Wigner Measures and Quantum Control

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This paper considers Wigner functions and measures for infinite-dimensional open quantum systems; important examples of such systems are objects of quantum control theory (see [1, 2]). An axiomatic definition of coherent quantum feedback is proposed.

A representation of the states of quantum systems in terms of Wigner measures is possible for systems having classical analogues; it is similar to the representation of the states of classical Hamiltonian systems in terms of probability measures on the phase space. In both cases, the passage to a description of the state of a subsystem of some larger quantum system is implemented by means of the projection operation, because the phase space of the classical analogue of the ambient quantum system, being the union of some subsystems, is the Cartesian product of the phase spaces of the classical analogues of these subsystems.

If the dimension of the phase space is finite, then, instead of Wigner measures, we can consider their densities with respect to the Liouville measure, which are classical Wigner functions. However, on an infinite-dimensional phase space, there exists no Liouville measure, i.e., a Borel σ-additive σ-finite locally finite measure invariant with respect to symplectic transformations (this is a special case of a well-known theorem of Weil). In this case, we can either directly apply the Wigner measure or introduce some "sufficiently nice" measure instead of the Lebesgue measure; e.g., in the case of a linear phase space, we can use the Gaussian measure, as is done in the so-called white-noise analysis. After this, it becomes possible again to replace Wigner measures by "Wigner functions," i.e., by their densities with respect to the new measure. We shall consider Wigner measures and their densities in parallel.

The first section, which is of independent interest, considers properties of Wigner measures and functions; some of the results of this section can be regarded as an extension of results of [4] to Wigner measures. In the next section, an equation describing the evolution of the Wigner functions of quantum systems obtained by quantizing Hamiltonian systems with infinite-dimensional phase space is given; this equation is obtained as a consequence of a similar equation for the evolution of a Wigner measure (see [5]). (A Wigner measure is a signed cylindrical measure, and it would be interesting to estimate its variation and find countable additivity conditions; however, we do not discuss these issues here.) The last, third, section considers the evolution of the Wigner measures and functions of subsystems of quantum systems. In the same section, models of control of quantum systems are discussed and an axiomatic definition of coherent quantum feedback is given, which, as far as we know, has not been explicitly introduced in the literature so far. We consider largely algebraic aspects of the theory, omitting analytical assumptions.

1. WIGNER MEASURES AND FUNCTIONS

This section discusses properties of Wigner measures and their densities with respect to fixed measures on a classical phase space, that is, Wigner functions (precise definitions are given below). Let $E := Q \times P$ be the phase space of a Hamiltonian system, where Q and P are real locally convex spaces (LCSs), $P = Q^*$, and $Q = P^*$ (given an LCS X, by X^* we denote its dual endowed with a locally convex topology consistent with the duality between X and X^*); then $E^* = P \times Q$. Suppose also that $\langle \cdot, \cdot \rangle$: $P \times Q \to \mathbb{R}$ is the bilinear form of the duality between P and Q. Then the linear mapping J: $E \ni (q, p) \mapsto (p, q) \in E^*$ is an isomorphism, and we identify $h \in E$ with $Jh \in E^*$. In particular, for each

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$h \in E$, the symbol $\hat{h}$ denotes the pseudodifferential operator on $\mathcal{L}_2(Q, \mu)$ whose Weyl symbol[1] is the function $Jh \in E^*$. By μ we denote the P-cylindrical (Gaussian) measure on Q whose Fourier transform Φ_μ: $P \to \mathbb{R}$ is determined by $\Phi_\mu(p) := \exp\left(-\frac{1}{2}\langle p, B_\mu p\rangle\right)$, where B_μ: $P \to Q$ is a continuous linear mapping such that $\langle p, B_\mu p\rangle > 0$ for $p \neq 0$. By ν we denote a Q-cylindrical measure on P whose Fourier transform Φ_ν: $Q \to \mathbb{R}$ is defined by $\Phi_\nu(q) := \exp\left(-\frac{1}{2}\langle B_\mu^* q, q\rangle\right)$. In what follows, we assume that all LCSs are Hilbert, although the main results can be extended to the general case. We identify the space Q with Q^* and P with P^*, so that $B_\mu^* = B_\mu$ and $B_\mu > 0$; note also that μ and ν are σ-additive if the operator B_μ is nuclear.

The Weyl operator $\mathcal{W}(h)$ generated by an element $h \in E$ is defined by $\mathcal{W}(h) := e^{-i\hat{h}}$. The Weyl function corresponding to a density operator T is the function $\mathfrak{W}_T$: $E \to \mathbb{R}$ defined by $\mathfrak{W}_T(h) := \mathrm{tr}(T\mathcal{W}(h))$ (see [4]); it does not depend on μ.

Definition 1 (see [5]). The Wigner measure corresponding to a density operator T is an E^*-cylindrical measure W_T on E determined by the relation

$$\int_{Q \times P} e^{i(\langle p_1, q_2\rangle + \langle p_2, q_1\rangle)} W_T(dq_1, dp_1) = \mathfrak{W}_T(h)(q_2, p_2).$$

In other words, W_T is the (inverse) Fourier transform of the function $\mathfrak{W}_T(h)$. Thus, we have $W_T(dq, dp) = \int_Q\int_P \mathfrak{W}_T(h)(q_2, p_2) F_{E \times E}(dq_2, dp_2, dq, dp)$, where $F_{E \times E}$ is the Hamiltonian Feynman pseudomeasure on $E \times E$.

The Feynman pseudomeasure $F_{\mathcal{H}}$ on a Hilbert space is a distribution (in the sense of the theory of Sobolev–Schwartz generalized functions) on $\mathcal{H}$, i.e., a continuous (in an appropriate sense) linear functional on some function space on $\mathcal{H}$. It is convenient to specify such a functional $F_{\mathcal{H}}$, as well as an ordinary measure, in terms of its Fourier transform $\tilde{F}_{\mathcal{H}}: \mathcal{H} \ni z \mapsto F_{\mathcal{H}}(\varphi_z) \in \mathbb{C}$, where φ_z: $\mathcal{H} \to \mathbb{C}$ is defined as $\varphi_z(x) := e^{i\langle z, x\rangle}$.

If $\mathcal{H} = E = Q \times P$ and $\tilde{F}_{\mathcal{H}}(q, p) = e^{i\langle q, p\rangle}$, then $F_{\mathcal{H}}$ is said to be a Hamiltonian Feynman pseudomeasure; it is convenient for defining the Fourier transform that on functions given on infinite-dimensional spaces and maps them to measures. Actually, the Hilbert space structure is not important here; a Feynman pseudomeasure, as well as a Gaussian measure, can be defined on any LCS; in particular, a Hamiltonian Feynman pseudomeasure can be defined on any symplectic LCS (additional information is contained in [3, 9, 11]).

Proposition 1 (see [5]). *If G is the Weyl symbol of a pseudodifferential operator on $\mathcal{L}_2(Q, \mu)$, then*

$$\int_Q\int_P G(q, p) W_T(dq, dp) = \mathrm{tr}(T\hat{G}).$$

This proposition can also be used as a definition (cf. [4, Definition 3], where it is, however, assumed that $\dim Q = \dim P < \infty$ and, for this reason, only Wigner function, rather than measures, are considered).

Definition 2. The density Φ_T of the Wigner measure W_T with respect to a measure η on $Q \times P$ (if this density exists) is called the Wigner η-function (if $\dim Q = \dim P < \infty$ and η is a Lebesgue measure on $Q \times P$, then the Wigner η-function is the classical Wigner function).

In what follows, we assume that $\eta = \mu \otimes \nu$ but refer to the Wigner $(\mu \otimes \nu)$-function simply as the Wigner function.

Corollary 1. *If the assumptions of Proposition* 1 *hold, then*

$$\int_Q\int_P G(q, p)\Phi_T(q, p)\mu \otimes \nu(dq, dp) = \mathrm{tr}(T\hat{G}).$$

Proposition 2. *The following relation holds:*

$$\Phi_T(q, p) := e^{\frac{1}{2}(\langle p_1, B_\mu^{-1} p_1\rangle + \langle q_1, B_\mu^{-1} q_1\rangle)}$$
$$\times \int_{Q \times P} e^{-i(\langle p_1, q_2\rangle + \langle p_2, q_1\rangle)} \mathcal{W}_T(h)(q_2, p_1) e^{\frac{1}{2}(\langle p_2, B_\mu^{-1} p_2\rangle + \langle q_2, B_\mu^{-1} q_2\rangle)}$$
$$\times (\mu \otimes \nu)(dq_2, dp_2).$$

The function $(q, p) \mapsto e^{-\frac{1}{2}(\langle p, B_\mu^{-1} p\rangle + \langle q, B_\mu^{-1} q\rangle)}$ is the generalized density of the Gaussian measure $\mu \otimes \nu$ (see [8] and the references therein). The relations given above and those similar to them can be obtained by using the following heuristic rule. First, we write the corresponding formulas for the case where $\dim Q < \infty$, replacing Gaussian measures by their densities with respect to Lebesgue (=Liouville) measures on the spaces Q and $Q \times P$; in turn, these formulas are obtained by using the standard isomorphisms between the spaces of functions square integrable with respect to the Lebesgue measure and the spaces of functions square integrable with respect to the Gaussian measures. After this, we pass to the infinite-dimensional case, for which purpose we replace the Gaussian density with respect to the Lebesgue measures by generalized densities. It should be borne in mind that the generalized densities of Gaussian measures are defined only up to multiplication by a positive number, so that the above method for extending formulas to the infinite-dimensional case applies only to formulas invari-

[1] The definition of a pseudodifferential operator $\hat{F}$ on $\mathcal{L}_2(Q, \mu)$ with symbol F can be found in [5].

ant with respect to the multiplication of Gaussian densities by positive numbers.

The following propositions can be regarded as definitions of Wigner measures and functions similar to those given in [4].

Proposition 3. *For any density operator T on $\mathcal{L}_2(Q, \mu)$ and any $\varphi \in \mathcal{L}_2(Q, \mu)$, the following relations hold:*

$$(T\varphi)(q) = e^{\frac{1}{4}\langle B_\mu^{-1}q, q\rangle}\int_P\int_Q e^{-i\langle p, q_1 - q\rangle}\varphi(q_1)e^{-\frac{1}{4}\langle B_\mu^{-1}q_1, q_1\rangle}$$

$$\times W_T\left(\frac{dq_1 + q}{2}, dp\right);$$

$$(T\varphi)(q) = e^{\frac{1}{4}\langle B_\mu^{-1}q, q\rangle}\int_P\int_Q e^{-i\langle p, q_1 - q\rangle}\varphi(q_1)e^{\frac{1}{4}\langle B_\mu^{-1}q_1, q_1\rangle}$$

$$\times \Phi_T\left(\frac{q_1 + q}{2}, p\right)e^{\frac{1}{2}\langle B_\mu^{-1}p, p\rangle}(\mu \otimes \nu)(dq, dp).$$

The notation in the first formula means that the mapping $q \mapsto W_T\left(\frac{dq_1 + q}{2}, dp\right)$ is a function, while the mapping $(dq_1, dp) \mapsto W_T\left(\frac{dq_1 + q}{2}, dp\right)$ is a measure. The function $q \mapsto e^{-\frac{1}{2}\langle B_\mu^{-1}q, q\rangle}$ is a generalized density of the Gaussian measure μ, and $p \mapsto e^{-\frac{1}{2}\langle B_\mu^{-1}p, p\rangle}$ is a generalized density of the measure ν.

Let ρ_T^1 be the integral kernel of a density operator T on $\mathcal{L}_2(Q, \mu)$ defined by

$$(T\varphi)(q) = e^{\frac{1}{4}\langle B_\mu^{-1}q, q\rangle}\int_Q e^{\frac{1}{4}\langle B_\mu^{-1}q_1, q_1\rangle}\varphi(q_1)\rho_T^1(q, q_1)\mu(dq_1).$$

Proposition 4. *For any $\varphi \in \mathcal{L}_2(Q, \mu)$, the relation* $\Phi_T(q, p) = e^{\frac{1}{2}(\langle B_\mu q, q\rangle + \langle B_\mu p, p\rangle)}\int_Q 1\rho_T\left(q - \frac{1}{2}r, q + \frac{1}{2}r\right)e^{i\langle r, p\rangle}e^{\frac{1}{2}\langle B_\mu^{-1}r, r\rangle}\mu(dr)$ *holds.*

Let ρ_T^2 be the integral kernel of a density operator T on $\mathcal{L}_2(Q, \mu)$ defined by $(T\varphi)(q) = e^{\frac{1}{4}\langle B_\mu^{-1}q, q\rangle}\int_Q \varphi(q_1)e^{-\frac{1}{4}\langle B_\mu^{-1}q_1, q_1\rangle}\rho_T^2(q, dq_1)$. Thus, ρ_T^2 is a function of a point with respect to the first argument and a measure with respect to the second argument.

It follows from Proposition 1 that $\rho_T^2(q, dq_1) = \int_P e^{-i\langle p, q_1 - q\rangle} W_T\left(\frac{dq_1 + q}{2}, dp\right)$. Setting $s - r = q$ and $s + r = q_1$ and using the change of variable formula, we obtain $\rho_T^2(s - r, ds + r) = \int_P e^{-i\langle p, 2r\rangle} W_T(ds, dp)$, or $\rho_T^2\left(q - \frac{r}{2}, dq + \frac{r}{2}\right) = \int_P e^{-\langle p, r\rangle} W_T(dq, dp)$, which means that the "measure" $dp \mapsto W_T(dq, dp)$ is the inverse Fourier transform of the function $r \mapsto \rho_T^2\left(q - \frac{r}{2}, dq + \frac{r}{2}\right)$. This implies the following proposition.

Proposition 5. *Let F_E be a Hamiltonian pseudomeasure Feynman on $E := Q \times P$. Then*

$$W_T(dq, dp) = \int_Q \rho_T^2\left(q - \frac{r}{2}, dq + \frac{r}{2}\right)F_E(dr, dp).$$

Here, the integration with respect to the "measure" $dq \mapsto W_T(dq, dp)$ requires using the so-called Kolmogorov integral[2].

2. EVOLUTION OF WIGNER FUNCTIONS AND MEASURES

We use the assumptions and notation of the preceding section. Suppose that, for each $t \in \mathbb{R}$, $W_T(t)$ is the Wigner measure describing the state of a quantum system at time t (thus, in this section, $W_T(\cdot)$ denotes a function of a real argument whose values are Wigner measures, while in the preceding section, the symbol W_T denotes a Wigner measure). Then $W_T(\cdot)$ satisfies the equation [5]

$$\dot{W}_T(t) = 2\sin\left(\frac{1}{2}\mathcal{L}_{\mathfrak{H}}^*(W_T(t))\right), \qquad (1)$$

where, for each $a \in \mathbb{R}$, $\sin(a\mathcal{L}_{\mathfrak{H}}^*)$ is the linear operator acting on the space $\mathfrak{H}$ of E^*-cylindrical measures on E and conjugate to the operator $\sin(a\mathcal{L}_{\mathfrak{H}})$ acting on the function space on E by the rule

$$\sin(a\mathcal{L}_{\mathfrak{H}}) := \sum_{n=1}^{\infty}\frac{a^{2n-1}}{(2n-1)!}\mathcal{L}_{\mathfrak{H}}^{(2n-1)}.$$

Here, $\mathcal{L}_{\mathfrak{H}}^{(n)}$ is defined as follows: for each function Ψ: $E \to \mathbb{R}$ and each $n \in \mathbb{N}$, $\mathcal{L}_{\mathfrak{H}}^{(n)}\Psi(x) := \{\Psi, \mathfrak{H}\}^{(n)}(x)$ for

[2] The Kolmogorov integral is the trace on the tensor product of the space of functions on Q and the space of measures on Q; ρ_T^2 is an element of this space (the initial definition, in which neither tensor product nor trace are mentioned, can be found in [7]).

$x \in E$, where $\{\Psi, \mathfrak{H}\}^{(n)}(x) := \Psi^{(n)}(x)I^{\otimes n}\mathfrak{H}^{(n)}(x)$, $\Psi^{(n)}$ and $\mathfrak{H}^{(n)}$ denote the nth derivatives of the functions Ψ and $\mathfrak{H}$, respectively, and $I^{\otimes n}$ is the nth tensor power of the operator I determining the symplectic structure on the phase space E [5].

Relation (1) implies an equation describing the evolution of the Wigner μ-function. To obtain it, is suffices to recall that, for any function Φ: $E \to \mathbb{R}$, the nth derivative of the product $\Phi^n\mu$ can be calculated by the Leibniz rule and that the derivatives of the Gaussian measure μ can be calculated as follows. If h, h_1, $h_2, \ldots \in B_\mu^{\frac{1}{2}}Q$, then $\mu'h = -\langle B_\mu^{-1}h, \cdot\rangle\mu$; $\mu''h_1h_2 = -\langle B_\mu^{-1}h_1, h_2\rangle\mu + \langle B_\mu^{-1}h_1, \cdot\rangle\langle B_\mu^{-1}h_2, \cdot\rangle\mu$, etc.

These relations are versions of the Wick formulas. For each $k \in B_\mu^{\frac{1}{2}}Q$, the symbol $\langle B_\mu^{-1}k, \cdot\rangle$ denotes a function defined μ-almost everywhere on Q with the following properties (see [10]):

(i) its domain is a measurable vector subspace of Q of full measure;

(ii) this function is linear on its domain;

(iii) if $x \in B_\mu^{\frac{1}{2}}Q$, then $\langle B_\mu^{-1}k, x\rangle = \langle B_\mu^{-\frac{1}{2}}k, B_\mu^{-\frac{1}{2}}x\rangle$ (such a function exists and any two functions with properties (i)–(iii) coincide μ-almost everywhere (see [10])).

For each $a > 0$, the operator $\sin(aL_{\mathfrak{H}}^*)$ acting on functions given on E is defined by $\sin(aL_{\mathfrak{H}}^*)\varphi(\mu \otimes \nu) := (\sin(a\mathcal{L}_{\mathfrak{H}}^*))(\varphi\mu \otimes \nu)$. Suppose also that, for each $t \in \mathbb{R}$, $\Phi_T(t)$ is the Wigner μ-function describing the state of a quantum system at time t.

Theorem 1. *The mapping $\Phi_T(\cdot)$ taking values in the set of Wigner μ-functions satisfies the equation*

$$\dot{\Phi}_T(t) = 2\sin\Big(\frac{1}{2}L_{\mathfrak{H}}^*(\Phi_T(t))\Big).$$

3. REDUCED EVOLUTION OF WIGNER MEASURES

Let ρ_T^1 and ρ_T^2 be the above-introduced integral kernels of a density operator T of a quantum system being the quantum version of a classical Hamiltonian system with phase space $E_1 \times E_2$, where $E_1 = Q_1 \times P_1$ and $E_2 = Q_2 \times P_2$. Then, for the integral kernels of the reduced density operator T_1 acting on $\mathcal{L}_2(Q_i, \mu_i)$, $i = 1, 2$ (here and in what follows, we use the natural generalizations of the above notation and assumptions), we have

$$\rho_{T_1}^1(q_1^1, q_2^1) = \int_{Q_2}\rho_T^1(q_1^1, q_2^1, q^1, q^2) \times e^{\frac{1}{2}\langle B_{\mu_1\otimes\mu_1}(q^1, q^2), (q^1, q^2)\rangle}\mu_2(dq_2),$$

$$\rho_{T_1}^1(q^1, dq_2^1) = \int_{Q_2}\rho_T^2(q^1, dq_2^1, q^2, dq^2);$$

the last integral is again a Kolmogorov integral. Therefore, Propositions 4 and 5 imply the following theorem.

Theorem 2. *Let W_T and Φ_T be the Wigner measure and function of the quantum system with Hilbert space $\mathcal{L}_2(Q_1 \times Q_2, \mu_1 \otimes \mu_2)$. Then the Wigner measure W_{T_1} and the Wigner function Φ_{T_1} of its subsystem with Hilbert space $\mathcal{L}_2(Q_1, \mu_1)$ are determined by the relations $W_{T_1}(dq_1, dp_1) = \int_{Q_2\times P_2} W_T(dq_1, dp_1, dq_2, dp_2)$ and $\Phi_T(q_1, p_1) = e^{\frac{1}{2}(\langle B_{\mu_1}^{-1}q_1, q_1\rangle + \langle B_{\mu_1}^{-1}p_1, p_1\rangle)}\int_{Q_2\times P_2} e^{\frac{1}{2}(\langle B_{\mu_2}^{-1}q_2, q_2\rangle + \langle B_{\mu_2}^{-1}p_2, p_2\rangle)}\Phi_T(q_1, p_1, q_2, p_2)(\mu_2 \otimes \nu_2)(dq_2, dp_2)$.*

Now, consider the models mentioned in the introduction. Throughout the rest of the paper, given any Hilbert space $\mathcal{T}$, we use $\mathcal{L}_s(\mathcal{T})$ to denote the set of all self-adjoint operators on $\mathcal{T}$.

Thus, let $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{C}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ be Hilbert spaces. We assume that $\mathcal{P}$ is the Hilbert space of a quantum control system, which we call a quantum plant (QF), and let $\mathcal{C}$ be the Hilbert space of another quantum control system, which we call a quantum controller (QC); suppose that $\mathcal{P}_j$ and $\mathcal{C}_j$, $j = 1, 2$, are the Hilbert space of parts of the QP and QC, respectively. Let $\mathcal{H} := \mathcal{P} \otimes \mathcal{C}$ be the Hilbert space of the united quantum system. Consider $\hat{\mathfrak{H}}_{\mathcal{P}} \in \mathcal{L}_s(\mathcal{P})$, $\hat{\mathfrak{H}}_{\mathcal{C}} \in \mathcal{L}_s(\mathcal{C})$, $\hat{\mathcal{K}}_{\mathcal{P}_1\otimes\mathcal{C}_1} \in \mathcal{L}^s(\mathcal{P}_1 \otimes \mathcal{C}_1)$, and $\hat{\mathcal{K}}_{\mathcal{P}_2\otimes\mathcal{C}_2} \in \mathcal{L}^s(\mathcal{P}_e \otimes \mathcal{C}_2)$. We set $\hat{\mathfrak{H}}_{\text{feedback}} := \hat{\mathfrak{H}}_{\mathcal{P}} \otimes \mathcal{I}d_{\mathcal{C}} + \mathcal{I}d_{\mathcal{P}} \otimes \hat{\mathfrak{H}}_{\mathcal{C}} + \hat{\mathcal{K}}_{\mathcal{P}_1\otimes\mathcal{C}_1} \otimes \mathcal{I}d_{\mathcal{P}_2\otimes\mathcal{C}_2} + \mathcal{I}d_{\mathcal{P}_1\otimes\mathcal{C}_1} \otimes \hat{\mathcal{K}}_{\mathcal{P}_2\otimes\mathcal{C}_2} \in \mathcal{L}^s(\mathcal{H})$, where $\mathcal{I}d_{\mathcal{P}} \in \mathcal{L}^s(\mathcal{P})$, $\mathcal{I}d_{\mathcal{C}} \in \mathcal{L}^s(\mathcal{C})$, $\mathcal{I}d_{\mathcal{P}_1\otimes\mathcal{C}_1} \in \mathcal{L}^s(\mathcal{P}_1 \otimes \mathcal{C}_1)$, and $\mathcal{I}d_{\mathcal{P}_2\otimes\mathcal{C}_2} \in \mathcal{L}^s(\mathcal{P}_2 \otimes \mathcal{C}_2)$ are the identity operators on the corresponding spaces. The first term in the expression for $\hat{\mathfrak{H}}_{\text{feedback}}$ describes the evolution of an isolated QP, the second term describes the evolution of the isolated QC, and the last two terms describe the (coherent) quantum feedback. It is worth mentioning that the definition of $\hat{\mathcal{H}}_{\text{feedback}}$ is symmetric with respect to QP, QC, and the feedback.

The more general Hamiltonian $\hat{\mathfrak{H}} := \hat{\mathfrak{H}}_{\mathcal{P}} \otimes \mathcal{I}d_{\mathcal{C}} + \mathcal{I}d_{\mathcal{P}} \otimes \hat{\mathfrak{H}}_{\mathcal{C}} + \hat{\mathcal{K}}$, where $\hat{\mathcal{K}} \in \mathcal{L}^s(\mathcal{P} \otimes \mathcal{C})$ (see [6]), may describe coherent quantum control both with and without feedback. In particular, if $\hat{\mathcal{K}} = \hat{\mathcal{K}}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \otimes \mathcal{I}d_{\mathcal{P}_2 \otimes \mathcal{C}_2} + \mathcal{I}d_{\mathcal{P}_1 \otimes \mathcal{C}_1} \otimes \hat{\mathcal{K}}_{\mathcal{P}_2 \otimes \mathcal{C}_2}$), then we obtain the previous model. On the other hand, if $\hat{\mathcal{K}} := \hat{\mathcal{K}}_1 \otimes \mathcal{I}d_{\mathcal{P}_2 \otimes \mathcal{C}_2}$, then we obtain a model of (coherent) quantum control without feedback.

If the QP and QC are obtained by quantizing Hamiltonian systems, then we can assume that, in the natural notation, $\mathcal{P}_j = \mathcal{L}_2(Q_{\mathcal{P}_j}, \mu_j)$, $\mathcal{C}_j = \mathcal{L}_2(Q_{\mathcal{C}_j}, \nu_j)$, $\mathcal{P} = \mathcal{L}_2(Q_{\mathcal{P}_1} \times Q_{\mathcal{P}_2}, \mu_1 \otimes \mu_2)$, $\mathcal{C} = \mathcal{L}_2(Q_{\mathcal{C}_1} \otimes Q_{\mathcal{C}_2}, \nu_1 \otimes \nu_2)$ for $j = 1, 2$. In this case, the Wigner function and measure of the union of the QP and the QC are defined on the space $Q_{\mathcal{P}_1} \times Q_{\mathcal{P}_2} \times Q_{\mathcal{C}_1} \times Q_{\mathcal{C}_2}$, and their evolution is described by the equations of the second section. To obtain the dynamics of the Wigner function and measure of the QP (which are defined on $Q_{\mathcal{P}_1} \times Q_{\mathcal{P}_2}$), we must apply Theorem 2.

Remark 1. Obtaining the dynamics of a quantum control system (QP) requires finding the Hamiltonians $\mathcal{K}_1$ and $\mathcal{K}_2$ (or $\mathcal{K}$) (in appropriate classes of Hamiltonians). This problem is similar to the simpler problem of choosing a time dependent Hamilton function $\mathcal{K}_1(\cdot)$ on $Q_{\mathcal{P}}$ to which the required dynamics in $\mathcal{L}_2(Q_{\mathcal{P}}, \mu)$ corresponds under the assumption $\hat{\mathcal{H}} = \hat{\mathfrak{H}}_1 + \widehat{\mathcal{K}_1(t)}$, where $\hat{\mathfrak{H}}_1 \in \mathcal{L}^s(\mathcal{P})$ and $\widehat{\mathcal{K}(t)} \in \mathcal{L}^s(\mathcal{P})$. Although this model is not a special case of any of the models described above, we expect that it can be obtained as the limit of an appropriate sequence of these models.

Remark 2. We can extend our model, assuming that the QP interacts also with one more quantum system perturbing the dynamics of the control system. Of course, we can also assume that the source of perturbations is a part of the QP.

Remark 3. The approach presented in the first two sections applies directly to quantum systems obtained by applying Schrödinger quantization to classical Hamiltonian systems. To consider more general cases, such as spin system, we must extend our approach by methods of superanalysis. We expect that all our results can be generalized to this case.

Remark 4. Feedback for classical Hamiltonian systems can be defined in a similar way.

Remark 5. The internal dynamics of the QP and QC in our quantum model with (coherent) feedback can be described in more detail. In particular, it can be assumed that $\hat{\mathcal{H}} = (\hat{\mathcal{H}}_{\mathcal{P}_1} \otimes \mathcal{I}d_{\mathcal{P}_2} + \mathcal{I}d_{\mathcal{P}_1} \otimes \hat{\mathcal{H}}_{\mathcal{P}_2}) \otimes \mathcal{I}d_{\mathcal{C}} + \mathcal{I}d_{\mathcal{P}} \otimes (\hat{\mathcal{H}}_{\mathcal{C}_1} \otimes \mathcal{I}d_{\mathcal{C}_2} + \mathcal{I}d_{\mathcal{C}_1} \otimes \hat{\mathcal{H}}_{\mathcal{C}_2}) + \hat{\mathcal{K}}_{\mathcal{P}_1 \otimes \mathcal{P}_2} \otimes \mathcal{I}d_{\mathcal{C}_1 \otimes \mathcal{C}_2} + \mathcal{I}d_{\mathcal{P}_1 \otimes \mathcal{P}_2} \otimes \hat{\mathcal{K}}_{\mathcal{C}_1 \otimes \mathcal{C}_2} + \hat{\mathcal{K}}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \otimes \mathcal{I}d_{\mathcal{P}_2 \otimes \mathcal{C}_2} + \hat{\mathcal{K}}_{\mathcal{P}_2 \otimes \mathcal{C}_2} \otimes \mathcal{I}d_{\mathcal{P}_1 \otimes \mathcal{C}_1}$.

In the above relation, the parts of the Hamiltonian describing the QP and the QC and the interaction between them are again symmetric.

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