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ON THE EXISTENCE OF APPROXIMATE EQUILIBRIA AND SHARING RULE SOLUTIONS IN DISCONTINUOUS GAMES

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This paper studies the existence of equilibrium solution concepts in a large class of economic models with discontinuous payoff functions. The issue is well understood for Nash equilibria, thanks to Reny's better-reply security condition (Reny (1999)), and its recent improvements (Barelli and Meneghel (2013); McLennan et al. (2011); Reny (2009, 2011)). We propose new approaches, related to Reny's work, and obtain tight conditions for the existence of approximate equilibria and of sharing rule solutions in pure and mixed strategies (Simon and Zame (1990)). As byproducts, we prove that many auction games with correlated types admit an approximate equilibrium, and that many competition models have a sharing rule solution.

Keywords: Discontinuous games, better-reply security, sharing rules, approximate equilibrium, Reny equilibrium, strategic approximation, auctions, timing games.

JEL classification: C02, C62, C72.

1. INTRODUCTION

Many economic interactions are modeled as games with discontinuous payoff functions. For example, in timing games, price and spatial competitions, auctions, bargaining, preemption games or wars of attrition, discontinuities occur

‡We thank Panayotis Mertikopoulos and Christina Pawlowitsch for their useful comments. We thank the anonymous reviewers and the associated editor for many insightful comments and suggestions. We thank many participants of the following seminars or conferences for their valuable comments: Conference on Economic Theory, June 2012 (Cowles, Yale), International Conference on Game Theory, July 2012 (Stony Brook), SAET conference, July 2012 (Brisbane), Workshop on Dynamic Game Theory, December 2012 (Valparaiso), Roy-Adres Seminar, September 2011 (Paris), CSEF Seminar, November 2011 (Napoli). This research was supported by grant ANR-10-BLAN 0112 (Laraki).

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when firms choose the same price, location, bid or acting time. The objective of this paper is to extend and link conditions under which Nash, approximate and sharing rule equilibria exist in such games.

Nash equilibrium is a strategy profile where each agent is reacting optimally to other players' plans. Mathematically, it is a fixed point of the best-response correspondence. When payoff functions are continuous and quasiconcaves, an application of Kakutani's fixed point theorem leads to the Nash-Glicksberg theorem (Glicksberg (1952); Nash (1950a,b)). In discontinuous games, the Kakutani approach cannot be directly applied because a player may have no optimal reply or because his best choice jumps as a function of the choices of the other players.[1]

A natural issue then is to identify regularity conditions on payoffs, which combined with quasiconcavity of the payoff functions, guarantee the existence of a Nash equilibrium. The first existence conditions are given by the seminal papers of Dasgupta and Maskin (1986b,a). A significant breakthrough[2] is achieved by Reny (1999) via the better-reply security approach.

Quoting Reny, *"A game is better-reply secure if for every nonequilibrium strategy x^* and every payoff vector limit u^* resulting from strategies approaching x^*, some player i has a strategy yielding a payoff strictly above u_i^* even if the others deviate slightly from x^*"*.

Reny's paper generated a large literature. For instance, Barelli and Meneghel (2013) and McLennan et al. (2011) proposed relaxations that cover non-quasiconcave preferences. Reny (2009, 2011) proposed new refinements for games in mixed strategies using a strategic approximation methodology. Recently, Barelli et al. (2014) applied Reny's better-reply security and strategic approximation techniques to prove existence of the value in Colonel Blotto game.

But what if a Nash equilibrium does not exist? Two related relaxations of Nash equilibrium have been studied in the literature: endogenous sharing rules and approximate equilibria.

In many discontinuous games, the *exogenously* given tie-breaking rule leads to games without pure Nash equilibria (e.g., asymmetric Bertrand duopoly, Hotelling location game) or without mixed Nash equilibria (e.g., 3-player preemption games (Laraki et al. (2005)), auctions with correlated types or values (Fang and Morris (2006); Jackson (2009)). However, the existence of a Nash equilibrium is restored if the tie-breaking rule is chosen *endogenously* (Andreoni et al. (2007); Maskin and Riley (2000); Simon and Zame (1990)). For example, in an asymmetric Bertrand duopoly, a pure Nash equilibrium exists if ties are broken in favor of the lower-cost firm. In first-price auctions with complete information, a pure Nash equilibrium exists if ties are broken in favor of the bidder with the

[1] Another approach, not considered in this paper, is to use ordered fixed point theory (e.g., Tarski's theorem) to obtain the existence of Nash equilibria in supermodular games (Topkis (1979)).

[2] Carmona (2009) gives an extension of Dasgupta and Maskin's results, which is unrelated to Reny's approach.

highest value. Under mild topological conditions, Simon and Zame (1990) proved that to every game, one could associate an auxiliary game that admits a Nash equilibrium in *mixed strategies* and where payoffs in the two games only differ at discontinuity points (see Section 2 for a formal definition). Jackson et al. (2002) remark that their "*results concern only the existence of solutions [sharing rule] in mixed strategies*" and that they "*have little to say about the existence of solutions in pure strategies*". We prove existence of a sharing rule solution in pure strategies in every quasiconcave and compact game (see Theorem 3.4).

An alternative solution for games without a Nash equilibrium is the notion of approximate equilibrium. It is a limit strategy profile x^* and a limit payoff vector u^* of ε-Nash equilibria x^ε with associated payoff vector $u(x^\varepsilon)$, as ε goes to 0.

There are many games without a Nash equilibrium but with a reasonable approximate equilibrium. In first-price auctions with complete information, for example, a natural approximate equilibrium arises if the player with the highest value proposes a bid slightly above the second highest value, and if the other players bid exactly their value. Another example is Bertrand duopoly with asymmetric costs, where in every approximate equilibria in pure strategies the most efficient firm proposes a price slightly below the marginal cost of the opponent.

There are few results in the literature establishing the existence of approximate equilibria, one of which is due to Reny (1996) and Prokopovych (2011).[3] While theoretically interesting, it requires assumptions that are not always satisfied in applications (as will be seen).

In this paper, we define a game G to be *approximately better-reply secure* if for every non-approximate equilibrium strategy profile x^* and every payoff vector limit u^* resulting from strategies approaching x^*, some player i has a strategy yielding a payoff strictly above u_i^*, even if the others deviate slightly from x^*.

We prove that every approximately better-reply secure quasiconcave compact game admits an approximate equilibrium. An example is given by the class of diagonal games, which encompasses many models of competition (in price, time, location or quantity): each player, $i = 1, ..., N$, chooses a real number x_i in $[0, 1]$. The payoff of player i is $f_i(x_i, \phi(x_{-i}))$ if $x_i < \phi(x_{-i})$, $g_i(x_i, \phi(x_{-i}))$ if $x_i > \phi(x_{-i})$, and $h_i(x_i, x_{-i})$ if $x_i = \phi(x_{-i})$, where $f_i : [0, 1]^2 \to \mathbf{R}$, $g_i : [0, 1]^2 \to \mathbf{R}$ and $\phi : [0, 1]^{N-1} \to \mathbf{R}$ are continuous functions. For example, in first-price auctions, $f_i = 0$, $g_i(x_i, x_{-i}) = v_i - x_i$, and $\phi(x_i, x_{-i}) = \max_{j \neq i} x_j$. In second-price auctions, $g_i(x_i, x_{-i}) = v_i - \max_{j \neq i} x_j$ (where v_i is the value of the object for player i).

The paper is organized as follows. In Section 2, we recall the main results for the existence of a solution in discontinuous games: Reny's better-reply security (Reny (1999)) for existence of Nash equilibria, the sharing rule solution of Simon and Zame (1990), and Reny and Prokopovych's conditions (Reny (1996), Prokopovych (2011)) for the existence of an approximate equilibrium. Section

[3]Historically, Reny proved this result in a working and unpublished paper (Reny (1996)). Independently, Prokopovych (2011) proved the same result with a different technique.

3 is dedicated to quasiconcave compact games in *pure strategies*. We introduce the new concept of Reny solution, which weakens Nash equilibrium concept, and we prove its existence for every discontinuous game. This solution is used to construct pure sharing rule solutions and approximate equilibria. The results are illustrated in the class of diagonal games. Section 4 is dedicated to compact metric games in *mixed strategies*. We prove that the intersection of the sets of Reny solutions and sharing rule solutions is nonempty and contains the set of approximate equilibria. In addition, we prove that in every approximately better-reply secure game, approximate equilibria may be obtained as limits of Nash equilibria of an *endogenously* chosen sequence of discretizations of the game. This is a natural extension of a similar result established by Reny (2009, 2011) for Nash equilibria. As applications, we prove the existence of a mixed approximate equilibrium in multiplayer auctions with correlated types and in two-player auctions with interdependent types and values.

2. THREE STANDARD APPROACHES TO DISCONTINUOUS GAMES

A game in strategic form $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$ is given by a finite set N of players, and for each player $i \in N$, a nonempty set X_i of pure strategies, and a payoff function $u_i : X = \prod_{i\in N} X_i \to \mathbf{R}$. This paper assumes G to be *compact*, that is, for every i, X_i is a compact subset of a topological vector space, and u_i is bounded. We let $V_i(x_{-i}) := \sup_{d_i\in X_i} u_i(d_i, x_{-i})$ denote the greatest payoff that player i can get against $x_{-i} = (x_j)_{j\neq i} \in X_{-i} := \Pi_{j\neq i} X_j$.

Definition 2.1 A pair $(x, v) \in X \times \mathbf{R}^N$ is a *Nash equilibrium* of G (and x is a *Nash equilibrium profile*) if $v = u(x)$ and for every player $i \in N$, $V_i(x_{-i}) = v_i$.

The game G is *quasiconcave* if for every player $i \in N$, X_i is convex and for every $x_{-i} \in X_{-i}$, the mapping $u_i(\cdot, x_{-i})$ is quasiconcave. The game is *continuous* if for every $i \in N$, u_i is a continuous function.[4]

Theorem 2.2 (Glicksberg (1952)) Every continuous, quasiconcave and compact game admits a Nash equilibrium.

The rest of the section presents three extensions of this result when payoffs are discontinuous. Our paper combines them into one general idea.

2.1. *Better-Reply Secure Game*

In many discontinuous games, a Nash equilibrium exists (symmetric Bertrand competition, auctions with private values, wars of attrition, among many). Reny's theorem provides an explanation for this (Reny (1999)). Formally, we let $\Gamma = \{(x, u(x)) : x \in X\}$ denote the graph of G and $\overline{\Gamma}$ be the closure of Γ. Since X is

[4] X is endowed with the product topology.

compact and u bounded, $\overline{\Gamma}$ is a compact subset of $X \times \mathbf{R}^N$. Define the "secure payoff level" of player i at $(d_i, x_{-i}) \in X$ as follows:

$$\underline{u_i}(d_i, x_{-i}) := \liminf_{x'_{-i} \to x_{-i}} u_i(d_i, x'_{-i}) := \sup_{V \in \mathcal{V}(x_{-i})} \inf_{x'_{-i} \in V} u_i(d_i, x'_{-i}),$$

where $\mathcal{V}(x_{-i})$ denotes the set of neighborhoods of x_{-i}. This is the payoff that d_i can almost guarantee to player i if his opponents play any profile close enough to x_{-i}. We let $\underline{V_i}(x_{-i}) := \sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i})$ denote the largest payoff that player i can secure against x_{-i}.

Definition 2.3 A game G is *better-reply secure* if whenever $(x, v) \in \overline{\Gamma}$ and x is not a Nash equilibrium profile, some player $i \in N$ can secure[5] a payoff strictly above v_i, i.e. $\underline{V}_i(x_{-i}) > v_i$.

Theorem 2.4 (Reny (1999)) Every better-reply secure, quasiconcave and compact game admits a Nash equilibrium in pure strategies.

Since every continuous game is obviously better-reply secure, this extends Glicksberg's theorem. In his paper, Reny gives two practical conditions which, together, imply better-reply security (see Theorem 2.6 below).

Definition 2.5
(i) G is payoff secure if $V_i(x_{-i}) = \underline{V_i}(x_{-i})$.
(ii) G is reciprocally upper semicontinuous if, whenever $(x, v) \in \overline{\Gamma}$ and $u_i(x) \leq v_i$ for every i, then $u(x) = v$.

Equivalently, G is payoff secure if for every $x \in X$, for every $\varepsilon > 0$, every player $i \in N$ can secure a payoff above $u_i(x) - \varepsilon$.

Theorem 2.6 Every payoff secure and reciprocally upper semicontinuous game is better-reply secure.

2.2. *Approximate Equilibrium*

In first-price auctions with complete information, bidding slightly above the second highest evaluation for the bidder with the highest evaluation yields an approximate equilibrium. One of the main goals of this paper is to develop theoretical tools to extend this existence result to a large class of auctions.

Definition 2.7 A pair $(x, v) \in \overline{\Gamma}$ is an *approximate equilibrium* (and x is an approximate equilibrium profile) if there exists a sequence $(x^n)_{n \in \mathrm{N}}$ of X and a

[5]The following definitions are standard: Player i can secure a payoff above $\alpha \in \mathrm{R}$ if there exists $d_i \in X_i$ and a neighborhood V_{-i} of x_{-i} such that for every $x'_{-i} \in V_{-i}$, $u_i(d_i, x'_{-i}) \geq \alpha$. Player i can secure a payoff strictly above $\alpha \in \mathrm{R}$ if he can secure a payoff above $\alpha + \varepsilon$ for some $\varepsilon > 0$. We give equivalent formulations using V_i and $\underline{V}_i$.

sequence $(\varepsilon_n)_{n\in \mathrm{N}}$ of positive real numbers, converging to 0, such that:
(i) for every $n \in \mathrm{N}$, x^n is an ε_n-equilibrium: $\forall i \in N$, $\forall d_i \in X_i$, $u_i(d_i, x^n_{-i}) \leq u_i(x^n) + \varepsilon_n$,
(ii) the sequence $(x^n, u(x^n))$ converges to (x, v).

In zero-sum games, the existence of an approximate equilibrium is equivalent to the existence of the value. Every Nash equilibrium is obviously an approximate equilibrium (take a constant sequence in the definition above). Let us state one of the very few existing results in the literature.[6]

DEFINITION 2.8 A game G has the marginal continuity property at $x \in X$ if for every $i \in N$, $V_i(x_{-i})$ is a continuous function at x_{-i}. If this holds for every x, the game has the marginal continuity property.

THEOREM 2.9 (Reny (1996) and Prokopovych (2011)) Every payoff secure, quasiconcave compact game that has the marginal continuity property admits an approximate equilibrium.

This theorem applies to first-price auctions and asymmetric Bertrand's duopoly. However, the following location game (Simon and Zame (1990)) is not payoff secure, but admits an approximate equilibrium.

EXAMPLE 2.10 *California Location Game*:
This example was introduced by Simon and Zame (1990). The length interval $[0, 4]$ represents an interstate highway. The strategy set of player 1 (a psychologist from California) is $X = [0, 3]$ (representing the Californian highway stretch). The strategy set of player 2 (a psychologist from Oregon) is $Y = [3, 4]$ (the Oregon part of the highway). The payoff function of player 1 is $u_1(x, y) = \frac{x+y}{2}$ if $x < y$ and $u_1(3, 3) = 2$. The payoff function of player 2 is $u_2(x, y) = 4 - u_1(x, y)$. The strategy profile $x_n = (3 - \frac{1}{n}, 3)$, corresponding to the vector payoff $v_n = (3 - \frac{1}{2n}, 1 + \frac{1}{2n})$, is a $\frac{1}{2n}$-equilibrium. Consequently, $(x = (3, 3), v = (3, 1))$ is an approximate equilibrium. However, the game is not payoff secure for player 2 at $x = (3, 3)$.

2.3. *Sharing Rule Solutions*

Simon and Zame show that even if a game does not have a Nash equilibrium, it is always possible to slightly change the payoffs at discontinuity points so that the new game has a mixed Nash equilibrium.

EXAMPLE 2.11 *California Location Game, Continued.*
In the California location game, define a new payoff function q as follows: $q(x) = u(x)$ for every $x \neq (3, 3)$ and $q(3, 3) = (3, 1)$. The pure strategy profile

[6] Ziad (1997) proposes another existence theorem of approximate equilibria, unrelated to our work. See also Carmona (2010, 2011) and Reny (1996).

$(3,3)$ with payoff $(3,1)$ is a Nash equilibrium of the game defined by q. The new sharing rule at $x = (3,3)$ has a simple interpretation: it corresponds to giving each psychologist his/her natural market share. Moreover, this is exactly the prediction of the approximate equilibrium in Example 2.10. We will prove that this property is very general: every approximate equilibrium is a sharing rule equilibrium (see Theorem 3.10).

To prove the existence of a solution, Simon and Zame do not require the game to be quasiconcave. However, they allow the use of *mixed strategies*. Formally, G is *metric* if strategy sets are Hausdorff and metrizable and payoff functions are measurable. Denote by $M_i = \Delta(X_i)$ the set of Borel probability measures on X_i (usually called the set of mixed strategies of player i). This is a compact Hausdorff metrizable set under the weak* topology. Let $M = \Pi_i M_i$.

Definition 2.12 A *mixed Nash equilibrium* of G is a pure Nash equilibrium of its mixed extension $G' = ((M_i)_{i\in N}, (u_i)_{i\in N})$, where payoff functions are extended multi-linearly to M.

Definition 2.13 A pair (m, q) is a *mixed sharing rule solution* of G if $m \in M$ is a mixed Nash equilibrium of the auxiliary game $\widetilde{G} = ((X_i)_{i\in N}, (q_i)_{i\in N})$, where the auxiliary measurable payoff functions $q = (q_i)_{i\in N}$ must satisfy the condition:

$$\text{(SR)}: \forall x \in X,\ q(x) \in co\overline{\Gamma}_x,$$

where, $\overline{\Gamma}_x = \{v \in \mathbf{R}^N : (x,v) \in \overline{\Gamma}\}$ is the x-section of $\overline{\Gamma}$, and co stands for the convex hull.

Condition (SR) has two implications: if u is continuous at x, $q(x) = u(x)$; if $\sum_{i\in N} u_i(x)$ is continuous, then $\sum_{i\in N} q_i(x) = \sum_{i\in N} u_i(x)$ (justifying the terminology "sharing rule").

Theorem 2.14 (Simon and Zame (1990)) Every compact metric game admits a mixed sharing rule solution.

Jackson et al. (2002) extend Simon and Zame's theorem to games with incomplete information. In their paper, they interpret a tie-breaking rule as a proxy for the outcome of an unmodeled second stage game. As example, they recall the analysis of first-price auctions with incomplete information for a single indivisible object. Maskin and Riley (2000) add to the sealed-bid stage a second stage where bidders with the greatest bid in the first stage play a Vickrey auction. In the private value setting, their dominant strategy is to bid their true values. Consequently, the second stage induces a tie-breaking rule where the bidder with the highest value wins the object. More generally, a tie-breaking rule may be implemented by asking players to send a cheap message (their private values in auctions), in addition to their strategies (bids). The messages will be used only to break ties (as in the second stage of Maskin and Riley's mechanism).

When the game is continuous, the new and the original games coincide, and so we recover Nash-Glicksberg's theorem in mixed strategies.

THEOREM 2.15 (Nash-Glicksberg's Theorem in mixed strategies) Every continuous, metric compact game admits a mixed Nash equilibrium.

In the next section, we prove the existence of a pure sharing rule solution, defined now.

DEFINITION 2.16 A pair (x, q) is a *pure sharing rule solution* if $x \in X$ is a pure Nash equilibrium of the auxiliary game $\tilde{G} = ((X_i)_{i\in N}, (q_i)_{i\in N})$, where the auxiliary payoff functions $q = (q_i)_{i\in N}$ must satisfy the following condition:[7]

$$\text{(SR strong)}: \forall y \in X,\ q(y) \in \overline{\Gamma}_y,$$

If G is metric, condition (SR strong) requires that for every strategy profile y, there exists a sequence (y_n) converging to y such that $q(y) = \lim_n u(y_n)$. On one hand, our condition is stronger than the original condition (SR) because one always has $\overline{\Gamma}_y \subset co\overline{\Gamma}_y$. On the other hand, to prove the existence of a pure sharing rule solution, we need payoff functions to be quasiconcave.

To allow comparison between sharing rule solution and approximate equilibrium, we introduce the following terminology.

DEFINITION 2.17 A pair $(m, v) \in M \times \mathbf{R}^N$ (resp. $(x, v) \in X \times \mathbf{R}^N$) is called a mixed (resp. pure) sharing rule equilibrium if (m, q) is a mixed (resp. pure) sharing rule solution for some q and $q(m) = v$.

The proof of the existence of a pure sharing rule equilibrium is a direct consequence of the existence of a Reny solution, defined in the next subsection.

3. EXISTENCE OF SOLUTIONS FOR GAMES IN PURE STRATEGIES

As discussed above, sharing rule and approximate equilibrium concepts are alternative solutions for games without a Nash equilibrium. Both are defined on $\overline{\Gamma}$ (the closure of the graph of the game). To prove their existence, we use a new concept, Reny solution, defined on $\overline{\Gamma}$.

3.1. *Existence of a Reny solution*

In the following definition, recall that $\underline{V_i}(x_{-i}) := \sup_{d_i\in X_i} \underline{u_i}(d_i, x_{-i})$.

[7] If (m, q) is a mixed sharing rule solution and $m \in X$ is also a pure strategy, then (m, q) may not be a pure sharing rule solution (because q may not satisfy (SR strong)), but the converse implication is always true.

Definition 3.1 A pair $(x, v) \in \overline{\Gamma}$ is a *Reny solution* if for every $i \in N$, $\underline{V_i}(x_{-i}) \leq v_i$. The strategy profile $x \in X$ is a Reny solution profile if $(x, v) \in \overline{\Gamma}$ is a *Reny solution* for some v.

Example 3.2 *Two-player first-price auctions*

Both players $i = 1, 2$ choose a bid $x_i \in [0, 1]$, and receive a payoff:

$$u_i(x_i, x_j) = \begin{cases} w_i - x_i & \text{if } x_i > x_j, \\ \frac{w_i - x_i}{2} & \text{if } x_i = x_j, \\ 0 & \text{if } x_i < x_j. \end{cases}$$

If $w_1 \in]0, 1[$ (the value of player 1) is higher than $w_2 \in]0, 1[$ (the value of player 2), then the above game is quasiconcave, and every $(x_1, x_2, v_1, v_2) = (y, y, w_1 - y, 0)$ is a Reny solution whenever $y \in [w_2, w_1]$. To see this, note first that the game is payoff secure, thus a Reny solution $(x, v) = (x_1, x_2, v_1, v_2) \in \overline{\Gamma}$ satisfies

(1) $\sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \ i = 1, 2$

Since this game has no Nash equilibrium, x_1 is equal to x_2 (otherwise u_i would be continuous at $x = (x_1, x_2)$, and equation (1) would imply that x is a Nash equilibrium). Moreover, each player can get a payoff of at least 0 by playing 0. Consequently, v_1 and v_2 are non-negative. From $(x, v) \in \overline{\Gamma}$, $(v_1, v_2) = \lim_{n \to +\infty}(u_1(x^n), u_2(x^n))$ for some sequence of profiles $x^n = (x_1^n, x_2^n)$ converging to (x_1, x_1). There are three cases (up to a subsequence), depending on whether the sequence converges to x from above, along the diagonal, or from below. In the two first cases, we get $v = (0, w_2 - x_1)$ or $v = (\frac{w_1 - x_1}{2}, \frac{w_2 - x_1}{2})$, thus $x_2 = x_1 \leq w_2 < 1$ (since v_2 is non-negative). Then, playing slightly above x_1 gives a payoff strictly above v_1 for player 1, which contradicts equation (1). In the last case, $v = (w_1 - x_1, 0)$, thus $x_1 \leq w_1$. Then, equation (1) implies that $x_1 \geq w_2$ (otherwise player 2 could do better than 0 by playing slightly above x_1). Conversely, it is easy to check that $(y, y, w_1 - y, 0)$ is Reny solution when $y \in [w_2, w_1]$.

In this example, the set of Reny solutions coincides with the set of approximate equilibria (playing $y \in [w_2, w_1]$ for player 2 and slightly above for player 1 is an ε-equilibrium). Note that there are several Reny solutions and thus several approximate equilibria, but multiplicity also happens for Nash equilibria: in this example with a second-price auction mechanism, playing y for player 2 and w_1 for player 1 is a Nash equilibrium for all $y \in [0, w_1)$.

Theorem 3.3 For every quasiconcave and compact game G, the set of Reny solutions is nonempty and compact, and it contains the set of Nash equilibria. Moreover, G is better-reply secure if and only if Nash equilibrium profiles and Reny solution profiles coincide.

Observe that a Nash equilibrium $(x, u(x))$ is a Reny solution because $\underline{V_i}(x_{-i}) \leq V_i(x_{-i})$, and by Nash conditions, $V_i(x_{-i}) \leq u_i(x)$. Moreover, if the game is continuous, then Reny solutions and Nash equilibria coincide because $\underline{V_i}(x_{-i}) = V_i(x_{-i})$ and $u(x) = v$.

PROOF: The existence of a Reny solution is a straightforward consequence of Reny's theorem (Reny (1999)). Indeed, assume, by contradiction, that there is no Reny solution. This implies that the game is better-reply secure. Consequently, by Reny's theorem, there exists a Nash equilibrium, which is a Reny solution: a contradiction. Compactness of the set of Reny solutions is due to the lower semi-continuity of $\underline{V_i}$, and the last assertion of Theorem 3.3 is a consequence of the definition of better-reply security. Actually, if the game is better reply secure, a Reny solution profile must be a Nash equilibrium profile because otherwise, better reply security contradicts Reny solution condition. Conversely, if Nash equilibrium profiles and Reny solution profiles coincide, then for every $x \in X$ that is not a Nash equilibrium profile, x not a Reny solution profile, thus better-reply security condition is satisfied. *Q.E.D.*

As an illustration, we can revisit the result of Reny in Theorem 2.6 and prove that whenever a game is payoff secure and reciprocally upper semicontinuous, it is better-reply secure. Actually, assume (x, v) to be a Reny solution. Thus, for every $i \in N$, $\underline{V_i}(x_{-i}) \leq v_i$. Since the game is payoff secure, $V_i(x_{-i}) = \underline{V_i}(x_{-i}) \leq v_i$. Since $u_i(x) \leq V_i(x_{-i})$, one has $u_i(x) \leq v_i$ for every $i \in N$. By reciprocal upper semicontinuity, $v = u(x)$, and so $V_i(x_{-i}) \leq u_i(x)$ for every $i \in N$. Consequently, (x, v) is a Nash equilibrium.

Two major applications of Reny solution are presented in the next subsections. In Subsection 3.2., Reny solution allows to prove the existence of a pure sharing rule equilibrium in every quasiconcave compact game. In Subsection 3.3., Reny solution is used to prove the existence of approximate equilibria in a number of economic models.

3.2. *Existence of a Sharing Rule Equilibrium*

The existence of a Reny solution allows to solve the open problem in Jackson et al. (2002).

THEOREM 3.4 Every Reny solution is a pure sharing rule equilibrium. In particular, every quasiconcave and compact game G admits a pure sharing rule solution.

REMARK 3.5 Observe that a pure sharing rule solution (m, q') of G' (the mixed extension of G) is not a mixed sharing rule solution à la Simon Zame of G because the new payoff profile q' is defined on M and not on X, and q' is not necessarily the multilinear extension of a pure strategy payoff profile. Thus, our result does not imply Simon-Zame's theorem.

PROOF: To prove Theorem 3.4, consider a Reny solution $(x, v) \in \overline{\Gamma}$. Then, we can build a sharing rule solution as follows. For every $i \in N$ and $d_i \in X_i$, denote by $\underline{\mathcal{S}}(d_i, x_{-i})$ the space of sequences[8] $(x_{-i}^n)_{n\in\mathbf{N}}$ of X_{-i} converging to x_{-i} such that $\lim_{n\to+\infty} u_i(d_i, x_{-i}^n) = \underline{u}_i(d_i, x_{-i})$. Then, define $q : X \to \mathbf{R}^N$ by

$$q(y) = \begin{cases} v & \text{if } y = x, \\ \text{any limit point of } (u(d_i, x_{-i}^n))_{n\in\mathbf{N}} & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N, d_i \neq x_i, (x_{-i}^n)_{n\in\mathbf{N}} \in \underline{\mathcal{S}}(d_i, x_{-i}), \\ q(y) = u(y) & \text{otherwise.} \end{cases}$$

Now, let us prove that (x, q) is a pure sharing rule solution. Since $(x, v) \in \overline{\Gamma}$, and by definition of q, condition (SR strong) of Definition 2.16 is satisfied at x (this would be false if the definition of q in the second case above was $q_j(d_i, x_{-i}) = \underline{u}_j(d_i, x_{-i})$, $j = 1, ..., n$). Obviously, it is satisfied at every y different from x for at least two components, and also at every (d_i, x_{-i}) with $d_i \neq x_i$, from the definition of $q(d_i, x_{-i})$ in this case. *Q.E.D.*

Remark 3.6 Thus, Reny solution refines pure sharing rule equilibrium, and the refinement is strict as the following better-reply secure game shows. A player maximizes over $[0, 1]$ the following discontinuous payoff function: $u(x) = 0$ if $x < 1$, and $u(1) = 1$. If $q(y) = 0$ for every y, then (x, q) is a pure sharing rule solution for every $x \in [0, 1]$. Yet, the only Reny solution is $(x, v) = (1, 1)$, and it coincides with the unique Nash equilibrium.

Remark 3.7 An important question, raised by De Castro (2011) and Carmona and Podczeck (2014), is whether for every game G one can define a new payoff function $q(y) \in co\overline{\Gamma}(y)$ inducing a better-reply secure game. The mapping q defined in the proof of Theorem 3.4 does not answer the question: if $(x, v) \in \overline{\Gamma}$ is the Reny solution considered in this proof, then q is equal to the initial payoff mapping u at every $y \in X$ different from x for at least two components. Since u is arbitrarily discontinuous, there is no hope, in general, that q defines a better-reply secure game.

De Castro (2011) proposes a first answer: he introduces a regularity property, a weakening of better-reply security, as follows: a game is regular if for every $(x, v) \in \overline{\Gamma}$, if $v_i \geq V_i(x_{-i})$ for all $i \in N$ then $u_i(x) = V_i(x_{-i})$ for all $i \in N$. Then he proves that every discontinuous game has a measurable selection $q(y) \in co\overline{\Gamma}(y)$ which induces a regular game. This result cannot be applied to prove Theorem 3.4 because a regular game may have no Nash equilibrium.

Another approach is given by Carmona and Podczeck (2014) throughout the concept of virtual continuity. However, they consider the case of better-reply security *in mixed strategies*. Consequently, this is not directly related to Theorem 3.4.

Remark 3.8 Actually, the pure sharing rule solution (x, q) built in the proof of Theorem 3.4 satisfies the additional property:[9] $q_i(d_i, x_{-i}) \geq \underline{u_i}(d_i, x_{-i})$ for every

[8] If X is not first countable, one should consider the space of nets instead of sequences.

[9] Indeed, if $d_i = x_i$, then $q_i(d_i, x_{-i}) = q_i(x) = v_i \geq \underline{u_i}(x)$ because (x, v) is a Reny solution. If $d_i \neq x_i$ then $q_i(d_i, x_{-i}) = \underline{u}_i(d_i, x_{-i})$.

$i \in N$ and every $d_i \in X_i$. This property says that q remains above the secure payoff level in the original game.

Application 3.9 *Shared Resource Games*

The payoff of each player $i \in N$ can be written as $u_i(x_i, x_{-i}) = F_i(x_i, S_i(x_i, x_{-i}))$, where $F_i : X_i \times \mathbf{R} \to \mathbf{R}$ and $S_i : X \to \mathbf{R}$ (the shared resource of player i). The total amount of the resource $\sum_{i=1}^N S_i$ is a (possibly discontinuous) function of the strategy profile $x \in X$. This game G was introduced to model fiscal competition for mobile capital (Rothstein (2007)).

A sharing rule of G is defined to be a family $(\tilde{S}_i)_{i\in N}$ of functions from X to $\mathbf{R}$ such that for every strategy profile $x \in X$, there is a sequence (x_n) converging to x such that for every player i, $\tilde{S}_i(x) = \lim_{n\to\infty} S_i(x_n)$. Theorem 3.4 implies the following extension of Rothstein's results. Assuming G to be quasiconcave and compact, F_i continuous and S_i bounded for every player i, we get the existence of a new sharing rule $(\tilde{S}_i)_{i\in N}$ whose associated game $\tilde{u}_i(x_i, x_{-i}) = F_i(x_i, \tilde{S}_i(x_i, x_{-i}))$ admits a pure Nash equilibrium. Moreover, under the following assumptions, every Nash equilibrium of $\tilde{G}$ is a Nash equilibrium[10] of G:

A1) For all $x_i \in X_i$, $F_i(x_i, s_i)$ is nondecreasing in s_i.

A2) For all $x_{-i} \in X_{-i}$, $\sup\limits_{d_i \in X_i : (d_i, x_{-i}) \in \mathcal{C}_i} u_i(d_i, x_{-i}) = \sup\limits_{d_i \in X_i} u_i(d_i, x_{-i})$, where $\mathcal{C}_i$ is the set of continuity points of S_i.

A3) If $x \notin \cap_{i\in N}\mathcal{C}_i$, $\sup\limits_{d_i \in X_i} u_i(d_i, x_{-i}) > F_i\left(x_i, \dfrac{\limsup_{x'\to x} \sum_{i=1}^N S_i(x')}{N}\right)$ for every $i \in N$.

In his paper, Rothstein says "the work of Simon and Zame (1990) is directly applicable, but only to establish the existence of a Nash equilibrium in mixed strategies with an endogenous sharing rule." By Theorem 3.4, the game has a pure endogenous sharing rule equilibrium.

3.3. *Existence of an Approximate Equilibrium*

Theorem 3.10 Every approximate equilibrium is a Reny solution, and, therefore, also a pure sharing rule equilibrium.

Proof: The proof is as follows. Let $(x^n)_{n\in\mathrm{N}}$ be a sequence of ε_n-equilibria such that $(x^n, u(x^n))$ converges to (x, v). By definition, $u_i(d_i, x^n_{-i}) \leq u_i(x^n) + \varepsilon_n$ for every $n \in \mathrm{N}$, every player $i \in N$ and every deviation $d_i \in X_i$. Passing to the infimum limit when n tends to infinity, we obtain $\underline{u_i}(d_i, x_{-i}) \leq v_i$. Thus, (x, v)

[10] Indeed, for every Nash equilibrium x of $\tilde{G}$, one has $\sup_{d_i\in X_i} u_i(d_i, x_{-i}) \leq \sup_{d_i\in X_i} F_i(d_i, \tilde{S}_i(d_i, x_{-i})) \leq F_i(x_i, \tilde{S}_i(x))$ for every player i. The first inequality is a consequence of A2 and the definition of $\tilde{S}_i$. If $x \in \mathcal{C}_i$ for every i, then $\tilde{S}_i(x) = S_i(x)$, and x is a Nash equilibrium of the initial game G. Otherwise, from A1 and A3, we get $\frac{\limsup_{x'\to x} \sum_{i=1}^N S_i(x')}{N} < \tilde{S}_i(x)$ for every $i \in N$. Summing these inequalities contradicts the definition of $\tilde{S}_i$.

is a Reny solution, and also a pure sharing rule equilibrium by Theorem 3.4. *Q.E.D.*

This leads to the following definition.

DEFINITION 3.11 A game G is *approximately better-reply secure* if whenever $(x, v) \in \overline{\Gamma}$ and x is not an approximate equilibrium profile, some player i can secure a payoff strictly above v_i.

The existence of a Reny solution implies the following result.

THEOREM 3.12 Every approximately better-reply secure quasiconcave and compact game admits an approximate equilibrium.

California location game is approximately better-reply secure. This theorem provides a local version of Reny-Prokopovych's theorem (described in Subsection 2.2).

COROLLARY 3.13 If (x, v) is a Reny solution and if, at x, the game is payoff secure and has the marginal continuity property, then (x, v) is an approximate equilibrium.

PROOF: Actually, if (x, v) is a Reny solution, then

$$\sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) = \sup_{d_i \in X_i} u_i(d_i, x_{-i}) \leq v_i,$$

the equality being a consequence of payoff security at x. Since $v = \lim_{x^n \to x} u(x^n)$ for some sequence x^n, the local continuity of $\sup_{d_i \in X_i} u_i(d_i, x_{-i})$ with respect to x guarantees that (x, v) is an approximate equilibrium. *Q.E.D.*

This corollary implies Reny-Prokopovych's theorem, and is useful in practice as the following application shows.

APPLICATION 3.14 *Diagonal Games.*

For every $i \in N$, we let f_i, g_i be *continuous* mappings from $[0,1] \times [0,1]$ to $\mathbf{R}$, and $h_i : [0,1]^N \to \mathbf{R}$ be a bounded mapping. The payoff of player i is:

$$u_i(x_i, x_{-i}) = \begin{cases} f_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) > x_i, \\ g_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) < x_i, \\ h_i(x_i, x_{-i}) & \text{if } \phi(x_{-i}) = x_i, \end{cases}$$

where $\phi : [0,1]^{N-1} \to [0,1]$ is a continuous aggregation function, i.e. a function that aggregates the $N-1$ strategies of the opponents into one strategy in $[0,1]$. This aggregation function must satisfy[11] for every $(y, z) \in [0,1]^{N-1} \times [0,1]^{N-1}$ and $t \in [0,1]$:

[11] For every $(x, y) \in [0,1]^{N-1} \times [0,1]^{N-1}$, $x << y$ (resp. $x \leq y$) means $x_i < y_i$ (resp. $x_i \leq y_i$) for every $i \in \{1, ..., N-1\}$.

<u>*Monotonicity:*</u> if $y \leq z$ then $\phi(y) \leq \phi(z)$; if $y << z$ then $\phi(y) < \phi(z)$.

<u>*Anonymity:*</u> for every permutation σ of $\{1, ..., N-1\}$, we have $\phi(y_1, ..., y_{N-1}) = \phi(y_{\sigma(1)}, ..., y_{\sigma(N-1)})$.

<u>*Unanimity:*</u> $\phi(t, ..., t) = t$.

<u>*Representativity:*</u>[12] if $\phi(y) > 0$ and $y_i > 0$ for some i, then $\phi(z_i, y_{-i}) > 0$ for every $z_i > 0$. Similarly, if $\phi(y) < 1$ and $y_i < 1$ for some i, then $\phi(z_i, y_{-i}) < 1$ for every $z_i < 1$.

These four properties are satisfied, for example, when ϕ is one of the following functions:

- $\phi_1(y) = \max\{y_1, y_2, ..., y_{N-1}\}$,
- $\phi_2(y) = \min\{y_1, y_2, ..., y_{N-1}\}$,
- $\phi_3(y) = \dfrac{1}{N-1} \sum_{j=1}^{N-1} y_j$,
- $\psi_k(y) = \{k\text{-th highest value of } \{y_1, ..., y_{N-1}\}\}$, $k = 1, ..., N-1$.

Diagonal games include many models of competition with complete information. For example, in one-unit auctions, $\phi = \phi_1$, in k-unit auctions $\phi = \psi_k$, in wars of attrition, preemption or Bertrand competition, $\phi = \phi_2$.

THEOREM 3.15 Every quasiconcave diagonal game satisfying condition (C) below is approximately better-reply secure, thus it possesses an approximate equilibrium.

(C) there is $\alpha \in]0, \frac{1}{2}[$ such that for every $x \in [0,1]^N$ and $i \in N$, if $x_i = \phi(x_{-i})$ then there is $\alpha_i(x) \in]\alpha, 1-\alpha[$ such that
$$h_i(x) = \alpha_i(x) f_i(x_i, \phi(x_{-i})) + (1 - \alpha_i(x)) g_i(x_i, \phi(x_{-i})).$$

Condition (C) means that h_i is a strict convex combination of g_i and f_i with weights that are bounded below. In auctions, for example, the winner is usually decided uniformly among highest bidders, thus the payoff of a player in case of ties is a strict convex combination between his payoff if he wins, g_i, and if he looses, f_i. The coefficient of the convex combination depends on how many players are tied, inducing a discontinuity on h_i. The probability of being selected or not selected is bounded below by $\frac{1}{N} = \alpha$.

PROOF: Hereafter, we give a sketch of the proof. See appendix 6.1 for a detailed proof.

Under Assumption (C), the game is payoff secure. Consequently, if $(x, v) \in \overline{\Gamma}$ is a Reny solution then

$$\sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \; i \in N.$$

To prove that x is an approximate equilibrium profile, one has to check four different cases: first, if $x_i \neq \phi(x_{-i})$ for every i, then the payoff functions are

[12] We denote $(z_i, y_{-i}) := (y_1, ...y_{i-1}, z_i, y_{i+1}, ..., y_{N-1})$.

continuous at x, $v = u(x)$, and the equation above defining Reny solution implies that (x, v) is a Nash equilibrium. Second, if there exists i such that $x_i = \phi(x_{-i}) \in]0, 1[$, then anonymity, representativity, and monotonicity give $\phi(x_{-j}) \in]0, 1[$ for every j. Then, the marginal continuity property is satisfied at x, and from Corollary 3.13, (x, v) is an approximate equilibrium. Third, assume there is a player i such that $x_i = \phi(x_{-i}) = 0$. By anonymity and monotonicity, $\phi(x_{-j}) = 0$ for every $j \in N$. Let $(x^n)_{n\in \mathrm{N}}$ be a sequence such that $(x^n, u(x^n)) \to (x, v)$. Define a sequence of profiles $(y^n)_{n\in \mathrm{N}}$ as follows: we let $j \in N$ be any player; if $v_j \leq f_j(0, 0)$ and $x_j = 0$, define $y^n_j := 0$ for every n. Otherwise, define $y^n_j := x^n_j$ for every n. We check that y^n is an ε^n-equilibrium for some $\varepsilon^n \to 0$. In the last case, there is a player $i \in N$ such that $x_i = \phi(x_{-i}) = 1$: this is similar to the third case. *Q.E.D.*

4. EXISTENCE OF SOLUTIONS FOR GAMES IN MIXED STRATEGIES

We let G be a metric compact game, G' its mixed extension and $\overline{\Gamma'}$ the closure of the graph of G'. As previously defined, $M_i = \Delta(X_i)$ is the set of mixed strategies of player i. This is a compact Hausdorff metrizable set under the weak* topology. Let $M = \Pi_i M_i$.

A Reny solution (resp. an approximate equilibrium) of G' is called a mixed Reny solution (resp. approximate mixed equilibrium) of G. A mixed Reny solution always exists from Theorem 3.3 because G' is compact and quasiconcave.

In the next subsection, we establish a formal link between the set of Reny solutions of G' and the set of mixed sharing rule equilibria (as defined by Simon and Zame originally, see Definition 2.13 and Definition 2.17 above) by proving that the intersection of these sets is nonempty. Then, we prove that if G' is approximately better-reply secure, some approximate equilibria can be obtained as limits of Nash equilibria of finite discretizations of the initial game. The approximation methodology is illustrated in auctions with correlated types or values.

4.1. *Linking Approximate and Simon-Zame Equilibria with Reny solutions*

The intersection of mixed Reny solutions and mixed sharing rule equilibria permits to localize mixed approximate equilibria, as proves the following theorem (see the proof in Appendix 6.2).

THEOREM 4.1 Mixed approximate equilibria are always in the intersection of mixed Reny solutions and mixed sharing rule equilibria.[13]

Importantly, this intersection is always nonempty, which is a consequence of a simple limit argument we now explain. We let $\mathcal{D}_0$ be the set of all finite sub-

[13]From Section 3, this is also true in pure strategies. Nevertheless, it cannot be directly proved applying Section 3 to G', simply because a pure sharing rule equilibrium of G' may not be a mixed sharing rule equilibrium of G.

sets $\Pi_{i\in N}D_i$ of M. Consider the inclusion relationship on $\mathcal{D}_0$: it is reflexive, transitive and binary. Then, each pair $\Pi_{i\in N}D_i$ and $\Pi_{i\in N}D'_i$ in $\mathcal{D}_0$ has an upper bound $\Pi_{i\in N}(D_i \cup D'_i)$ in $\mathcal{D}_0$. The pair $(\mathcal{D}_0, \subset)$ is called a directed set. To every $D = \Pi_{i\in N}D_i \in \mathcal{D}_0$, we can associate $(m^D, u(m^D))$, where m^D is a mixed Nash equilibrium of the finite game restricted to D. This defines a mapping[14] from $\mathcal{D}_0$ to $\overline{\Gamma'}$, called a net (of $\overline{\Gamma'}$). A limit point $(m, v) \in \overline{\Gamma'}$ of this net, denoted $(m^D, u(m^D))_{D\in\mathcal{D}_0}$, is defined by the following property: for every neighborhood $V_{m,v}$ of (m, v) and every $D = \Pi_{i\in N}D_i \in \mathcal{D}_0$, there exists $D' \in \mathcal{D}_0$ with $D \subset D'$ such that $(m^{D'}, u(m^{D'})) \in V_{m,v}$.

Definition 4.2 A pair $(m, v) \in \overline{\Gamma'}$ is a limit-equilibrium of G' if it is a limit point
of a net $(m^D, u(m^D))_{D\in\mathcal{D}_0}$ of mixed Nash equilibria of the finite game restricted to D.

Theorem 4.3 Every compact metric game has a limit-equilibrium. Every limit-equilibrium is a mixed Reny solution and is a mixed sharing rule equilibrium. Consequently, the intersection between mixed Reny solutions and mixed sharing rule equilibria is nonempty, and if G' is better-reply secure then every limit-equilibrium is a mixed Nash equilibrium.

Existence of a limit-equilibrium is obvious and is a consequence of the compactness of $\overline{\Gamma'}$. The rest of the proof is presented in the appendix and is an adaptation of the arguments of Simon and Zame.

Remark 4.4 Observe that this provides a very short and constructive proof of Reny's existence result for games in mixed strategies (Reny (1999)) because if G' is better-reply secure, every limit-equilibrium is a mixed Nash equilibrium.

4.2. *Weak Strategic Approximation*

The idea of using a sequence of finite games to detect Nash equilibria goes back to Dasgupta and Maskin (1986b). This has been formalized by Reny (2011) in the class of better-reply secure games via the notion of strategic approximation. We can extend this method to approximately better-reply secure games.

Definition 4.5 A game G admits a weak strategic approximation if there is a sequence of finite sets $D_n \subset M$ such that all accumulation points of mixed Nash equilibria
of the game restricted to D_n are approximate equilibria of G'.

[14] By definition, $m^D = (m_i^D)_{i\in N}$ is an element of $\Pi_{i\in N}\Delta(\Delta(X_i))$. More precisely, it is a profile of probability measures on finite subsets of $\Delta(X_i)$, where $i \in N$. Given $i \in N$, let $\{\sigma_1, ..., \sigma_K\}$ be the support of m_i^D and $p_1, ..., p_K$ the associated weights. By abuse of notation, we can define $m_i^D = \sum_{k=1}^K p_k.\sigma_k$, which is now an element of $\Delta(X_i)$. Up to this identification, $(m^D, u(m^D))$ can be seen as an element of $\overline{\Gamma'}$.

Theorem 4.6 If a compact metric game G' is approximately better-reply secure, it has a weak strategic approximation. Moreover, if a compact metric game admits a weak strategic approximation, it has an approximate equilibrium.

The proof of the first part (in the appendix) is an adaptation of Reny's arguments (Reny (2011)), thanks to the notion of limit-equilibrium. The second part is straightforward, from the existence of accumulation points for every sequence in a compact set.

Remark 4.7 A condition that guarantees the existence of a weak strategic approximation was given by Blackwell and Girshick (1954) (see Theorem 2.3.3) in zero-sum 2-player games $G = (X_1, X_2, u_1, u_2)$. It could be stated as follows: for every $\epsilon > 0$, there exists a finite set $D_1 \subset X_1$ such that, for every $x_1 \in X_1$, there exists a mixture σ_1 of elements of D_1 such that $u_1(\sigma_1, x_2) \geq u_1(x_1, x_2) - \varepsilon$ for every $x_2 \in X_2$. Assume this is true. From boundedness of payoffs, there exists a finite subset $D_2 \subset X_2$ such that for every $x_2 \in X_2$, there is $d_2 \in D_2$ with: $\forall d_1 \in D_1, \mid u_2(d_1, x_2) - u_2(d_1, d_2) \mid \leq \varepsilon$. Then every mixed Nash equilibrium of the restriction of G to $D_1 \times D_2$ is an $\varepsilon-$Nash equilibrium of G (see Blackwell and Girshick (1954) for more details). Thus, this game admits a weak strategic approximation.

The following diagonal game does not satisfy the assumption of Blackwell and Girshick, but satisfies the assumption of Theorem 4.8 below (and thus admits a weak strategic approximation): consider $X_1 = X_2 = [-1, 1]$, $u_1(x_1, x_2) = 0$ if $x_1 \neq x_2$, $u_1(x_1, x_2) = x_1$ otherwise, and $u_2 = -u_1$. If there exists a finite subset $D_1 \subset X_1$ satisfying Blackwell and Girshick condition, taking $x_2 \in (0, 1]$ which is not in D_1 and $x_1 = x_2$, we get a contradiction for $\varepsilon > 0$ small enough.

4.3. *Applications*

Non Quasiconcave Two Player Diagonal Games

Theorem 4.8 Every two-player diagonal game in which h is continuous admits a weak strategic approximation (and thus an approximate equilibrium in mixed strategies).

The proof of theorem 4.8 consists of constructing a weak strategic approximation (see Appendix 6.5). Interestingly, the approximation is endogenous (i.e. game dependent). The multi-player case is investigated in the next application.

Example 4.9 Sion-Wolfe's zero-sum game shows that Theorem 4.8 can be false for games with two lines of discontinuities instead of one (see Sion and Wolfe (1957)).

Some particular cases covered by Theorem 4.8 follow.

Example 4.10 *Bertrand Duopoly with Discontinuous Costs*

Hoernig (2007) introduced the following modification of Bertrand's game: each firm $i = 1, 2$ chooses a price $p_i \in [0, 1]$; the demand is $D(p_1, p_2) = \max\{0, 1 - \min(p_1, p_2)\}$; the total (symmetric) cost for each firm is $C(q) = \tilde{C} \in (0, \frac{1}{4})$ if the production q is positive, and $C(0) = 0$ otherwise. Assuming equal sharing at ties, Hoernig (2007) proved that the game has no mixed Nash equilibrium. By Theorem 4.8, it has an approximate equilibrium.

Example 4.11 *Bertrand-Edgeworth Duopoly with Capacity constraints*

There are two firms. Firm i has an endowment of C_i units of the commodity (the capacity of a zero-cost technology). Firms choose prices (p_1 and p_2). The firm choosing the lowest price (say p) serves the entire market $D(p)$ up to its capacity. The residual demand $D(p) - C_i$ is met by the other firm (up to its capacity as well). If the duopolists set the same price they share the market according to some rule h. If h shares the market in proportion to the capacities, Dasgupta and Maskin (1986a) proved the existence of a mixed equilibrium. Theorem 4.8 proves the existence of an approximate equilibrium for every continuous h.

Bayesian Diagonal Games and Auctions with Correlated Types

In many economic models, such as auctions, players do not have full knowledge about other player's evaluations. This leads naturally to the following class of Bayesian diagonal games. At stage 0, a type $t = (t_1, ..., t_N) \in T = T_1 \times ... \times T_N$ is drawn according to some joint probability distribution p, and each player i is informed of his own type t_i (correlations between types are allowed). At stage 1, each player i is asked to choose an element $x_i \in [0, 1]$ (interpreted as a bid). The payoff of player i is:

$$u_i(t, x_i, x_{-i}) = \begin{cases} f_i(t, x_i, \phi_i(x_{-i})) & \text{if } \phi_i(x_{-i}) > x_i, \\ g_i(t, x_i, \phi_i(x_{-i})) & \text{if } \phi_i(x_{-i}) < x_i, \\ h_i(t, x) & \text{if } \phi_i(x_{-i}) = x_i, \end{cases}$$

where $f_i(t, \cdot, \cdot)$ and $g_i(t, \cdot, \cdot)$ are two continuous mappings on $[0, 1] \times [0, 1]$, and $\phi_i : \mathbf{R}^{N-1} \to [0, 1]$ is a monotone aggregation function (see application 3.14). The mapping h is the tie-breaking rule and may be discontinuous (but is measurable). Actually, in first price auctions for example, the tie-breaking rule may depend on how many players offer the highest bid. That's why in our definition, h_i may be discontinuous and have more arguments than f_i and g_i. The game is called a game of private values if for every i, u_i depends only on its own type t_i and does not depend on t_{-i}.

To avoid additional notations and measurability issues, we assume the type space T to be finite. This assumption could be relaxed (see Remark 6.2 in Appendix 6.6).

Theorem 4.12 Every Bayesian diagonal game admits a weak strategic approximation (and so a mixed approximate equilibrium)[15] if for every $i \in N$ and $t \in T$ one has:
(a) $f_i(t,0,0) \leq h_i(t,0,..,0) \leq g_i(t,0,0)$;
(b1) $f_i(t,1,1) \geq h_i(t,1,...,1) \geq g_i(t,1,1)$ **or**[16] (b2) there is $\eta > 0$ such that there is always a best response of each type in $[0, 1-\eta[$;
(c1) there are only two players **or** (c2) values are private.

Example 4.13 *One unit first-pay, Second-pay and All-Pay Auctions*

Take any N-player auction where the winner is the player with maximal bid. More precisely, suppose that player i's value for the object is $v_i(t) \in]0,1[$. If $x_i > \bar{x}_{-i} := \max_{j \neq i} x_j$, i wins the auction and pays a price $p_i(x_i, \bar{x}_{-i}) \geq 0$. His final payoff is then $g_i(t, x_i, \bar{x}_{-i}) = v_i(t) - p_i(x_i, \bar{x}_{-i})$ where p_i is continuous, non decreasing in both variables and $p_i(y,y) = y$ for every y. If $x_i < \bar{x}_{-i}$, player i looses the auction, and pays a transfer $\tau_i(x_i) \geq 0$. His payoff is then $f_i(t, x_i, \bar{x}_{-i}) = -\tau_i(x_i)$ where $\tau_i(x_i)$ is continuous, non decreasing and $\tau_i(0) = 0$. In case of a tie ($x_i = \bar{x}_{-i}$), the winner is selected uniformly among the set of players with maximum bid. For example, in first-price and second-price auctions, $\tau_i = 0$. In all-pay auctions, $\tau_i = -x_i$. In first-pay and all-pay auctions, $p_i(x_i, \bar{x}_{-i}) = \max\{x_i, \bar{x}_{-i}\}$, in second-price auctions, $p_i(x_i, \bar{x}_{-i}) = \bar{x}_{-i}$. In this general model, $0 = f_i(t,0,0) < g_i(t,0,0) = v_i(t)$ and $0 \leq h_i(t,x) \leq v_i(t)$, so that (a) is satisfied. Condition (b1) is satisfied in first-price and second-price auctions, but not in all-pay auctions. However, Condition (b2) is satisfied in these three types of auctions because player i has always a best response in $[0, max_t v_i(t) + \varepsilon[$. Thus Theorem 4.12 applies when there are two players or values are private.

Example 4.14 *Multi-unit Auctions*

Consider the previous model with the following modification: K homogeneous units of an indivisible good are sold, but each bidder $i = 1, ..., N$ ($N \geq K$) can buy only one unit of the good. Player i wins if his bids is among the K highest bids. In case of a tie, the remaining winners are chosen at random among the tie-players. Theorem 4.12 applies, and here $\phi(x_1, ..., x_{N-1})$ is simply the K-th highest of $x_1, ..., x_{N-1}$.

Example 4.15 *Double Auction*

Suppose player 1 is a buyer with a value $v(t_1, t_2) \in]0,1[$ and player 2 is a seller with a cost $c(t_1, t_2) \in]0,1[$. Player 1 chooses a maximum bid $x_1 \in [0,1]$ and player 2 a minimum price x_2. If $x_1 < x_2$, there is no trade (so that $f_1(t, x_1, x_2) =$

[15] Importantly, for every $\varepsilon > 0$, the ε-equilibria we build in the proof are tie-breaking rule independent.

[16] Conditions b1 and b2 are boundary conditions at 1. Condition b1 is satisfied by first-price, second-price, multi-unit and double auctions, but not for all-pay auctions. Condition b2 is true for all-pay, first-price, second-price and multi-unit auctions, but not for double auctions (see Examples 4.13, 4.14 and 4.15)

$g_2(t, x_1, x_2) = 0$). If $x_1 \geq x_2$, there is a trade at price $p = \frac{x_1+x_2}{2}$, so that $h_1(t, x_1, x_2) = g_1(t, x_1, x_2) = v(t_1, t_2) - \frac{x_1+x_2}{2}$ and $h_2(t, x_1, x_2) = f_2(t, x_1, x_2) = \frac{x_1+x_2}{2} - c(t_1, t_2)$. Consequently, $f_1(t, 0, 0) = 0 < h_1(t, 0, 0) = g_1(t, 0, 0) = v(t)$ and $f_2(t, 0, 0) = -c(t) = h_2(t, 0, 0) < g_2(t, 0, 0) = 0$: condition (a) is satisfied. Condition (b1) is satisfied similarly, but Condition (b2) is not satisfied for the seller. Since the game has only two players, Theorem 4.12 applies.

An open question is whether Theorem 4.12 holds for N-player diagonal games without condition (c2). But without conditions (a) or (b), approximate equilibria may not exist as the following zero-sum example shows.

Example 4.16 Consider a zero-sum timing game, viewed as a diagonal game with constant payoff functions f, g and h. Each player should decide when to stop the game between 0 and 1. The game stops at the first moment when one of the two players stops. If both players stop simultaneously before the exit time $t = 1$ or no player stops before time $t = 1$, then there is a tie (payoff is given by h). Player 2 has two types A and B with equal probabilities. Player 1 has only one type. If player 1 stops first, he gets $f = 1$. If player 1 stops second he gets $g = -1$. The payoffs depend on the type of player 2 only when the players stop simultaneously. If his type is A, player 1 has an advantage and gets the payoff $h = 3$, and if his type is B, player 1 has a disadvantage and gets the payoff $h = -2$. We can prove that $\max\min \leq -\frac{1}{2}$ and that $\min\max \geq -\frac{1}{4}$, so that the game has no value and so no approximate equilibrium (See Appendix 6.7).

Remark 4.17 Theorem 4.12 is to be compared with the one in Jackson and Swinkels (2005). They show existence of a Nash equilibrium which is tie-breaking-rule independent in multi-unit auctions with private values and uncorrelated types. Recalling that when types are correlated, a Nash equilibrium may not exist (Fang and Morris (2006)), the existence of an approximate equilibrium is the best one can hope.

Remark 4.18 Existence of Nash equilibria for Bayesian games is well understood for Bayesian games with continuous payoffs (Balder (1988), Milgrom and Weber (1985)). Recently, it has been extended to discontinuous payoffs under a uniform payoff security assumption (e.g., Carbonell-Nicolau and McLean (2015)), with application to auctions.

5. CONCLUSION

Our paper proposes a unifying framework to study the existence of approximate and sharing rule equilibria in discontinuous games, which links Simon-Zame and Reny approaches in pure and mixed strategies.

In the first part, we focus on quasiconcave compact games in *pure strategies*. A new relaxation of Nash equilibrium notion (Reny solution) is shown to always exist. It provides tight conditions, in the spirit of Reny's conditions, that guarantee the existence of an approximate equilibrium. Reny solution is also used to solve an open problem in Jackson et al. (2002), namely the existence of a sharing rule equilibrium in pure strategies (up to now, existence was known only for games in mixed strategies). As applications, we prove the existence of sharing rule equilibria in many economic models with discontinuous preferences and approximate equilibria in a large class of diagonal games.

In the second part, we concentrate on metric compact games in *mixed strategies*. We prove that the intersection between the sets of Simon-Zame and Reny solutions contains the set of approximate equilibria. Moreover, this intersection is nonempty. This shows that the three main solution concepts discussed in this paper are strongly connected. As application, we prove the existence of an approximate equilibrium in a large class of auctions with interdependent types and values.

6. APPENDIX

6.1. *Proof of Theorem 3.15*

Under Assumption (C), the game is payoff secure: indeed, if $x_i \neq \phi(x_{-i})$, u_i is continuous at x, thus x_i is secure for player i. If $x_i = \phi(x_{-i})$, then player i can secure his payoff (up to an arbitrary $\varepsilon > 0$), increasing or decreasing x_i slightly, or keeping it constant. Consequently, if $(x, v) \in \overline{\Gamma}$ is a Reny solution of the game, then:

$$(2) \qquad \sup_{d_i \in [0,1]} \underline{u_i}(d_i, x_{-i}) = \sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \ \forall i \in N.$$

Now, we prove that x is an approximate equilibrium profile by checking 4 different cases. In the *first case*, assume for every player $i \in N$, $x_i \neq \phi(x_{-i})$. Thus, payoffs are continuous at x and $v = u(x)$. From equation (2), $(x, v) = (x, u(x))$ is a Nash equilibrium. In the *second case*, there exists a player i such that $x_i = \phi(x_{-i}) \in]0, 1[$. Then $\phi(x_{-j}) \in]0, 1[$ for every j. Indeed, for every $j \neq i$, either $x_j \geq x_i > 0$, and anonymity and representativity imply that $\phi(x_{-j}) > 0$ and monotonicity that $\phi(x_{-j}) \leq \phi(x_{-i}) < 1$, or $x_j \leq x_i < 1$, and we get similarly $\phi(x_{-j}) \in]0, 1[$. Thus, the marginal continuity property is satisfied at x, since for every player $j \in N$, we have

$$\sup_{d_j \in [0,1]} u_j(d_j, x_{-j}) = \max\{\sup_{d_j < \phi(x_{-j})} f_j(d_j, \phi(x_{-j})), \sup_{d_j > \phi(x_{-j})} g_j(d_j, \phi(x_{-j}))\},$$

from assumption (C), and from continuity of ϕ, f_j and g_j. Thus, Corollary 3.13 implies that (x, v) is an approximate equilibrium. In the *third case*, there exists a player i such that $x_i = \phi(x_{-i}) = 0$. Then $\phi(x_{-j}) = 0$ for every player j: indeed, anonymity gives $\phi(x_{-j}) = 0$ for every j such that $x_j = 0$, and monotonicity gives $\phi(x_{-j}) = 0$ for every j such that $x_j > 0$. Now, let $(x^n)_{n \in \mathrm{N}}$ be a sequence of profiles such that $(x^n, u(x^n)) \to (x, v)$. For every player j such that $x_j = 0$ and $v_j \leq f_j(0, 0)$, we let $y_j^n := 0$ for every integer n, and $(y_j^n)_{n \in \mathrm{N}} := (x_j^n)_{n \in \mathrm{N}}$: otherwise. This defines a sequence of profiles $(y^n)_{n \in \mathrm{N}}$ converging to x. Let us prove that y^n is an ε^n-equilibrium for some $\varepsilon^n \to 0$. From continuity of ϕ, $\lim_{n\to+\infty} \phi(y^n_{-j}) = \phi(x_{-j}) = 0$ for every player j. In particular, from equation (2), and since g_j is continuous, we get

(3) $\sup_{d_j\in[0,1]} g_j(d_j, 0) \leq v_j.$

Fix some player $j \in N$, and consider 3 subcases:

- First, assume $f_j(0,0) = g_j(0,0)$. In this case, from assumption (C), $f_j(0,0) = g_j(0,0) = h(0, x_{-j})$, and thus $d_j \to u_j(d_j, x_{-j})$ is continuous. In particular, $v_j = u_j(x)$ and equation (2) implies that player j plays optimally at x_j. By continuity, player j is ε_n-optimizing by playing y^n_j against y^n_{-j} (because y^n converges to x), for some sequence of positive reals $(\varepsilon_n)_{n\in\mathrm{N}}$ converging to 0.
- Second, assume $v_j \leq f_j(0,0)$ and $f_j(0,0) \neq g_j(0,0)$.
  From equation (3), $g_j(0,0) \leq v_j \leq f_j(0,0)$, and since $f_j(0,0) \neq g_j(0,0)$, we get $f_j(0,0) > g_j(0,0)$. But this implies that $g_j(.,0)$ is non increasing on $[0,1]$: indeed, otherwise, for $\varepsilon > 0$ small enough, we would get a contradiction with the quasiconcavity of $y_j \to u_j(y_j, \varepsilon, ..., \varepsilon)$ and the inequality $u_j(0, \varepsilon, ..., \varepsilon) = f_j(0, \varepsilon) > u_j(2\varepsilon, \varepsilon, ..., \varepsilon) = g_j(2\varepsilon, \varepsilon)$.
  Consequently, we finally get
  (4) $\sup_{d_j\in[0,1]} g_j(d_j, 0) = g_j(0,0).$
  Moreover, since $u_j(0, x_{-j}) = h_j(0, x_{-j}) \in]g_j(0,0), f_j(0,0)[$ (because of assumption (C)), the fact that $g_j(.,0)$ is non increasing on $[0,1]$ also implies that $x_j = 0$ (otherwise $g_j(0,0) < u_j(0, x_{-j}) \leq v_j = u_j(x) = g_j(x_j, 0)$ and equation (4) would be false). In particular, by definition, this implies $y^n_j = 0$ for every n.
  From $g_j(0,0) < f_j(0,0)$, assumption (C), the continuity of g_j and f_j, equation (4) and the fact that $\phi(y^n_{-j})$ converges to 0, there exists a real sequence $\varepsilon_n \to 0$ such that one has
  (5) $g_j(\phi(y^n_{-j}), \phi(y^n_{-j})) - \varepsilon_n \leq \sup_{d_j\in[0,\phi(y^n_{-j})]} f_j(d_j, \phi(y^n_{-j})) - \varepsilon_n \leq f_j(0, \phi(y^n_{-j}))$
  (6) $\sup_{d_j\in[0,1]} g_j(d_j, \phi(y^n_{-j})) - \varepsilon_n \leq g_j(\phi(y^n_{-j}), \phi(y^n_{-j})) \leq h_j(\phi(y^n_{-j}), y^n_{-j})$
  and
  (7) $g_j(\phi(y^n_{-j}), \phi(y^n_{-j})) < h_j(\phi(y^n_{-j}), y^n_{-j}) < f_j(\phi(y^n_{-j}), \phi(y^n_{-j})),$
  the last equation being true because the inequality $g_j(0,0) < f_j(0,0)$ remains true on a neighborhood of $(0,0)$, and from Assumption (C).
  This implies that player j is $2\varepsilon_n$-optimizing by playing $y^n_j = 0$ against y^n_{-j}: indeed if $\phi(y^n_{-j}) > 0$, this is a consequence of equations (5), (6), (7); if $\phi(y^n_{-j}) = 0$, this is a consequence of equation (6).
- Third, assume $f_j(0,0) \neq g_j(0,0)$ and $v_j > f_j(0,0)$ (thus $y^n_j = x^n_j$ for every n).
  From assumption (C) and the definition of v_j, we deduce that $f_j(0,0) < h_j(0, x_{-j}) < g_j(0,0)$, but from equation (3), $g_j(0,0) \leq v_j$, and we finally

get

(8) $\quad f_j(0,0) < h_j(0,x_{-j}) < g_j(0,0) = v_j$.

Now, first assume $x_j = 0$.

Thus, we should have $u_j(x^n) = g_j(x_j^n, \phi(x_{-j}^n))$ for n large enough, which requires $x_j^n > \phi(x_{-j}^n)$ for n large enough. By monotonicity of ϕ and since $x_i^n \geq y_i^n$ for every player i, we get $y_j^n = x_j^n > \phi(x_{-j}^n) \geq \phi(y_{-j}^n)$. Thus, $u_j(y^n) = g_j(y_j^n, \phi(y_{-j}^n))$ and by continuity of g_j, this converges to $g_j(0,0) = v_j$, when n tends to infinity.

From equation (3) and equation (8), we get

(9) $\quad \max\{f_j(0,0), h_j(0,x_{-j}), \sup_{d_j \in [0,1]} g_j(d_j,0)\} \leq g_j(0,0)$

Since $u_j(y^n)$ converges to $g_j(0,0)$, we get that y_j^n is an ε'_n-best-response against y_{-j}^n for some sequence of positive reals $(\varepsilon'_n)_{n\in\mathrm{N}}$ converging to 0 (use continuity of f_j, g_j and assumption (C)).

Second, assume $x_j > 0$.

Since $\phi(x_{-j}) = 0$, u_j is continuous at x and $v_j = g_j(x_j, 0) = u_j(x)$. From equation (2), x_j is a best-response against x_{-j}, thus we finally have

(10) $\quad \max\{f_j(0,0), h_j(0,x_{-j}), \sup_{d_j \in [0,1]} g_j(d_j,0)\} \leq u_j(x)$

and, similarly to the previous case, from the continuity of f_j and g_j, assumption (C) and the continuity of u_j around x, we obtain that $y_j^n = x_j^n$ is an ε'_n-best-response against y_{-j}^n for some sequence of positive reals $(\varepsilon'_n)_{n\in\mathrm{N}}$ converging to 0.

This ends the three subcases, and x is an approximate equilibrium profile associated to the sequence $(y^n)_{n\in\mathrm{N}}$.

In the *last case*, we assume there exists $i \in N$ such that $x_i = \phi(x_{-i}) = 1$: this can be proved as in the third case.

6.2. *Proof of Theorem 4.1*

The first inclusion is a consequence of Theorem 3.10 applied to G'.

For the second inclusion, consider a sequence $(m^n, u(m^n))_{n\in\mathrm{N}^*}$ of $\frac{1}{n}$-mixed Nash equilibria converging to (m, v). To prove that it is a mixed sharing rule equilibrium, we show how to adapt Simon-Zame's proof.

Note that the sequence $(m^n)_{n\in\mathrm{N}}$ converges (weakly) to m. We let E be the space of $\mathbf{R}^N$-valued vector measures on X, endowed with the weak* topology. Consider the sequence $(u \cdot m^n)_{n\in\mathrm{N}}$ of the compact space E (here, $u \cdot m^n$ denotes the $\mathbf{R}^N$-valued measure on X defined by $u \cdot m^n(F) = \int_F u \, dm^n$ for every Borelian set F of X). Without any loss of generality, up to a subsequence, this sequence converges to some measure ν. From Lemma 2, p.867 (Simon and Zame (1990)), there exists a Borel measurable selection q of Q, the multivalued function from X to $\mathbf{R}^N$, defined by $Q(x) = \mathrm{co}\,\overline{\Gamma_x}$, such that $\nu = q \cdot m$ (remark that the proof of this lemma does not use the support of m^n, but only the fact that u is a selection of Q). Define, for every player i and every integer $k > 0$, $H_i^k = \{x_i \in$

$X_i : \int q_i d(\delta_{x_i} \times m_{-i}) > \int q_i d(m_i \times m_{-i}) + \frac{1}{k}\}$. Let us prove that $m_i(H_i^k) = 0$ for every integer $k > 0$.

Otherwise, consider $K \subset H_i^k \subset U$, where K is compact, U open, $m_i(U-K) < \varepsilon$ with $\varepsilon > 0$, and with $m_i(K) > 0$. We let $f : X_i \to [0,1]$ be a continuous function which is identically equal to 1 on K and 0 on the complement of U. Consider the strategy $\beta_i^n = \frac{f m_i^n}{\int f dm_i^n}$. Then, $u_i(\beta_i^n, m_{-i}^n) > u_i(m^n) + \frac{1}{2k}$ for n large enough and $\varepsilon > 0$ small enough (see Step 4 in Simon and Zame), which contradicts the fact that m^n is a $\frac{1}{n}$-Nash equilibrium (for n large enough). Thus $m_i(H_i^k) = 0$ for every integer $k > 0$. Now, let $H_i = \{x_i \in X_i : \int q_i d(\delta_{x_i} \times m_{-i}) > \int q_i d(m_i \times m_{-i})\}$. Since $H_i = \cup_k H_i^k$, we get $m_i(H_i) = 0$.

From Step 5 in Simon and Zame, there exists a modification $\tilde{q}$ of q, such that $q = \tilde{q}$ except on a set of $m-$measure 0, such that m is a Nash equilibrium of the game $\tilde{G} = ((X_i)_{i\in N}, (\tilde{q}_i)_{i\in N})$, and $\tilde{q}(m) = q(m)$. More precisely, take $\tilde{p}^i$ a Borel measurable selection from Q which minimizes the i-th component of Q, define $T = \{x \in X : x_i \in H_i$ for at least two indices $i \in N\}$, define $\tilde{q}(x) = \tilde{p}^i(x)$ if $x \in H_i \times X_{-i}$ but $x \notin T$, and $\tilde{q}(x) = q(x)$ otherwise. To prove that m is a Nash equilibrium of $\tilde{G}$, we follow Simon and Zame. By contradiction, assume that some player i has a profitable deviation x_i, that is $\tilde{q}_i(x_i, m_{-i}) > \tilde{q}_i(m) + \eta$ for some $\eta > 0$. Then the case $x_i \notin H_i$ is not possible by definition of H_i (see Step 6, Case 1 in Simon and Zame). The case $x_i \in H_i$ is as in Step 6, Case 2 in Simon and Zame, and implies that $u_i(x_i, m_{-i}^n) > u_i(m^n) + \frac{\eta}{2}$ for n large enough, a contradiction.

6.3. *Proof of Theorem 4.3*

A limit-equilibrium (m, v) exists by compactness of $\overline{\Gamma'}$. First let us prove that it is a Reny solution of Γ'. Fix $d \in M$. The definition of a limit-equilibrium implies that for every neighborhood V of (m, v), there exists $m^V \in M$ and D' a finite subset of M containing d such that (1) $\forall i \in N, \forall d' \in D', u_i(d'_i, m_{-i}^V) \leq u_i(m^V)$ and (2) $(m^V, u(m^V)) \in V$. Shrinking V to (m, v) implies that $\underline{u_i}(d_i, m_{-i}) \leq v_i$, and since this is true for every $d \in M$, (m, v) is a Reny solution of Γ'.

Now, let us prove that (m, v) induces a solution à la Simon-Zame. Since M is a compact metric set, there exists a countable decreasing basis of neighborhoods V^n of (m, v) in $\overline{\Gamma'}$. Since X is a compact metric set, there exists a sequence $D^n = \Pi_{i\in N} D_i^n$ of finite sets of pure strategies converging to X for the Hausdorff distance. By definition of a limit-equilibrium (Definition 4.2), for every integer n, there exists a sequence of finite sets $D'^n = \Pi_{i\in N} D_i'^n$ of mixed strategies containing D^n, and a probability m^n, which is a Nash equilibrium of the game restricted to D'^n such that $(m^n, u(m^n)) \in V^n$. Recall that Simon and Zame's existence proof consists in approximating the game by a finite game in pure-strategies (here D^n), and in considering a weak limit of a sequence $(m^n)_{n\in \mathbb{N}}$ of Nash equilibria of this approximation. We cannot apply Simon and Zame's proof directly to the Nash equilibria m^n of the finite games D'^n, because D'^n

may contain mixed strategies. But $D'^n \supset D^n$: thus, no player i has a profitable deviation in D_i^n against m^n, and we shall prove that this property is sufficient to adapt Simon-Zame's proof.

Let $\tilde{G}$, $\tilde{q}$ and H_i be defined as in the proof of Theorem 4.1 (Section 6.2 just above). To prove that m is a Nash equilibrium of $\tilde{G}$, assume that some player i has a a profitable deviation x_i to m_{-i} that is, $\tilde{q}_i(x_i, m_{-i}) > \tilde{q}_i(m)$. Then the case $x_i \notin H_i$ yields a contradiction (Step 6 Case 1 in Simon and Zame). For the second case, simply consider a sequence x_i^n converging to x_i such that $x_i^n \in D_i^n$ (here, we use that $D^n = \Pi_{i \in N} D_i^n$ converges to X for the Hausdorff distance). A limit argument (Step 6 Case 2 in Simon and Zame) proves that $u_i(x_i^n, m_i^n) > u_i(m^n)$ for n large enough, a contradiction, because m^n is a Nash of D'^n and because $x_i^n \in D_i^n \subset D_i'^n$.

6.4. *Proof of Theorem 4.6*

Let $V \subset \overline{\Gamma'}$ be the set of non Reny solutions of G'. By definition, for every $(m, v) \in V$, there exists $V^{(m,v)}$ (a neighborhood of (m, v)), $d^{(m,v)} \in M$ and a player $i \in N$ such that $u_i(d_i^{(m,v)}, m'_{-i}) > v'_i$ for every $(m', v') \in V^{(m,v)}$. This yields a collection of pairs $(V^{(m,v)}, d^{(m,v)})$ for every $(m, v) \in V$. Since the set of Reny solutions is compact in $\overline{\Gamma'}$, V is open in $\overline{\Gamma'}$, thus there is a countable family of pairs $\{(V^n, d^n)\}_n$ defined as above such that $V = \cup_{n \geq 0} V^n$. Define $D_i^n = \{d_i^1, ..., d_i^n\}$ and $D^n = \Pi_{i \in N} D_i^n$ for every integer n, and let us prove that it is a weak strategic approximation of the initial game. We only have to prove that if $(m, v) = \lim_{n \to +\infty}(m^n, u(m^n))$, where $(m^n, u(m^n))$ is a Nash equilibrium of the game restricted to D^n (for every integer $n \geq 0$), then (m, v) is a Reny solution (which implies that it is an approximate equilibrium, since the game is approximately better-reply secure). By contradiction, assume (m, v) is not a Reny solution. By definition of V and the V^n $(n \geq 0)$, there exists some integer k such that $(m, v) \in V^k$ and $(m^n, u(m^n)) \in V^k$ for every n large enough. By definition of V^k, there is some player i such that $u_i(d_i^k, m'_{-i}) > v'_i$ for every $(m', v') \in V^k$. In particular, $u_i(d_i^k, m_{-i}^n) > u_i(m^n)$ for n large enough, which contradicts the fact that $(m^n, u(m^n))$ is a mixed Nash equilibrium of the game G' restricted to D^n, since $d_i^k \in D_i^n$ for every $n \geq k$.

6.5. *Proof of Theorem 4.8*

The proof of Theorem 4.8 and Theorem 4.12 uses the same principle: for every $\varepsilon > 0$, we construct a finite approximation G'_ε of G' such that for every player i and every mixed strategy of players $j \neq i$ in G'_ε, player i has an ε-best response which belongs to G'_ε. This proves that every mixed Nash equilibrium of the finite approximation is an $\varepsilon-$Nash equilibrium of the initial game. The approximation shall depend on the structure of the game, and in particular on the behavior of the payoffs in a neighborhood of the boundary.

First note that in two player diagonal games, necessarily $\phi(y) = y$ for every $y \in [0,1]$ (by the unanimity condition). Call $x \in [0,1]$ a right local equilibrium if $h_i(x,x) > g_i(x,x)$ for both $i = 1,2$ and a left local equilibrium if $h_i(x,x) > f_i(x,x)$ for both $i = 1,2$. Thus, if players are supposed to play (x,x) and if x is a right local equilibrium, no player has an interest to deviate to some strategy in some right neighborhood of x (but he may have a profitable deviation outside that neighborhood) and similarly for left local equilibria.

We let x_0 be the largest element in $[0,1]$ such that all $x < x_0$ are right local equilibria and y_0 be the smallest element in $[0,1]$ such that all $y > y_0$ are left local equilibria. Note that x_0 may be 0 and y_0 could be 1. By continuity of f, g and h, if $x_0 < 1$ then $h_i(x_0,x_0) \leq g_i(x_0,x_0)$ for some $i \in \{1,2\}$ and similarly, if $y_0 > 0$ then $h_j(y_0,y_0) \leq f_j(y_0,y_0)$ for some $j \in \{1,2\}$. Depending on the relative position of x_0 and y_0, we consider the three following cases.

First case. $x_0 > y_0$. In this case, the finite approximated game is simply defined by some finite discretization D of $[0,1]$ containing 0 and 1 and σ a mixed strategy of the game restricted to D . Without any loss of generality, taking the mesh of this discretization smaller than some $\eta > 0$, we can assume that the payoff functions f and g do not change by more than $\frac{\varepsilon}{2}$ if a player moves by no more than η, and such that if $x < x_0$ is in D, then $h_i(x,x) > g_i(y,x)$ for all $x < y < x + \eta$, and if $x > y_0$ is in D, then $h_i(x,x) > f_i(y,x)$ for all $x > y > x - \eta$. We let $y \in [0,1]$ be some $\varepsilon/2$-best reply to σ_j of player i which is not in D (if such strategy does not exist, we are done with the proof). Then either $y < x_0$ or $y > y_0$. In the first case, denote by z the greatest element in D smaller than y, so that $h_i(z,z) > g_i(y,z)$ by assumption of the discretization and since z is a right equilibrium. Since player j plays a probability distribution supported on D, moving from y to z for player i induces for him a greater payoff from the event associated to player j playing z and at most a change of $\frac{\varepsilon}{2}$ on the events where player j is playing a strategy in D different from z. Thus, z is an ε-best reply for player i. A similar argument applies to $y > y_0$ (use the left equilibrium property).

Second case. $x_0 < y_0$, which implies that $h_k(x_0,x_0) \leq g_k(x_0,x_0)$ and $h_l(y_0,y_0) \leq f_l(y_0,y_0)$ for some $k \in \{1,2\}$ and $l \in \{1,2\}$. By continuity, we get $\eta > 0$ small enough such that $h_k(x_0,x_0) < g_k(x,x_0) + \frac{\varepsilon}{4}$ for every $x \in]x_0, x_0+\eta[$ and $h_l(y_0,y_0) < f_l(y,y_0) + \frac{\varepsilon}{4}$ for every $y \in]y_0-\eta, y_0[$. Thus, there are four cases to check, depending on the values of k and l. Let us solve explicitly the case $k = 1$ and $l = 2$. The same idea of construction could be done in the other cases, with a small adaptation in the definition of the weak strategic approximation.

Fix $\varepsilon > 0$ and define $x_0 = t_0 < s_0 < t_1, ... < s_{K-1} < t_K = y_0$, a discretization of $[x_0, y_0]$ with a mesh smaller than some $\eta > 0$ so that payoff functions f and g do not change by more than $\varepsilon/4$ if the pure strategy moves by no more than η. As in the first case, we let D be a finite discretization of $[0, x_0[\cup]y_0, 1]$ with a mesh smaller than $\eta > 0$ so that payoff functions f and g do not change by more than $\frac{\varepsilon}{2}$ if the pure strategy moves by no more than η and such that if $x < x_0$ is in D, then $h_i(x,x) > g_i(y,x)$ for all $x < y < x+\eta$ and if $x > y_0$ is in D, then $h_i(x,x) > f_i(y,x)$ for all $x > y > x+\eta$. Now, the finite approximation of G' is defined as follows: player 1 is restricted to play in D or uniformly on one of the intervals $[t_k, s_k]$, $k = 0, ..., K-1$, or to choose $t_K = y_0$. Player 2 is restricted to play in D or uniformly on one of the intervals $[s_k, t_{k+1}]$, $k = 0, ..., K-1$, or to choose $t_0 = x_0$. Observe that the intervals where players are uniformly mixing are disjoint and alternate. We let σ be some strategy of player 2 in the restricted game. Let y be some $\varepsilon/4$ pure best response of player 1 in G, which is not in the discretization D (again, if it does not exist, this is finished). Several subcases have to be examined. *First subcase*, if $y < x_0$ or $y > y_0$, we proceed as in the first case to construct an ε-best reply in D. *Second subcase*, if y is in some interval $]s_k, t_{k+1}[$ of player 2 ($k \in \{0, 1, ..., K-1\}$), and if player 2 is choosing that interval with positive probability, an easy computation proves that the payoff of player 1 coming from that interval is, up to $\varepsilon/4$, a convex combination of his payoff if he chooses t_{k+1} and his payoff if he chooses s_k. But, the payoff of player 1 coming from Player 2 playing in the other intervals or in D changes by no more that $\varepsilon/4$ when he moves in the interval $[s_k, t_{k+1}]$. Consequently, player 1 has a $\frac{3}{4}\varepsilon$-best response at the extreme points s_k or t_{k+1} of the interval, a case which is treated in the next subcase: *Third subcase*, let $z \in [t_k, s_k]$ being a $\frac{3}{4}\varepsilon$-best response, for some $k \in \{0, 1, ..., K-1\}$. If $k > 0$, by assumption, there is zero probability that player 2 stops in that interval and so player 1's payoff does not move by more that $\varepsilon/4$ if he plays uniformly in $[t_k, s_k]$ (which is authorized for player 1) instead of playing z. This gives an ε-best response. If $k = 0$, if player 2 is playing x_0 with positive probability and player 1 is playing $z = x_0$, then player 1 does not lose more than $\varepsilon/4$ by playing slightly more than x_0 instead of x_0 (since $h_1(x_0, x_0) < g_1(x, x_0) + \frac{\varepsilon}{4}$ for every $x \in]x_0, x_0+\eta[$). Then remains the case where z belongs to the interval $]t_0, s_0[$. But, again, since his payoff moves continuously in that interval, playing uniformly in it is an ε-best response. The proof for player 2 is similar (we use the fact that $h_2(y_0, y_0) < f_2(y, y_0) + \frac{\varepsilon}{4}$ for every $y \in]y_0 - \eta, y_0[$).

The three remaining cases for k and l are solved similarly, by a judicious choice of who of the two players is allowed to stop at x_0 and y_0: if $k = 2$ and $l = 1$, then player 1 can stop at x_0 and player 2 at y_0; if $k = 2$ and $l = 2$, (only) player 1 is allowed to stop at both x_0 and y_0; if $k = 1$ and $l = 1$, only player 2 is allowed to stop at both points. If some player can stop at x_0 then it is the other player who is authorized to stop uniformly in the small interval of the discretization just after x_0, and the intervals in which players can stop (by mixing uniformly) alternate until the point y_0, and the last interval belongs to the player who is not allowed to stop at y_0.

Third case. $x_0 = y_0$, implying $h_k(x_0, x_0) < g_k(x, x_0) + \frac{\varepsilon}{4}$ for $x \in]x_0, x_0+\eta[$ and $h_l(x_0, x_0) < f_l(x, x_0) + \frac{\varepsilon}{4}$ for $x \in]x_0 - \eta, x_0[$ for some $k \in \{1, 2\}$ and $l \in \{1, 2\}$ (if x_0 is 0 or 1, then only one of the inequalities holds). Suppose for example that $h_1(x_0, x_0) < g_1(x, x_0) + \frac{\varepsilon}{4}$ for $x \in]x_0, x_0+\eta[$. We let $D_1 = \{0 = t_0 < ... < t_K\}$

be a discretization on the left of x_0, not including x_0, and empty if $x_0 = 0$; let $D_2 = \{s_0 < ... < s_K = 1\}$ be a discretization on the right of y_0, not including y_0, and empty if $y_0 = 1$. Again, without any loss of generality, assume that the mesh of the discretizations is smaller than $\eta > 0$, so that payoff functions f and g do not change by more than $\frac{\varepsilon}{2}$ if a player moves by no more than η. Consider a strategic approximation where Player 2 is allowed to play in $D_1 \cup D_2 \cup \{x_0\}$ and player 1 to play in $D_1 \cup D_2$ or to mix uniformly in the length $[x_0, s_0]$. Let $y \in [0,1]$ be some $\varepsilon/2$-best reply of player 1 to some mixed strategy of player 2 which is not in D_1 (if such strategy does not exist, this is finished). If $y < x_0$, moving from y to the greatest element in D_1 smaller than y gives an ε-best reply for player 1. If $y > x_0$, moving from y to the smallest element in D_1 larger than y gives an ε-best reply for player 1. Last, if $y = x_0$, playing uniformly in $[x_0, s_0]$ instead of playing x_0 is an ε-best reply for player 1, because of $h_1(x_0, x_0) < g_1(x, x_0) + \frac{\varepsilon}{4}$ for $x \in]x_0, x_0 + \eta[$. We treat in a similar way the case of player 2, and the case $k = 2$.

6.6. *Proof of Theorem 4.12*

Recall that for simplicity, we assume the type space T to be finite.

Case c2: the multiplayer private value setting

Define a weak strategic approximation of the initial game G as follows: for each integer m, a strategy (in M_i) of player i (whatever his type) is some element of the finite set D^m of uniform distributions on $I_m^k = [\frac{k}{m}, \frac{k+1}{m}]$ ($k \in \{0, 1, ..., m-1\}$). By Nash theorem, this finite (Bayesian) game has a mixed Nash equilibrium σ^m. We shall prove that if players $j \neq i$ play according to σ_{-i}^m, each type t_i of player i has some ε-best response (in G') which belongs to D^m. This proves that σ^m is an ε-Nash equilibrium of G' for m large enough.

Consider $\varepsilon > 0$, and suppose m is large enough so that for every $t \in T$, $f_i(t, ., y)$ and $g_i(t, ., y)$ do not move by more than ε in the interval $[\frac{k}{m}, \frac{k+2}{m}]$ ($k = 0, ..., m-2$) uniformly in y. If player i of type t_i chooses a pure strategy $x \in [0,1]$ and if the *realized* strategy profile of its opponents is x_{-i}, then its payoff can be written $w_i(t_i, x, \phi_i(x_{-i}))$, where $w_i(t_i, x, \phi_i(x_{-i}))$ is almost surely equal to either $f_i(t_i, x, \phi_i(x_{-i}))$ or $g_i(t_i, x, \phi_i(x_{-i}))$ (ties have zero-probability), depending on the position of $\phi_i(x_{-i})$ with respect to x. This is because the image probability measure of σ_{-i}^m by ϕ_i has no atoms.[17] It also implies that the (expected) payoff

[17]To prove that, consider the event $[\phi_N(x_1, ..., x_{N-1}) = \alpha]$ for some $\alpha \in [0,1]$. Let $S = \{(x_1, ..., x_{N-1}) \in [0,1]^{N-1} : \sum_{k=1}^{N-1} x_k = 1\}$ be the $(N-2)$-simplex. The monotonicity assumption guarantees that for every $(x_1, ..., x_{N-1}) \in S$, there is no more than one $y \in \mathbf{R}$ such that $\phi_N((x_1, ..., x_{N-1}) + y(1, ..., 1)) = \alpha$. For every $x = (x_1, ..., x_{N-1}) \in S$, define $\psi(x_1, ..., x_{N-1}) = y$ if such y exists, and $\psi(x_1, ..., x_{N-1}) = 0$ otherwise. One can identify $S \times \mathbf{R}$ to a subset of $\mathbf{R}^{N-1}$ (which contains $[0,1]^{N-1}$) throughout the mapping $i : S \times \mathbf{R} \to \mathbf{R}^{N-1}$ defined by $i((x_1, ..., x_{N-1}), y) = (x_1, ..., x_{N-1}) + y(1, ..., 1)$. With this identification, the graph of ψ contains the event $[\phi_N(x_1, ..., x_{N-1}) = \alpha]$. But ψ is measurable, and thus its graph has

of player i, fixing the strategies of the opponents, is a continuous function of his own strategy x. Consequently, there exists $x^* \in [0,1]$, a pure best response of player t_i (in the original game G'). From x^*, one can construct an ε-best response in D^m as follows: if $x^* \in [0, \frac{1}{m}]$, from assumption (a), replacing x^* by the uniform distribution on I_m^1 is a ε-best response for m large enough. If $\frac{k}{m} < x^* < \frac{k+1}{m}$ for some $k = 1, ..., m-2$ then, either $g_i(t_i, x^*, x^*) \geq f_i(t_i, x^*, x^*)$ and then replace x^* with the uniform strategy on I_m^{k+1} (it is better for i to choose a higher strategy), or $g_i(t_i, x^*, x^*) < f_i(t_i, x^*, x^*)$, and then replace x^* with the uniform distribution on I_m^{k-1} (it is better for i to choose a lower strategy). In both cases, this gives an ε-best response in D^m for m large enough. Last if Assumption b1) is satisfied and not b2) and $x^* \in [1 - \frac{1}{m}, 1]$, then replace x^* with the uniform distribution on I_m^{m-1}.

Remark 6.1 Note that this proof works also when the payoff of type t_i depends also on (t_i, t_{-i}) if we add the following assumption: for every player i and every $x^* \in [0,1]$, if $g_i(t, x^*, x^*) \leq f_i(t, x^*, x^*)$ is true for one $\bar{t}_{-i}$ then it is true for every t_{-i}, and similarly for the inequality $g_i(t, x^*, x^*) \geq f_i(t, x^*, x^*)$. Finally, remark that the proof only requires the (strict) Monotonicity of ϕ_i, and not the other properties.

Case c1: the two-player general value setting

When there are two players, by unanimity $\phi_i(y) = y$. Now, we mimic the proof and the approximation scheme of the second case of Theorem 4.8 with $x_0 = 0$ and $y_0 = 1$, proving that if σ is some mixed strategy profile of player 2 in the approximated game, then every type t_1 has an ε-best response against σ in the full game that belongs to his set of authorized strategies. That is, take the following discretization of $[0,1]$: $0 = s_0 < t_0 < s_1 < t_1 < ... < t_K < s_{K+1} = 1$. Player 1 is restricted to play uniformly on one of the intervals $[s_k, t_k]$, $k = 0, ..., K$, or to choose $x = 1$. Player 2 is restricted to play uniformly on one of the intervals $[t_k, s_{k+1}]$, $k = 0, ..., K$, or to choose $x = 0$. Observe that the intervals where players are mixing are disjoint and alternate (player 1 can stop uniformly in the first interval, player 2 in the second, player 1 in the third, etc.).

Remark 6.2 In both cases (two players or private value with N players), by construction, in the weak strategic approximation, ties have zero probability. Consequently, the Nash equilibria are independent on the tie-breaking rule h. Also, examining the proofs, one can see that the result of Theorem 4.12 holds when the type space T is a compact metric set, functions $f_i(t, a, b)$, $g_i(t, a, b)$ are jointly continuous in $(t, a, b) \in T \times [0,1] \times [0,1]$, and $h_i(t, x)$ is measurable and continuous in t uniformly in x. In that case, for every $\varepsilon > 0$, one can discretize the type space to obtain a Bayesian diagonal game with finitely many types

a 0-Lebesgue-measure in $\mathbf{R}^{N-1}$ (from Fubini theorem). The assertion follows from the fact that σ_{-i}^m is a convex combination of uniform probabilities whose supports are rectangles with nonempty interiors.

$t_\varepsilon \in T_\varepsilon$. This game admits an ε-equilibrium (by Theorem 4.12) from which one can construct a 2ε-equilibrium in the original game by asking each type t_i in the original game to play a strategy of a closest type t_i^ε in the game with finitely many types. If the discretization is well chosen so that $\sup_x |u_i(t_i, x) - u_i(t_i^\varepsilon, x)| \leq \varepsilon$, then type t_i is playing a 2ε-best response in the original game.

6.7. *Proof of Example 4.16*

Start with the maxmin. We let α be the probability with which player 1 stops at $x = 0$ (so with probability $(1-\alpha)$ he stops after zero). If $\alpha = 0$, player 2 gets 1 by stoping at time zero (and so player 1 gets -1). If $\alpha > 0$, type A for player 2 can stop uniformly between 0 and some ε where ε is very small so that with high probability, if the game has not been stopped at time zero, it is stopped by player 2 (just after zero). Assume that type B of player 2 stops at time zero. Payoff of player 1 is thus very close to $\alpha(\frac{1}{2} \times 1 + \frac{1}{2} \times -2) + (1-\alpha) \times -1$. Consequently, the best strategy for player 1 against such behavior by player 2 is to stop at $t = 0$ with probability 1 so that $\max\min \leq -\frac{1}{2}$.

Let us now compute the $\min\max$. Let us restrict player 1 to playing best-replies to the following kind of strategies : (1) to stop at time $t = 0$ or (2) to stop uniformly between 0 and some ε very small, which depends of course on the strategy of player 2. Knowing this behavior, type B must stop at time zero. We let β be the probability that type A stops at time zero. The payoff of player 1 if he stops at 0 (choose option 1) is $\frac{1}{2} \times -2 + \frac{1}{2} \times (\beta \times 3 + (1-\beta) \times 1) = \frac{-1}{2} + \beta$, while if he chooses option 2 his payoff is close to $\frac{1}{2} \times -1 + \frac{1}{2}(\beta \times -1 + (1-\beta) \times 1) = -\beta$. Thus, the optimal β for type B against this behavior of player 1 must be equalizing and so is $\beta = \frac{1}{4}$. Consequently, $\min\max \geq -\frac{1}{4}$.

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