Growth of quotients of groups acting by isometries on Gromov hyperbolic spaces

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GROWTH OF QUOTIENTS OF GROUPS ACTING BY ISOMETRIES ON GROMOV HYPERBOLIC SPACES

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Abstract. We show that every group G with no cyclic subgroup of finite index that acts properly and cocompactly by isometries on a proper geodesic Gromov hyperbolic space X is growth tight. In other words, the exponential growth rate of G for the geometric (pseudo)-distance induced by X is greater than the exponential growth rate of any of its quotients by an infinite normal subgroup. This result unifies and extends previous works of Arzhantseva-Lysenok and Sambusetti using a geometric approach.

1. Introduction

In this article, we investigate the asymptotic geometry of some discrete groups (G, d) endowed with a left-invariant metric through their exponential growth rate. The *exponential growth rate* of (G, d), also called entropy or critical exponent, is defined as

$$\omega(G, d) = \limsup_{R \to +\infty} \frac{\log \operatorname{card} B_G(R)}{R} \tag{1.1}$$

where $B_G(R)$ is the ball of radius R formed of the elements of G at distance at most R from the neutral element e. (Some authors define the exponential growth rate of (G, d) as the exponential of $\omega(G, d)$.) The quotient group $\bar{G} = G/N$ of G by a normal subgroup N inherits the quotient distance $\bar{d}$ given by the least distance between representatives. The distance $\bar{d}$ is also left-invariant. Clearly, we have $\omega(\bar{G}, \bar{d}) \leq \omega(G, d)$. The metric group (G, d) is said to be *growth tight* if

$$\omega(\bar{G}, \bar{d}) < \omega(G, d) \tag{1.2}$$

for any quotient $\bar{G}$ of G by an infinite normal subgroup $N \lhd G$. In other words, (G, d) is growth tight if it can be characterized by its exponential growth rate among its quotients by an infinite normal subgroup. Observe that if the normal subgroup N is finite, then the exponential growth rates of G and $\bar{G}$ clearly agree.

The notion of growth tightness was first introduced by R. Grigorchuk and P. de la Harpe [GrH97] for word metrics on finitely generated groups. In this context, A. Sambusetti [Sa02b] showed that every nontrivial free product of groups, different from the infinite dihedral group, and every

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amalgamated product of residually finite groups over finite subgroups are growth tight with respect to any word metric. In another result, G. Arzhantseva and I. Lysenok [AL02] gave an affirmative answer to the question about growth tightness of word hyperbolic groups posed by R. Grigorchuk and P. de la Harpe [GrH97]. More precisely, they proved that every non-elementary word hyperbolic group is growth tight for any word metric. Recently, W. Yang [Ya] extended this result to non-elementary relatively hyperbolic groups (and more generally to groups with nontrivial Floyd boundary), still for any word metric. On the other hand, it is not difficult to check that the direct product $F_1 \times F_2$ of two free groups of rank at least 2 is not growth tight for the word metric induced by the natural basis of $F_1 \times F_2$ obtained from two free basis of F_1 and F_2, *cf.* [GrH97].

Applications of growth tightness to geometric group theory in connection with the Hopf property and the minimal growth of groups can be found in [GrH97, Sa01, AL02, Sa02a, Sa02b, Sa04, CSS04].

Word metrics are not the only natural metrics which arise on groups. For instance, let G be the fundamental group of a closed Riemannian manifold (M, g) with basepoint x_0. The group G acts properly and cocompactly by isometries on the Riemannian universal cover $(\tilde{M}, \tilde{g})$ of (M, g). The distance on G induced by g between two elements $\alpha, \beta \in G$, denoted by $d_g(\alpha, \beta)$, is defined as the length of the shortest loop based at x_0 representing $\alpha^{-1}\beta \in G = \pi_1(M, x_0)$. Every quotient group $\bar{G} = G/N$ by a normal subgroup $N \lhd G$ is the deck transformation group of the normal cover $\bar{M} = \tilde{M}/N$ of M. The quotient distance $\bar{d}_g$ on $\bar{G}$ agrees with the distance $d_{\bar{g}}$ on $\bar{G}$ induced by the lift $\bar{g}$ on $\bar{M}$ of the Riemannian metric g on M (here, we take for a basepoint on $\bar{M}$ any lift of x_0). Furthermore, the exponential growth rate of $(\bar{G}, d_{\bar{g}})$ agrees with the one of the Riemannian cover $(\bar{M}, \bar{g})$ defined as

$$\omega(\bar{M}, \bar{g}) = \lim_{R \to +\infty} \frac{\log \operatorname{vol} B_{\bar{g}}(R)}{R}$$

where $B_{\bar{g}}(R)$ is the ball of radius R in $\bar{M}$ centered at the basepoint (the limit exists since M is compact). In other words, we have $\omega(\bar{G}, d_{\bar{g}}) = \omega(\bar{M}, \bar{g})$. Note that the exponential growth rates of $\bar{G}$ and $\bar{M}$ do not depend on the choice of the basepoint. By definition, the exponential growth rate of the Riemannian universal cover $(\tilde{M}, \tilde{g})$ is the volume entropy of (M, g).

In [Sa08], A. Sambusetti proved the following Riemannian analogue of G. Arzhantseva and I. Lysenok's result [AL02].

Theorem 1.1 ([Sa08])**.** *Every Riemannian normal cover $\bar{M}$ of a closed negatively curved Riemannian manifold M, different from the universal cover $\tilde{M}$, satisfies $\omega(\bar{M}) < \omega(\tilde{M})$.*

As pointed out in [Sa08], even though the fundamental group of M is a word hyperbolic group in the sense of Gromov, its geometric distance is only quasi-isometric to any word metric. Since the exponential growth rate of the fundamental group of M (hence, *apriori*, its growth tightness) is not invariant under quasi-isometries, it is not clear how to derive this theorem from its group-theoretical counterpart.

Clearly, this theorem does not extend to nonpositively curved manifolds: the product of a flat torus with a closed hyperbolic surface provides a simple counterexample. In [Sa04], A. Sambusetti asks if growth tightness holds for any Riemannian metric, without curvature assumption, on a closed negatively curved manifold. We affirmatively answer this question in a general way unifying different results on the subject, *cf.* Theorem 1.3.

The following classical definitions will be needed to state our main result.

Definition 1.2. A metric space X is *proper* if all its closed balls are compact and *geodesic* if there is a geodesic segment joining every pair of points of X. Following E. Rips's definition, a geodesic metric space X is *δ-hyperbolic* if each side of a geodesic triangle of X is contained in the 4δ-neighborhood of the union of the other two sides (we refer to Section 4 for the original definition of δ-hyperbolicity). A geodesic metric space is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. A group G is *elementary* if it contains a (finite or infinite) cyclic subgroup of finite index. A group G acts *properly* on a metric space X if for any compact set $K \subset X$, there are only finitely many $\alpha \in G$ such that $\alpha(K)$ intersects K. A group G acts *cocompactly* on a metric space X if the quotient space X/G is compact.

Let G be a group acting by isometries on a metric space $X = (X, |\cdot|)$ with origin O. The distance on X induces a left-invariant pseudo-metric d on G given by

$$d(\alpha, \beta) = |\alpha(O)\beta(O)| \tag{1.3}$$

for every $\alpha, \beta \in G$, where $|xy|$ represents the distance between a pair of elements $x, y \in X$. The notion of exponential growth rate for $G = (G, d)$ extends to the pseudo-distance d and does not depend on the choice of the origin O.

In this article, we consider a non-elementary group G acting properly and cocompactly by isometries on a proper geodesic δ-hyperbolic metric space X. This implies that G is finitely generated. Furthermore, its Cayley graph with respect to any finite generating set is quasi-isometric to X and so is Gromov hyperbolic. Thus, G is a word hyperbolic group in the sense of Gromov. In particular, the exponential growth rate of G is positive for the (pseudo)-metric induced by the distance on X and any word distance since non-elementary hyperbolic groups have non-abelian free subgroups. Note also in this case that the limit-sup (1.1) is a true limit, *cf.* [Co93].

We can now state our main result.

Theorem 1.3. *Let G be a non-elementary group acting properly and cocompactly by isometries on a proper geodesic δ-hyperbolic metric space X. Then G is growth tight for the (pseudo)-metric induced by the distance on X.*

A quantitative version of this result can be found in the last section. Note that the exponential growth rate of a quotient group $\bar{G}$ of G can vanish. In this case, the relation (1.2) is clearly satisfied.

Theorem 1.3 yields an alternative proof of the growth tightness of non-elementary word hyperbolic groups in the sense of Gromov for word metrics, *cf.* [AL02]. Indeed, every word hyperbolic group in the sense of Gromov acts properly and cocompactly by isometries on its Cayley graph (with respect to a given finite generating set) which forms a proper geodesic Gromov hyperbolic space.

Since the universal cover of a closed negatively curved n-manifold M is a Gromov hyperbolic space, we recover Theorem 1.1 as well by taking $G = \pi_1(M)$ and $X = \tilde{M}$. Actually, Theorem 1.3 allows us to extend this result in three directions as it applies to

- Riemannian metrics on M without curvature assumption (and even to Finsler metrics);
- manifolds with a different topology type than M, such as the non-aspherical manifolds $M\#(S^1 \times S^{n-1})$ or $M\#M_0$, where M_0 is any simply connected closed n-manifold different from S^n;
- intermediate covers and more generally covering towers, whereas Theorem 1.1 only deals with the universal cover of M, see the following corollary for an illustration.

Corollary 1.4. *Let $\hat{M}$ be a Riemannian normal cover of a closed Riemannian manifold M. Suppose that $\hat{M}$ is Gromov hyperbolic and that its boundary at infinity contains more than two points. Then every Riemannian normal cover $\hat{M} \to \bar{M} \to M$ with $\hat{M} \neq \bar{M}$ satisfies $\omega(\bar{M}) < \omega(\hat{M})$.*

Obviously, Corollary 1.4 applies when M is diffeomorphic to a closed locally symmetric manifold of negative curvature. One may wonder if the conclusion holds true when M is diffeomorphic to a closed irreductible higher rank locally symmetric manifold of noncompact type. This question finds its answer in Margulis' normal subgroup theorem [Ma91]. Indeed, in the higher rank case, the only normal covers of M are either compact (their exponential growth rate is zero) or are finitely covered by the universal cover $\tilde{M}$ of M (their exponential growth rate agrees with the exponential growth rate of $\tilde{M}$).

In [DPPS11, §5.1], building upon constructions of [DOP00], the authors show that there exists a noncompact complete Riemannian manifold M with pinched negative curvature and finite volume which does not satisfy the conclusion of Corollary 1.4 even for $\hat{M} = \tilde{M}$. This shows that we cannot replace the Gromov hyperbolic space X by a relatively hyperbolic metric space in Theorem 1.3.

Finally, we mention that the gap between the exponential growth rates of G and $\bar{G}$ can be arbitrarily small. For free groups, this follows from [Sh99], see also [Ta05, Lemma 3]. Even for word hyperbolic groups in the sense of Gromov, this is also true. Indeed, it has recently been established in [Cou] that the exponential growth rate of the periodic quotient G/G^n of a non-elementary torsion-free word hyperbolic group G is arbitrarily close to the exponential growth rate of G, for every odd integer n large enough. Here, G^n represents the (normal) subgroup generated by the n-th powers of the elements of G.

The strategy used to prove Theorem 1.3 follows and extends the approach initiated by A. Sambusetti and used in [Sa01, Sa02a, Sa02b, Sa03, Sa04, Sa08, DPPS11]. However, the nature of the problem leads us to adopt a more global point of view, avoiding the use of any hyperbolic trigonometric comparison formula, which cannot be extended in the absence of curvature assumption and without a control of the topology of the spaces under consideration. We present an outline of the proof in the next section.

Notation 1.5. Given $A, B, C \geq 0$, it is convenient to write $A \simeq B \pm C$ for $B - C \leq A \leq B + C$.

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2. Outline of the proof

In this section, we review the approach initiated by A. Sambusetti and developed in this article.

Let G be a non-elementary group acting properly and cocompactly by isometries on a proper geodesic δ-hyperbolic metric space X. Every quotient group $\bar{G} = G/N$ by a normal subgroup $N \lhd G$ acts properly and cocompactly by isometries on the quotient metric space $\bar{X} = X/N$. We will assume that $\omega(\bar{G})$ is nonzero, otherwise the main result clearly holds. This implies that $\bar{X}$ is unbounded.

Fix an origin $O \in X$. The left-invariant pseudo-distance d, *cf.* (1.3), induces a semi-norm $||\cdot||_G$ on G given by

$$||\alpha||_G = d(e, \alpha)$$

for every $\alpha \in G$. By definition, a semi-norm on G is a nonnegative function $||\cdot||$ defined on G such that

$$\begin{aligned} ||\alpha^{-1}|| &= ||\alpha|| \\ ||\alpha\beta|| &\leq ||\alpha||\,||\beta|| \end{aligned}$$

for every $\alpha, \beta \in G$. Semi-norms and left-invariant pseudo-distances on a given group are in bijective correspondence. Similarly, we define a semi-norm $||\cdot||_{\bar{G}}$ on $\bar{G}$ from the quotient pseudo-distance $\bar{d}$. For the sake of simplicity and despite the risk of confusion, we will drop the prefixes pseudo- and semi- in the rest of this article and simply write "distance" and "norm".

For $\lambda \geq 0$, consider the norm $||\cdot||_\lambda$ on the free product $\bar{G} * \mathbb{Z}_2$ where

$$||\gamma_1 * 1 * \cdots * \gamma_{m+1}||_\lambda = \left(\sum_{i=1}^{m+1} ||\gamma_i||_{\bar{G}} \right) + m\lambda$$

for every element $\gamma_1 * 1 * \cdots * \gamma_{m+1} \in \bar{G} * \mathbb{Z}_2$, with $\gamma_i \in \bar{G}$, in reduced form (that is, with m minimal). The norm $||\cdot||_\lambda$ induces a left-invariant distance, denoted by d_λ, on $\bar{G} * \mathbb{Z}_2$. We will write $\omega(\bar{G} * \mathbb{Z}_2, \lambda)$ for the exponential growth rate of $(\bar{G} * \mathbb{Z}_2, d_\lambda)$. Clearly, $\omega(\bar{G} * \mathbb{Z}_2, \lambda + \lambda') \leq \omega(\bar{G} * \mathbb{Z}_2, \lambda)$ for every $\lambda, \lambda' \geq 0$.

A direct combinatorial computation [Sa02b, Proposition 2.3] shows that

$$\omega(\bar{G} * \mathbb{Z}_2, \lambda) > \omega(\bar{G}).$$

More precisely, the following estimate holds

Proposition 2.1 (see Proposition 2.3 in [Sa02b])**.**
For every $\lambda > \mathrm{diam}(X/G)$*, we have*

$$\omega(\bar{G} * \mathbb{Z}_2, \lambda) \geq \omega(\bar{G}) + \frac{1}{4\lambda} \log(1 + e^{-\lambda\omega(\bar{G})}).$$

Strictly speaking this estimate has been stated for true distances, but it also applies to pseudo-distances.

If we could construct an injective 1-Lipschitz map

$$(\bar{G} * \mathbb{Z}_2, ||\cdot||_\lambda) \to (G, ||\cdot||_G),$$

we could claim that the R-ball $B_{\bar{G}*\mathbb{Z}_2,\lambda}(R)$ of $\bar{G} * \mathbb{Z}_2$ for d_λ injects into an R-ball of G. This would imply that $\omega(\bar{G} * \mathbb{Z}_2, \lambda) \leq \omega(G)$. Combined with Proposition 2.1, the main theorem would follow. Here, we do not construct such a nonexpanding embedding, but derive a slightly weaker result which still leads to the desired result.

Fix $\rho > 0$. Let $\bar{G}_\rho$ be a subset of $\bar{G}$ containing the neutral element $\bar{e} \in \bar{G}$ such that the elements of $\bar{G}_\rho$ are at distance greater than ρ from each other and every element of $\bar{G}$ is at distance at most ρ from an element of $\bar{G}_\rho$.

Example 2.2. Let G be the free group $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ endowed with the word metric induced by the canonical generators of the $\mathbb{Z}$ factors. For $\rho = 5/4$, the set $\bar{G}_\rho$ is formed of all words of even length.

Consider the subset $\bar{G}_\rho * \mathbb{Z}_2$ of $\bar{G} * \mathbb{Z}_2$ formed of the elements $\gamma_1 * 1 * \cdots * \gamma_{m+1}$ for $m \in \mathbb{N}$ with $\gamma_i \in \bar{G}_\rho$.

Suppose we can construct an injective 1-Lipschitz map

$$\Phi : (\bar{G}_\rho * \mathbb{Z}_2, ||\cdot||_\lambda) \to (G, ||\cdot||_G) \tag{2.1}$$

for λ large enough. Then, as previously, the R-ball $B_{\bar{G}_\rho*\mathbb{Z}_2,\lambda}(R)$ injects into an R-ball of G and so

$$\omega(\bar{G}_\rho * \mathbb{Z}_2, \lambda) \leq \omega(G). \tag{2.2}$$

To derive the main theorem, we simply need to compare $\omega(\bar{G}_\rho * \mathbb{Z}_2, \lambda)$ with $\omega(\bar{G} * \mathbb{Z}_2, \lambda)$. This is done in the next section, *cf.* Proposition 3.1, where we show that

$$\omega(\bar{G} * \mathbb{Z}_2, 2\lambda + \lambda') \leq \omega(\bar{G}_\rho * \mathbb{Z}_2, \lambda) \tag{2.3}$$

for λ' large enough. The combination of Proposition 2.1, applied to $2\lambda + \lambda'$, with (2.3) and (2.2) allows us to conclude.

Thus, the key argument in the proof of the main theorem consists in constructing and deriving the properties of the nonexpanding map Φ, *cf.* (2.1). This is done in Proposition 9.1 for λ large enough, without assuming that the action of G on X is cocompact.

3. Exponential growth rate of lacunary subsets

The goal of this section is to compare the exponential growth rates of $\bar{G} * \mathbb{Z}_2$ and $\bar{G}_\rho * \mathbb{Z}_2$. We will use the notations (and obvious extensions) previously introduced without further notice.

Given $\sigma > 0$, there exists $r_\sigma > 0$ such that

$$\mathrm{card} B_{\bar{G}_\rho}(r_\sigma) \geq \mathrm{card} B_{\bar{G}}(\sigma). \tag{3.1}$$

An explicit value for r_σ can be obtained from a (naive) packing argument based on the following observation: every point of $\bar{X}$ is at distance at most Δ from a point of the $\bar{G}$-orbit of $\bar{O}$ and so at distance at most $\Delta + \rho$ from some point $\mathring{\gamma}(\bar{O})$, where $\mathring{\gamma} \in \bar{G}_\rho$. Here, $\Delta = \mathrm{diam}(\bar{X}/\bar{G}) = \mathrm{diam}(X/G)$ and $\bar{O}$ represents the projection of O to $\bar{X}$.

More precisely, let $\epsilon > 0$. As $\bar{X}$ is unbounded, we can take $\mathrm{card} B_{\bar{G}}(\sigma)$ points $(\bar{x}_i)$ on a minimizing ray of $\bar{X}$ of length $\ell = 2(\Delta + \rho)\,\mathrm{card} B_{\bar{G}}(\sigma) + \epsilon$ based at $\bar{O}$ with $d(\bar{x}_i, \bar{x}_j) > 2(\Delta + \rho)$ for $i \neq j$. From the previous observation, every point $\bar{x}_i$ is at distance at most $\Delta + \rho$ from some $\mathring{\gamma}_i(\bar{O})$, where $\mathring{\gamma}_i \in \bar{G}_\rho$. By construction, the elements $\mathring{\gamma}_i$ are disjoint and lie in $B_{\bar{G}_\rho}(\ell + \Delta + \rho)$. Thus, the bound (3.1) holds with

$$r_\sigma = 3(\Delta + \rho)\,\mathrm{card} B_{\bar{G}}(\sigma).$$

By applying the previous observation to a point of $\bar{X}$ at distance $\Delta + \rho + \epsilon$ from $\bar{O}$, we can also show that there exists $\mathring{\theta} \in \bar{G}_\rho$ different from $\bar{e}$ with $||\mathring{\theta}||_{\bar{G}} \leq 2(\Delta + \rho)$.

Proposition 3.1. *We have*

$$\omega(\bar{G} * \mathbb{Z}_2, \lambda + \lambda') \leq \omega(\bar{G}_\rho * \mathbb{Z}_2, \tfrac{\lambda}{2})$$

where $\lambda \geq 0$ and $\lambda' \geq R_\rho = 6(\Delta + \rho)\,\mathrm{card} B_{\bar{G}}(3(\Delta + \rho))$.

Proof. Let $\lambda' \geq r_\sigma + \sigma$ where $\sigma = 3(\Delta + \rho)$. For every $\gamma \in \bar{G}$, we define $\mathring{\gamma} \in \bar{G}_\rho$ as follows. If $\gamma \in B_{\bar{G}}(\rho)$, then $\mathring{\gamma} = \mathring{\theta}$. Otherwise, we choose for $\mathring{\gamma}$ an element of $\bar{G}_\rho$ at distance at most ρ from γ. By construction, we have

$$\bar{d}(\gamma, \mathring{\gamma}) \leq 2(\Delta + \rho) + \rho \leq 3(\Delta + \rho) = \sigma.$$

Now, we consider the map $\varphi : \bar{G} * \mathbb{Z}_2 \to \bar{G}_\rho * \mathbb{Z}_2$ defined as

$$\varphi(\gamma_1 * 1 * \cdots * \gamma_{m+1}) = \mathring{\gamma}_1 * 1 * \cdots * \mathring{\gamma}_{m+1}.$$

As no $\mathring{\gamma}_i$ agrees with $\bar{e}$ (this is the reason for introducing $\mathring{\theta}$), the product $\mathring{\gamma}_1 * 1 * \cdots * \mathring{\gamma}_{m+1}$ is in reduced form.

It follows from the bound $\bar{d}(\gamma, \mathring{\gamma}) \leq \sigma$ that

$$\begin{aligned}
||\mathring{\gamma}_1 * 1 * \cdots * \mathring{\gamma}_{m+1}||_{2\lambda+\lambda'-\sigma} &= \left(\sum_{i=1}^{m+1} ||\mathring{\gamma}_i||_{\bar{G}} \right) + m(2\lambda + \lambda' - \sigma) \\
&\leq \left(\sum_{i=1}^{m+1} ||\gamma_i||_{\bar{G}} \right) + (m+1)\sigma + m(2\lambda + \lambda' - \sigma) \\
&\leq ||\gamma_1 * 1 * \cdots * \gamma_{m+1}||_{2\lambda+\lambda'} + \sigma.
\end{aligned}$$

Therefore, the map φ sends $B_{\bar{G}*\mathbb{Z}_2, 2\lambda+\lambda'}(R)$ to $B_{\bar{G}_\rho*\mathbb{Z}_2, 2\lambda+\lambda'-\sigma}(R + \sigma)$.

Furthermore, the bound $\bar{d}(\gamma, \mathring{\gamma}) \leq \sigma$ also implies that every element $\mathring{\gamma}_1 * 1 * \cdots * \mathring{\gamma}_{m+1} \in \bar{G}_\rho * \mathbb{Z}_2$ in reduced form has at most

$$\mathrm{card} B_{\bar{G}}(\mathring{\gamma}_1, \sigma) \times \cdots \times \mathrm{card} B_{\bar{G}}(\mathring{\gamma}_{m+1}, \sigma) = (\mathrm{card} B_{\bar{G}}(\sigma))^{m+1}$$

preimages by φ.

Hence, the cardinal of $B_{\bar{G}*\mathbb{Z}_2, 2\lambda+\lambda'}(R)$ is bounded by the following sum

$$\sum_{m=0}^{\infty} \mathrm{card}\{\mathring{\gamma}_1 * 1 * \cdots * \mathring{\gamma}_{m+1} \in B_{\bar{G}_\rho * \mathbb{Z}_2, 2\lambda+\lambda'-\sigma}(R+\sigma)\} \; (\mathrm{card} B_{\bar{G}}(\sigma))^{m+1}.$$

Now, since $\mathrm{card} B_{\bar{G}}(\sigma) \leq \mathrm{card} B_{\bar{G}_\rho}(r_\sigma)$, each term of this sum is bounded by the number of elements

$$\mathring{\gamma}_1 * 1 * \cdots * \mathring{\gamma}_{m+1} * 1 * \mathring{c}_1 * 1 * \cdots * \mathring{c}_{m+1} \in \bar{G}_\rho * \mathbb{Z}_2$$

such that

$$\left(\sum_{i=1}^{m+1} ||\mathring{\gamma}_i||_{\bar{G}} \right) + m(2\lambda + \lambda' - \sigma) \leq R + \sigma$$

and $\mathring{c}_i \in B_{\bar{G}_\rho}(r_\sigma)$. As all these elements satisfy

$$\left(\sum_{i=1}^{m+1} ||\mathring{\gamma}_i||_{\bar{G}} + \sum_{i=1}^{m+1} ||\mathring{c}_i||_{\bar{G}} \right) + m(2\lambda + \lambda' - \sigma - r_\sigma) \leq R + \sigma + r_\sigma$$

and so

$$\left(\sum_{i=1}^{m+1} ||\mathring{\gamma}_i||_{\bar{G}} + \sum_{i=1}^{m+1} ||\mathring{c}_i||_{\bar{G}} \right) + (2m+1)\,(\lambda + \frac{\lambda' - \sigma - r_\sigma}{2}) \leq R + \lambda + \lambda'$$

since $\lambda' \geq r_\sigma + \sigma$, we derive that

$$\mathrm{card} B_{\bar{G}*\mathbb{Z}_2, 2\lambda+\lambda'}(R) \leq \mathrm{card} B_{\bar{G}_\rho * \mathbb{Z}_2, \lambda + \frac{\lambda'-\sigma-r_\sigma}{2}}(R + \lambda + \lambda').$$

Therefore,

$$\omega(\bar{G} * \mathbb{Z}_2, 2\lambda + \lambda') \leq \omega(\bar{G}_\rho * \mathbb{Z}_2, \lambda + \frac{\lambda' - \sigma - r_\sigma}{2}) \leq \omega(\bar{G}_\rho * \mathbb{Z}_2, \lambda)$$

from the nonincreasing property of $\omega(\bar{G}_\rho * \mathbb{Z}_2, \cdot)$. □

4. Classical results about Gromov hyperbolic spaces

In this section, we recall the definition of Gromov hyperbolic spaces and present some well-known results. Classical references on the subject include [Gr87, GH90, CDP90].

Definition 4.1. Let $X = (X, |\cdot|)$ be a metric space. The *Gromov product* of $x, y \in X$ with respect to a basepoint $w \in X$ is defined as

$$(x|y)_w = \frac{1}{2}(|xw| + |yw| - |xy|). \tag{4.1}$$

By the triangle inequality, it is nonnegative.

Fix $\delta \geq 0$. A metric space X is *δ-hyperbolic* if

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - \delta$$

for every $x, y, z, w \in X$. Equivalently, a metric space X is *δ-hyperbolic* if

$$|xy| + |zw| \geq \max\{|xz| + |yw|, |yz| + |xw|\} + 2\delta \tag{4.2}$$

for every $x, y, z, w \in X$.

A metric space is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. A finitely generated group is *word hyperbolic* in the sense of Gromov if its Cayley graph with respect to some (or equivalently any) finite generating set is Gromov hyperbolic.

Example 4.2. Gromov hyperbolic spaces include complete simply connected Riemannian manifolds of sectional curvature bounded away from zero and their convex subsets, metric trees and more generally CAT(-1) spaces.

Remark 4.3. In this definition, the metric space X is not required to be geodesic. However, from [BS00], every δ-hyperbolic metric space isometrically embeds into a complete geodesic δ-hyperbolic metric space.

Without loss of generality, we will assume in the sequel that X is a complete geodesic δ-hyperbolic metric spaces.

The following result about approximation maps can be found in [Gr87], [GH90, Ch. 2, §3] and [CDP90].

Lemma 4.4. *Given a geodesic triangle Δ in X, there exists a map $\Phi : \Delta \to T$ to a possibly degenerated tripode T (i.e., a metric tree with at most three leaves) such that*

1. *the restriction of Φ to each edge of Δ is an isometry,*
2. *for every $x, y \in X$,*
$$|xy| - 4\delta \leq |\Phi(x)\Phi(y)| \leq |xy|.$$

In this lemma, we also denoted by $|\cdot|$ the metric on T. We will refer to the map Φ as the *approximation map* of the geodesic triangle Δ.

Remark 4.5. This result implies the Rips condition: each side of a geodesic triangle of X is contained in the 4δ-neighborhood of the union of the other two sides.

Remark 4.6. Given $x, y, z \in X$, let Φ be the approximation map of a geodesic triangle $\Delta = \Delta(x, y, z)$ with vertices x, y and z. From Lemma 4.4.(1), the Gromov product $(x|y)_z$ is equal to the distance between $\Phi(z)$ and the center of the tripode T.

The following lemma is a simple version of the Local-to-Global theorem, *cf.* [Gr87, GH90, CDP90].

Lemma 4.7. *Let $x, y, p, q \in X$ such that $(x|q)_p \leq 4\delta$ and $(y|p)_q \leq 4\delta$. Then*
$$|xy| \geq |xp| + |pq| + |qy| - 14\delta.$$

Proof. From the definition of the Gromov product, *cf.* (4.1), the bounds $(x|q)_p \leq 4\delta$ and $(y|p)_q \leq 4\delta$ yield the following two estimates
$$|xq| \geq |xp| + |pq| - 8\delta$$
$$|yp| \geq |yq| + |qp| - 8\delta.$$

Now, from (4.2), we have

$$\begin{aligned}|xy| + |pq| &\geq \max\{|xp| + |yq|, |yp| + |xq|\} + 2\delta \\ &\geq |xq| + |yp| + 2\delta.\end{aligned}$$

Combined with the previous two estimates, we obtain the desired lower bound for $|xy|$. □

The following simple fact will be useful in the sequel.

Lemma 4.8. *Let $x \in X$. Consider the projection p of x to a given geodesic line or geodesic segment τ (the projection may not be unique). Then, for every $q \in \tau$,*

$$(x|q)_p \leq 4\delta.$$

In particular,

$$|xq| \geq |xp| + |pq| - 8\delta.$$

Proof. Consider an approximation map $\Phi : \Delta \to T$ of a geodesic triangle $\Delta = \Delta(x, p, q)$ with vertices x, p and q. The center of the tripode T has three preimages by Φ, one in each segment $[x, p]$, $[p, q]$ and $[x, q]$. Let x_1 and x_2 be the preimages of the center of T in $[xp]$ and $[pq]$. Note that $|xx_1| = |xp| - |x_1p|$. From Lemma 4.4.(2), the points x_1 and x_2 are at distance at most 4δ , that is, $|x_1x_2| \leq 4\delta$. Since p is the projection of x to τ and x_2 lies in τ, we have $|xp| \leq |xx_2|$. Combining these estimates with the triangular inequality, we obtain

$$\begin{aligned}|xp| \leq |xx_2| &\leq |xx_1| + |x_1x_2| \\ &\leq |xp| - |x_1p| + 4\delta.\end{aligned}$$

That is, $|x_1p| \leq 4\delta$. Now, we observed in Remark 4.6 that $|x_1p| = (x|q)_p$. The result follows. □

The following classical result follows from the previous lemma.

Lemma 4.9. *Let $[x, y]$ be a geodesic segment joining x to y in X. Consider an arc γ in X with the same endpoints as $[x, y]$ such that*

$$\text{length}(\gamma) \leq |xy| + \ell. \tag{4.3}$$

Then the arc γ lies in the $\left(\frac{\ell}{2} + 8\delta\right)$-neighborhood of $[x, y]$.

Proof. Let p be the projection to $[x, y]$ of a point z in γ. From Lemma 4.8, we have the following two estimates

$$\begin{aligned}|xz| &\geq |xp| + |pz| - 8\delta \\ |zy| &\geq |zp| + |py| - 8\delta.\end{aligned}$$

Since p lies between x and y, we have $|xy| = |xp| + |py|$. Hence,

$$\begin{aligned}\text{length}(\gamma) &\geq |xz| + |zy| \\ &\geq |xp| + |pz| - 8\delta + |zp| + |py| - 8\delta \\ &\geq 2\,|zp| + |xy| - 16\delta.\end{aligned}$$

Substituting this inequality into the upper bound (4.3), we obtain

$$d(z, [x, y]) = |zp| \leq \frac{\ell}{2} + 8\delta.$$

□

5. Traveling along hyperbolic orbits

In this section, we introduce some definitions, notations and constructions which will be used throughout this article.

The following set of definitions and properties can be found in [Gr87, GH90, CDP90].

Definition 5.1. Let X be a proper geodesic δ-hyperbolic metric space. The *boundary at infinity* of X, denoted by ∂X, is defined as the equivalence classes of geodesic rays of X, where two rays are equivalent if they are at finite Hausdorff distance. Similarly, it can be defined as the equivalence classes of sequences of points $(x_i)_{i\geq 1}$ of X whose Gromov product $(x_i|x_j)_p$ with respect to any point p goes to infinity. Here, two such sequences (x_i) and (y_j) are equivalent if $(x_i|y_j)_p$ goes to infinity. Thus, ∂X can be considered as a space of limit points. The space $X \cup \partial X$, endowed with the natural topology extending the initial topology on X, is compact and contains X as an open dense subset. Given two distinct points a and b in ∂X, there is a geodesic line τ in X joining a to b, that is, $\tau(-\infty) = a$ and $\tau(\infty) = b$. Every isometry of X uniquely extends to a homeomorphism of $X \cup \partial X$. An isometry α of X is *hyperbolic* if for some (or equivalently any) point $x \in X$, the map $n \mapsto \alpha^n(x)$ is a quasi-isometric embedding of $\mathbb{Z}$ into X. Alternatively, an isometry of X is hyperbolic if and only if it is of infinite order. Every hyperbolic isometry of X has exactly two fixed points on ∂X. An *axis* of a hyperbolic isometry of X is a geodesic line of X joining the two fixed points of the isometry. The *minimal displacement* of an isometry α of X is defined as

$$\mathrm{dis}(\alpha) = \inf_{x\in X} |x\alpha(x)|.$$

The minimal displacement of a hyperbolic isometry of X is positive.

Example 5.2. Let G be a group acting properly and cocompactly by isometries on a proper geodesic δ-hyperbolic metric space X. We know that G is a word hyperbolic group in the sense of Gromov (see the paragraph preceding Theorem 1.3). The argument leading to this result also shows that an element of G is hyperbolic as an isometry of X if and only if it is hyperbolic as an isometry of the Cayley graph of G with respect to any finite generating set (and so if and only if it is of infinite order). The boundary at infinity of the word hyperbolic group G, defined as the boundary at infinity of its Cayley graph with respect to some (and so any) finite generating set, agrees with ∂X. The group G is non-elementary if and only if ∂X contains more than two points. In this case, G contains a hyperbolic element. More generally, every infinite subgroup N of a word hyperbolic group G in the sense of Gromov contains a hyperbolic element.

Let G be a finitely generated group acting by isometries on a proper geodesic δ-hyperbolic metric space X. Suppose that G contains a hyperbolic isometry ξ of X, *cf.* Example 5.2. By taking a large enough power of ξ if necessary, we can assume that the minimal displacement of ξ, denoted by $L = \mathrm{dis}(\xi)$, is at most 300δ, *i.e.*, $L \geq 300\delta$. Fix an axis τ of ξ.

Definition 5.3. The *stable norm* of the fixed hyperbolic element ξ is defined as the limit of the subadditive sequence $\frac{||\xi^k||}{k}$. It is denoted by $||\xi||_\infty$. From [CDP90, §10, Proposition 6.4], the stable norm of ξ is positive and satisfies

$$||\xi||_\infty \leq L = \mathrm{dis}(\xi) \leq ||\xi||_\infty + 16\delta.$$

Observe that $||\xi||_\infty \geq 0.9L$ as $L \geq 300\delta$.

Given $\epsilon \in (0, \delta)$, let $O \in X$ be an origin with $|O\xi(O)| \leq L + \epsilon$.

Lemma 5.4. *The point O is at distance at most 28δ from the axis τ of ξ.*

Proof. Let p and q be the projections of O and $\xi(O)$ to τ (they may not be unique but we choose some). Let also q_- be the projection of $\xi^{-1}(q)$ to τ. From [GH90, Corollaire 7.3], the image $\xi^{-1}(\tau)$, which is a geodesic line with the same endpoints at infinity as τ, is at distance at most 16δ from τ. Thus,

$$|\xi^{-1}(q)q_-| \leq 16\delta. \tag{5.1}$$

From this relation and Lemma 4.8, we derive

$$|Op| + |pq_-| - 8\delta \leq |Oq_-| \leq |O\xi^{-1}(q)| + 16\delta.$$

That is,

$$|pq_-| \leq |\xi(O)q| - |Op| + 24\delta. \tag{5.2}$$

Now, from Lemma 4.7, we have

$$\begin{aligned}
|O\xi(O)| &\geq |Op| + |pq| + |q\xi(O)| - 14\delta \\
&\geq |Op| + |\xi^{-1}(q)q| - |\xi^{-1}(q)p| + |q\xi(O)| - 14\delta \\
&\geq |Op| + |q\xi(q)| - |pq_-| - 16\delta + |q\xi(O)| - 14\delta \\
&\geq |q\xi(q)| + 2|Op| - 54\delta
\end{aligned}$$

where the third inequality follows from (5.1) and the last one from (5.2). By definition of O, we obtain

$$|Op| \leq 27\delta + \frac{\epsilon}{2} < 28\delta.$$

□

Let $x_1 = O$. For $i \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, we denote by x_i the point $\xi^{i-1}(x_1)$ if $i > 0$ and $\xi^i(x_1)$ if $i < 0$ (this choice of indices may not seem natural, but it allows us to consider fewer cases in forecoming arguments). As $||\xi||_\infty \geq 0.9L$, we derive that

$$\mathrm{dis}(\xi^k) \geq ||\xi^k||_\infty = |k|\, ||\xi||_\infty \geq L$$

for every $k \in \mathbb{Z}^* \setminus \{\pm 1\}$. Thus, $\mathrm{dis}(\xi^k) \geq L$ for every $k \in \mathbb{Z}^*$. That is,

$$|x_i x_j| \geq L$$

for every $i, j \in \mathbb{Z}^*$ with $i \neq j$.

We also define p_i as the projection of x_i to τ (again, it may not be unique but we choose one). Since ξ is an isometry, the points x_i of the ξ-orbit of x_1 attain the minimal displacement of ξ up to ϵ. Thus, from Lemma 5.4, we derive

$$|x_i p_i| \leq 28\delta. \tag{5.3}$$

For every pair of distinct indices $i, j \in \mathbb{Z}^*$, we obtain

$$|p_i p_j| \geq |x_i x_j| - 56\delta \geq L - (56\delta + \epsilon).$$

If the indices i and j are adjacent, we actually have

$$\begin{aligned} |p_i p_j| &\simeq |x_i x_j| \pm 56\delta \\ &\simeq L \pm (56\delta + \epsilon). \end{aligned} \tag{5.4}$$

using the notation 1.5. Therefore, since $L > 168\delta + 3\epsilon$, we deduce that the points p_i lie in τ in the order induced by $\mathbb{Z}^*$.

Consider the Voronoi cells of the ξ-orbit (x_i) of O, namely

$$D_i = \{x \in X \mid |xx_i| \leq |xx_j| \text{ for every } j \in \mathbb{Z}^*\}.$$

Note that $\xi(D_i) = D_{i+1}$ if $i \in \mathbb{Z}^* \setminus \{-1\}$ and $\xi(D_{-1}) = D_1$.

Lemma 5.5. *Given $x \in D_i$, let p be a projection of x to τ. Then p strictly lies between the points p_{i_-} and p_{i_+} of $\{p_j \mid j \in \mathbb{Z}^*, j \neq i\}$ adjacent to p_i.*

In particular,

$$|p_i p| \leq L + 56\delta + \epsilon.$$

Proof. By contradiction, we assume that there is an index $j \in \mathbb{Z}^*$ adjacent to i such that p_j lies between p_i and p, or is equal to p. From (5.4), we have

$$\begin{aligned} |pp_j| &= |pp_i| - |p_i p_j| \\ &\leq |pp_i| - L + 56\delta + \epsilon. \end{aligned}$$

From the triangular inequality and the bound (5.3), this estimate leads to

$$\begin{aligned} |xx_j| &\leq |xp| + |pp_j| + |p_j x_j| \\ &\leq |xp| + |pp_i| - L + 56\delta + \epsilon + 28\delta. \end{aligned}$$

On the other hand, from Lemma 4.8, we have

$$|xp| + |pp_i| \leq |xp_i| + 8\delta \leq |xx_i| + |x_i p_i| + 8\delta.$$

Hence, with the help of (5.3) and the inequality $L > 120\delta + \epsilon$, we obtain

$$|xx_j| \leq |xx_i| + 28\delta + 8\delta - L + 56\delta + \epsilon + 28\delta < |xx_i|.$$

Thus, x does not lie in D_i, which is absurd. Hence the first part of the lemma.

The second part of the lemma follows from (5.4). □

Remark 5.6. Lemma 5.5 also shows that two domains D_i and D_j corresponding to non-adjacent indices are disjoint. Thus, the Voronoi cells are ordered by their indices.

Lemma 5.7. *Let x and y be two points of X separated by $D_{\pm 1}$ or $D_{\pm 2}$. Then*

$$|xy| \geq |Ox| + |Oy| - 4L - 280\delta - 4\epsilon.$$

Proof. Let p and q be the projections of x and y to τ. By assumption, there exist two indices $i, j \in \mathbb{Z}^*$ such that $x \in D_i$ and $y \in D_j$ with $D_{\pm 1}$ or $D_{\pm 2}$ separating D_i and D_j.

From Lemma 5.5, $p_{\pm 1}$ or $p_{\pm 2}$ separates p and q. This point between p and q will be denoted by r. From the bounds (5.4) and (5.3), we derive that $|Or| \leq 2L + 140\delta + 2\epsilon$. Hence

$$\begin{aligned} |pq| &= |pr| + |rq| \\ &\geq |Op| + |Oq| - 2\,|Or| \\ &\geq |Op| + |Oq| - 2(2L + 140\delta + 2\epsilon). \end{aligned}$$

□

6. Geometric properties of symmetric elements

Using the previous notations and constructions, we introduce a notion of symmetric element in G and establish some (almost) norm-preserving properties.

Definition 6.1. Let $\beta \in G$. Denote by $j(\beta)$ the smallest index $j \in \mathbb{Z}^*$ such that $\beta(O) \in D_j$. We say that β is *positive* if $j(\beta) > 0$ and *negative* if $j(\beta) < 0$.

We define $\beta_\pm \in G$ as follows

$$\beta_+ = \beta$$

and

$$\beta_- = \begin{cases} \xi^{-2j-1}\beta & \text{if } \beta \text{ is positive} \\ \xi^{-2j+1}\beta & \text{if } \beta \text{ is negative} \end{cases}$$

where $j = j(\beta)$.

We will think of β_- as the symmetric element of β with respect to a symmetry line perpendicular to the direction of the ξ-shift.

Consider the following norm on G defined for every $\beta \in G$ as

$$||\beta|| = |O\beta(O)|.$$

Proposition 6.2. *Let $\beta, \beta_1, \beta_2 \in G$ and $j = j(\beta)$.*

(1) *$\beta_-(O)$ lies in D_{-j-2} if $j > 0$ and in D_{-j+2} if $j < 0$.*
In particular, β_+ and β_- have opposite signs.

(2) *If $(\beta_1)_- = (\beta_2)_-$ then $\beta_1 = \beta_2$.*

(3) *We have*

$$||\beta_-|| \simeq ||\beta|| \pm \Delta_-$$

where $\Delta_- = \Delta_-(L, \delta, \epsilon) = 8L + 464\delta + 8\epsilon \leq 10L$.

Proof. The point (1) is obvious by construction of β_-.

For the second point, if $(\beta_1)_- = (\beta_2)_-$ then β_1 and β_2 have the same sign from the second assertion of (1). We derive from the definition of β_- that $\xi^k\beta_1 = \xi^k\beta_2$ for $k = -2j \pm 1$. Hence $\beta_1 = \beta_2$.

For the last point, let $x = \beta(O)$ and $x_- = \beta_-(O)_- = \xi^{-2j\mp 1}\beta(O)$, where the exponent $-2j \mp 1$ depends on the sign of β, see Definition 6.1. Fix $k = -j \mp 2$. By assumption, $x \in D_j$ and from the point (1), $x_- \in D_k$. Thus, $|xx_j| = |x_-x_k|$.

Let p and p_- be the projections of x and x_- to τ. From Lemma 5.5 and the bound (5.3), both $|x_j p|$ and $|x_k p_-|$ are bounded from above by $L + 84\delta + \epsilon$. Combined with

$$|xp| \simeq |xx_j| \pm |x_j p|$$
$$|x_- p_-| \simeq |x_- x_k| \pm |x_k p_-|$$

and the relation $|xx_j| = |x_- x_k|$, we obtain

$$|xp| \simeq |x_- p_-| \pm (2L + 168\delta + 2\epsilon). \tag{6.1}$$

On the other hand, from (5.3) and the previous bound on $|x_j p|$, we derive

$$\begin{aligned} |p_1 p| &\simeq |x_1 x_j| \pm (|p_1 x_1| + |x_j p|) \\ &\simeq |x_1 x_j| \pm (L + 112\delta + \epsilon) \end{aligned} \tag{6.2}$$

Similarly,

$$|p_1 p_-| \simeq |x_1 x_k| \pm (L + 112\delta + \epsilon). \tag{6.3}$$

Now, since

$$\begin{aligned} |x_1 x_j| &= |x_{-j+2} x_1| \\ &\simeq |x_k x_1| \pm 4(L + \epsilon) \end{aligned}$$

we obtain from (6.2) and (6.3) that

$$|p_1 p| \simeq |p_1 p_-| \pm (6L + 224\delta + 6\epsilon). \tag{6.4}$$

Now, from Lemma 4.8, we have

$$|xp_1| \simeq |xp| + |pp_1| \pm 8\delta$$
$$|x_- p_1| \simeq |x_- p_-| + |p_- p_1| \pm 8\delta$$

Combined with (6.1) and (6.4), we obtain

$$|xp_1| \simeq |x_- p_1| \pm (8L + 408\delta + 8\epsilon).$$

The result follows from the bound $|Op_1| \leq 28\delta$ derived in (5.3). $\square$

7. Geometric properties of twisted products

In this section, we introduce the twisted product and show that the norm on G is almost multiplicative with respect to the twisted product.

Definition 7.1. The twisted product of two elements α, β in G is defined as

$$\alpha \star \beta = \begin{cases} \alpha\beta_+ & \text{if } D_1 \text{ or } D_{-1} \text{ separates } \alpha^{-1}(O) \text{ and } \beta(O) \\ \alpha\beta_- & \text{otherwise} \end{cases}$$

Note that the twisted product $\star$ is not associative. Furthermore, it has no unit and is not commutative.

Recall that $||\alpha|| = |O\alpha(O)|$ for every $\alpha \in G$.

Proposition 7.2. *Let $\alpha, \beta, \beta' \in G$. Then*

1. *If $\alpha \star \beta = \alpha\beta_\varepsilon$ with $\varepsilon = \pm$ then $D_{\pm 1}$ or $D_{\pm 2}$ separates $\alpha^{-1}(O)$ and $\beta_\varepsilon(O)$.*
2. *If $\alpha \star \beta = \alpha \star \beta'$ with β and β' of the same sign, then $\beta = \beta'$.*

(3) *We have*

$$||\alpha \star \beta|| \simeq ||\alpha|| + ||\beta|| \pm \Delta_\star$$

where $\Delta_\star = \Delta_\star(L, \delta, \epsilon) = 12L + 744\delta + 12\epsilon \leq 15L$.

Proof. For the first point. If $\varepsilon = +$, then the result is clear by definition of the twisted product. Namely, $D_{\pm 1}$ separates $\alpha^{-1}(O)$ and $\beta_\varepsilon(O)$. If $\varepsilon = -$, then D_1 and D_{-1} do not separate $\alpha^{-1}(O)$ and $\beta(O)$. Assume that β is positive, that is, $\beta(O) \in D_j$ with $j > 0$ (the other case is similar). This implies that $\alpha^{-1}(O)$ lies in a domain D_i with $i \geq -1$, otherwise D_{-1} would separate the two points. Therefore, D_{-2} separates $\alpha^{-1}(O)$ and $\beta_-(O)$ since $\beta_-(O)$ lies in D_{-j-2} from Proposition 6.2.

For the second point, if $\alpha\beta = \alpha\beta'$ then $\beta = \beta'$. Similarly, if $\alpha(\beta)_- = \alpha(\beta')_-$ then $\beta = \beta'$ from Proposition 6.2.(2). Switching β and β' if necessary, we can assume that $\alpha\beta = \alpha(\beta')_-$, and so $\beta = (\beta')_-$. From Proposition 6.2.(1), this implies that β and β' have opposite signs, which is absurd.

For the last point. We have

$$||\alpha \star \beta|| = |O\, \alpha\beta_\varepsilon(O)| = |\alpha^{-1}(O)\beta_\varepsilon(O)|.$$

Using the point (1), we can apply Lemma 5.7 to derive

$$|O\alpha^{-1}(O)| + |O\beta_\varepsilon(O)| - 4L - 280\delta - 4\epsilon \leq ||\alpha \star \beta|| \leq |O\alpha^{-1}(O)| + |O\beta_\varepsilon(O)|.$$

That is,

$$||\alpha \star \beta|| \simeq ||\alpha|| + ||\beta_\varepsilon|| \pm (4L + 280\delta + 4\epsilon).$$

Hence, by Proposition 6.2, we obtain

$$||\alpha \star \beta|| \simeq ||\alpha|| + ||\beta|| \pm (12L + 744\delta + 12\epsilon).$$

□

8. Inserting hyperbolic elements

The proposition established in this section will play an important role in the proof of the injectivity of the nonexpanding map defined in Section 9.

Lemma 8.1. *Let* $\beta \in G$ *and* $\kappa \geq 4$.

(1)

$$(\xi^\kappa \star \beta)_+ = \begin{cases} \xi^\kappa \beta_+ & \text{if } \beta \text{ is positive} \\ \xi^\kappa \beta_- & \text{if } \beta \text{ is negative} \end{cases}$$

In particular, $(\xi^\kappa \star \beta)_+$ *is positive.*

(2)

$$(\xi^\kappa \star \beta)_- = \begin{cases} \xi^{-\kappa} \beta_- & \text{if } \beta \text{ is positive} \\ \xi^{-\kappa-4} \beta_+ & \text{if } \beta \text{ is negative} \end{cases}$$

In particular, $(\xi^\kappa \star \beta)_-$ *is negative.*

Therefore, for every $\alpha \in G$, *we have*

$$\alpha \star (\xi^\kappa \star \beta) = \alpha\, \xi^{\kappa_*} \beta_\varepsilon \tag{8.1}$$

with $\kappa_* = \pm\kappa$ *or* $-\kappa - 4$, *and* $\varepsilon = \pm$.

Proof. The first point is clear since $\xi^{-\kappa}(O) \in D_{-\kappa}$. We only check the second one. Let $j = j(\beta)$ be the index of β, *cf.* Definition 6.1.

Suppose that $j > 0$. We observe that the image of O by $\xi^\kappa \star \beta = \xi^\kappa \beta$ lies in $D_{j+\kappa}$ and that the index of $\xi^\kappa \star \beta$ is positive equal to $j + \kappa$. Thus, $(\xi^\kappa \star \beta)_- = \xi^{-2(j+\kappa)-1}\xi^\kappa\beta = \xi^{-\kappa}\beta_-$.

Suppose that $j < 0$. We observe that the image of O by $\xi^\kappa \star \beta = \xi^{\kappa-2j+1}\beta$ lies in $D_{-j+2+\kappa}$ and that the index of $\xi^\kappa \star \beta$ is positive equal to $-j+2+\kappa$. Thus, $(\xi^\kappa \star \beta)_- = \xi^{-2(-j+2+\kappa)-1}\xi^{\kappa-2j+1}\beta = \xi^{-\kappa-4}\beta$. ☐

Definition 8.2. An element $\alpha \in G$ is *η-minimal modulo a normal subgroup* $N \lhd G$, where $\eta \geq 0$, if for every $\alpha' \in \alpha N$,

$$||\alpha|| \leq ||\alpha'|| + \eta.$$

We denote by $\langle\langle \xi \rangle\rangle$ the normal subgroup of G generated by ξ.

Recall that $d(\alpha, \beta) = |\alpha(O)\beta(O)|$ for every $\alpha, \beta \in G$ and that the stable norm of ξ introduced in Definition 5.3 satisfies

$$||\xi||_\infty \leq L = \mathrm{dis}(\xi) \leq ||\xi||_\infty + 16\delta.$$

Proposition 8.3. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in G$ such that α_i is η-minimal modulo $\langle\langle \xi \rangle\rangle$ for some $\eta \geq 0$. Let $\kappa \geq 5$ be an integer such that*

$$\kappa > (6\Delta_\star + 3\Delta_- + 48\delta + \eta)/||\xi||_\infty.$$

If $\alpha_1 \star (\xi^\kappa \star \beta_1) = \alpha_2 \star (\xi^\kappa \star \beta_2)$, then

$$d(\alpha_1, \alpha_2) \leq 2\kappa\, L + 4(\Delta_\star + \frac{1}{2}\Delta_- + 8\delta).$$

Remark 8.4. Recall that $L \geq 300\delta$, $\Delta_- \leq 10L$ and $\Delta_\star \leq 15L$. Hence, when $\eta = \Delta_-$, we can take $\kappa = 140$.

Proof. Let $\gamma = \alpha_i \star (\xi^\kappa \star \beta_i) = \alpha_i \xi^{\kappa_i}(\beta_i)_{\varepsilon_i}$ where $\kappa_i = \pm\kappa$ or $-\kappa - 4$ and $\varepsilon_i = \pm$ according to the decomposition (8.1) of Lemma 8.1. The segments from O to $A_i = \alpha_i(O)$, from A_i to $B_i = \alpha_i(\xi^{\kappa_i}(O))$, from B_i to $\gamma(O)$, and from O to $\gamma(O)$ will be denoted by a_i, c_i, b_i and c (these segments may not be unique, but we choose some). We denote also by $\bar{A}_i$ and $\bar{B}_i$ the projections of A_i and B_i to c (which again may not be unique). Recall that $||\xi|| \leq L + \delta$ with $300\delta \leq L$ and observe that for $\kappa \geq 5$,

$$\kappa\, ||\xi||_\infty \leq |\kappa_i|\, ||\xi||_\infty \leq |A_i B_i| = ||\xi^{\kappa_i}|| \leq (\kappa + 4)||\xi|| \leq 2\kappa\, L. \tag{8.2}$$

From Propositions 6.2.(3) and 7.2.(3), we derive

$$\begin{aligned} \mathrm{length}(a_i \cup c_i \cup b_i) &= ||\alpha_i|| + ||\xi^{\kappa_i}|| + ||(\beta_i)_{\varepsilon_i}|| \\ &\simeq ||\alpha_i \star (\xi^\kappa \star \beta_i)|| \pm (2\Delta_\star + \Delta_-) \\ &\simeq \mathrm{length}(c) \pm (2\Delta_\star + \Delta_-). \end{aligned} \tag{8.3}$$

Thus, by Lemma 4.9, the arc $a_i \cup c_i \cup b_i$ lies at distance at most $\Delta_\star + \frac{1}{2}\Delta_- + 8\delta$ from the geodesic c with the same endpoints. In particular, we have

$$|A_i \bar{A}_i| \leq \Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta \tag{8.4}$$

$$|B_i \bar{B}_i| \leq \Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta \tag{8.5}$$

Fact 1. *The points O, $\bar{A}_i$ and $\bar{B}_i$ are aligned in this order along c.*

Indeed, from (8.3), we have

$$|OA_i| + |A_iB_i| + |B_i\gamma(O)| - 2\Delta_\star - \Delta_- \leq |O\gamma(O)| \leq |OB_i| + |B_i\gamma(O)|.$$

This implies that

$$|OB_i| \simeq |OA_i| + |A_iB_i| \pm (2\Delta_\star + \Delta_-)$$

and so

$$|O\bar{B}_i| \geq |O\bar{A}_i| + \kappa\, ||\xi||_\infty - (2\Delta_\star + \Delta_-) - 2(\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta)$$

using (8.4), (8.5) and (8.2). By our choice of κ, we have $|O\bar{B}_i| \geq |O\bar{A}_i|$.

Fact 2. *The points O, $\bar{A}_1$ and $\bar{B}_2$ are aligned in this order along c. Similarly, the points O, $\bar{A}_2$ and $\bar{B}_1$ are aligned in this order along c.*

Let us prove the first assertion. Arguing by contradiction, we can assume from Fact 1 that the points O, $\bar{A}_2$, $\bar{B}_2$ and $\bar{A}_1$ are aligned in this order along c. Combined with (8.4) and (8.5), this implies that

$$\begin{aligned} |OA_1| &\simeq |O\bar{A}_1| \pm (\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta) \\ &\simeq |O\bar{A}_2| + |\bar{A}_2\bar{B}_2| + |\bar{B}_2\bar{A}_1| \pm (\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta) \\ &\simeq |OA_2| + |A_2B_2| + |B_2A_1| \pm 6(\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta) \end{aligned} \tag{8.6}$$

Now, consider the projection

$$\pi : X \to X/\langle\langle\xi\rangle\rangle,$$

where $X/\langle\langle\xi\rangle\rangle$ is endowed with the quotient distance, still denoted by $|\cdot|$.

By definition, $A_i = \alpha_i(O)$ and $B_i = \alpha_i(\xi^{\kappa_i}(O)) = (\alpha_i\xi^{\kappa_i}\alpha_i^{-1})(\alpha_i(O))$. Hence, $\pi(A_i) = \pi(B_i)$ since $\alpha_i\xi^{\kappa_i}\alpha_i^{-1} \in \langle\langle\xi\rangle\rangle$.

Note also that

$$|OA_i| \simeq |\pi(O)\pi(A_i)| \pm \eta$$

since α_i is η-minimal modulo $\langle\langle\xi\rangle\rangle$.

Continuing with (8.6) and since $\pi(A_2) = \pi(B_2)$, we obtain

$$\begin{aligned} |OA_1| &\geq |\pi(O)\pi(A_2)| + \kappa\, ||\xi||_\infty + |\pi(B_2)\pi(A_1)| - 6(\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta) \\ &\geq |\pi(O)\pi(A_1)| + \kappa\, ||\xi||_\infty - 6(\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta) \\ &\geq |OA_1| - \eta + \kappa\, ||\xi||_\infty - 6(\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta). \end{aligned}$$

Hence a contradiction from our choice of κ. Similarly, we derive the second assertion of Fact 2.

From the order of the points $\bar{A}_i$ and $\bar{B}_i$ on c given by Facts 1 and 2, we deduce by (8.2), (8.4) and (8.5) that

$$|\bar{A}_1\bar{A}_2| \leq \max\{|\bar{A}_1\bar{B}_1|, |\bar{A}_2\bar{B}_2|\} \leq 2\kappa\, L + 2(\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta).$$

Therefore,

$$d(\alpha_1, \alpha_2) = |A_1A_2| \leq 2\kappa\, L + 4(\Delta_\star + \tfrac{1}{2}\Delta_- + 8\delta).$$

$\square$

9. A nonexpanding embedding

In this section, we finally establish the key proposition in the proof of the main theorem. Namely, we construct a nonexpanding map $\Phi : \bar{G} * \mathbb{Z}_2 \to G$ and show that it is injective in restriction to some separated subset $\bar{G}_\rho * \mathbb{Z}_2$.

Let G be a finitely generated group acting by isometries on a proper geodesic δ-hyperbolic metric space X. Let N be a normal subgroup of G. The quotient group $\bar{G} = G/N$, whose neutral element is denoted by $\bar{e}$, acts by isometries on the quotient metric space $\bar{X} = X/G$.

Suppose that N contains a hyperbolic isometry ξ of X, *cf.* Example 5.2. By taking a large enough power of ξ if necessary, we can assume that the minimal displacement L of ξ is at least 300δ.

Given $\epsilon \in (0, \delta)$, let $O \in X$ be an origin with $|O\xi(O)| \leq L + \epsilon$. Denote by $||\cdot||$ and $||\cdot||_{\bar{G}}$ the norms induced on G and $\bar{G}$ by the distance d, *cf.* (1.3), and the quotient distance $\bar{d}$. We also need to fix $\kappa = 140$, *cf.* Remark 8.4.

For every $\gamma \in \bar{G}$, we fix once and for all a representative α in G which is ν-minimal modulo N. (To avoid burdening the arguments by epsilontics, we will actually assume that $\nu = 0$.) This yields an embedding

$$\Phi : \bar{G} \hookrightarrow G.$$

We extend this embedding to a map

$$\Phi : \bar{G} * \mathbb{Z}_2 \to G$$

by induction on m with the relation

$$\Phi(\gamma_1 * 1 * \cdots * \gamma_{m+1}) = \Phi(\gamma_1)_\varepsilon \star (\xi^\kappa \star \Phi(\gamma_2 * 1 * \cdots * \gamma_{m+1}))$$

where $\gamma_1 * 1 * \cdots * \gamma_{m+1}$ is in reduced form and ε is the sign of $\Phi(\gamma_2 * 1 * \cdots * \gamma_{m+1})$, unless $\gamma_1 = \bar{e}$, in which case $\varepsilon = +$.

For $\lambda \geq 0$, consider the norm $||\cdot||_\lambda$ on the group $\bar{G} * \mathbb{Z}_2$ defined as

$$||\gamma_1 * 1 * \cdots * \gamma_{m+1}||_\lambda = \left(\sum_{i=1}^{m+1} ||\gamma_i||_{\bar{G}} \right) + m\lambda.$$

Let $\alpha_i = \Phi(\gamma_i)$. From Propositions 6.2.(3) and 7.2.(3), we derive

$$\begin{aligned} ||\Phi(\gamma_1 * 1 * \cdots * \gamma_{m+1})|| &\leq ||\alpha_1|| + ||\xi^\kappa|| + ||\Phi(\gamma_2 * 1 * \cdots * \gamma_{m+1})|| + 2\Delta_\star + \Delta_- \\ &\leq \left(\sum_{i=1}^{m+1} ||\alpha_i|| \right) + m(\kappa L + 2\Delta_\star + \Delta_-) \end{aligned}$$

Thus, since $||\alpha_i|| = ||\gamma_i||_{\bar{G}}$, the map Φ does not increase the "norms" if $\lambda \geq \kappa L + 2\Delta_\star + \Delta_-$.

For $\rho > 0$, consider a maximal system of disjoint (closed) balls of radius $\rho/2$ in $\bar{G}$ containing the ball of radius $\rho/2$ centered at $\bar{e}$. By maximality, the centers of these balls form a subset $\bar{G}_\rho$ of $\bar{G}$ such that

(1) the elements of $\bar{G}_\rho$ are at distance greater than ρ from each other;
(2) every element of $\bar{G}$ is at distance at most ρ from an element of $\bar{G}_\rho$.

Note that $||\gamma||_{\bar{G}} > \rho$ for every $\gamma \in \bar{G}_\rho$ with $\gamma \neq \bar{e}$.

Consider the subset $\bar{G}_\rho * \mathbb{Z}_2$ of $\bar{G} * \mathbb{Z}_2$ formed of the elements

$$\gamma_1 * 1 * \cdots * \gamma_{m+1}$$

where $m \in \mathbb{N}$ and $\gamma_i \in \bar{G}_\rho$. Note that $\bar{G}_\rho * \mathbb{Z}_2$ is not a group, as $\bar{G}_\rho$ itself is not a group.

Proposition 9.1. *Let $\lambda \geq \kappa L + 2\Delta_\star + \Delta_-$ and $\rho \geq 2\kappa L + 4(\Delta_\star + \frac{1}{2}\Delta_- + 8\delta)$. Then the nonexpanding map $\Phi : (\bar{G}_\rho * \mathbb{Z}_2, ||\cdot||_\lambda) \to (G, ||\cdot||)$ is injective.*

Remark 9.2. We can take $\lambda = 200L$ and $\rho = 500L$.

Proof. Assume

$$\Phi(\gamma_1 * 1 * \cdots * \gamma_{m+1}) = \Phi(\gamma'_1 * 1 * \cdots * \gamma'_{m'+1}). \tag{9.1}$$

Let $\alpha_1 = \Phi(\gamma_1)$ and ε be the sign of $\Phi(\gamma_2 * 1 * \cdots * \gamma_{m+1})$, unless $\gamma_1 = \bar{e}_1$ in which case $\varepsilon = +$. Similar definitions hold for α'_1 and ε' by replacing γ_i with γ'_i.

By Proposition 6.2.(3), for every $\alpha \in G$ which is minimal modulo N, and so modulo $\langle\langle \xi \rangle\rangle$, its symmetric α_- is η-minimal modulo $\langle\langle \xi \rangle\rangle$ with $\eta = \Delta_-$. Thus, from Proposition 8.3, we have

$$d((\alpha_1)_\varepsilon, (\alpha'_1)_{\varepsilon'}) \leq \rho.$$

By definition of the quotient distance,

$$\bar{d}(\gamma_1, \gamma'_1) \leq d(\xi^i \alpha_1, \xi^{i'} \alpha'_1)$$

for every $i, i' \in \mathbb{Z}$. Hence,

$$\bar{d}(\gamma_1, \gamma'_1) \leq d((\alpha_1)_\varepsilon, (\alpha'_1)_{\varepsilon'}) \leq \rho.$$

Since γ_1 and γ'_1 lie in a ρ-separated set, this shows that $\gamma_1 = \gamma'_1$ and so $\alpha_1 = \alpha'_1$.

Suppose $\varepsilon \neq \varepsilon'$. As ε and ε' play symmetric roles, we can assume that $\varepsilon = -$. In this case, $\alpha = \alpha_1$ is different from $\bar{e}$ and $d(\alpha, \alpha_-) \leq \rho$. Now, as $\alpha^{-1} \star \alpha = \alpha^{-1}\alpha_-$, we derive from Proposition 7.2.(3) that

$$d(\alpha, \alpha_-) = ||\alpha^{-1}\alpha_-|| = ||\alpha^{-1} \star \alpha|| \geq 2||\alpha|| - \Delta_\star \geq 2\rho - \Delta_\star > \rho.$$

From this contradiction, we deduce that $\varepsilon = \varepsilon'$.

Substituting this equality into (9.1), we derive

$$\alpha_\varepsilon \star (\xi^\kappa \star \beta) = \alpha_\varepsilon \star (\xi^\kappa \star \beta')$$

where $\beta = \Phi(\gamma_2 * 1 * \cdots * \gamma_{m+1})$ and with a similar definition for β' by replacing γ_i with γ'_i. As both $\xi^\kappa \star \beta$ and $\xi^\kappa \star \beta'$ are positive, *cf.* Lemma 8.1.(1), we obtain from Proposition 7.2.(2) that $\xi^\kappa \star \beta = \xi^\kappa \star \beta'$. Since β and β' are of the same sign (*i.e.*, $\varepsilon = \varepsilon'$), we similarly derive that $\beta = \beta'$.

We conclude by induction that $m = m'$ and $\gamma_i = \gamma'_i$. □

10. Conclusion

Consider a non-elementary group G acting properly and cocompactly by isometries on a proper geodesic δ-hyperbolic metric space with fixed origin O. Denote by Δ the diameter of X/G. Let N be an infinite normal subgroup of G and $\bar{G} = G/N$. We can assume that $\omega(\bar{G})$ is nonzero. Denote by L the maximal value between 300δ and the minimal norm of a hyperbolic element in N.

Combining Propositions 2.1, 3.1 and 9.1, we obtain

$$\omega(G) \geq \omega(\bar{G}_\rho * \mathbb{Z}_2, \lambda) \geq \omega(\bar{G} * \mathbb{Z}_2, \tilde{\lambda}) \geq \omega(\bar{G}) + \frac{1}{4\tilde{\lambda}} \log(1 + e^{-\tilde{\lambda}\omega(\bar{G})})$$

where $\lambda = 200L$, $\rho = 500L$ and $\tilde{\lambda} = 2\lambda + 6(\Delta + \rho)\,\mathrm{card} B_{\bar{G}}(3(\Delta + \rho))$. One could also replace $\mathrm{card} B_{\bar{G}}(3(\Delta+\rho))$ with $\mathrm{card} B_G(3(\Delta+\rho))$, since the latter cardinal is at least as large as the former.

Observe that if $\omega(\bar{G})$ is close to $\omega(G)$, then L is large, that is, the norm of every hyperbolic element of N is large.

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