# Quantum Anomalies and Logarithmic Derivatives of Feynman Pseudomeasures

J. E. Gough, T. S. Ratiu, O. G. Smolyanov

# Quantum Anomalies and Logarithmic Derivatives of Feynman Pseudomeasures

**J. Gough[a], T. S. Ratiu[b], and O. G. Smolyanov[c]**

**Abstract**—Connections between quantum anomalies and transformations of pseudomeasures of the type of Feynman pseudomeasures are studied. Mathematical objects related to the notion of the volume element in an infinite-dimensional space considered in the physics literature [1] are discussed.

A quantum anomaly is the violation of symmetry (see [4]) with respect to some transformations under quantization. At that, the quantization of a classical Hamiltonian system invariant with respect to some transformations yields a quantum system noninvariant with respect to the same transformations (see [2]).

The situation in related literature is fairly unusual. The points of view presented in monographs [1, 2], whose authors are well-known experts, contradict each other. On page 352 of the book [1], it is written that the description of reasons why quantum anomalies occur which is given in [2] is incorrect. This refers to the first, 2004, edition of [2]; however, in 2013, the second edition of [2] appeared, which did not differ from the first; in the new edition, the authors did not cited book [1] containing criticism of their book. In this paper, we show that the correct description of the reasons for the emergence of quantum anomaly is that given in [2].

We use the fact that the transformations of a functional Feynman integral (i.e., an integral with respect to a Feynman pseudomeasure; its definition is given below) are determined by transformations of two objects. One of them is the product of a Feynman pseudomeasure and a function integrable with respect to this pseudomeasure, which is, in turn, the product of the exponential of a part of the classical action and the initial condition. The exponential of the other part of the action determines the Feynman pseudomeasure. If the action and the initial condition are invariant with respect to some transformation, then this object is invariant with respect to this transformation as well.

The second object is a determinant, which plays the role of a Jacobian; this determinant may differ from 1 even in the case where the action and the initial condition are invariant with respect to phase transformations; of course, in this case, the Feynman integral is noninvariant as well. What is said above virtually coincides with what is said in [2]. At the same time, in [1], it was proposed to compensate this determinant by multiplying the measure with respect to which the integration is performed (that is, the counterpart of the classical Lebesgue measure, which does not exist in the infinite-dimensional case according to a well-known theorem of Weil) by an additional factor, which, of course, is equivalent to multiplying the integrand by the same factor.

In this paper, we consider families of transformations of the domain of a (pseudo)measure depending on a real parameter and show that such a compensation is impossible; for this purpose, we use differentiation with respect to this parameter.

The paper is organized as follows. First, we recall two basic definitions of the differentiability of a measure and, more generally, a pseudomeasure (distribution); then, we give explicit expressions for the logarithmic derivatives of measures and pseudo-measures with respect to transformations of the space on which they are defined. The application of these expressions[1] makes it possible to obtain a mathematically correct version of results of [2] concerning quantum anomalies. After this, we discuss the approach to explaining the same anomalies proposed in [1]. We also discuss mathematical objects related to the notion of the vol-

[a] *Institute of Mathematics, Physics, and Computer Sciences, Aberystwyth University, Great Britain*

[b] *Section de Mathématiques and Bernoulli Center, École Polytechnique Fédérale de Lausanne, CH-1015 Switzerland*
*e-mail: tudor.ratiu@epfl.ch*

[c] *Mechanics and Mathematics Faculty, Moscow State University, Moscow, 119991 Russia*
*e-mail: smolyanov@yandex.ru*

[1] Among other things, these formulas lead to infinite-dimensional versions of both theorems of Emmy Noether.

ume element in an infinite-dimensional space considered in physics literature (including [1]). We concentrate on the algebraic structure of problems, leaving aside most assumptions of analytical character.

## 1. DIFFERENTIATION OF MEASURES AND DISTRIBUTIONS

This section contains results (in fact, known) about differentiable measures and distributions on infinite-dimensional spaces in a form convenient for our purposes.

Given a locally convex space (LCS) $E$, by $\mathcal{B}_E$ we denote the σ-algebra of Borel subsets of $E$ and by $\mathfrak{M}_E$, the vector space of countably additive (complex) measures on $E$. We say that the vector space $C$ of bounded Borel functions on $E$ determines a norm if, whatever a measure $\mathfrak{M}_E$, its total variation $\mu \in \mathfrak{M}_E$ satisfies the condition $\|\mu\|_1 = \sup\{\int u\, d\mu\colon u \in C, \|u\|_\infty \le 1\}$, where $\|u\|_\infty = \sup\{|u(x)|\colon x \in E\}$.

A Hilbert subspace of an LCS $E$ is defined as a vector subspace $H$ of $E$ endowed with the structure of a Hilbert space such that the topology induced on $H$ by the topology of $E$ is weaker than the topology generated by the Hilbert norm.

A mapping $F$ of an LCS $E$ to an LCS $G$ is said to be smooth along a Hilbert subspace $H$ of $E$ (or $H$-smooth) if it is infinitely differentiable over $H$ and both the mapping $F$ and all of its derivatives are continuous on $E$, provided that the spaces in which the derivatives take values are endowed with the topologies of uniform convergence on compact subsets of $H$. A vector field on an LCS $E$ is a mapping $h\colon E \to E$; we denote the set of vector fields on $E$ by Vect($E$). The derivative along a vector field $h \in$ Vect($E$) of a function $f$ defined on $E$ is the function on $E$ denoted by $f'h$ and defined by $(f'h)(x) := f'(x)h(x)$ for $x \in E$, where $f'(x)$ is the Gâteaux derivative of $f$ at the point $x$.

Let $\varepsilon > 0$, and let $S$ be a mapping of the interval $(-\varepsilon, \varepsilon)$ to the set of $\mathcal{B}_E$-measurable self-mappings of $E$ for which $S(0) = \mathrm{id}$; suppose that $\tau$ is a topology on $\mathfrak{M}_E$ compatible with the vector space structure. A measure $\nu \in \mathfrak{M}_E$ is said to be τ-differentiable along $S$ if the function $f\colon (-\varepsilon, \varepsilon) \ni t \mapsto S(t)_*\nu := \nu(S(t)^{-1}) \in (\mathfrak{M}_E, \tau)$ is differentiable at $t = 0$ (the symbol $S(t)_*\nu$ denotes the image of ν under the mapping $S(t)$); in this case, we denote $f'(0)$ by $\nu'_S$ and call it the derivative of the measure ν along $S$. If, in addition, $f'(0) \ll f(0)$ (the measure $f'(0)$ is absolutely continuous with respect to $f(0)$), then its density with respect to the measure $f(0)$ is called the τ-logarithmic derivative of the measure ν along $S$ and denoted by $\beta_S^\nu$.

If $k \in E$ and $S(t)(x) := x - tk$, then a measure ν τ-differentiable along $S$ is said to be τ-differentiable along $k$, and $\nu'k$ is defined by $\nu'k = \nu'_S$; the τ-logarithmic derivative of the measure ν along $S$ is called the τ-logarithmic derivative of ν along $k$ and denoted by $\beta^\nu(k, \cdot)$. The τ-differentiability of a measure along a vector field $h$ and its τ-logarithmic derivative along $h$ (denoted by $\beta_h^\nu$) are defined in a similar way: we set $S(t)(x) := x - \tanh(x)$.

If a measure ν is τ-differentiable along each $k \in H$, then it can be shown that the mapping $H \ni k \mapsto \nu'k$ is linear; the corresponding vector-valued measure $\nu'$ : $\mathcal{B}_E \ni A \mapsto [k \mapsto (\nu'k)(A)]$ is called the τ-derivative of ν over the subspace $H$. If, for any $k \in H$, there exists a τ-logarithmic derivative measure ν along $k$, then the mapping $H \ni k \mapsto \beta^\nu(k, \cdot)$ is linear; it is called the τ-logarithmic derivative of ν over $H$ and denoted by $\beta^\nu$.

**Remark 1.** If the measure ν has a logarithmic derivative over a subspace $H$ and $h(x) \in H$ for all $x \in E$, then, against expectation, $\beta_S^\nu(x) \neq \beta^\nu(h(x), x)$ in the general case (see below).

**Remark 2.** If τ is the topology of convergence on all sets, then any measure τ-differentiable along $S$ has a logarithmic derivative along τ (see [12]); for weaker topologies, this may be not the case. An example is as follows. Given an LCS $E$ being a Radon space,[2] let $S$ be the space of bounded continuous functions on $E$, and let $\tau_C$ be the weak topology on $\mathfrak{M}_E$ determined by the duality between $C$ and $\mathfrak{M}_E$. Then a measure $\tau_C$-differentiable along $S$ may have no logarithmic derivative along $S$ (even in the case $E = \mathbb{R}^1$).

Let $C$ be a norm-defining vector space of $H$-smooth functions on $E$ bounded together with all derivatives. A measure ν is said to be $C$-differentiable along a vector field $h \in$ Vect($E$) if there exists a measure $\nu'_h$ such that, for any $\varphi \in C$, we have $\int \varphi'(x)h(x)\nu(dx) = -\int \varphi(x)(\nu'_h)(dx)$. The Radon–Nikodym density of $\nu'_h$ with respect to ν (if it exists) is called the $C$-logarithmic derivative of the measure ν along $h$; if $h(x) = h_0 \in E$ for all $x \in E$, then, as above, the $C$-logarithmic derivative of ν along $h$ is called the $C$-logarithmic derivative of ν along $h_0$ ($C$-logarithmic derivatives are denoted by the same symbols as τ-logarithmic derivatives introduced above).

Suppose that a vector field $h_S$ is determined by $h_S(x) := S'(0)x$. Then the following proposition is valid.

**Proposition 1.** *A measure ν is $\tau_C$-differentiable along $S$ if and only if it is $C$-differentiable along $h_S$. In this* case,

[2] A topological space $E$ is called a Radon space if any countably additive Borel measure ν on $E$ is Radon; this means that, for any Borel subset $A$ of $E$ and any $\varepsilon > 0$, there exists a compact set $K \subset A$ such that $\nu(A \backslash K) < \varepsilon$. If $E$ is a completely regular Radon space, then the space of all bounded continuous functions on $E$ is in natural duality with $\mathfrak{M}_E$.

$\beta^{\nu}_{h_S} = \beta^{\nu}_S$, *where* $\beta^{\nu}_{h_S}$ *is the C-logarithmic derivative of* $\nu$ *along* $h_S$ *and* $\beta^{\nu}_S$ *is the* $\tau_C$*-logarithmic derivative of* $\nu$ *along S.*

**Proof.** This follows from the change of variable formula. Suppose that $\varphi \in C$ and, as above, $f(t) := (S(t))_*\nu$. Then

$$\lim_{t \to 0} t^{-1} \int_E \varphi(x)(f(t))(dx) - \int_E \varphi(x)(f(0))(dx)$$

$$= \lim_{t \to 0} t^{-1} \int_E \varphi(x)((S(t))_*\nu)(dx) - \int_E \varphi(x)\nu(dx)$$

$$= \lim_{t \to 0} t^{-1} \int_E (\varphi(S(t)) - \varphi(x))\nu(dx) = \int_E \varphi'(x)S'(0)(x)\nu(dx),$$

which implies the required assertion.

**Corollary 1.** *Let* $S_1$ *be another mapping of the interval* $(-\varepsilon, \varepsilon)$ *to* $\mathcal{B}_E$ *with the same properties as S. If* $h_S = h_{S_1}$, *then the measure* $\nu$ *is* $\tau_C$*-differentiable along S if and only if it is* $\tau_C$*-differentiable along* $S_1$.

**Remark 2.** It is natural to say that the measure $\nu$ is invariant with respect to $S$ if $\beta^{\nu}_S = 0$.

**Theorem 1.** *Suppose that a measure* $\nu \in \mathfrak{M}_E$ *has a* $\tau_C$*-logarithmic derivative over a subspace H, and let h be a vector field on E taking values in H. Then* $\beta^{\nu}_h(x) = \beta^{\nu}(h(x), x) + \operatorname{tr} h'(x)$, *where h' is the derivative of the mapping h over the subspace H.*

**Proof.** Suppose that $\varphi \in C$ and $h$ is a vector field on $E$, and let $\mu$ be the $E$-valued measure defined by $\mu := h(\cdot)\varphi(\cdot)\nu$. Applying Leibniz' rule to the derivative of $\mu$ over the subspace $H$, we obtain $\mu' = h'(\cdot)\varphi(\cdot)\nu + \varphi(\cdot)\nu' \otimes h(\cdot) + \varphi'(\cdot) \otimes h(\cdot)\nu$. Each summand in this relation is a measure whose values are operators on $H$; calculating the traces of these operators, we obtain $\operatorname{tr}\mu' = (\varphi(\cdot)\operatorname{tr}h'(\cdot))\nu + \varphi(\cdot)\beta^{\nu}(h(\cdot), \cdot)\nu + \varphi'(\cdot)h(\cdot)\nu$. Since $\int_E \mu'(dx) = 0$ and, therefore, $\int_E (\operatorname{tr}\mu')(dx) = 0$, it follows that $\int_E \varphi'(x)h(x)\nu(dx) = -\int_E \varphi(x)[\beta^{\nu}(h(x), x) + \operatorname{tr}h'(x)]\nu(dx)$. This means that the required relation holds.

Both the definitions given above and the algebraic parts of proofs can be extended to distributions (in the Sobolev–Schwartz sense) defined as continuous linear functionals on appropriate spaces of test functions. The difference is that the integrals of functions with respect to measures should be replaced by values of these linear functionals at functions, and instead of the change of variables formula for integrals, the definition of the transformation of a distribution generated by a transformation of the space on which the test functions are defined should be used.

## 2. QUANTUM ANOMALIES

In fact, quantum anomalies arise because the second term in the relation of Theorem 1 proved above is the same for all measures. Indeed, by virtue of Leibniz' rule, the logarithmic derivative (both over a subspace and along a vector field) of the product of a function and a measure is the sum of the logarithmic derivatives of the factors; therefore, a measure $\nu$ whose logarithmic derivative along a vector field is given by the expression in Theorem 1 can formally be taken for the product of a function $\psi_\nu$ whose logarithmic derivative over the subspace $H$ coincides with the logarithmic derivative over the measure $\nu$ over this subspace and a measure $\eta$ whose logarithmic derivative over the same subspace vanishes. If $E$ is finite-dimensional and $H$ coincides with $E$, then such a function and a measure indeed exist; moreover, $\eta$ turns out to be the Lebesgue measure, and $\psi_\nu$ is the density of $\nu$ with respect to it.

But in the infinite-dimensional case, there exist no exact counterpart of the Lebesgue measure; nevertheless, an analogue of density, called the generalized density of a measure, does exist [3, 10, 12], although its properties are far from those of usual density, and the corresponding distribution can be regarded as an analogue of the Lebesgue measure. It is this distribution that should be considered as a formalization of the term "volume element" used in [1, p. 362]. We however emphasize that the contents of this paper depends on the properties of neither this distribution nor the generalized density.

Let $Q$ be a finite-dimensional vector space being the configuration space of a Lagrangian system with Lagrange function $L: Q \times Q \to \mathbb{R}$ defined by $L(q_1, q_2) := \eta(q_1, q_2) + b(q_2)$, where $b$ is a quadratic functional (the kinetic energy of the system). We assume that the Lagrange function $L$ is nondegenerate (hyperregular), i.e., the corresponding Legendre transform is a diffeomorphism, so that it determines a Hamiltonian system with Hamiltonian function $\mathcal{H}: Q \times P \to \mathbb{R}$, where $P = Q^*$.

For $t > 0$, by $E_t$ we denote the set of continuous functions on $[0, t]$ taking values in $Q$ and vanishing at zero and by $H_t$, the Hilbert subspace of $E_t$ consisting of absolutely continuous functions on $[0, t]$ with square integrable derivative; the Hilbert norm $\|\cdot\|_{H_t}$ on $E_t$ is defined by

$$\|f\|^2_{H_t} := \int_0^t \|f'(\tau)\|^2_Q \, d\tau,$$

where $f \in E_t$ and $\|\cdot\|_{Q_t}$ is the Euclidean norm on $Q$. Finally, by $\mathcal{S}(t)$ we denote the classical action defined as the functional on $H_t$ determined by the Lagrange function $L$ via $\mathcal{S}(t)(f) := \int_0^t L(f(\tau), \dot{f}(\tau))d\tau$.

The Schrödinger quantization of the Hamiltonian system generated by the Lagrangian system described above yields the Schrödinger equation $i\dot{\psi}(t) = \hat{\mathcal{H}}\psi(t)$, where $\hat{\mathcal{H}}$ is a self-adjoint extension of a pseudodifferential operator on $\mathcal{L}_2(Q)$ with symbol equal to the Hamiltonian function $H$ generated by the Lagrange function $L$. The solution of the Cauchy problem for this equation with initial condition $f_0$ is

$$\psi(t)(q)$$
$$= \int_{E_t} \exp\left(i\int_0^t \eta(\psi(\tau) + q, \dot{\psi}(\tau))d\tau\right) f_0(\psi(t) + q)\phi_t(d\psi), \tag{1}$$

where $\phi_t$ is the Feynman pseudomeasure on $E_t$ (the exponential under the integral sign is well defined on the space $H_t$).

Let $W_t$ be the pseudomeasure on $E_t$ being the product of the exponential in the above integral and the function $\phi_t$. The following theorem is valid.

**Theorem 2.** *The logarithmic derivative of the pseudomeasure $W_t$ along $H_t$ exists and is determined by*

$$\beta^{W_t}(k, \psi) = i\int_0^t [L_1'(\psi(\tau) + q, \dot{\psi}(\tau))k(\tau)$$
$$+ L_2'(\psi(\tau) + q, \dot{\psi}(\tau))\dot{k}(\tau)]d\tau;$$

*here*, $k \in H_t$, $\psi \in E_t$, *and* $L_1'$ *and* $L_2'$ *are the partial derivatives of L with respect to the first and the second argument.*

**Corollary 2.** *If h is a vector field on $E_t$ taking values in $H_t$, then the logarithmic derivative of the pseudomeasure $W_t$ along h is determined by*

$$\beta_h^{W_t}(\psi) = i\int_0^t \Big[L_1'\Big(h(\psi)(\tau) + q, \frac{dh(\psi)}{d\tau}(\tau)\Big)h(\psi)(\tau)$$
$$+ L_2'\Big(h(\psi)(\tau) + q, \frac{dh(\psi)}{d\tau}(\tau)\Big)\frac{dh(\psi)}{d\tau}(\tau)\Big]d\tau + \operatorname{tr} h'(\psi).$$

A similar assertion is valid for the logarithmic derivative along a family $S(\alpha)$ of transformations of the space $E_t$ depending on a parameter $\alpha \in (-\varepsilon, \varepsilon)$.

It follows from Corollary 1 that if the classical action $\mathcal{S}(t)$ is invariant with respect to a family $S(\alpha)$, $\alpha \in (-\varepsilon, \varepsilon)$ of transformations of the space $Q$, then the logarithmic derivative $\beta_S^W(\psi)$ does not necessarily vanish. In turn, this means that if a Lagrange function is invariant with respect to transformations $S(\alpha)$ of the configuration space, then the solution of the corresponding Schrödinger equation is not necessarily invariant with respect to the same transformations.

**Remark 3.** For each family $S(\alpha)$ of transformations of the space $E_t$, we can obtain an explicit expression for the transformations of the pseudomeasure $W_t$ generated by the transformations $S(\alpha)$ by solving the equation $\dot{g}(\alpha) = \check{\beta}_{S(\alpha)} g(\alpha)$ (see [10]). It follows [10] that if $\operatorname{tr}(h_{S(\alpha)})'(\psi) > 0$ for $\alpha \in [0, \alpha_0]$, then $\det(S(\alpha))' \neq 1$; this fact does not depend on the classical action.

**Remark 4.** Using the notion of the generalized density of a pseudomeasure (cf. [3, 10, 11], where only generalized densities of usual measures were considered), we can say that the pseudomeasure $W_t$ is determined by its generalized density being the exponential in the Feynman integral (1). Moreover, as mentioned above, the expression for the transformations of the pseudomeasure contain determinants, and the expressions for the corresponding logarithmic derivatives contain traces, which do not depend on the generalized densities. This can be interpreted by treating the Feynman pseudomeasure as the product of its generalized density and distribution whose transformations are described by the corresponding determinants and traces, and the logarithmic derivatives of this distribution along constant vectors vanish. In turn, this allows us to say that the distribution mentioned above corresponds to the volume element considered in [1].

**Remark 5.** Thus, the determinants and traces mentioned above cannot be eliminated by any choice of the integrand and the Feynman pseudomeasure if the corresponding Feynman integrals are required to represent the solutions of the corresponding Schrödinger equation. Clearly, this contradicts [1, p. 362].

**Remark 6.** If $E$ is a this superspace, then, instead of traces and determinants, we should use supertraces and superdeterminants.


## ACKNOWLEDGMENTS

T.S. Ratiu acknowledges the partial support of Swiss National Science Foundation, grant no. Swiss NSF 200021-140238. J. Gough and O.G. Smolyanov acknowledge the partial support of the Swiss Federal Institute of Technology in Lausanne (EPFL). O.G. Smolyanov acknowledges the support of the Russian Foundation for Basic Research, project no. 14-01-00516.


## REFERENCES


1. P. Cartier and C. DeWitt-Morette, *Functional Integration* (Cambridge Univ. Press, Cambridge, 2006).
2. K. Fujikawa and H. Suzuki, *Path Integrals and Quantum Anomalies* (Oxford Univ. Press, Oxford, 2004; 2nd ed., 2013).
3. A. I. Kirillov, Russ. Math. Surv. **49** (3), 43–95 (1994).

4. J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry* (Springer-Verlag, Berlin, 1994; 2nd ed., 2003).
5. M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Boston, 1995).
6. S. Weinberg, *Quantum Theory of Fields* (Cambridge Univ. Press, Cambridge, 1995–2000), Vols. 1–3.
7. O. G. Smolyanov, in *Trends in Stochastic Analysis* (Cambridge Univ. Press, Cambridge, 2009), pp. 283–302.
8. O. G. Smolyanov and A. Truman, Theor. Math. Phys. **119** (3), 677–686 (1999).
9. O. G. Smolyanov and H. von Weizsäcker, J. Funct. Anal. **118** (2), 454–476 (1993).
10. H. von Weizsäcker, C.R. Acad. Sci. Paris. Ser. I Math. **321** (1), 103–108 (1995).
11. J. Montaldi and O. G. Smolyanov, Russ. J. Math. Phys. **21** (3), 379–385 (2014).
12. O. G. Smolyanov and H. von Weizsäcker, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **2** (1), 51–78 (1999).

*Translated by O. Sipacheva*