_{\text{irr}}(G) \leq 4$.*

Proof. Let G be a counterexample to the claim that is minimal in terms of $|V(G)|+|E(G)|$. In other words, we have $\chi'_{\text{irr}}(G) > 4$, and every smaller connected strictly subcubic graph with even size has irregular chromatic index at most 4. Our proof consists in showing that G cannot contain certain configurations, until we get to the point where G is shown to be cubic, a contradiction.

Recall that a *bridge* of a graph refers to an edge whose deletion disconnects the graph. We start off by showing that G cannot contain non-pendant bridges, where, by a *pendant bridge*, we mean a bridge one of whose end is a 1-vertex. In other words, a pendant bridge is a pendant edge, and a non-pendant bridge is a bridge whose deletion results into two connected components having edges.

Claim 3.2. *The graph G has no non-pendant bridge.*

Proof. Assume, for a contradiction, that G has a non-pendant bridge, *i.e.*, an edge uv such that $G-uv$ has two connected components G_u and G_v with $|E(G_u)|, |E(G_v)| > 0$. Further assume that u belongs to G_u while v belongs to G_v. Since G has even size, we have that $|E(G_u)| + |E(G_v)|$ is odd. We may hence assume that G_u has even size, while G_v has odd size. Since G_u and $G_v + uv$ are smaller than G, are strictly subcubic and of even size, we have $\chi'_{\text{irr}}(G_u), \chi'_{\text{irr}}(G_v + uv) \leq 4$ due to the minimality of G. Hence, there exist a locally irregular 4-edge-colouring ϕ_u of G_u, and a locally irregular 4-edge-colouring ϕ_v of $G_v + uv$. Since $d_{G_u}(u) \leq 2$, and we can freely permute any two colours assigned by ϕ_u to the edges of G_u, we can make sure that ϕ_u assigns colours among $\{1, 2\}$ to the edges of G_u incident to u. Similarly, since $d_{G_v+uv}(u) = 1$, and we can freely permute the colours assigned by ϕ_v to the edges of $G_v + uv$, we can make sure that $\phi_v(uv) = 3$. Clearly, ϕ_u and ϕ_v give rise to a locally irregular 4-edge-colouring of G, a contradiction. □

We now show that G cannot contain pendant bridges as well. In the upcoming proof, and throughout this paper, whenever considering a subgraph obtained by removing edges, we also remove its isolated vertices, if any.

Claim 3.3. *The graph G has no 1-vertex.*

Proof. Assume the contrary, and let uv be an edge of G such that $d(u) = 1$. We must have $d(v) = 3$, as otherwise $d(v) = 2$ and the other edge incident to v would be a non-pendant bridge. Let w_1 and w_2 denote the two neighbours of v different from u. Consider the graph $G' := G - uv - vw_1$. Note that G' is connected as otherwise vw_1 would be a non-pendant bridge of G whose existence would contradict Claim 3.2. Hence G' is a strictly subcubic graph with even size, and smaller than G. There hence exists a locally irregular 4-edge-colouring of G'. By this edge-colouring, the vertices u, v and w_1, because $d_{G'}(u) = 0$,

$d_{G'}(v) = 1$ and $d_{G'}(w_1) \leq 2$, are incident to at most three different colours. A non-used colour can hence be assigned to uv and vw_1, resulting in a locally irregular 4-edge-colouring of G, again a contradiction. □

We gather previous Claims 3.2 and 3.3 in the following way:

Claim 3.4. *The graph G has no bridge.*

Our goal now is to show that G has no 2-vertex. To that aim, we first show that G cannot have small cycles, namely triangles (C_3's) and squares (C_4's).

Claim 3.5. *The graph G has no triangle.*

Proof. Assume the contrary, and let $C := uvwu$ be a triangle of G. If one vertex, say u, of C is a 2-vertex, then consider $G' := G - uv - uw$. That graph is a strictly subcubic graph, with even size and fewer vertices and edges than G, which hence admits a locally irregular 4-edge-colouring. Since $d_{G'}(v), d_{G'}(w) \leq 2$, at most two different colours are assigned to the edges incident to v and w in G'. This is because a locally irregular graph cannot include a connected component isomorphic to K_2. We can thus assign a non-used colour to uv and uw, resulting in a locally irregular 4-edge-colouring of G, a contradiction.

Assume now that $d(u) = d(v) = d(w) = 3$. We note that if removing any of the 2-paths vuw, uwv or uwv from G results in a connected graph, then we can deduce a locally irregular 4-edge-colouring of the remaining graph, having the additional property that at most three colours are assigned to the at most four remaining edges incident to u, v and w. This is again because a locally irregular graph cannot have a connected component isomorphic to K_2. Such a colouring can hence be extended to the removed 2-path using one of the non-used colours, hence to G, a contradiction. Thus, removing any two edges among $\{vu, uw, wv\}$ disconnects G. But this contradicts Claim 3.4, as this implies that every edge not in C and incident to C (there at three of them) is a bridge (either pendant or non-pendant). So C cannot exist. □

Claim 3.6. *The graph G has no square.*

Proof. Assume the contrary, and let $C := uvwxu$ be a square of G. First assume that C has at least one 2-vertex; without loss of generality, we may assume that $d(u) = 2$. Consider the graph $G' := G - ux - uv$; this graph is connected, has even size, and is smaller than G. Therefore, it admits a locally irregular 4-edge-colouring. If one of the four colours is not assigned to one of the at most four edges incident to x and v in G', then we can obtain a locally irregular 4-edge-colouring of G by assigning the non-used colour to ux and uv. So we may assume that $d_G(v) = d_G(x) = 3$, and that all four edges incident to v and x in G' are assigned different colours. But then, in the 4-edge-colouring, necessarily one of wx and wv is isolated in the subgraph induced by its assigned colour, implying that this subgraph is not locally irregular, thus that the 4-edge-colouring is not locally irregular, a contradiction. So, necessarily, one of the four colours does not appear around x and v in G', and the previous case applies.

Assume now that $d(u) = d(v) = d(w) = d(x) = 3$. We denote by u', v', w', x', respectively, the neighbour of u, v, w, x, respectively, which does not belong to C. Note that $G' := G - ux - uv$ remains connected as otherwise uu' would be a bridge in G (contradicting Claim 3.4). Since G' is of even size and is smaller than G, it admits a locally

irregular 4-edge-colouring ϕ. We show that ϕ can always be extended to a locally irregular 4-edge-colouring of G, a contradiction.

Similarly as in a previous case, we may assume that ϕ assigns each of the four colours to at least one edge incident to u, v and x in G'. Note that there are exactly five such edges, as G is simple and does not have triangles by Claim 3.5 (in particular, $v' \neq x$). Assume, without loss of generality, that $\phi(uu') = 1$. Note first that we cannot have $\phi(vw) = 1$ or $\phi(wx) = 1$. Indeed, in such a situation (say $\phi(wx) = 1$), so that all four colours appear in the neighbourhood of u, v, x, one would need, without loss of generality, $\phi(xx') = 2$, $\phi(vw) = 3$ and $\phi(vv') = 4$. But then either wx is an isolated edge in the 1-subgraph[1], or vw is an isolated edge in the 3-subgraph, contradicting the fact that ϕ is locally irregular.

So we may assume that $1 \notin \{\phi(vw), \phi(wx)\}$. We consider two cases depending on whether $\phi(vw)$ and $\phi(wx)$ are equal or not.

- **Case 1:** $\phi(vw) \neq \phi(wx)$**.**

  Without loss of generality, assume that $\phi(vw) = 3$ while $\phi(wx) = 2$, and also that $\phi(xx') = 4$ (since colour 4 appears in the neighbourhood of u, v, x). Because the 2-subgraph is locally irregular, we necessarily have $\phi(ww') = 2$, which implies, because the 3-subgraph is locally irregular, $\phi(vv') = 3$. Therefore, if u' is a 2-vertex in the 1-subgraph, then we can extend ϕ to G by setting $\phi(ux) = \phi(uv) = 1$. So assume u' is a 3-vertex in the 1-subgraph. Analogously, if v' is not a 3-vertex in the 3-subgraph, then we can extend ϕ to G by setting $\phi(uv) = 3$ and $\phi(ux) = 1$. So assume v' is a 3-vertex in the 3-subgraph. Now, if w' is not a 3-vertex in the 2-subgraph, then we can extend ϕ to G by setting $\phi(wv) = \phi(wx) = 2$, and $\phi(ux) = 1$ and $\phi(uv) = 3$. So assume that w' is a 3-vertex in the 2-subgraph. Again, ϕ can be extended to G by setting $\phi(wv) = \phi(wx) = 1$, and $\phi(ux) = \phi(uv) = 2$.

- **Case 2:** $\phi(vw) = \phi(wx)$**.**

  We may assume that $\phi(vw) = \phi(wx) = 2$, and that $\phi(vv') = 4$ and $\phi(xx') = 3$ (because all four colours appear around u, v, x). As in the previous case, we may assume that u' is a 3-vertex in the 1-subgraph. If w is a 3-vertex in the 2-subgraph, then ϕ can be extended to G by setting $\phi(ux) = 1$ and $\phi(uv) = 2$. So assume that w is a 2-vertex in the 2-subgraph. Similarly, if x' is a 3-vertex in the 3-subgraph, then we can extend the colouring by setting $\phi(xu) = 3$ and $\phi(uv) = 1$. So assume that x' is a 2-vertex in the 3-subgraph. A similar argument shows that we may as well assume that v' is a 2-vertex in the 4-subgraph. Now consider the value of $\phi(ww')$. On the one hand, if $\phi(ww') = 1$, then ϕ can be extended to G by setting $\phi(xw) = \phi(xu) = 3$, and $\phi(vw) = \phi(vu) = 4$. On the other hand, if $\phi(ww') \neq 1$, then ϕ can be extended to G by setting $\phi(ux) = \phi(uv) = 2$ and $\phi(wx) = \phi(wv) = 1$.

In each case, ϕ can be extended to a locally irregular 4-edge-colouring of G, a contradiction. So G cannot contain a square. □

We now focus on the 2-vertices of G, which exist, since G is strictly subcubic and has no 1-vertex (Claim 3.3).

Claim 3.7. *The graph G has no neighbouring 2-vertices.*

[1]Given any colour α assigned by an edge-colouring, when mentioning the *α-subgraph*, we refer to the subgraph whose edges are the ones assigned colour α.

Proof. Assume G has two adjacent 2-vertices u and v, and let $u'uvv'$ be the induced path of length 3 of G containing u and v. Here, we consider the graph $G' := G - u'u - uv$. This graph is connected, as otherwise $u'u$ would be a bridge of G, contradicting Claim 3.4. Furthermore, G' has even size and is smaller than G. Hence, there exists a locally irregular 4-edge-colouring of G'. Since, in G', the vertices u' and v are a 2^--vertex and a 1-vertex, respectively, that edge-colouring assigns at most three different colours to edges incident to u' and v in G'. So we can assign a non-used colour to $u'u$ and uv, which results in a locally irregular 4-edge-colouring of G, a contradiction. □

Claim 3.8. *The graph G has no 3-vertex adjacent to two 2-vertices.*

Proof. Assume, for contradiction, that G has a 3-vertex v adjacent to two 2-vertices u_1, u_2 and another 2^+-vertex w. Consider the graph $G' := G - vu_1 - vu_2$. If G' is not connected, then necessarily w belongs to the same connected component as one of u_1 and u_2 (as, otherwise, vw would be a bridge in G, contradicting Claim 3.4). Actually, w belongs to the same connected component as only one of u_1 and u_2, as otherwise G' would be connected. Assume without loss of generality that w and u_2 belong to the same connected component of G', while u_1 belongs to another connected component. But then vu_1 is a bridge in G, which contradicts Claim 3.4.

So G' is necessarily connected. Furthermore, it has even size and is smaller than G. Hence, there exists a locally irregular 4-edge-colouring of G'. Since $d_G(u_1) = d_G(u_2) = 2$, by that edge-colouring, at most three different colours are assigned to the edges incident to v, u_1 and u_2 in G'. There is thus a non-used colour that can be assigned to vu_1 and vu_2, resulting in a locally irregular 4-edge-colouring of G. This is a contradiction. □

We are now ready to conclude the proof, by raising a final contradiction. Since G is strictly subcubic and $\delta(G) > 1$, there is a 2-vertex v in G. Let u_1 and u_2 be the two neighbours of v in G. Because G has no triangle by Claim 3.5, the vertices u_1 and u_2 are not joined by an edge. Furthermore, since G has no 1-vertex by Claim 3.3, nor neighbouring 2-vertices by Claim 3.7, we have $d(u_1) = d(u_2) = 3$. So let w_1, w_2 denote the two neighbours of u_1 different from v, and w_3, w_4 denote the two neighbours of u_2 different from v. Since G has no square by Claim 3.6, we have $N(u_1) \cap N(u_2) = \{v\}$.

By symmetry, and because G has no bridge, we may assume that, in $G - \{v, u_1, u_2\}$, vertices w_1 and w_3 are in a same connected component, and w_2 and w_4 are in a same (possibly different) connected component. Also, the two paths $P_1 := w_1u_1vu_2w_3$ and $P_2 := w_1u_1vu_2w_4$ are symmetric, and it is easy to verify that, for some $i = 1, 2$, each connected component of $G - E(P_i)$ has an even number of edges.

Assume, without loss of generality, that P_1 has that property, and let $G' := G - E(P_1)$. Remember that G' can have up to two connected components, each of which has even size. Since G' is strictly subcubic, smaller than G, and is of even size, there exists a locally irregular 4-edge-colouring ϕ of G'. We extend ϕ to G, so that a contradiction is obtained.

Since $d_{G'}(w_1) \leq 2$ and $d_{G'}(u_1) = 1$, the vertices w_1 and u_1 are incident to at most three edge colours by ϕ, namely the colours assigned to u_1w_2 and to the at most two edges incident to w_1 in G'. So there is a colour $\alpha_1 \in \{1, 2, 3, 4\}$ such that, when assigning colour α_1 to w_1u_1 and u_1v, those two edges induce a path of length 2 in the α_1-subgraph. Analogously, there is a colour $\alpha_2 \in \{1, 2, 3, 4\}$ such that, when assigning colour α_2 to w_4u_2 and u_2v, those two edges induce a path of length 2 in the α_2-subgraph. If $\alpha_1 \neq \alpha_2$, then we get a locally irregular 4-edge-colouring of G by assigning colour α_1 to w_1u_1 and u_1v, and colour α_2 to w_4u_2 and u_2v.

Assume thus that $\alpha_1 = \alpha_2$. Let $\beta_1 := \phi(u_1w_2)$. Recall that $\beta_1 \neq \alpha_1$. We may assume that β_1 is not assigned to any edge incident to w_1 in G', as otherwise there would be another colour, different from α_2, that can be assigned to w_1u_1 and u_1v, and the previous extension strategy could be applied. We note that if w_2 is a 2-vertex in the β_1-subgraph of G' induced by ϕ, then a correct extension of ϕ is obtained by assigning colour β_1 to w_1u_1 and u_1v, and colour α_2 to w_4u_2 and u_2v. Analogously, we can deduce a correct extension when w_3 is a 2-vertex in the β_2-subgraph induced by ϕ, where $\beta_2 := \phi(u_2w_3)$ (unless β_2 appears on an edge incident to w_4, in which case there would be another colour, different from α_1, available to colour w_4u_2 and u_2v). Therefore, we may assume that w_2 is a 3-vertex in the β_1-subgraph induced by ϕ, and w_3 is a 3-vertex in the β_2-subgraph induced by ϕ. But, then, a locally irregular 4-edge-colouring of G is obtained by assigning colour β_1 to u_1w_1, colour β_2 to u_2w_4, and colour α_1 to vu_1 and vu_2. □

We now use Theorem 3.1 to derive corollaries for decomposable subcubic graphs with odd size, and cubic graphs with even size.

Corollary 3.9. *For every connected decomposable strictly subcubic graph G with odd size, we have $\chi'_{\text{irr}}(G) \leq 5$.*

Proof. According to Lemma 2.3, one can find, in G, a claw H with 0 or 2 of its edges subdivided such that $G' := G - E(H)$ has connected components with even size only. All connected components of G' are strictly subcubic. So, every connected component of G' is a strictly subcubic graph with even size. Hence, there exists a locally irregular 4-edge-colouring of G' according to Theorem 3.1. We can extend it to a locally irregular 5-edge-colouring of G by assigning colour 5 to all edges of H, which is locally irregular. □

Corollary 3.10. *For every connected cubic graph G, we have $\chi'_{\text{irr}}(G) \leq 5$.*

Proof. If G has odd size, then the proof can be conducted similarly as the proof of Corollary 3.9. So assume G has even size. Then, according to Lemma 2.2, one can find, in G, a path P with length 2 such that all connected components of $G' := G - E(P)$ have even size (just apply the lemma with any vertex). Again, all connected components of G' are strictly subcubic and of even size. So, similarly as in the proof of Corollary 3.9, we can deduce a locally irregular 4-edge-colouring of G' (from Theorem 3.1), which we can extend to the edges of P using colour 5, hence to G. □

We summarize Theorem 3.1 and Corollaries 3.9 and 3.10 in the following result, which improves Corollary 2.5 for subcubic graphs.

Theorem 3.11. *For every decomposable subcubic graph G, we have $\chi'_{\text{irr}}(G) \leq 5$.*

4 Locally irregular decompositions of subcubic graphs with maximum average degree less than $\frac{12}{5}$

In this section, we focus on decomposable graphs with maximum average degree less than $\frac{12}{5}$, where the *maximum average degree* of a given graph G is

$$\text{mad}(G) := \max\left\{\frac{2|E(H)|}{|V(H)|},\ H \text{ is a subgraph of } G\right\}.$$

More precisely, we again focus on connected subcubic graphs with even size, and prove the following, which is our main result in this section.

Theorem 4.1. *For every connected subcubic graph G with even size and* $\mathrm{mad}(G) < \frac{12}{5}$*, we have* $\chi'_{\mathrm{irr}}(G) \leq 3$*.*

The upcoming folklore lemma gives a relationship between the maximum average degree and the girth of a planar graph. We provide a short proof for the readers' convenience. Recall that the *girth* of a graph G is the length of a shortest cycle in G.

Lemma 4.2. *For every planar graph G with girth at least g, we have $mad(G) < \frac{2g}{g-2}$.*

Proof. Let G be a connected planar graph with girth g. Assume g is finite, as, otherwise, G would be a tree and the result holds. Let H be a subgraph of G. Note that H is planar and has girth at least g. Hence, $g|F(H)| \leq 2|E(H)|$, where $F(H)$ is the set of faces of H. From Euler's Formula, we obtain:

$$2g - g|V(H)| + g|E(H)| = g|F(H)| \leq 2|E(H)|.$$

Hence,

$$2g + (g-2)|E(H)| \leq g|V(H)|,$$

which yields

$$2|E(H)|(2g + (g-2)|E(H)|) \leq 2|E(H)|g|V(H)|,$$

and eventually that

$$\frac{2|E(H)|}{|V(H)|} \leq \frac{2g|E(H)|}{2g + (g-2)|E(H)|} < \frac{2g}{g-2}$$

holds. Since this is true for every subgraph H of G, the claim is proved. □

Hence, from Theorem 4.1 and Lemma 4.2, we deduce the following corollary.

Corollary 4.3. *For every connected planar subcubic graph G with even size and girth $g(G) \geq 12$, we have $\chi'_{\mathrm{irr}}(G) \leq 3$.*

Since edge removals cannot increase the maximum average degree of a graph, Theorem 4.1 can be combined with Theorem 2.1, which yields the following (improving Theorem 3.11 for some classes of decomposable subcubic graphs):

Theorem 4.4. *For every decomposable subcubic graph G with* $\mathrm{mad}(G) < \frac{12}{5}$*, we have* $\chi'_{\mathrm{irr}}(G) \leq 4$*.*

Before proceeding with the proof of Theorem 4.1, let us introduce a few definitions and notations that we use throughout. A 3_k*-vertex* is a 3-vertex adjacent to exactly k 2-vertices. A *bad* 2*-vertex* is a 2-vertex adjacent to another 2-vertex, while a *good* 2*-vertex* is a 2-vertex adjacent to two 3-vertices. A *light* 3*-vertex* is a 3-vertex adjacent to a 1-vertex, while a *heavy* 3*-vertex* is a 3-vertex adjacent to no 2^--vertex. A *bad* 3*-vertex* is a 3-vertex adjacent to two bad 2-vertices. A vertex is called *deficient* if it is a 2-vertex (bad or good) or a light 3-vertex.

Proof of Theorem 4.1. The proof is done by induction. Assuming there exists a minimum counterexample H to the claim, we prove that H cannot exist. To that aim, we go through two steps. The first step consists in proving the non-existence of some set $\mathcal{S}$ of subgraphs in H. Based on the resulting structural properties of H, we then, through a second step, use the discharging technique in order to obtain a contradiction to the fact that H has small

maximum average degree. More precisely, during this second step, we first define a weight function $\omega : V(H) \to \mathbb{R}$ with $\omega(v) := d(v) - \frac{12}{5}$. An important observation is that, by our hypothesis on the maximum average degree of H, the total sum of weights must be strictly negative, since

$$\sum_{v \in V(H)} \omega(v) = \sum_{v \in V(H)} d(v) - \frac{12}{5} \cdot |V(H)|$$

and

$$\sum_{v \in V(H)} d(v) \leq |V(H)| \cdot \mathrm{mad}(H) < \frac{12}{5} \cdot |V(H)|.$$

Next, we define discharging rules to redistribute weights among vertices, resulting, once the discharging process is finished, in a new weight function ω^*. During the discharging process, the total sum of weights is kept fixed. Nevertheless, by the non-existence of $\mathcal{S}$, it will follow that $\omega^*(v) \geq 0$ for all $v \in V(H)$. This will lead to the following contradiction

$$0 \leq \sum_{v \in V(H)} \omega^*(v) = \sum_{v \in V(H)} \omega(v) < 0,$$

contradicting the existence of H.

Structural properties

Let H be a counterexample to Theorem 4.1 minimizing $|E(H)| + |V(H)|$. So, in other words, the graph H has even size, verifies $\mathrm{mad}(H) < \frac{12}{5}$ and $\chi'_{\mathrm{irr}}(H) > 3$, and every proper subgraph H' of H with even size verifies $\chi'_{\mathrm{irr}}(H') \leq 3$. In particular, if we consider a subgraph $H' := H - E$ for some subset $E \subseteq E(H)$ such that all connected components of H' have even size, we get $\chi'_{\mathrm{irr}}(H') \leq 3$.

We start off by showing that H, because it is a minimal counterexample to Theorem 4.1, cannot contain certain structures.

Claim 4.5. *The graph H satisfies the following:*

1. *H does not contain a non-pendant bridge.*
2. *H does not contain a 1-vertex adjacent to a 2-vertex.*
3. *H does not contain a 3-vertex adjacent to a 1-vertex and a 2^--vertex.*
4. *H does not contain a path uvw where u, v, w are 2-vertices.*
5. *H does not contain two adjacent light 3-vertices.*
6. *H does not contain a 3-vertex adjacent to three 2-vertices.*
7. *H does not contain a 3-vertex adjacent to a bad 2-vertex and to two deficient vertices.*
8. *H does not contain two adjacent 3-vertices, such that one of them is adjacent to two bad 2-vertices, while the other one is adjacent to one deficient vertex.*

Proof. We consider each of these structural properties separately.

1. It can easily be checked that the proof of Claim 3.2 can be mimicked in the current context, and still applies, despite we here use three colours only.

2. This just follows from the fact that H has no non-pendant bridge (Claim 4.5.1).

3. Assume H has a 3-vertex v adjacent to a 1-vertex u_1 and a 2^--vertex u_2. Consider $H' := H - vu_1 - vu_2$. We note that H' remains connected as otherwise vu_2 would be a non-pendant bridge in H, contradicting Claim 4.5.1. So H' has even size, verifies $\text{mad}(H') < \frac{12}{5}$, and is smaller than H. It hence admits a locally irregular 3-edge-colouring. Now, because $d_{H'}(v) = 1$ and $d_{H'}(u_2) \leq 1$, there are, by that edge-colouring, at most two different colours assigned to the edges incident to v and u_2 in H'. So we can freely extend this locally irregular 3-edge-colouring to H by assigning to vu_1 and vu_2 one colour non-assigned to any edge incident to v or u_2 in H'. This is a contradiction.

4. We consider $H' := H - uv - vw$. Note that H' remains connected as otherwise all four edges incident to u, v, w would be bridges of H, contradicting Claim 4.5.1 or 4.5.2. Now, a locally irregular 3-edge-colouring of H' can be extended to H by assigning a same colour to uv and vw that does not appear around u or w in H'. This is a contradiction.

5. Assume H has two adjacent light 3-vertices v_1 and v_2. Let u_1 and u_2, respectively, denote the 1-vertex adjacent to v_1 and to v_2, respectively. Let further w denote the third neighbour of v_1 different from u_1 and v_2. By Claim 4.5.3, we know that $d(w) = 3$. Consider $H' := H - v_1v_2 - v_2u_2$. Again, H' is connected as otherwise v_1v_2 would be a non-pendant bridge in H, contradicting Claim 4.5.1. Thus, there exists a locally irregular 3-edge-colouring of H'. To see that it can be extended to v_1v_2 and v_2u_2, hence to H, we just note that, by that edge-colouring, necessarily u_1v_1 and v_1w are assigned the same colour. This is because $d_{H'}(u_1) = 1$ and $d_{H'}(v_1) = 2$, and a locally irregular graph cannot include a connected component isomorphic to K_2. So, by the edge-colouring of H', there are at most two different colours assigned to the edges incident to v_1 and v_2. Therefore, a non-used colour can freely be assigned to v_1v_2 and v_2u_2, resulting in a locally irregular 3-edge-colouring of H, a contradiction.

6. Assume H has a 3-vertex v whose three neighbours u_1, u_2, u_3 are 2-vertices. Let further w_1, w_2, w_3, respectively, denote the neighbour of u_1, u_2, u_3, respectively, different from v. Consider $H' := H - vu_2 - vu_3$. First, we claim that H' remains connected. Assume the contrary. Note that the connected component C containing v must also contain one of u_2 and u_3 as otherwise vu_1 would be a non-pendant bridge in H, contradicting Claim 4.5.1. So C contains v and, say, u_2, while it does not contain u_3. But then vu_3 has to be a non-pendant bridge in H, contradicting Claim 4.5.1. So H' is indeed connected.

   Because H' has even size, verifies $\text{mad}(H') < \frac{12}{5}$, and is smaller than H, there is a locally irregular 3-edge-colouring ϕ of H'. We extend ϕ to H, in the following way. First, if one of the three colours does not appear in the neighbourhood of u_2, u_3 and v, then we can freely assign that colour to both vu_2 and vu_3. So, without loss of generality, we may assume $\phi(u_1v) = 1$, $\phi(u_2w_2) = 2$ and $\phi(u_3w_3) = 3$. Because ϕ is locally irregular, necessarily we have $\phi(u_1w_1) = \phi(u_1v) = 1$. In particular, u_1 is a 2-vertex in the 1-subgraph induced by ϕ. So we can extend ϕ to H by just assigning colour 1 to vu_2 and vu_3. This is correct as v then becomes a 3-vertex in the 1-subgraph while its neighbours are 2^--vertices. Hence, we get a contradiction.

7. The proof of this claim is a bit tedious as it cannot be treated using a common argument for all cases. So, we basically have to consider all possible combinations of deficient vertices. For the sake of legibility, we describe, for each of these cases, the edges which should be removed from H (resulting in H'), and how to extend a locally irregular 3-edge-colouring ϕ of H' to H. In particular, checking whether H' remains connected can be done similarly as in the previous claim.

   Let v be a 3-vertex of H, and u_1 be a bad 2-vertex adjacent to v. We denote by u_2 and u_3 the two deficient neighbours of v different from u_1. Recall that u_2 and u_3 cannot both be 2-vertices as otherwise v would contradict Claim 4.5.6. So, there are, essentially, two cases to consider:

   (a) Both u_2 and u_3 are light 3-vertices. Consider $H' := H - vu_2 - vu_3$. If a colour of ϕ is not assigned to any of the edges incident to u_1, u_2, u_3 in H', then we assign that colour to vu_2 and vu_3. Note further that, for each of u_1, u_2, u_3, its two incident edges in H' are assigned a same colour by ϕ (as otherwise it would not be locally irregular). So we may assume that the two edges incident to u_1 are assigned colour 1, the two edges incident to u_2 are assigned colour 2, and the two edges incident to u_3 are assigned colour 3. Then ϕ can be extended to H by assigning colour 1 to vu_2 and vu_3.

   (b) The vertex u_2 is a light 3-vertex while u_3 is a 2-vertex. Consider $H' := H - vu_1 - vu_2$. Again, if a colour by ϕ does not appear around v, u_1 and u_2, then we assign that colour to the two removed edges. Otherwise, we again get the property that, for each of u_2, u_3 and the neighbour u_1' of u_1 different from v, the two incident edges in H' are assigned the same colour. So, without loss of generality, we may assume that the two edges incident to u_1' in H' are assigned colour 1, the two edges incident to u_2 are assigned colour 2, and the two edges incident to u_3 are assigned colour 3. Then ϕ can be extended to H' by assigning colour 3 to vu_1 and vu_2.

8. In the previous case, we have highlighted the fact that, if uv is an edge of H such that $d(u) > 1$ and v is deficient, then, in a locally irregular edge-colouring of a subgraph H' of H not containing uv, the at most two edges incident to v in H' are assigned the same colour.

   Assume H has two adjacent 3-vertices v_1 and v_2 such that v_1 has a deficient neighbour u_1, while v_2 is adjacent to two bad 2-vertices u_2 and u_3. We further denote by w the neighbour of v_1 different from u_1 and v_2. Due to the fact that u_2 and u_3 are bad 2-vertices, the only possible triangle in $H[u_1v_1, v_1v_2, v_2u_2, v_2u_3]$ is formed by v_2, u_2, u_3. If this triangle exists, then we consider $H' := H - u_3u_2 - u_2v_2$, and deduce a locally irregular 3-edge-colouring of H', which can easily be extended to H. So assume that $H[u_1v_1, v_1v_2, v_2u_2, v_2u_3]$ has not triangle, and consider $H' := H - u_1v_1 - v_1v_2 - v_2u_2 - v_2u_3$. First assume that H' remains connected. Then H' has even size, satisfies $\mathrm{mad}(H) < \frac{12}{5}$, and is smaller than H. It hence admits a locally irregular 3-edge-colouring, which we extend to H as follows. The idea is to colour, if possible, u_1v_1 and v_1v_2 with a same colour, and v_2u_2 and v_2u_3 with a same colour. Note that $d_{H'}(u_1) \leq 2$ and $d_{H'}(v_1) = 1$; there is thus a non-used colour α that can freely be assigned to u_1v_1 and v_1v_2. Similarly, there is also a non-used colour α' that can be assigned to v_2u_2 and v_2u_3. We now note that, even if $\alpha = \alpha'$, we get a locally

irregular 3-edge-colouring of H by assigning colour α to v_1u_1 and v_1v_2, and colour α' to v_2u_2 and v_2u_3.

Lastly, assume that H' is not connected. The rest of the proof now goes quite similarly as the proof of Theorem 3.1. Using similar arguments, it can be checked that H' has exactly two connected components C_1 and C_2. In particular, each of the C_i's contains two of v_1, u_1, u_2, u_3 (note that if $d_H(w) = 1$, then the configuration can easily be treated by removing the edges v_2u_2 and v_2u_3 off H). If C_1 and C_2 both have even size, then induction can be invoked, locally irregular 3-edge-colourings of C_1 and C_2 yield a locally irregular 3-edge-colouring of H', which can be extended to H as previously. So assume that C_1 and C_2 both have odd size. It can be checked that, under all those structural properties, H can be decomposed into two graphs H_1 and H_2, such that $V(H_1) \cap V(H_2) = \{v_1, v_2\}$, and v_1 and v_2 are 2-vertices in, say, H_1, and 1-vertices in H_2. Since v_1v_2 cannot be a non-pendant bridge by Claim 4.5.1, the two cases to consider, in order to construct H_1 and H_2, are the following:

- C_1 includes u_1 and u_2 (while C_2 includes v_1 and u_3): we add u_1v_1, v_1v_2 and v_2u_2 to C_1 to obtain H_1, and add v_2u_3 to C_2 to obtain H_2.
- C_1 includes u_1 and u_3 (while C_2 includes v_1 and u_2): we add u_1v_1, v_1v_2 and v_2u_3 to C_1 to obtain H_1, and add v_2u_2 to C_2 to obtain H_2.

Then H_1 and H_2, which have even size, verify $\mathrm{mad}(H_1), \mathrm{mad}(H_2) < \frac{12}{5}$, and are smaller than H, admit locally irregular 3-edge-colourings ϕ_1 and ϕ_2 (where ϕ_i is that of H_i), respectively. Note that, in H_1, if we have $\phi_1(v_1v_2) = \alpha_1$, then α_1 is also assigned to one of the two edges adjacent to v_1v_2 in H_1. In other words, by ϕ_1, there are only two distinct colours α_1, α_2 assigned to the edges incident to v_1 or v_2. Furthermore, we have, without loss of generality, that v_1 is only incident to edges assigned colour α_1, while v_2 is incident to one edge assigned colour α_1, and one edge assigned colour α_2.

We would now like to permute some of the colours assigned by ϕ_2, so that ϕ_1 and ϕ_2 yield a locally irregular 3-edge-colouring of H. Recall that $V(H_1) \cap V(H_2) = \{v_1, v_2\}$ and that $d_{H_2}(v_1) = d_{H_2}(v_2) = 1$. We start by possibly permuting two colours assigned by ϕ_2, so that the edge incident to v_2 in H_2 is assigned a colour β different from α_1 and α_2. We then finish the permutation process, by, if needed, permuting the two colours by ϕ_2 different from β, so that the edge incident to v_1 in H_2 is assigned a colour different from α_1. Clearly, three colours are sufficient in order to obtain a correct permutation verifying all these constraints. So we end up with a locally irregular 3-edge-colouring of H, a contradiction. □

To lighten the upcoming discharging process, we will not work directly on H but rather on a subgraph H^- of H. More precisely, H^- is the graph obtained from H by removing all 1-vertices of H, *i.e.*, $H^- := H - \{v \in V(H),\ d_H(v) = 1\}$. Clearly, H^- is connected and $\mathrm{mad}(H^-) < \frac{12}{5}$. Furthermore, from the structural properties of H exhibited in Claim 4.5, one can easily derive the following properties of H^-.

Claim 4.6. *According to Claim 4.5, the graph H^- satisfies the following:*

1. *$\delta(H^-) \geq 2$ (Claims 4.5.2 and 4.5.3).*

2. *H^- does not contain a path uvw where u, v, w are 2-vertices (Claims 4.5.3, 4.5.4 and 4.5.5).*

3. *H^- does not contain a 3_3-vertex adjacent to at least one bad 2-vertex (Claims 4.5.3, 4.5.5, 4.5.6 and 4.5.7).*

4. *A bad 2-vertex of H^- is also a bad 2-vertex of H (Claims 4.5.3 and 4.5.5).*

Discharging procedure

To each vertex v of H^-, we assign an initial charge $w(v) := d_{H^-}(v) - \frac{12}{5}$. We then carry out the discharging procedure in two steps:

Step 1. We here just apply, in H^-, the following rule:

(R0) Every heavy 3-vertex gives $\frac{1}{5}$ to each adjacent bad 3-vertex.

Once Step 1 is finished, a new weight function ω' results. We proceed then with Step 2:

Step 2. We here apply, in H^-, the following two rules:

(R1) Every 3-vertex gives $\frac{2}{5}$ to each adjacent bad 2-vertex.

(R2) Every 3-vertex gives $\frac{1}{5}$ to each adjacent good 2-vertex.

Recall that we denote by ω^* the resulting weight function. Let $v \in V(H^-)$ be a k-vertex. By Claim 4.6.1, we have $k \geq 2$. Now, consider the following cases:

- $\mathbf{k = 2}$. Observe that $\omega(v) = -\frac{2}{5}$. Suppose v is a bad 2-vertex. By Claim 4.6.2, the vertex v is adjacent to a 3-vertex. Hence, by (R1), we have $\omega^*(v) = -\frac{2}{5} + \frac{2}{5} = 0$. If v is a good 2-vertex, then $\omega^*(v) = -\frac{2}{5} + 2 \times \frac{1}{5} = 0$ by (R2).

- $\mathbf{k = 3}$. Observe that $\omega(v) = \frac{3}{5}$. To simplify the analysis, we distinguish two cases:

  - Suppose first that v is adjacent to a bad 2-vertex u_1. By Claim 4.6.3, all neighbours of v cannot be 2-vertices, so v is adjacent to at most two 2-vertices (including u_1). If u_1 is the only 2-vertex neighbouring v, then, by (R1), we have $\omega^*(v) \geq \frac{3}{5} - 1 \times \frac{2}{5} = \frac{1}{5} > 0$. Now assume v is adjacent to a second 2-vertex u_2. If u_2 is a good 2-vertex, then, by (R1) and (R2), we have $\omega^*(v) \geq \frac{3}{5} - 1 \times \frac{2}{5} - 1 \times \frac{1}{5} = 0$. Now, if u_2 is a bad 2-vertex, then the third neighbour (different from u_1 and u_2) of v is a heavy 3-vertex, as otherwise H would contain, according to Claim 4.6.4, the configuration described in Claim 4.5.8. So, by (R0), we have $\omega'(v) = \frac{4}{5}$. Hence, by (R1), we get $\omega^*(v) = \frac{4}{5} - 2 \times \frac{2}{5} = 0$.
  - Finally, if v is not adjacent to a bad 2-vertex, then $\omega^*(v) \geq \frac{3}{5} - 3 \times \frac{1}{5} = 0$ by (R0) and (R2).

Therefore, H^- cannot exist and consequently H does not exist either. This completes the proof. □

5 K_2-irregular decompositions of subcubic graphs

In this section, and in Section 6 as well, we focus on two relaxations of Conjecture 1.2 considered by Bensmail and Stevens [5]. In particular, we completely verify these two relaxations for subcubic graphs.

The idea is to study how easier it is, for proving Conjecture 1.2, to allow any locally irregular decomposition to also include additional regular components. In this section, we focus on *K_2-irregular decompositions* (or, analogously, *K_2-irregular edge-colourings*), which are decompositions in which every part induces connected components that are either locally irregular or isomorphic to K_2. In this definition, it should be understood that, in every subgraph induced by a part of the decomposition, there may be locally irregular connected components, and some connected components isomorphic to K_2 as well. For a given graph G, we denote by $\chi'_{\mathrm{K_2-irr}}(G)$ the smallest number of colours in a K_2-irregular edge-colouring of G. Note that $\chi'_{\mathrm{K_2-irr}}(G)$ is defined for every graph G as every proper edge-colouring is K_2-irregular.

Clearly, we have $\chi'_{\mathrm{K_2-irr}}(G) \leq \chi'_{\mathrm{irr}}(G)$ for every decomposable graph G. Hence, Conjecture 1.2, if true, would imply that $\chi'_{\mathrm{K_2-irr}}(G) \leq 3$ holds for every graph G, unless G is exceptional. One may thus wonder whether even $\chi'_{\mathrm{K_2-irr}}(G) \leq 2$ is true for every graph G. This is actually not the case, as, for example, $\chi'_{\mathrm{K_2-irr}}(K_4) = 3$. So, in the context of K_2-irregular edge-colourings, the conjecture that is analogous to Conjecture 1.2 should be the next one, which stands as a relaxation of Conjecture 1.2.

Conjecture 5.1. *For every graph G, we have $\chi'_{\mathrm{K_2-irr}}(G) \leq 3$.*

In the following result, we show that Conjecture 5.1 admits an easy proof in the context of subcubic graphs. Recall that this result remains best possible even in this context because of the complete graph K_4.

Theorem 5.2. *For every subcubic graph G, we have $\chi'_{\mathrm{K_2-irr}}(G) \leq 3$.*

Proof. We prove the claim by induction on $|V(G)| + |E(G)|$. As the claim can easily be verified whenever G is small, we proceed with the general case. Consider any vertex v of G and denote by $u_1, \ldots, u_k$ its neighbours, where $k \leq 3$. Set $G' := G - \{vu_1, \ldots, vu_k\}$. Since G' is smaller than G, there exists a K_2-irregular 3-edge-colouring of G'. Since $d_{G'}(u_1), \ldots, d_{G'}(u_k) \leq 2$, there are, by the edge-colouring, at most two different colours assigned to the edges incident to each u_i in G'. For each vu_i, let α_i denote a colour not assigned to an edge incident to u_i in G'.

Extending the 3-edge-colouring of G' to a K_2-irregular 3-edge-colouring of G can then be done by assigning, for every $i \in \{1, \ldots, k\}$, colour α_i to vu_i, for the following reasons. First of all, because, for each u_i, edge vu_i has been assigned a colour not incident to u_i in G', no conflict involving two vertices of G' may arise. This is because the degrees of the u_i's in the 1-, 2-, and 3-subgraphs of G' that contain them are not altered by the extension. Then, since each u_i is a 1-vertex in the α_i-subgraph induced by the resulting edge-colouring of G, it cannot be that v and u_i are involved in a conflict: the only situation where v and u_i have the same degree in the α_i-subgraph is when this degree is exactly 1, in which case v and u_i belong to a component isomorphic to K_2 in the α_i-subgraph. Thus, we necessarily end up with a K_2-irregular 3-edge-colouring of G. □

6 Regular-irregular decompositions of subcubic graphs

In this section, we focus on *regular-irregular decompositions* (or, analogously, *regular-irregular edge-colourings*), which are more general than K_2-irregular decompositions considered in Section 5. Here, we allow every subgraph induced by a part of a decomposition to have connected components being either locally irregular or regular. So, K_2-irregular decompositions are nothing but regular-irregular decompositions where one requires all induced regular subgraphs to be 1-regular. For a given graph G, we denote by $\chi'_{\mathrm{reg-irr}}(G)$ the smallest number of colours in a regular-irregular edge-colouring of G. Since we have $\chi'_{\mathrm{reg-irr}}(G) \leq \chi'_{\mathrm{K_2-irr}}(G)$ for every graph G, again every graph is decomposable in that manner. Note further that if G is regular, then $\chi'_{\mathrm{reg-irr}}(G) = 1$.

Regular-irregular decompositions were considered by Bensmail and Stevens [5], who conjectured the following.

Conjecture 6.1 (Bensmail, Stevens [5])**.** *For every graph G, we have $\chi'_{\mathrm{reg-irr}}(G) \leq 2$.*

Conjecture 6.1 is known to hold for a few classes of graphs, including trees and some other classes of bipartite graphs [5]. We here give further evidence to the conjecture by showing it to hold for subcubic graphs as well.

Theorem 6.2. *For every subcubic graph G, we have $\chi'_{\mathrm{reg-irr}}(G) \leq 2$.*

Proof. The proof consists in edge-colouring with colours red and green two edge-disjoint subgraphs $\mathcal{C}$ and F of G, in the following way:

1. We consider, as $\mathcal{C}$, a collection of vertex-disjoint cycles of G, and assign colour, say, red, to all edges of $\mathcal{C}$.

2. Set $F := G - E(\mathcal{C})$. Then, we edge-colour F in a regular-irregular way with colours red and green, in such a way that all edges in F being adjacent, in G, to edges of $\mathcal{C}$ are assigned colour green.

If F can be edge-coloured as described, then we note that the connected components of the red subgraph induced by the edge-colouring of $\mathcal{C}$ are disjoint, in G, from the connected components of the red subgraph induced by the edge-colouring of F. So the 2-edge-colourings of $\mathcal{C}$ and F yield a regular-irregular 2-edge-colouring of G.

Start from $\mathcal{C}$ being empty, and, until this procedure cannot be repeated, pick any cycle C of $G - E(\mathcal{C})$ and move the edges of C to $\mathcal{C}$. Once this process stops, the following holds, basically because G is subcubic.

Claim 6.3. *The subgraph $F := G - E(\mathcal{C})$ is a forest. Furthermore, for every vertex v of G having incident edges in $\mathcal{C}$ and incident edges in F, we have $d_{\mathcal{C}}(v) = 2$ and $d_F(v) = 1$.*

Assign colour red to all edges in $\mathcal{C}$. When referring to a *leaf edge* of F, we mean an edge that is incident to a leaf of F. We note that there are, in F, some leaves that are special in the sense that they have both incident edges in $\mathcal{C}$ (two edges) and in F (one leaf edge). We refer to these leaves as *frontier leaves*, and derive this concept to *frontier leaf edges*, which are leaf edges of F whose at least one end is a frontier leaf. Note that a connected component of F can be isomorphic to K_2, in which case this connected component is a frontier leaf edge which potentially joins two frontier leaves.

Following the explanations above, we assign colour green to all frontier leaf edges of F. Note that F might have non-frontier leaf edges. We assign colour green to these edges as well. It now remains to show that the non-coloured (*i.e.*, non-leaf) edges of F can be assigned colours red and green, without modifying the pre-colouring we have described, in a regular-irregular way. In other words, we now want to prove the following:

Claim 6.4. *Every subcubic tree T admits a regular-irregular 2-edge-colouring, such that all leaf edges are assigned colour 1.*

Proof. All along this proof, we see T as a tree whose leaf edges have been pre-assigned colour 1, and we extend this pre-colouring until a regular-irregular 2-edge-colouring is attained.

The proof is by induction on the size of T. As base cases, we note that the claim is true whenever $|E(T)| \leq 3$. Indeed, if T has diameter at most 2, then T is a star on at most three edges being all assigned colour 1. The 1-subgraph is then exactly T, which is either regular (one edge) or locally irregular (two or three edges). On the other hand, if T has diameter 3, then T is the path of length 3 whose two end-edges are assigned colour 1. We here get a regular-irregular 2-edge-colouring (with the desired additional property) of T by assigning colour 2 to the middle-edge.

Assume thus that the claim holds whenever $|E(T)|$ is smaller than some value, and consider the next value of $|E(T)|$. To begin with, if $\Delta(T) \leq 2$, then T is a path whose two end-edges are assigned colour 1. If the length of T is odd, then we obtain the desired regular-irregular 2-edge-colouring of T by assigning colours 1 and 2 alternately, from one end-edge to the other. When the length of T is even, the claimed edge-colouring can be obtained by applying this colouring scheme starting from the second edge of T. In particular, the first two edges of T get assigned colour 1 and thus induce a path of length 2, which is locally irregular, in the 1-subgraph.

We may thus assume that $\Delta(T) = 3$ since T is subcubic. By a *pendant path* of T, we refer to a maximal path $u_1u_2 \dots u_k$ of T such that u_1 is a leaf, all internal vertices $u_2, \dots, u_{k-1}$ are 2-vertices, and u_k is a 3-vertex. Since T has 3-vertices, there are at least three pendant paths in T. If T has a pendant path P with length at least 3, then the desired regular-irregular 2-edge-colouring of T can be obtained in the following way. Let $P := u_1 \dots u_k$ where $d(u_1) = 1$ and $d(u_k) = 3$. Due to the length of P, we have $k \geq 4$. We consider $T' := T - u_1u_2 - u_2u_3$ and assign colour 1 to u_3u_4 in T'. Since T' is subcubic, smaller than T, and has all its leaf edges assigned colour 1, there is, by the induction hypothesis, a regular-irregular 2-edge-colouring of T' which is as claimed. This edge-colouring can be extended to the claimed regular-irregular 2-edge-colouring of T by assigning colour 2 to u_2u_3 and colour 1 to u_1u_2.

We may thus assume that all pendant paths of T have length 1 or 2. If T has only one vertex v with $d(v) = 3$, then T is a subdivided claw all of whose leaf edges are assigned colour 1. We here extend the pre-colouring by just assigning colour 2 to all non-coloured edges of T. Note that these edges are edges that are incident to v and belong to pendant paths with length 2. The resulting edge-colouring is clearly regular-irregular since the 1- and 2-subgraphs include stars only.

Now assume that T has at least two 3-vertices, and let r denote any of them. We designate r as the root of T, which defines, in the usual way, a (virtual) orientation of T from its root to its leaves. Following that orientation, we say that a vertex $v \neq r$ of T is a *multifather* if v has exactly two children (and is hence a 3-vertex as v also has a father).

A multifather of T is said *last* if all of its descendants are 2^--vertices. In other words, a last multifather is a 3-vertex with two pendant paths attached (which are of length 1 or 2). Furthermore, a last multifather is said *deepest* if it is at maximum distance from r in T.

We first claim that if T has a deepest last multifather v such that at least one of its two attached pendant paths P_1 and P_2 has length 2, then we can deduce the desired regular-irregular 2-edge-colouring of T. This follows from the following arguments. First assume that $P_1 := u_1u_2v$ and $P_2 := u'_1u'_2v$ have length 2. In that case, we consider $T' := T - u_1u_2 - u_2v - u'_1u'_2 - u'_2v$. Assuming $f(v)$ denotes the father of v in T, we assign colour 1 to $vf(v)$ in T'. Since T' is subcubic, smaller than T, and has all of its leaf edges assigned colour 1, we can deduce a regular-irregular 2-edge-colouring of T' which is as required. This edge-colouring can be extended to T by assigning colour 2 to vu_2 and vu'_2 (and still assigning colour 1 to u_1u_2 and $u'_1u'_2$). Now assume that $P_2 := u'_1v$ has length 1 (while P_1 is as previously). We here consider $T' := T - u_1u_2 - u_2v - u'_1v$ in which $vf(v)$ is assigned colour 1, and a regular-irregular 2-edge-colouring of T' (with the additional property). We now extend that edge-colouring to T. If, by assigning colour 2 to u_2v (and still assigning colour 1 to u_1u_2 and u'_1v), we do not get a regular-irregular edge-colouring of T, that is only because, in the resulting 1-subgraph, $f(v)$ and v are 2-vertices. In that situation, the desired regular-irregular 2-edge-colouring of T is obtained by assigning colour 1 to u_2v.

Hence, we may assume that $P_1 := u_1v$ and $P_2 := u'_1v$ have length 1. Note that if $f(v) = r$, then, by definition of a deepest last multifather, every vertex of T is at distance at most 2 from r. In that situation, again, by assigning colour 2 to all non-leaf edges of T, we directly get a regular-irregular 2-edge-colouring which is as desired. So assume $f(v) \neq r$, meaning that $f(v)$ has a father $f(f(v))$ in T. In case $f(v)$ is a 2-vertex, *i.e.*, is not a multifather, we consider $T' := T - vu_1 - vu'_1 - vf(v)$, in which the edge $f(v)f(f(v))$ is assigned colour 1. Here, a regular-irregular 2-edge-colouring of T is obtained by assigning colour 2 to $vf(v)$ and colour 1 to vu_1 and vu'_1.

When $d(f(v)) = 3$, there are, according to all assumptions we have made so far, three possibilities concerning the child v' of $f(v)$ different from v: either 1) v' is a leaf, 2) v' has one child w_1 being a leaf, or 3) v' is a deepest last multifather with two children w_1 and w'_1 that are leaves.

In case 1), we consider $T' := T - vu_1 - vu'_1 - vf(v)$. According to the induction hypothesis, T' admits a regular-irregular 2-edge-colouring which is as desired. Recall that $v'f(v)$ is assigned colour 1 by that colouring. On the one hand, if $f(v)f(f(v))$ is assigned colour 1, then we can extend the colouring to T by assigning colour 2 to $vf(v)$ and colour 1 to vu_1 and vu'_1. On the second hand, if $f(v)f(f(v))$ is assigned colour 2, then we get a correct extension by assigning colour 1 to all of $vf(v)$, vu_1 and vu'_1.

In case 2), we consider $T' := T - vu_1 - vu'_1$ and assign colour 1 to the leaf edge $vf(v)$ of T'. Again, according to the induction hypothesis, we can find a regular-irregular 2-edge-colouring of T' which is as desired. Note that if $f(v)$ is not a 3-vertex in the 1-subgraph induced by that edge-colouring, then we can extend the edge-colouring to T by assigning colour 1 to both vu_1 and vu'_1. So we may assume that all three edges incident to $f(v)$ in T' are assigned colour 1. In that case, by assigning colour 1 to vu_1 and vu'_1, and modifying the colour of $vf(v)$ and $v'f(v)$ to 2, we get a 2-edge-colouring of T which is regular-irregular and as desired. In particular, the connected component of the 1-subgraph that contains $f(v)$ remains locally irregular, or becomes a K_2.

Finally, in case 3), we again consider $T' := T - vu_1 - vu'_1$ in which the leaf edge $vf(v)$ is assigned colour 1. Note that $f(v)$ cannot be a 3-vertex in the 1-subgraph induced by any

given regular-irregular 2-edge-colouring of T' since otherwise $f(v)$ and v' would be adjacent 3-vertices in the 1-subgraph. So, necessarily, $f(v)$ is a 2^--vertex in the 1-subgraph, and the edge-colouring can be extended to T by assigning colour 1 to vu_1 and vu'_1.

Thus, a regular-irregular 2-edge-colouring of T with the desired additional property always exists. This concludes the proof. □

Following Claim 6.4, there is thus a regular-irregular edge-colouring of F with colours red and green, such that all frontier leaf edges are green. Together with the edges of $\mathcal{C}$ being assigned colour red, this yields the claimed regular-irregular 2-edge-colouring of G, hence our conclusion. □

7 Conclusion

In this work, we have studied locally irregular decompositions in subcubic graphs. Although we did not manage to prove Conjecture 1.2 for decomposable subcubic graphs, we have showed that they decompose into at most 5 locally irregular subgraphs, which improves by 2 the straight upper bound given by Corollary 2.5.

One first direction for future work could be to try pushing this bound further down. As pointed out in the introduction, our bound has been recently improved down to 4 by Lužar, Przybyło and Soták [7]. The next step would thus be to completely prove Conjecture 1.2 for decomposable subcubic graphs, or at least subclasses of decomposable subcubic graphs. We actually made a first step towards this direction when we considered subcubic graphs with bounded maximum average degree, and proved the conjecture for some of them. As examples, let us mention that the cases of subcubic bipartite graphs and subcubic planar graphs sound quite appealing to us. It might be interesting studying how locally irregular decompositions behave in these graphs.

Another direction for future work could be to consider locally irregular decompositions of graphs with larger, but fixed, maximum degree. Recall that we have provided an upper bound on their irregular chromatic index in Corollary 2.5. As a first step, it could be interesting to investigate how lower this bound can be pushed down for decomposable graphs with maximum degree 4. More generally, it could also be interesting to improve the method in the proof of Observation 2.4, in order to obtain better bounds on the irregular chromatic index of bounded-degree graphs.

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