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Julie Decaup, Adrien Dubouloz

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JULIE DECAUP AND ADRIEN DUBOULOZ

Abstract. We classify closed curves isomorphic to the affine line in the complement of a smooth rational projective plane conic Q. Over a field of characteristic zero, we show that up to the action of the subgroup of the Cremona group of the plane consisting of birational endomorphisms restricting to biregular automorphisms outside Q, there are exactly two such lines: the restriction of a smooth conic osculating Q at a rational point and the restriction of the tangent line to Q at a rational point. In contrast, we give examples illustrating the fact that over fields of positive characteristic, there exist exotic closed embeddings of the affine line in the complement of Q. We also determine an explicit set of birational endomorphisms of the plane whose restrictions generates the automorphism group of the complement of Q over a field of arbitrary characteristic.

Introduction

A famous theorem of Abhyankar and Moh [1] asserts that over a field k of characteristic zero, all closed embeddings of the affine line $\mathbb{A}^1_k$ into the affine plane $\mathbb{A}^2_k$ are equivalent under the action of the group $\mathrm{Aut}_k(\mathbb{A}^2_k)$ of algebraic k-automorphisms of $\mathbb{A}^2_k$: for any two such closed embeddings with images A and A', there exists $\Psi \in \mathrm{Aut}_k(\mathbb{A}^2_k)$ such that $A' = \Psi(A)$. In this article, we consider the classification of equivalence classes of closed embeddings of the affine line into another smooth affine surface very similar to the affine plane: the complement of a smooth k-rational conic Q in the projective plane $\mathbb{P}^2_k$.

In the complex case, such a smooth affine surface $S = \mathbb{P}^2_{\mathbb{C}} \setminus Q$ has divisor class group $\mathrm{Cl}(S) = \mathbb{Z}_2$, integral homology groups $H_0(S;\mathbb{Z}) = \mathbb{Z}$, $H_1(S;\mathbb{Z}) = \mathbb{Z}_2$ and $H_i(S;\mathbb{Z}) = 0$ for every $i \geq 2$, and its logarithmic Kodaira dimension $\overline{\kappa}(S) = \overline{\kappa}(\mathbb{P}^2, K_{\mathbb{P}^2} + Q)$ (see [10]) is equal to $-\infty$. It is thus very close to the affine plane from both algebraic and topological points of view. It also contains many closed curves isomorphic to the affine line $\mathbb{A}^1$. For instance, for every point $p \in Q$, Q and twice its tangent line T_pQ at Q generate a pencil $\mathcal{P}_p \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$ whose members, except for the one $2T_pQ$, are smooth conics intersecting Q with multiplicity 4 at p. The intersections with S of all members of $\mathcal{P}_p$ except Q are thus isomorphic to $\mathbb{A}^1$. The subgroup $\mathrm{Aut}(\mathbb{P}^2, Q)$ of $\mathrm{Aut}(\mathbb{P}^2)$ consisting of automorphism preserving Q acts transitively on Q, and for a given point $p_0 \in Q$, the action on $\mathcal{P}_{p_0} \setminus Q$ of the subgroup $\mathrm{Aut}(\mathbb{P}^2, Q, p_0)$ of $\mathrm{Aut}(\mathbb{P}^2, Q)$ consisting of automorphisms fixing p_0 has exactly two orbits: a fixed point $T_{p_0}Q$ and its complement $\mathcal{P}_{p_0} \setminus (Q \cup T_{p_0}Q)$. Viewing $\mathrm{Aut}(\mathbb{P}^2, Q)$ as a subgroup of $\mathrm{Aut}(S)$ via the natural restriction homomorphism, it follows in particular that $\mathrm{Aut}(S)$ acts on the set of so-defined affine lines in S with at most two orbits: the one of $T_{p_0}Q \cap S$ and the one of $Q_1 \cap S$ for a fixed member Q_1 of $\mathcal{P}_{p_0} \setminus (Q \cup T_{p_0}Q)$. But since $\mathrm{Cl}(S \setminus (S \cap T_{p_0}Q_1))$ is trivial while $\mathrm{Cl}(S \setminus (S \cap Q_1)) \simeq \mathbb{Z}_2$, it follows that $T_{p_0}Q \cap S$ and $Q_1 \cap S$ cannot belong to a same orbit of the action of $\mathrm{Aut}(S)$ on the set of closed curves in S isomorphic to $\mathbb{A}^1$. So in contrast with the case of $\mathbb{A}^2_{\mathbb{C}}$, the best we can hope for is that the action of $\mathrm{Aut}(S)$ on the set of such closed curves has precisely two orbits. Our main result just below implies that this is exactly the case:

Theorem 1. *Let k be a field of characteristic zero, let $Q \subset \mathbb{P}^2_k$ be a smooth conic and let $S = \mathbb{P}^2_k \setminus Q$. Suppose that $A \subset S$ is a closed curve isomorphic to $\mathbb{A}^1_k$ and let $\overline{A} \subset \mathbb{P}^2_k$ be its closure. Then Q is k-rational and for every given k-rational point $p_0 \in Q$ and every smooth k-rational member $Q_1 \neq Q$ of the pencil $\mathcal{P}_{p_0}$ generated by Q and $2T_{p_0}Q$, there exists a birational map $\Phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ defined over k, restricting to an automorphism of S, such that*

$$\Phi_*(\overline{A}) = \begin{cases} Q_1 & \text{if } \deg \overline{A} \text{ is even} \\ T_{p_0}Q & \text{if } \deg \overline{A} \text{ is odd.} \end{cases}$$

In particular, there are precisely two classes of closed curves isomorphic to $\mathbb{A}^1_k$ in S up to the action of $\mathrm{Aut}_k(S)$.

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Note that the dichotomy depending on the degree of $\overline{A}$ follows from the fact that the divisor classes of Q and $\overline{A}$ either generate $\mathrm{Cl}(\mathbb{P}^2_k)$ when $\deg \overline{A}$ is odd, or generate a proper subgroup of index 2 when $\deg \overline{A}$ is even, so that $\mathrm{Cl}(S \setminus A) = \{0\}$ or $\mathbb{Z}_2$ according to $\deg \overline{A}$ is odd or even.

Recall that by a theorem of Jung and van der Kulk [11, 15], the automorphism group of the affine plane $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$ over an arbitrary field k is the free product of the group of affine automorphisms and of the group of automorphisms of the form $(x, y) \mapsto (ax + b, cy + s(x))$, where $a, c \in k^*$, $b \in k$ and $s \in k[t]$, amalgamated over their intersection. Viewing $\mathbb{A}^2_k$ as the complement of the line at infinity $L = \{z = 0\}$ in $\mathbb{P}^2_k = \mathrm{Proj}(k[x, y, z])$, these two subgroups coincide respectively with the restriction to $\mathbb{A}^2_k$ of the group $\mathrm{Aut}(\mathbb{P}^2_k, L)$ and with the group $\mathrm{Aut}(\mathbb{A}^2_k, \mathrm{pr}_1)$ of automorphisms preserving globally the $\mathbb{A}^1$-fibration $\mathrm{pr}_1 : \mathbb{A}^2_k \to \mathbb{A}^1_k$ induced by the restriction of the pencil of lines through the point $[0 : 1 : 0]$. Our second result consists of an analogous presentation of the automorphism group of the complement of a smooth k-rational conic in $\mathbb{P}^2_k$, providing in particular a complete description of the birational maps $\Phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ which can occur in Theorem 1.

Theorem 2. *Let k be a field of arbitrary characteristic and let S be the complement of a smooth conic $Q \subset \mathbb{P}^2_k$ with a k-rational point p. Let $\mathrm{Aut}(S, \rho_p)$ denote the subgroup of $\mathrm{Aut}_k(S)$ consisting of automorphisms which preserve globally the $\mathbb{A}^1$-fibration $\rho_p : S \to \mathbb{A}^1_k$ induced by restriction of the pencil $\mathcal{P}_p \subset \left|\mathcal{O}_{\mathbb{P}^2_k}(2)\right|$ generated by Q and twice its tangent line T_pQ at p. Then $\mathrm{Aut}_k(S)$ is the free product of $\mathrm{Aut}(\mathbb{P}^2_k, Q)|_S$ and $\mathrm{Aut}(S, \rho_p)$ amalgamated along their intersection.*

The scheme of the article is the following: in the first section, we review standard material on projective completions of smooth quasi-projective surfaces and certain rational fibrations on them. Section two is devoted to the proof of Theorem 1, which proceeds through the analysis of the structure of the total transform of the divisor $Q \cup \overline{A}$ in a minimal log-resolution of the pair $(\mathbb{P}^2_k, Q \cup \overline{A})$. Our argument, inspired by a recent alternative proof of the Abhyankar-Moh and Lin-Zaidenberg theorems due to Palka [14], uses techniques and classification results from the theory of $\mathbb{Q}$-*acyclic complex surfaces*, that is, normal complex surfaces with trivial reduced rational homology groups. A standard reference for most of these results is [13], to which we refer the reader for a more complete picture of the theory of non complete algebraic surfaces. Theorem 2 is proved in the third section, in which we give in addition an explicit set of generators of $\mathrm{Aut}_k(S)$ for a suitably chosen model of Q up to projective equivalence. We also derive from this description examples illustrating that similarly to the situation for the affine plane, Theorem 1 does not hold over fields of positive characteristic.

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1. Preliminaries and notations

In what follows, the term k-variety refers to a geometrically integral scheme X of finite type over a base field k of arbitrary characteristic. A morphism of k-varieties is a morphism of k-schemes. A surface V is a k-variety of dimension 2, and by a curve on a surface, we mean a geometrically reduced closed sub-scheme $C \subset V$ of pure codimension 1 defined over k.

1.1. SNC divisors and smooth completions.

(i) An *SNC divisor* on a smooth projective surface X is a curve B on X with smooth irreducible components and ordinary double points only as singularities. Equivalently, for every closed point $p \in B$, the local equations of the irreducible components of B passing through p form a part of regular sequence in the maximal ideal $\mathfrak{m}_{S,p}$ of the local ring $\mathcal{O}_{S,p}$ of S at p.

An SNC divisor B on X is said to be *SNC-minimal* if there does not exist any strictly birational projective morphism $\tau : X \to X'$ onto a smooth projective surface X' with exceptional locus contained in B such that $\tau_*(B)$ is SNC. If k is algebraically closed, then this property is equivalent to the fact that any (-1)-curve E contained in B is *branching*, i.e. meets at least three other irreducible components of B.

(ii) A *smooth completion* of a smooth quasi-projective surface V is a pair (X, B) consisting of a smooth projective surface X and an SNC divisor $B \subset V$ such that $X \setminus B \simeq V$.

1.2. **Rational trees and rational chains.**

(i) A *geometrically rational tree* B on a smooth projective surface X is an SNC divisor whose irreducible components are geometrically rational curves and such that the dual graph of the base extension $B_{\overline{k}}$ of B to an algebraic closure $\overline{k}$ of k is a tree. A *geometrically rational chain* B is a geometrically rational tree such that the dual graph of $B_{\overline{k}}$ is a chain. A *rational tree* (resp. *rational chain*) is a geometrically rational tree (resp. geometrically rational chain) whose irreducible components are all k-rational.

The irreducible components $B_0, \ldots, B_r$ of a rational chain B can be ordered in such a way that $B_i \cdot B_j = 1$ if $|i-j| = 1$ and 0 otherwise. A rational chain B with such an ordering on the set of its irreducible components is said to be *oriented*. The components B_0 and B_r are called respectively the left and right boundaries of B, and we say by extension that an irreducible component B_i of B is on the left of another one B_j when $i < j$. The sequence of self-intersections $[B_0^2, \ldots, B_r^2]$ is called the *type* of the oriented rational chain B. An *oriented subchain* of an oriented rational chain B is a rational chain Z whose support is contain in that of B. We say that an oriented rational chain B is composed of subchains $D_1, \ldots, D_s$, and we write $B = D_1 \triangleright \cdots \triangleright D_s$, if the D_i are oriented subchains of B whose union is B and the irreducible components of D_i precede those of D_j for $i < j$.

(ii) An oriented rational chain $F \triangleright C \triangleright E$ where F and C are irreducible and E is an oriented subchain, possibly empty, is said to be *m-standard, $m \in \mathbb{Z}$,* if it is of type $[0, -m]$ or $[0, -m, -a_1, \ldots, -a_r]$ where $a_i \geq 2$ for every $i = 1, \ldots, r$. It is an elementary exercise (see e.g. [3]) to check that every chain B with non negative definite intersection matrix can be transformed by a sequence of blow-ups and blow-downs whose centers are contained in the successive total transforms of B either into a 0-curve, or into a chain of type $[0, 0, 0]$, or into an m-standard chain for every $m \in \mathbb{Z}$.

(iii) In particular, every affine surface S non isomorphic to $\mathbb{A}^1_k \times (\mathbb{A}^1_k \setminus \{0\})$ admitting a smooth completion (X_0, B_0) for which B_0 is a rational chain, admits a smooth completion (X, B) for which $B = F \triangleright C \triangleright E$ is m-standard chain (see e.g. [4, Lemma 2.7]). Furthermore, it follows from a result of Danilov and Gizatullin [3, Corollary 2] that the type of the subchain E is an invariant of S, in the sense that if (X', B') is another smooth completion of S by an m'-standard chain $B' = F' \triangleright C' \triangleright E'$ then the type of E' is either equal to that of E or to that of E equipped with the reversed orientation. For instance, if $S = \mathbb{P}^2_k \setminus Q$ is the complement of a smooth k-rational conic Q in $\mathbb{P}^2$, then for every smooth completion (X, B) of S by a m-standard chain $B = F \triangleright C \triangleright E$, the subchain E has type $[-2, -2, -2]$.

1.3. **Recollection on $\mathbb{P}^1$, $\mathbb{A}^1$ and $\mathbb{A}^1_*$ -fibrations.** We review some basic properties of $\mathbb{P}^1$-fibrations on smooth projective surfaces and their restrictions to certain of their open subsets, see e.g [13, Chapter 3] for more details.

(i) By a $\mathbb{P}^1$-fibration on a smooth projective surface X, we mean a surjective morphism $\overline{\rho} : X \to \overline{Z}$ onto a smooth projective curve $\overline{Z}$ whose generic fiber is isomorphic to the projective line over the function field of $\overline{Z}$. It is well known that every $\mathbb{P}^1$-fibration $\overline{\rho} : X \to \overline{Z}$ is obtained from a Zariski locally trivial $\mathbb{P}^1$-bundle over $\overline{Z}$ by a finite sequence of blow-ups of points. In particular, every such $\mathbb{P}^1$-fibration has a section, and its singular fibers are supported by geometrically rational trees on X. If X is k-rational, then it follows from the Riemann-Roch Theorem, that for every smooth k-rational curve F with self-intersection 0, the complete linear system $|F|$ defines a $\mathbb{P}^1$-fibration $\overline{\rho}_{|F|} : X \to \mathbb{P}^1_k$ having F as a smooth fiber.

(ii) An $\mathbb{A}^1$-fibration on a smooth quasi-projective surface V is a surjective morphism $\rho : V \to Z$ onto a smooth curve Z whose generic fiber is isomorphic to the affine line over the function field of Z. Every $\mathbb{A}^1$-fibration is the restriction of a $\mathbb{P}^1$-fibration $\overline{\rho} : X \to \overline{Z}$ over the smooth projective model $\overline{Z}$ of Z on a smooth completion (X, B) of V. Furthermore, one can always find such a smooth completion for which B has the form $B = \bigcup_{z \in \overline{Z} \setminus Z} F_z \cup C \cup \bigcup_{z \in Z} H_z$ where, $F_z = \overline{\rho}^{-1}(z) \simeq \mathbb{P}^1_{\kappa(z)}$ for every $z \in \overline{Z} \setminus Z$, C is a section of $\overline{\rho}$, and where for every $z \in Z$, H_z is an SNC-minimal geometrically rational subtree of $\overline{\rho}^{-1}(z)$, possibly empty, the support of the fiber $\overline{\rho}^{-1}(z)$ being equal to the union of H_z and of the closure in X of the support of $\rho^{-1}(z)$.

If in addition V is affine, then every nonempty H_z contains a $\kappa(z)$-rational irreducible component intersecting C, the closure in X of every irreducible component of $\rho^{-1}(z)$ is isomorphic to the projective line over a finite extension κ' of $\kappa(z)$, and it intersects H_z transversally in a unique κ'-rational point. A scheme theoretic closed fiber $\rho^{-1}(z)$ of $\rho : V \to Z$ which is not isomorphic to $\mathbb{A}^1_{\kappa(z)}$ is said to be *degenerate*.

(iii) An $\mathbb{A}^1_*$-fibration on smooth quasi-projective surface V is a surjective morphism $\xi : V \to Z$ onto a smooth affine curve Z whose geometric generic fiber is isomorphic to the punctured affine line $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$ over an algebraic closure of the function field of Z. We say that ξ is twisted if the generic fiber of ξ is a nontrivial form of $\mathbb{A}^1_*$ over the function field of Z, and untwisted otherwise.

2. Affine lines in $\mathbb{P}^2 \setminus Q$

This section is devoted to the proof of Theorem 1. A field k of characteristic zero being fixed throughout, we let S be the complement of a smooth conic Q in $\mathbb{P}^2_k$, we let $A \subset S$ be a closed curve isomorphic to $\mathbb{A}^1_k$ and we let $\overline{A}$ be its closure in $\mathbb{P}^2$. Let $\nu : C \to \overline{A}$ be the normalization of $\overline{A}$. Since $A \simeq \mathbb{A}^1_k$, C is isomorphic to $\mathbb{P}^1_k$ and $C \setminus \nu^{-1}(A)$ consists of a unique point. So $\overline{A} \setminus A$ consists of a unique point $p = Q \cap \overline{A}$, which is thus necessarily k-rational, at which $\overline{A}$ has a unique local analytic branch. In particular, Q is k-rational. Since $\mathrm{Aut}(\mathbb{P}^2_k)$ acts transitively on the set of pairs (Q, p) consisting of a smooth k-rational conic and a k-rational point on it and since the stabilizer $\mathrm{Aut}(\mathbb{P}^2_k, Q, p) \subset \mathrm{Aut}(\mathbb{P}^2_k)$ of a given pair (Q, p) acts transitively on Q, we are reduced to establish the following:

Proposition 3. *There exists a smooth k-rational conic $Q' \subset \mathbb{P}^2_k$ and a birational map $\Psi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ restricting to an isomorphism $\psi : S = \mathbb{P}^2_k \setminus Q \xrightarrow{\sim} \mathbb{P}^2_k \setminus Q'$ mapping $\overline{A}$ either to a smooth k-rational conic intersecting Q' in a unique k-rational point p' or to the tangent line $T_{p'}Q'$ of Q' at a k-rational point p'.*

The proof of this proposition if given in § 2.1 and § 2.2 below.

2.1. **Log-resolution setup and preliminary observations.** Let $\sigma : (X', D') \to (\mathbb{P}^2_k, Q \cup \overline{A})$ be the minimal log-resolution of the pair $(\mathbb{P}^2_k, Q \cup \overline{A})$. Recall that by definition, X' is smooth, σ is a projective birational morphism restricting to an isomorphism over $\mathbb{P}^2_k \setminus Q \cup \overline{A}$, and minimal for the property that $D' = \sigma^{-1}(Q \cup \overline{A})_{\mathrm{red}}$ is an SNC divisor. Note that if $p = Q \cap \overline{A}$ is a singular point of $\overline{A}$ then σ is in particular a log-resolution of the singularity of $\overline{A}$. Since $\overline{A} \cdot Q \geq 2$ and p is k-rational, σ consists of the blow-up of p followed by a sequence of blow-ups of k-rational points supported on the successive total transforms of $Q \cup \overline{A}$. Since $\overline{A}$ has a unique analytic branch at p, D' is a rational tree of the form $\overline{A} \cup D'_1 \cup E' \cup D'_2$, where $D'_1 \cup E' \cup D'_2 = \sigma^{-1}(Q)_{\mathrm{red}}$ consists of a rational tree D'_1 containing the proper transform of Q, a nonempty SNC-minimal rational chain D'_2 with negative definite intersection matrix and a (-1)-curve E' such that $D'_1 \cap E'$, $D'_2 \cap E'$ and $\overline{A}' \cap E'$ all consist of a unique point.

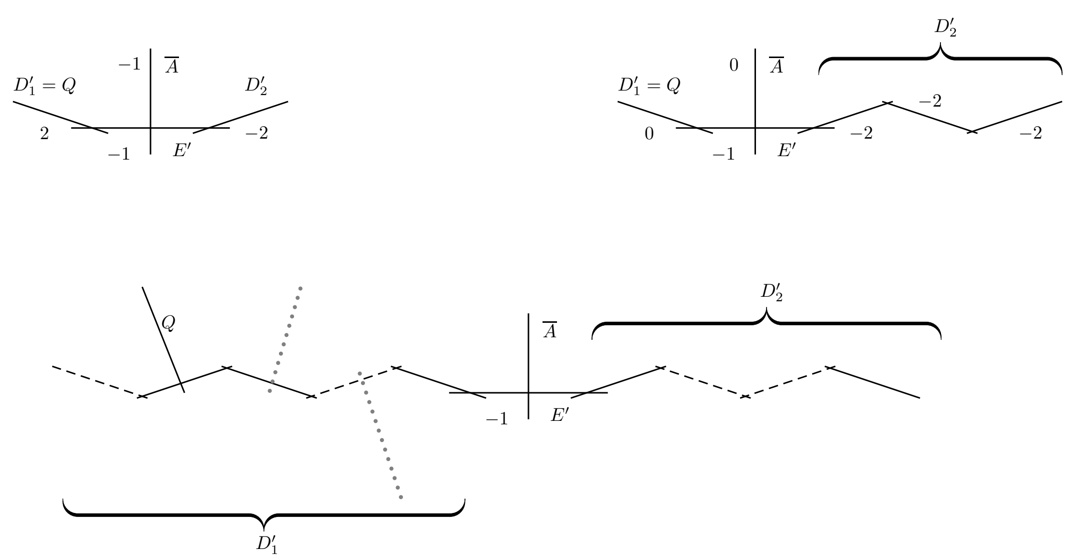

Figure 2.1. Structure of the divisor D' in the case where $\overline{A}$ is a line, a rational conic, and a general curve respectively. The gray dotted lines represent rational subtrees of D'_1.

The proper transform of Q is the unique possible non-branching (-1)-curve in D'_1 and we let $\sigma' : (X', D') \to (X, D)$ be the map consisting of the contraction all successive non-branching (-1)-curves in D'_1 . The image of D' by σ' is again a rational tree $D = \tilde{A} \cup D_1 \cup E \cup D_2$ where $\tilde{A} = \sigma_*(\overline{A}) \simeq \mathbb{P}^1_k$, $D_1 = \sigma'_*(D_1)$ is an SNC-minimal rational tree, $D_2 = \sigma_*(D'_2)$ is a rational chain isomorphic to D_2, and $E = \sigma'_*(E')$. By construction $S = \mathbb{P}^2_k \setminus Q$ is isomorphic to $X \setminus (D_1 \cup E \cup D_2)$.

We now establish two crucial auxiliary results which will serve for the analysis of the case where the self-intersection of the proper transform $\tilde{A}$ of $\overline{A}$ in X is negative.

Lemma 4. *If $\tilde{A}^2 < 0$ then the following hold:*

a) The rational tree D_1 is not empty, and its intersection matrix is not negative definite.

b) Every closed irreducible curve C in X distinct from $\tilde{A}$ or an irreducible component of D_2 intersects D_1.

Proof. These properties are invariant under extension and restriction of the base field k. So by first replacing k by a subfield $k_0 \subset k$ of finite transcendence degree over $\mathbb{Q}$ over which the projective surface X, the divisor $D = \tilde{A} \cup D_1 \cup E \cup D_2$ and the curve C are defined and then taking base extension by an embedding $k_0 \hookrightarrow \mathbb{C}$, we may assume from the beginning that $k = \mathbb{C}$. Then since $S \simeq X \setminus (D_1 \cup E \cup D_2) \simeq \mathbb{P}^2 \setminus Q$ is $\mathbb{Q}$-acyclic, the classes E and of the irreducible components of D_1 and D_2 form a basis of $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ [13, Lemma 4.2.1]. In $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, we may thus write $\tilde{A} \equiv (\tilde{A}^2)E + R$, where R is the class of a $\mathbb{Q}$-divisor supported on $D_1 \cup D_2$ and since $\tilde{A}^2 \neq 0$, it follows that the classes of $\tilde{A}$ and of the irreducible components of D_1 and D_2 also form a basis of $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\tau : X \to \tilde{X}$ be the birational morphism onto a normal surface with at most cyclic quotient singularities obtained by contracting $\tilde{A}$ and the negative definite rational chain D_2. The irreducible components of $\tau(D_1)$ then form a basis of $\mathrm{Cl}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and since $\tau(D_1)$ is connected, it follows that $\tilde{X} \setminus \tau(D_1)$ is a normal $\mathbb{Q}$-acyclic surface, hence an affine surface by virtue [7]. So by [8], $\tau(D_1)$ is the support of an ample divisor on $\tilde{X}$, in particular, $\tau(D_1)$ is not empty, its intersection matrix is not negative definite and it intersects every proper curve in $\tilde{X}$. Because D_1 is disjoint from D_2 and $\tilde{A}$, τ restricts to an isomorphism in an open neighborhood of D_1, and so the assertion follows. □

Lemma 5. *If $E^2 = -1$ and $\tilde{A}^2 < 0$ then $\tilde{A}^2 = -1$.*

Proof. Similarly as in the proof of the previous lemma, the assertion is invariant under restriction to a subfield $k_0 \subset k$ of finite transcendence degree over $\mathbb{Q}$ over which X and D are defined and extension to $\mathbb{C}$ via the choice of an embedding $k_0 \hookrightarrow \mathbb{C}$. So we may again assume that $k = \mathbb{C}$. The following argument is inspired from [14]. Suppose for contradiction $\tilde{A}^2 < -1$. Then $\Delta = D_1 \cup \tilde{A} \cup D_2$ is an SNC-minimal divisor on X with three connected components, whose complement is a smooth quasi-projective surface V such that $V \setminus (E \cap V) \simeq S \setminus A$. The theory of minimal models of log-surfaces (see [13, Chapter 3] for a detailed account) asserts the existence of a sequence of projective birational morphisms

$$f = f_n \circ \cdots \circ f_1 : (X, \Delta) = (X_0, \Delta_0) \xrightarrow{f_1} (X_1, \Delta_1) \xrightarrow{f_2} \cdots \xrightarrow{f_n} (X_n, \Delta_n) \tag{2.1}$$

with the following properties:

a) For every $i = 1, \ldots, n$, X_i is a smooth projective surface and Δ_i is an SNC-minimal reduced divisor

b) For every $i = 1, \ldots, n$, $f_i : (X_{i-1}, \Delta_{i-1}) \to (X_i, \Delta_i)$ is the contraction of a (-1)-curve $\ell_i \not\subset \Delta_{i-1}$ intersecting Δ_{i-1} transversally in at most two smooth points and each connected component of Δ_{i-1} at most once and such that $\overline{\kappa}(X_i, \Delta_i) = \overline{\kappa}(X_i, \Delta_i \cup \ell_i)$, followed by the SNC-minimalization Δ_i of the push-forward $(f_i)_*\Delta_{i-1}$ of Δ_i. Furthermore, if ℓ_i intersects precisely two connected components of Δ_{i-1} then one of these components is rational chain with negative definite intersection matrix.

c) The pair (X_n, Δ_n) is *almost-minimal*, meaning that every log Minimal Model Program ran from (X_n, Δ_n) terminates with a log terminal projective surface (X_{n+1}, Δ_{n+1}) and the induced birational morphism $f_{n+1} : (X_n, \Delta_n) \to (X_{n+1}, \Delta_{n+1} = (f_{n+1})_*\Delta_n)$ contracts only irreducible components of Δ_n onto quotient singularities of X_{n+1}. Equivalently, (X_{n+1}, Δ_{n+1}) is a log-terminal pair which does not contain any proper curve ℓ such that $\ell^2 < 0$ and $\ell \cdot (K_{X_{n+1}} + \Delta_{n+1}) < 0$ and $f_{n+1} : (X_n, \Delta_n) \to (X_{n+1}, \Delta_{n+1})$ is its minimal log-resolution.

Since $\Delta_0 = D_1 \cup \tilde{A} \cup D_2$ has three connected components, Δ_n has at most three connected components, and since by Lemma 4 a) the intersection matrix of D_1 is not negative definite, the connected component of Δ_n containing the image of D_1 is not contracted to a point by f_{n+1}. So the exceptional locus $\mathrm{Exc}(f_{n+1})$ consists of at most two connected components of Δ_n, and since Δ_n is SNC-minimal, $f_{n+1}(\mathrm{Exc}(f_{n+1}))$ consists of singular points of X_{n+1}. In particular, the local fundamental group G_p at every point $p \in f_{n+1}(\mathrm{Exc}(f_{n+1}))$ has order at least 2. An elementary calculation shows that the topological Euler characteristic of the surface $X_{i-1} \setminus \Delta_{i-1}$ increases at a step if and only if the curve ℓ_i contracted by f_i intersects two connected components of Δ_{i-1} and the union of ℓ_i with these components is contracted by f_i to a smooth point of X_i. If such a curve existed, then by Lemma 4 b), one of these connected components would necessarily be the one containing the image of D_1, which would imply in turn that the intersection matrix of D_1 is definite negative, a contradiction to Lemma 4 a). So for every $i = 1, \ldots, n$,

$$\chi(X_i \setminus \Delta_i) \leq \chi(V) = \chi(S \setminus A) + \chi(E \cap V) = -1.$$

Now suppose that $\overline{X} \setminus \Delta$ has non-negative logarithmic Kodaira dimension $\overline{\kappa}(X, \Delta) \geq 0$. Then $\overline{\kappa}(X_n, \Delta_n) \geq 0$ and since (X_n, Δ_n) is almost-minimal, it follows from the logarithmic Bogomolov-Miyaoka-Yau inequality

[12] that

$$0 \leq \chi(X_n \setminus \Delta_n) + \sum_{p \in f_{n+1}(\mathrm{Exc}(f_{n+1}))} \frac{1}{|G_{p_i}|} \leq \chi(X_n \setminus \Delta_n) + \frac{1}{2}\pi_0(\mathrm{Exc}(f_{n+1})).$$

The only possibility is thus that $\chi(X_n \setminus \Delta_n) = \chi(V) = -1$ and that $f_{n+1}(\mathrm{Exc}(f_{n+1}))$ consists of two points, with local fundamental group $\mathbb{Z}_2$. The corresponding connected components of Δ_n are thus simply (-2)-curves, and we deduce in turn from Lemma 4 a) that D_2 and $\tilde{A}$ are (-2)-curves themselves. Since $E^2 = -1$ by hypothesis, the complete linear system $|D_2 + 2E + \tilde{A}|$ on X defines a $\mathbb{P}^1$-fibration $\overline{\xi} : X \to \mathbb{P}^1$ having the irreducible component $D_{1,E}$ of D_1 intersecting E as a 2-section. The restriction of ξ to $S = X \setminus (D_1 \cup E \cup D_2)$ is thus a twisted $\mathbb{A}^1_*$-fibration $\xi : S \to Z$ over an open subset Z of $\mathbb{P}^1$. Since S is a $\mathbb{Q}$-acyclic, it follows from [13, Lemma 4.5.1] that $Z = \mathbb{A}^1$. The fiber $\overline{\xi}$ over the point $\infty = \mathbb{P}^1 \setminus Z$ is supported by a connected component F_∞ of $D_1 - D_{1,E}$, and since D_1 is a rational tree, $D_{1,E}$ intersects F_∞ transversally in a unique point. Since D_1 is SNC-minimal, we infer that $F_\infty = F_{\infty,1} \triangleright L \triangleright F_{\infty,2}$ is a chain of type $[-2,-1,-2]$ intersecting $D_{1,E}$ along L. Since $S = \mathbb{P}^2 \setminus Q$ admits a smooth completion by a rational curve, it follows from [4, Theorem 2.16] that by contracting successively E, D_2 and then all successive non-branching (-1)-curves in D_1, the image of D in the corresponding smooth projective surface Y is an SNC-minimal chain B such that $Y \setminus B \simeq S$. Since the irreducible components $F_{\infty,1}$ and $F_{\infty,2}$ are untouched during this process, B must be equal to the image of F_∞, which is a chain of type $[-2, a, -2]$ for some $a \geq 0$. But one checks that no such chain can be transformed into one of type $[0,-1,-2,-2,-2]$, a contradiction to § 1.2 (iii).

So $\overline{\kappa}(X, \Delta) = -\infty$, and hence $\overline{\kappa}(X_n, \Delta_n) = -\infty$. Since $\chi(X_n \setminus \Delta_n) \leq -1$ and the connected component of Δ_n containing the image of D_1 is not contracted by f_{n+1}, it follows from Theorem 3.15.1 and Theorem 5.1.2 in [13] that $X_n \setminus \Delta_n$ $\mathbb{A}^1$-ruled, i.e. contains a Zariski open subset of the form $C \times \mathbb{A}^1$ for a certain smooth rational curve C. It follows in turn that V is $\mathbb{A}^1$-ruled, and we let $q : V \dashrightarrow \mathbb{P}^1$ and $\overline{q} : X \dashrightarrow \mathbb{P}^1$ be the rational maps induced by the projection pr_C. By virtue of Lemma 4 b), the closure F in X of a general fiber of q intersects D_1. So $q : V \to Z$ is a well defined $\mathbb{A}^1$-fibration over an open subset Z of $\mathbb{P}^1$, and if $\overline{q}$ is not regular, then its unique proper base point is supported on D_1. Furthermore, D_2 and $\tilde{A}$ are necessarily contained in fibers of $\overline{q} : X \dashrightarrow \mathbb{P}^1$. So $\overline{q}$ is well defined on S, restricting to an $\mathbb{A}^1$-fibration $\rho : S \to \mathbb{A}^1$ containing A in one of its fibers. Let $\delta : Y \to X$ be the minimal resolution of the indeterminacies of $\overline{q}$, so that $\overline{\rho} = \overline{q} \circ \sigma : Y \to \mathbb{P}^1$ is an everywhere defined $\mathbb{P}^1$-fibration, say with section $C \subset \delta^{-1}(D_1)$. Since the closures in X of the general fibers of q intersect D_1, the proper transform in Y of $D_2 \cup E \cup \tilde{A}$ is contained in $\overline{\rho}^{-1}(\rho(A))$ and since by [13, Theorem 4.3.1], all fibers of $\rho : S \to \mathbb{A}^1$ are irreducible, $\overline{\rho}^{-1}(\rho(A))$ is the union of the proper transform of $D_2 \cup E \cup \tilde{A}$, the proper transform of a subset of irreducible components of D_1, possibly empty, and a subset of exceptional divisors of σ, again possibly empty. Since D_2 is a nonempty chain with negative definite intersection matrix and $\tilde{A}^2 < -1$, at the step of the contraction of $\overline{\rho}^{-1}(\rho(A))$ onto a smooth fiber of a $\mathbb{P}^1$-fibration, the image of E would have to become a (-1)-curve intersecting the images of D_2 and $\tilde{A}$, and either an irreducible component of the image of $\overline{\rho}^{-1}(\rho(A))$ or the image of the section C, which is impossible. This is absurd, and so $\tilde{A}^2 = -1$ necessarily. □

2.2. **Proof of Proposition 3.**
Proposition 3, and hence Theorem 1, are now consequences of the following lemma:

Lemma 6. *With the notation of §2.1 above, the following alternative holds:*

1) Either $\tilde{A}^2 = 0$ and then there exists a birational map $\psi : X \dashrightarrow \mathbb{P}^2$ restricting to an isomorphism between $S = X \setminus D$ and the complement of a smooth conic Q' and mapping $\tilde{A}$ to a smooth conic intersecting Q' with multiplicity 4 at a single k-rational point p'.

2) Or $\tilde{A}^2 = -1$ and then there exists a birational map $\psi : X \dashrightarrow \mathbb{P}^2$ restricting to an isomorphism between $S = X \setminus D$ and the complement of a smooth conic Q' and mapping $\tilde{A}$ to the tangent line to Q' at a k-rational point p'.

Proof. We consider the two cases $\tilde{A}^2 \geq 0$ and $\tilde{A}^2 < 0$ separately.

Case 1). $\tilde{A}^2 \geq 0$. Suppose first that $a = \tilde{A}^2 > 0$. Then by performing a minimal sequence of blow-ups of k-rational points supported on the successive proper transforms of $\tilde{A}$, starting with that of $E \cap \tilde{A}$ and then continuing with those of the intersection point of the proper transform of $\tilde{A}$ with the previous exceptional divisor produced, we obtain a surface $\beta : Y \to X$ in which the proper transform of $\tilde{A}$ has self-intersection 0, while its reduced total transform is a rational chain $\tilde{A} \triangleright H \triangleright D_3$ where H is a (-1)-curve and D_3 is a chain of $a - 1$ curves with self-intersection -2 connecting H to E. The complete linear system $|\tilde{A}|$ then defines a $\mathbb{P}^1$-fibration $\overline{q} : Y \to \mathbb{P}^1_k$ having $\tilde{A}$ as a smooth fiber and H as a section.

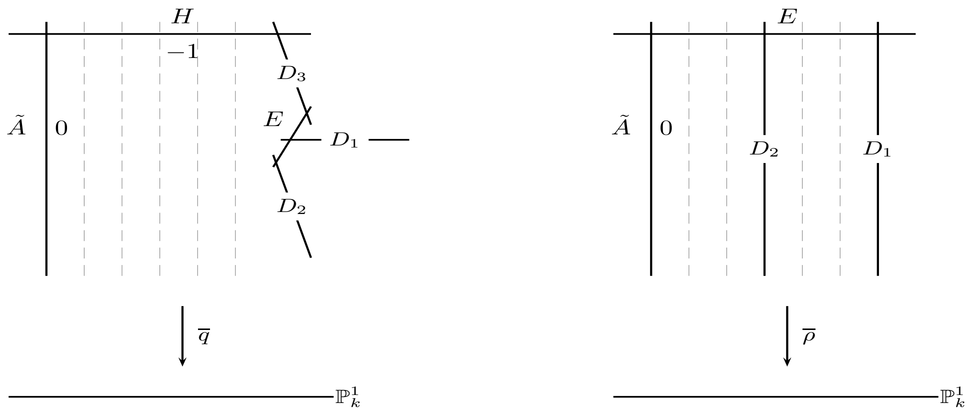

FIGURE 2.2. The $\mathbb{P}^1$-fibrations $\overline{q} : Y \to \mathbb{P}^1_k$ and $\overline{\rho} : X \to \mathbb{P}^1_k$ respectively.

The surface Y, the rational chain $\tilde{A} \rhd H \rhd D_3$ and the $\mathbb{P}^1$-fibration are all defined over a subfield $k_0 \subset k$ of finite transcendence degree over $\mathbb{Q}$, and applying [13, Theorem 4.3.1] to the complex surface obtained by base change via an embedding $k_0 \hookrightarrow \mathbb{C}$, we conclude that the restriction of $\overline{q}$ to $S \simeq Y \setminus \beta^{-1}(D)$ is an $\mathbb{A}^1$-fibration $q : S \to \mathbb{A}^1_k$ with a unique degenerate fiber. Since the union of the proper transform of $D_1 \cup E \cup D_2$ with D_3 is connected, it would be fully contained in a unique degenerate fiber of $\overline{q} : Y \to \mathbb{P}^1_k$ hence equal to it, and q would be an $\mathbb{A}^1$-fibration without any degenerate fiber, a contradiction.

So $\tilde{A}^2 = 0$, and the $\mathbb{P}^1$-fibration $\overline{\rho} : X \to \mathbb{P}^1_k$ defined by the complete linear system $|\tilde{A}|$ restricts to an $\mathbb{A}^1$-fibration $\rho : S \to \mathbb{A}^1_k$ having $\tilde{A} \cap S \simeq A$ as a fiber. Since E is a section of $\overline{\rho}$, for the same reason as before, either D_1 or D_2 supports a full fiber F of $\overline{\rho} : X \to \mathbb{P}^1_k$, and since the intersection matrix of D_2 is negative definite and D_1 is SNC minimal, it must be that $F = D_1$ is a (0)-curve. So $D = D_1 \rhd E \rhd D_2$ is a $(-E^2)$-standard chain, and by §1.2 (iii), D_2 is thus a chain of type $[-2,-2,-2]$. After performing elementary transformations with center on E if necessary to reach a smooth completion (X', B') of S by a rational chain $F_\infty \rhd E \rhd D_2$ of type $[0,-1,-2,-2,-2]$, the images of F_∞ and $\tilde{A}$ by the contraction $\tau : X' \to X''$ of $E \cup D_2$ are curves of self-intersection 4 intersecting each others with multiplicity 4 in a single k-rational point. Since $X'' \setminus \tau(F_\infty) \simeq S$ and $\mathrm{Cl}(S) \simeq \mathbb{Z}_2$, $\mathrm{Cl}(X'') \otimes_{\mathbb{Z}} \mathbb{Q}$ is freely generated by the class of $\tau(F_\infty)$, and since X'' is a smooth k-rational surface, we conclude that $X'' \simeq \mathbb{P}^2_k$ and that $\tau(F_\infty)$ and $\tau(\tilde{A})$ are smooth k-rational conics.

Case 2). $\tilde{A}^2 < 0$. By Lemma 4 a), D_1 is not empty. We consider two subcases according to the self-intersection of E.

Subcase 1). $E^2 = -1$. By virtue of Lemma 5, $\tilde{A}^2 = -1$. It follows that the complete linear system $|E + \tilde{A}|$ on X defines a $\mathbb{P}^1$-fibration $\overline{\xi} : X \to \mathbb{P}^1_k$ having the irreducible components $D_{1,E}$ and $D_{2,E}$ of D_1 and D_2 intersecting E as disjoint sections. The restriction of $\overline{\xi}$ to $S = X \setminus (D_1 \cup E \cup D_2)$ is thus an untwisted $\mathbb{A}^1_*$-fibration $\xi : S \to Z$ over a smooth curve $Z \subset \mathbb{P}^1_k$ having $A = \tilde{A} \cap S$ as a degenerated fiber of multiplicity 1. Let again $k_0 \subset k$ be a subfield of finite transcendence degree over $\mathbb{Q}$ over which X, D and $\overline{\xi}$ are defined, denote by $X_{\mathbb{C}}$, $D_{\mathbb{C}}$ and $\overline{\xi}_{\mathbb{C}}$ the corresponding surface, divisor and morphism obtained by base extension via an embedding $k_0 \hookrightarrow \mathbb{C}$, and let $S_{\mathbb{C}} = X_{\mathbb{C}} \setminus (D_{1,\mathbb{C}} \cup E_{\mathbb{C}} \cup D_{2,\mathbb{C}}) \simeq \mathbb{P}^2_{\mathbb{C}} \setminus Q_{\mathbb{C}}$. It follows from Lemma 4.5.1 and Theorem 4.6.2 in [13] applied to $S_{\mathbb{C}}$ that $Z_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$ and that $\xi_{\mathbb{C}} = \overline{\xi}_{\mathbb{C}} \mid_{S_{\mathbb{C}}}$ has at most a second degenerate fiber, whose support F is isomorphic to $\mathbb{A}^1_*$. Since $(D_{2,E})^2_{\mathbb{C}} = D^2_{2,E} \leq -2$ and $H_1(S_{\mathbb{C}};\mathbb{Z}) = \mathbb{Z}_2$ we deduce from § 4.5.2 (5) and Theorem 4.6.1 (2) in [13] that $\overline{\xi}_{\mathbb{C}}^{-1}(\overline{\xi}_{\mathbb{C}}(E_{\mathbb{C}} \cup \tilde{A}))$ is actually the unique degenerate fiber of $\overline{\xi}_{\mathbb{C}}$. It follows in turn that $D_{1,\mathbb{C}} = (D_{1,E})_{\mathbb{C}}$ and $D_{2,\mathbb{C}} = (D_{2,E})_{\mathbb{C}}$ and hence that $D_1 = D_{1,E}$ and $D_2 = D_{2,E}$. Thus D is a chain $D_1 \rhd E \rhd D_2$ of type $[D_1^2, -1, D_2^2]$, where $D_2^2 \leq -2$ and where, by virtue of Lemma 4 a), $D_1^2 \geq 0$ because the intersection matrix of D_1 is not negative definite. Such a chain has a 1-standard form of type $[0,-1,-2,-2,-2]$ if and only if $D_1^2 = 2$ and $D_2^2 = -2$, and then the images of D_1 and $\tilde{A}$ by the contraction $\tau : X \to \mathbb{P}^2_k$ of E and D_2 are respectively a smooth k-rational conic Q' and its tangent line $T_{p'}Q'$ at the k-rational point $p' = \tau(E \cup D_2)$.

Subcase 2). $E^2 \geq 0$. Since D_1 is SNC-minimal, D is SNC minimal, and since the boundary of every SNC-minimal completion of S is a rational chain by virtue of [4, Theorem 2.16], D is a rational chain. So D_1 is an SNC-minimal rational chain with non negative definite intersection matrix, and hence it contains an irreducible component with non negative self-intersection. If D_1 is a (0)-curve, then the $\mathbb{P}^1$-fibration $\overline{q} : X \to \mathbb{P}^1_k$ defined by $|D_1|$ has E as a section, hence restricts to an $\mathbb{A}^1$-fibration on S containing $A = \tilde{A} \cap S$

in one of its fibers. Since D_2 and $\tilde{A}$ both intersect E, they are contained in two different fibers of $\overline{q}$. But since $\tilde{A}^2 < 0$, $\tilde{A}$ would be properly contained in a degenerate fiber of $\overline{q}$, which is impossible by virtue of §1.3 (ii). So D_1 is either reducible or irreducible with positive self-intersection. By elementary birational transformations whose centers blown-up and curves contracted are k-rational and supported on D_1 and its successive images, we obtain a smooth completion W of S for which the reduced total transform $\tilde{D} = W \setminus S$ of D is a rational chain $\tilde{D}_1 \triangleright E \triangleright D_2$, where the reduced total transform $\tilde{D}_1 = F_\infty \triangleright C \triangleright D_3$ of D_1 is a 1-standard chain. The complete linear system $|F_\infty|$ on Y defines a $\mathbb{P}^1$-fibration $\overline{\rho} : W \to \mathbb{P}^1_k$ with section C, containing $D_3 \cup E \cup D_2 \cup \tilde{A}$ in one of its fibers.

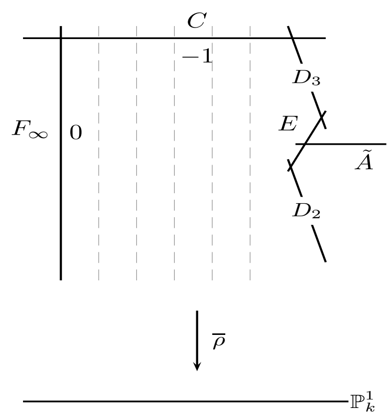

FIGURE 2.3. The $\mathbb{P}^1$-fibration $\overline{\rho} : W \to \mathbb{P}^1_k$.

So $E^2 \leq -1$ and since E intersects D_2, $\tilde{A}$ and either an irreducible component of D_3 if D_3 is not empty or C otherwise, we have $E^2 \leq -2$ necessarily. The chain $D_3 \triangleright E \triangleright D_2$ is thus SNC-minimal, hence of type $[-2, -2, -2]$ by §1.2 (iii). By applying [13, Theorem 4.3.1] to the surface $S_{\mathbb{C}}$ defined in a similar way as in the previous subcase, we deduce that $A_{\mathbb{C}}$ must be the support of the unique degenerate fiber of the restriction $\rho_{\mathbb{C}} : S_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ of $\overline{\rho}_{\mathbb{C}}$. So $\overline{\rho}_{\mathbb{C}}^{-1}(\rho_{\mathbb{C}}(A_{\mathbb{C}}))_{\mathrm{red}} = D_{3,\mathbb{C}} \cup E_{\mathbb{C}} \cup D_{2,\mathbb{C}} \cup \tilde{A}_{\mathbb{C}}$ implying that $\tilde{A}^2_{\mathbb{C}} = -1$. Thus $\tilde{A}^2 = -1$ and the images of F_∞ and $\tilde{A}$ by the contraction of $C \cup D_3 \cup E \cup D_2$ to a k-rational point p' are then respectively a smooth k-rational conic Q' in $\mathbb{P}^2_k$ and its tangent line $T_{p'}Q'$ at p'. □

As a consequence of Proposition 6 and of the fact that $\mathrm{Aut}(\mathbb{P}^2_k)$ acts transitively on the set of pairs (Q, p) consisting of a smooth k-rational conic and a k-rational point on it, we obtain the following:

Corollary 7. *Let k be a field of characteristic 0 and let $S = \mathbb{P}^2_k \setminus Q$ be the complement of a smooth k-rational conic $Q \subset \mathbb{P}^2_k$. Then the following hold:*

a) Every closed curves $A \simeq \mathbb{A}^1_k$ is equal to the support of a fiber of an $\mathbb{A}^1$-fibration $\rho : S \to \mathbb{A}^1_k$. More precisely, there exists a smooth completion $(\mathbb{P}^2_k, Q')$ of S by a smooth k-rational conic Q' such that A is the support of a fiber of the $\mathbb{A}^1$-fibration induced by the restriction of the pencil generated by Q and twice its tangent line at a k-rational point p.

b) There exists a unique equivalence class of $\mathbb{A}^1$-fibrations $\rho : S \to \mathbb{A}^1_k$ on S up to automorphisms, in the sense that every two such $\mathbb{A}^1$-fibrations $\rho : S \to \mathbb{A}^1_k$ and $\rho' : S \to \mathbb{A}^1_k$ fit into a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\Psi"] \arrow[d, "\rho"'] & S \arrow[d, "\rho'"] \\ \mathbb{A}^1_k \arrow[r, "\psi"] & \mathbb{A}^1_k, \end{tikzcd}$$

for some automorphisms Ψ and ψ of S and $\mathbb{A}^1_k$ respectively.

Remark 8. In the complex case, it was established more generally in [9, Theorem 2.1] that on a smooth $\mathbb{Q}$-acyclic surface S admitting a smooth completion (X, B) by a chain of rational curves, every closed curve isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ is the support of a fiber of an $\mathbb{A}^1$-fibration $\rho : S \to \mathbb{A}^1_{\mathbb{C}}$. But every such $\mathbb{Q}$-acyclic surface different from $\mathbb{A}^2_{\mathbb{C}}$ or $\mathbb{P}^2_{\mathbb{C}} \setminus Q$ turns out to have more than one equivalence class of $\mathbb{A}^1$-fibrations up to action of its automorphism groups. Indeed, by [5, Theorem 5.6], a $\mathbb{Q}$-acyclic surface as above different from $\mathbb{A}^2_{\mathbb{C}}$ is isomorphic to the quotient $S_{m,q}$ of a smooth surface $S_m = \{xz = y^m - 1\} \subset \mathbb{A}^3_{\mathbb{C}}$, $m \geq 2$, by a free action of the group μ_m of complex m-th roots of unity of the form $(x,y,z)\mapsto(\varepsilon x,\varepsilon^q y,\varepsilon^{-1}z)$ where $q\in\{1,\ldots,m-1\}$ and $\gcd(m,q)=1$. Note that for every m and every such q, the involution $(x,y,z)\mapsto(z,y,x)$ of S_m descends to isomorphism between $S_{m,q}$ and $S_{m,m-q}$. The $\mathbb{A}^1$-fibration $\mathrm{pr}_x: S_m\to\mathbb{A}^1_{\mathbb{C}}$ descends to an $\mathbb{A}^1$-fibration $\rho_{m,q}: S_{m,q}\to\mathbb{A}^1_{\mathbb{C}}$ having $\rho_{m,q}^{-1}(0)$ as unique degenerate fiber, isomorphic to $\mathbb{A}^1_{\mathbb{C}}$, of multiplicity m, whose support $F_{m,q}$ generates the divisor class group $\mathrm{Cl}(S_{m,q})\simeq\mathbb{Z}_m$ of $S_{m,q}$. Furthermore, the μ_m-invariant regular 2-form $x^{m-q}(\frac{dx\wedge dy}{x})$ on S_m descends to a regular 2-form vanishing at order $m-q$ along $F_{m,q}$ and nowhere else, implying that $K_{S_{m,q}}\sim(m-q)F_{m,q}$. It follows that if $m\geq 3$, then the $\mathbb{A}^1$-fibrations $\rho_{m,q}$ and $\rho_{m,m-q}$ on $S_{m,q}\simeq S_{m,-q}$ are not equivalent under the action of $\mathrm{Aut}(S_{m,q})$. Indeed, otherwise there would exist an isomorphism $\Psi: S_{m,q}\xrightarrow{\sim}S_{m,m-q}$ and an automorphism ψ of $\mathbb{A}^1_{\mathbb{C}}$ fixing the origin such that $\rho_{m,m-q}\circ\Psi=\psi\circ\rho_{m,q}$, and we would have the relation $(m-q)F_{m,q}=qF_{m,q}$ in $\mathrm{Cl}(S_{m,q})$, in contradiction with the fact that $F_{m,q}$ has order m in $\mathrm{Cl}(S_{m,q})$.

3. Automorphisms of $\mathbb{P}^2\setminus Q$ and exotic affine lines

In this section, we fix a base field k of arbitrary characteristic $p\geq 0$. Recall that a smooth k-rational conic $Q\subset\mathbb{P}^2_k$ is projectively equivalent to that $Q_0\subset\mathbb{P}^2_k$ defined by the equation $q_0=xz+y^2=0$ and that the induced action on Q_0 of the stabilizer $\mathrm{Aut}(\mathbb{P}^2_k,Q_0)$ of Q_0 in $\mathrm{Aut}(\mathbb{P}^2_k)$ is transitive on the set of k-rational points of Q_0. We let $S_0=\mathbb{P}^2\setminus Q_0$, $p_0=[0:0:1]$ and we denote by

$$\rho_0: S_0\to\mathbb{A}^1_k=\mathrm{Spec}(k[t]),\quad [x:y:z]\mapsto\frac{x^2}{q_0}$$

the $\mathbb{A}^1$-fibration induced by the restriction to S_0 of the rational map $\overline{\rho}_0:\mathbb{P}^2_k\dashrightarrow\mathbb{P}^1_k$ defined by the pencil $\mathcal{P}_{p_0}\subset\left|\mathcal{O}_{\mathbb{P}^2_k}(2)\right|$ generated by Q_0 and twice its tangent line $T_{p_0}Q_0$ at p_0. We denote by $\mathrm{Aut}(S_0,\rho_0)$ the group of k-automorphisms of S_0 preserving ρ_0 globally, that is, automorphisms $\Psi\in\mathrm{Aut}_k(S_0)$ for which there exists $\psi_{\rho_0}\in\mathrm{Aut}(\mathbb{A}^1_k)$ such that $\rho_0\circ\Psi=\psi_{\rho_0}\circ\rho_0$.

3.1. Automorphisms of S_0.

This subsection is devoted to the proof of the following more precise version of Theorem 2.

Proposition 9. *With the notation above, the following hold:*

1) The group $\mathrm{Aut}_k(S_0)$ is isomorphic to the free product of $\mathrm{Aut}(\mathbb{P}^2_k,Q_0)\mid_{S_0}$ and $\mathrm{Aut}(S_0,\rho_0)$ amalgamated along their intersection.

2) The group $\mathrm{Aut}(\mathbb{P}^2_k,Q_0)\subset\mathrm{PGL}_3(k)$ is isomorphic to $\mathrm{PGL}_2(k)$, generated by the following automorphisms:

a) $[x:y:z]\mapsto[x:y+bx:z-2by-b^2x]$, $b\in k$,

b) $[x:y:z]\mapsto[ax:y:a^{-1}z]$, $a\in k^*$,

c) $[x:y:z]\mapsto[z:-y:x]$.

3) The group $\mathrm{Aut}(S_0,\rho_0)$ is generated by the restrictions to S_0 of birational endomorphisms of $\mathbb{P}^2_k$ of the form

$$[x:y:z]\mapsto\left[x:y+s(\frac{x^2}{q_0})x:z-2ys(\frac{x^2}{q_0})-xs(\frac{x^2}{s_0})^2\right],\qquad s\in k[t]$$

and of the elements of the subgroup $\mathrm{Aut}(\mathbb{P}^2,Q_0,p_0)$ of $\mathrm{Aut}(\mathbb{P}^2_k,Q_0)$ consisting of automorphisms fixing the point $p_0\in Q_0$.

Proof. 1) The assertion already appeared in [6, §4.1.3] in the case $k=\mathbb{C}$. It extends readily to the case of an arbitrary base field k thanks to the techniques developed in [2], so we just sketch the argument for the convenience of the reader. Every automorphism φ of S_0 uniquely extends to a birational map $\overline{\varphi}:\mathbb{P}^2_k\dashrightarrow\mathbb{P}^2_k$. If $\overline{\varphi}$ is biregular then it is an automorphism preserving S_0, hence its complement Q_0, and so $\overline{\varphi}\in\mathrm{Aut}(\mathbb{P}^2_k,Q_0)$. Otherwise if $\overline{\varphi}$ is strictly birational, it lifts in a unique way to a strictly birational endormorphism of the 1-standard completion (X_0,B_0) of S_0 by a chain $Q_0\triangleright C\triangleright E$ of type $[0,-1,-2,-2,-2]$ obtained by taking the minimal resolution of $\overline{\rho}_0:\mathbb{P}^2_k\dashrightarrow\mathbb{P}^1_k$. By virtue of Theorem 3.0.2 in [2] this lift factors in a unique way into a finite sequence of two particular types of birational maps between 1-standard completions $(X_i,B_i=Q_i\triangleright C_i\triangleright E_i)$ of S_0, called *fibered modifications* and *reversions*. In our case, a fibered modification $(X_{i-1},B_{i-1})\dashrightarrow(X_i,B_i)$ descends through the contractions of the rational sub-chains $C_{i-1}\triangleright E_{i-1}$ and $C_i\triangleright E_i$ in X_{i-1} and X_i onto k-rational points $x_{i-1}\in Q_{i-1}$ and $y_i\in Q_i$ to a birational endomorphism $\overline{\varphi}_i:(\mathbb{P}^2_k,Q_{i-1})\dashrightarrow(\mathbb{P}^2_k,Q_i)$ with the following properties:

a) x_{i-1} and y_i are the unique proper base points of $\overline{\varphi}_i$and $\overline{\varphi}_i^{-1}$ respectively,

b) $\overline{\varphi}_i$ maps the pencil $\mathcal{P}_{x_{i-1}} \subset \left|\mathcal{O}_{\mathbb{P}^2_k}(2)\right|$ generated by the smooth k-rational conic Q_{i-1} and $2T_{x_{i-1}}Q_{i-1}$ onto the pencil $\mathcal{P}_{y_i} \subset \left|\mathcal{O}_{\mathbb{P}^2_k}(2)\right|$ generated by the smooth k-rational conic Q_i and $2T_{y_i}Q_i$,

c) $\overline{\varphi}_i$ restricts to an isomorphism between $\mathbb{P}^2_k \setminus Q_{i-1}$ and $\mathbb{P}^2_k \setminus Q_i$.

On the other hand, the definition of a reversion $(X_{i-1}, B_{i-1}) \dashrightarrow (X_i, B_i)$ (see [2, Definition 2.3.1]) implies that such a map descends via the same contractions to an isomorphism of pairs $(\mathbb{P}^2_k, Q_{i-1}) \xrightarrow{\sim} (\mathbb{P}^2_k, Q_i)$. As a consequence, every strictly birational endomorphism $\overline{\varphi} : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ restricting to an automorphism of $S_0 = \mathbb{P}^2_k \setminus Q_0$ admits a decomposition into a finite sequence of strictly birational maps of pairs

$$\overline{\varphi} = \overline{\varphi}_n \circ \cdots \circ \overline{\varphi}_2 \circ \overline{\varphi}_1 : (\mathbb{P}^2_k, Q_0) \overset{\overline{\varphi}_1}{\dashrightarrow} (\mathbb{P}^2_k, Q_1) \overset{\overline{\varphi}_2}{\dashrightarrow} \cdots \overset{\overline{\varphi}_n}{\dashrightarrow} (\mathbb{P}^2_k, Q_n) = (\mathbb{P}^2_k, Q_0)$$

satisfying properties a), b) and c) above. Now for every $i = 1, \ldots, n$, there exists an automorphism $\alpha_i : (\mathbb{P}^2_k, Q_0) \xrightarrow{\sim} (\mathbb{P}^2_k, Q_{i-1})$ of $\mathbb{P}^2_k$ mapping Q_0 onto Q_{i-1} and p_0 onto the proper base point x_{i-1} of $\overline{\varphi}_i$ and an automorphism $\beta_i : (\mathbb{P}^2_k, Q_0) \xrightarrow{\sim} (\mathbb{P}^2_k, Q_i)$ mapping Q_0 onto Q_i and p_0 onto the proper base point y_i of $\overline{\varphi}_i^{-1}$.

$$\begin{tikzcd}
\cdots \arrow[r, dashed, "\overline{\varphi}_{i-1}"] & (\mathbb{P}^2_k, Q_{i-1}) \arrow[r, dashed, "\overline{\varphi}_i"] \arrow[d, shift right, "\beta_{i-1}^{-1}"'] & (\mathbb{P}^2_k, Q_i) \arrow[r, dashed, "\overline{\varphi}_{i+1}"] \arrow[d, shift right, "\beta_i^{-1}"'] & \cdots \\
\cdots \arrow[r, dashed] & (\mathbb{P}^2_k, Q_0) \arrow[r, dashed, "\overline{\psi}_i"] \arrow[u, shift right, "\alpha_i"'] & (\mathbb{P}^2_k, Q_0) \arrow[r, dashed] \arrow[u, shift right, "\alpha_{i+1}"'] & \cdots
\end{tikzcd}$$

The composition $\overline{\psi}_i = \beta_i^{-1} \circ \overline{\varphi}_i \circ \alpha_i$ is then a birational map of pairs $(\mathbb{P}^2_k, Q_0) \dashrightarrow (\mathbb{P}^2_k, Q_0)$ mapping the pencil $\mathcal{P}_{p_0}$ onto itself, and restricting to an automorphism ψ_i of $S_0 = \mathbb{P}^2 \setminus Q_0$ preserving the $\mathbb{A}^1$-fibration $\rho_0 : S_0 \to \mathbb{A}^1$ globally. Writing

$$\begin{aligned}
\overline{\varphi} &= \overline{\varphi}_n \circ \cdots \circ \overline{\varphi}_2 \circ \overline{\varphi}_1 = (\beta_n \circ \overline{\psi}_n \circ \alpha_n^{-1}) \circ \cdots \circ (\beta_2 \circ \overline{\psi}_2 \circ \alpha_2^{-1}) \circ (\beta_1 \circ \overline{\psi}_1 \circ \alpha_1^{-1}) \\
&= \beta_n \circ \overline{\psi}_n \circ (\alpha_n^{-1} \circ \beta_{n-1}) \circ \cdots \circ (\alpha_3^{-1} \circ \beta_2) \circ \overline{\psi}_2 \circ (\alpha_2^{-1} \circ \beta_1) \circ \overline{\psi}_1 \circ \alpha_1^{-1}
\end{aligned}$$

we obtain a decomposition of $\overline{\varphi}$ into an alternating sequence of automorphisms β_n, $(\alpha_{i+1}^{-1} \circ \beta_i)_{i=1,\ldots n-1}$, α_1^{-1} of the pair $(\mathbb{P}^2_k, Q_0)$ and birational endomorphisms $\overline{\psi}_i$ of $\mathbb{P}^2_k$ restricting to elements of the group $\mathrm{Aut}(S_0, \rho_0)$. This shows that $\mathrm{Aut}(S_0)$ is generated by the subgroups $\mathrm{Aut}(\mathbb{P}^2_k, Q_0)|_{S_0}$ and $\mathrm{Aut}(S_0, \rho_0)$. The existence of an amalgamated product structure follows from general properties of the above decompositions into birational maps, see [6, §3, Proposition 16] and [2, Lemma 3.2.4].

2) The description of the generators of $\mathrm{Aut}(\mathbb{P}^2_k, Q_0)$ follows from the classical faithful representation of $\mathrm{Aut}(Q_0) = \mathrm{PGL}_2(k)$ as the special orthogonal group $\mathrm{SO}_3(\mathrm{q}_0) \subset \mathrm{GL}_3(k)$ of the quadratic form $q_0 = xz + y^2$, defined by the action $\sigma : \mathrm{PGL}_2(k) \times T \to T$ of $\mathrm{PGL}_2(k)$ by conjugation on the space $T \simeq k^3$ of 2×2 matrices of trace zero. Explicitly, the representation $\gamma : \mathrm{PGL}_2(k) \to SO_3(q_0)$ is given by

$$\mathrm{PGL}_2(k) \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{1}{ad - bc} \begin{bmatrix} a^2 & -2ab & -b^2 \\ -ac & ad + bc & bd \\ -c^2 & 2cd & d^2 \end{bmatrix} \in \mathrm{SO}_3(q_0),$$

and the listed generators of $\mathrm{Aut}(\mathbb{P}^2_k, Q_0)$ coincide with the respective images in $\mathrm{PGL}_3(k)$ of the generators

$$\begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}, b \in k, \quad \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, a \in k^* \quad \text{and} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

of $\mathrm{PGL}_2(k)$ by γ.

3) The generators of $\mathrm{Aut}(S_0, \rho_0)$ can be determined as follows. The correspondence which maps every $\Psi \in \mathrm{Aut}(S_0, \rho_0)$ to the unique element $\psi_{\rho_0} \in \mathrm{Aut}(\mathbb{A}^1_k)$ such that $\rho_0 \circ \Psi = \psi_{\rho_0} \circ \rho_0$ defines a group homomorphism $d : \mathrm{Aut}(S_0, \rho_0) \to \mathrm{Aut}(\mathbb{A}^1_k)$. Since $\rho_0^{-1}(0)$ is the unique degenerate fiber of ρ_0, ψ_{ρ_0} necessarily fixes the origin, hence belongs to the sub-torus $\mathbb{G}_{m,k} \times \{0\}$ of $\mathrm{Aut}(\mathbb{A}^1_k) = \mathbb{G}_{m,k} \ltimes \mathbb{G}_{a,k}$. Conversely, the existence of the homomorphism $\mathbb{G}_{m,k} \to \mathrm{Aut}(S_0, \rho_0) \cap \mathrm{Aut}(\mathbb{P}^2_k, Q_0, p_0)$, $a \mapsto [ax : y : a^{-1}z]$ implies that we have a split exact sequence

$$0 \to \mathrm{Aut}_0(S_0, \rho_0) \to \mathrm{Aut}(S_0, \rho_0) \xrightarrow{d} \mathbb{G}_{m,k} \to 0,$$

and it remains to describe the elements of the group $\mathrm{Aut}_0(S_0, \rho_0)$ of automorphisms of S_0 preserving ρ_0 fiber wise. Every such automorphism Ψ restricts to an automorphism of the complement of $\rho_0^{-1}(0)$ in S_0. Under the isomorphisms

$$\begin{aligned} S_0 \setminus \rho_0^{-1}(0) &\simeq \mathbb{P}^2 \setminus (Q_0 \cup T_{p_0}Q_0) \\ &\simeq \mathrm{Spec}(k[Y,Z]) \setminus \{Z+Y^2=0\} \\ &\simeq \mathrm{Spec}(k[Y,t^{\pm 1}]), \end{aligned}$$

where $Y = y/x$, $Z = z/x$ and $t = x^2q_0^{-1} = (Z+Y^2)^{-1}$, $\Psi|_{S_0\setminus\rho_0^{-1}(0)}$ coincides with a $\mathrm{Spec}(k[t^{\pm1}])$-automorphism of $\mathrm{Spec}(k[Y,t^{\pm1}])$ which is thus of the form $(t,Y) \mapsto (t, \lambda t^n Y + s(t))$ for some $\lambda \in k^*$, $n \in \mathbb{Z}$ and $s(t) \in k[t^{\pm1}]$. It follows that Ψ is induced by the restriction of a birational endomorphism $\overline{\Psi}$ of $\mathbb{P}^2_k$ of the form

$$[x:y:z] \mapsto [x : \lambda(\frac{x^2}{q_0})^n y + s(\frac{x^2}{q_0})x : z + (1-\lambda^2(\frac{x^2}{q_0})^{2n})\frac{y^2}{x} - 2\lambda(\frac{x^2}{q_0})^n s(\frac{x^2}{q_0})y - s(\frac{x^2}{q_0})^2 x].$$

If $(1-\lambda^2(\frac{x^2}{q_0})^{2n}) \neq 0$ or $s(t) \in k[t^{\pm1}] \setminus k[t]$ then such a birational endomorphism $\overline{\Psi}$ contracts the tangent line $T_{p_0}Q_0 = \{x=0\}$ to the point $[0:0:1]$, hence is not the extension of any automorphism of S_0. So $n=0$, $\lambda = \pm 1$, $s(t) \in k[t]$ necessarily. Conversely, every $\overline{\Psi}$ of the form

$$[x:y:z] \mapsto \left[x : \lambda y + s(\frac{x^2}{q_0})x : z - 2\lambda y s(\frac{x^2}{q_0}) - xs(\frac{x^2}{s_0})^2\right]$$

where $\lambda = \pm 1$ and $s \in k[t]$ is the composition of an element $[x:y:z] \mapsto [x:\pm y:z]$ of $\mathrm{Aut}(\mathbb{P}^2_k, Q_0, p_0)$ and of a birational endomorphism of the desired type, which indeed restricts to an automorphism of $S_0 = \mathbb{P}^2 \setminus Q_0$. □

3.2. **Exotic affine lines in positive characteristic.** A well known consequence of the structure of $\mathrm{Aut}(\mathbb{A}^2_k)$ is that if an embedded affine line $A \simeq \mathbb{A}^1_k$ in $\mathbb{A}^2_k$ with parametrization $t \mapsto (x(t), y(t))$ belongs to the $\mathrm{Aut}(\mathbb{A}^2_k)$-orbit of the coordinate line $\{x=0\}$, then either $\deg_t(x(t))$ divides $\deg_t(y(t))$ or $\deg_t(y(t))$ divides $\deg_t(x(t))$. Letting $L_0 = \{x=0\}$ and $L_1 = \{x^2 - q_0 = 0\}$ be the reduced fibers of the $\mathbb{A}^1$-fibration

$$\rho_0 : S_0 \to \mathbb{A}^1_k = \mathrm{Spec}(k[t]), \quad [x:y:z] \mapsto \frac{x^2}{q_0}$$

over the closed points 0 and 1 of $\mathbb{A}^1_k$ respectively, the description of $\mathrm{Aut}(S_0)$ given in Proposition 9 leads to the following analogue for closed embeddings of $\mathbb{A}^1_k$ in S_0:

Lemma 10. *Let $j : \mathbb{A}^1_k \hookrightarrow S_0$, $t \mapsto [x(t):y(t):z(t)]$ be a closed embedding with image A. If A belongs to the $\mathrm{Aut}(S_0)$-orbit of L_0 or L_1, then up to composition by the involution $[x:y:z] \mapsto [z:y:x]$, the following hold:*

a) $\deg_t(x(t)) < \deg_t(y(t)) < \deg_t(z(t))$,

b) If $\deg_t(x(t)) \neq -\infty$, *then it divides* $\deg_t(y(t))$ *and* $\deg_t(z(t))$.

Proof. This holds for the parametrizations $t \mapsto [0:1:t]$ and $t \mapsto [1:t:1-t^2]$ of L_0 and L_1 respectively, and both properties are preserved under the application of any of the generator of $\mathrm{Aut}(S_0)$ listed in Proposition 9. □

As a consequence of the above lemma, we obtain in every characteristic $p \geq 3$ the existence of closed embeddings $j : \mathbb{A}^1_k \hookrightarrow S_0$ whose image does not belong to the $\mathrm{Aut}(S_0)$-orbit of L_0 or L_1, a phenomenon similar to the failure of the Abhyankar-Moh Theorem in positive characteristic. Namely, we have the following family of examples:

Proposition 11. *Let k be a field of characteristic $p \geq 3$. Then the morphism*

$$j : \mathbb{A}^1_k \hookrightarrow S_0,\ t \mapsto [t^{p^2} : t^{p^2}(t^{p^2+p}+t)+1 : -t^{p^2}(t^{p^2+p}+t)^2 - 2(t^{p^2+p}+t)]$$

is a closed embedding whose image does not belong to the $\mathrm{Aut}(S_0)$-orbit of L_0 or L_1.

Proof. Once we show that j is indeed a closed embedding, the conclusion follows immediately from Lemma 10 above. Letting $\tilde{S}_0 \subset \mathbb{A}^3_k = \mathrm{Spec}(k[x,y,z])$ be the smooth affine surface with equation $xz+y^2-1=0$, j is the composition of the morphism

$$\tilde{j} : \mathbb{A}^1_k \to \tilde{S}_0, \quad t \mapsto (t^{p^2}, t^{p^2}(t^{p^2+p}+t)+1, -t^{p^2}(t^{p^2+p}+t)^2 - 2(t^{p^2+p}+t))$$

with the étale Galois double cover $\pi : \tilde{S}_0 \to S_0$, $(x,y,z) \mapsto [x:y:z]$. Letting $\tilde{A}$ be the image of $\tilde{j}$, $\pi^{-1}(\pi(\tilde{A}))$ is the disjoint union of $\tilde{A}$ with its image by the action $(x,y,z) \mapsto (-x,-y,-z)$ of the Galois group of π. So π induces an isomorphism between $\tilde{A}$ and the image of j. Since $\tilde{S}_0$ is affine, every embedding of $\mathbb{A}^1_k$ into it is necessarily closed, and hence, it now suffices to show that $\tilde{j}$ is an embedding. Noting that $\tilde{A}$ is contained in the complement V of the curve with equation $\{x = y+1 = 0\} \subset \tilde{S}_0$ and that V is isomorphic to $\mathbb{A}^2_k$ via the restriction of the rational map

$$\alpha : \tilde{S}_0 \dashrightarrow \mathbb{A}^2_k = \mathrm{Spec}(k[x,v]),\ (x,y,z) \mapsto (x, x^{-1}(1-y) = (y+1)^{-1}z),$$

we are reduced to check that $\alpha \circ \tilde{j} : \mathbb{A}^1_k \to \mathbb{A}^2_k,\ t \mapsto (x(t), v(t)) = (t^{p^2}, t^{p^2+p} + t)$ is an embedding. This follows from the identity $t^{p(p+1)} = (v(t)^p - x(t)^{p+1})^{p+1}$ which implies that the inclusion $k[x(t), v(t)] \subset k[t]$ is an equality. □

Remark 12. The morphism $\mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k,\ t \mapsto (t^{p^2}, t^{p^2+p} + t)$ used in the proof of the proposition above is a typical example of closed embedding of the line in $\mathbb{A}^2_k$ whose image, as a consequence of the Jung and van der Kulk Theorem, does not belong to the $\mathrm{Aut}(\mathbb{A}^2_k)$-orbit of the coordinate line $\{x = 0\}$.

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Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 9, France
E-mail address: jdecaup@math.univ-toulouse.fr

IMB UMR5584, CNRS, Univ. Bourgogne Franche-Comté, F-21000 Dijon, France.
E-mail address: adrien.dubouloz@u-bourgogne.fr