Structure-preserving adiabatic elimination for open bipartite quantum systems

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Abstract

We consider a quantum system composed of a fast quantum subsystem coupled to a slow one. We provide expressions of the approximate reduced model describing the slow subsystem perturbed by the fast one (adiabatic elimination), based on an asymptotic expansion that we solve up to second order. The specificity of our expressions is to preserve the quantum structure in the reduced model: we provide the reduced dynamics in Lindblad form, and the mapping defining the slow manifold as a completely positive map in Kraus form.

1 Introduction

Very often in quantum physics we are dealing with complex connected systems where we want to characterize the evolution of one subsystem, e.g. an engineered information encoding device, coupled to various external influences and to quantum communication buses. When the time scales in the subsystem of interest are much slower than in the other subsystems, one can derive a reduced model with rigorous approximation guarantees, by eliminating the fast variables with so-called *adiabatic elimination* techniques. We would essentially treat the slow dynamics of interest as a small perturbation to the fast dynamics, and solve for the effect of this perturbation. One of the key issues of adiabatic elimination is to ensure a physical meaning for the reduced model. For closed quantum systems, described by the Schrödinger master equation, the quantum evolution is unitary and regular perturbation theory is routinely applied ([15]). The case of open quantum systems, described by a Lindblad master equation ([5]), is much more complicated and involves singular perturbation theory.

A lot of particular examples of open quantum systems have been successfully treated in the literature. [6], [14], [13] study models with excited states decaying fast towards several ground states. For systems with Gaussian dynamics an adiabatic elimination technique is presented in [10]. [1] investigate a specific atom-optics system. Some more generic methods have also been proposed. A generalization of the Schrieffer-Wolff formalism for Lindblad dynamics is developed in [11]. [4] derive a reduced dynamics as the speed of the fast subsystem tends to infinity, without giving the order of approximation nor the perturbation of the slow manifold. However, as pointed out by recent quantum experiments aiming at strong indirect stabilization of quantum systems ([12]), a better knowledge of the order of validity of such approximations is becoming necessary.

In the present paper, we use a geometric approach to perform adiabatic elimination for composite open quantum systems. Our main contribution is to combine two key features. First, we provide an asymptotic expansion that allows to choose the order of approximation, both for the dynamics and for the characterization of the slow manifold, as a function of the time-scale separation between the fast dynamics and its perturbation. Second, we preserve the structural properties of open quantum systems, and hence allow a physical interpretation of the reduced model:

- The reduced dynamics follows a Lindblad master equation.
- The parameterization of the slow manifold is explicitly given as a trace preserving completely positive map, also called Kraus map, see [8].

To our knowledge, ensuring a Kraus map for the slow manifold is novel in adiabatic elimination.

Our approach is based on center manifold techniques ([7]) and geometric singular perturbation theory ([9]), to derive a recurrence relation between the corrections at various orders. We made a first contribution along those lines in [2], deriving formulas only for the first order approximation, for a general quantum system with fast convergence of part of the variables. Here we provide the second-order expressions as well, which leads to a conjecture about extending the structured computation to any orders, when the perturbing dynamics is Hamiltonian.

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We also consider a more specific quantum model, where the slow and fast variables correspond to two different interacting quantum subsystems. The slow and fast manifolds are hence factored in tensor product form, rather than the cartesian product which is standard in classical dynamical systems. We can then interpret the dynamics of the reduced quantum subsystem. As expected, the first order approximation corresponds to the well-known Zeno Hamiltonian. It also shows that entanglement already appears at this order; to our knowledge, this is the first time that such systematic first order entanglement is shown in adiabatic elimination. The second order approximation emphasizes how a Hamiltonian coupling to an open system introduces decoherence in the slow dynamics. It also shows that the second order decoherence operators are linear combinations of the operators in the interaction Hamiltonian, and that there are precisely as many decoherence channels as Hermitian operators in the interaction Hamiltonian.

The paper is organized as follows. Section 2 presents the model of composite open quantum system, our assumptions on the dynamics, and some key properties of our particular model. Section 3 introduces the asymptotic expansion for adiabatic elimination. Section 4 contains our main results, i.e. the structured expressions for the first and second order terms of the asymptotic expansion. Section 5 is devoted to an illustrative example of two qubits, a fast one relaxing to a driven mixed state and a slow qubit dispersively coupled to the fast one.

2 Bipartite systems

Let $\mathcal{H}_\mathcal{F}$ and $\mathcal{H}_S$ be two Hilbert spaces of finite dimension associated with two quantum subsystems. The Hilbert space associated with their composite system is the tensor product space $\mathcal{H}_F \otimes \mathcal{H}_S$, whose dimension is the product of the individual dimensions. Denoting with $\{|f_j\rangle\}_j, \{|s_j\rangle\}_j$ some orthonormal bases for $\mathcal{H}_\mathcal{F}$ and $\mathcal{H}_S$ respectively, an orthonormal basis for $\mathcal{H}_F \otimes \mathcal{H}_S$ is given by $\{|f_j\rangle \otimes |s_k\rangle\}_{j,k}$.

The state of an open quantum system on $\mathcal{H}$ is described by a density operator ρ, which is a linear Hermitian nonnegative operator from $\mathcal{H}$ to $\mathcal{H}$ ($\simeq$ a matrix acting on the complex vector space) whose trace equals 1. For all other operators on Hilbert spaces we use bold letters. We will denote $\mathcal{D}$ (resp. $\mathcal{D}_F$,$\mathcal{D}_S$) the "state space", i.e. the compact convex set of all density operators on $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_S$ (resp. $\mathcal{H}_F$, $\mathcal{H}_S$). A given state ρ of the composite system on $\mathcal{H}$ cannot always be described in terms of one state on $\mathcal{H}_F$ and one on $\mathcal{H}_S$ – the dimensions of the Hilbert spaces readily shows that the composite state can contain more information. To extract from some $\rho \in \mathcal{H}$ the maximum of information concerning the state of the subsystem in $\mathcal{H}_S$, we take the partial trace over $\mathcal{H}_F$, i.e. $\mathrm{Tr}_F(\rho_0) = \sum_j \langle f_j|\rho|f_j\rangle$. Here, with a slight abuse of notation, we mean $\langle f_j|(|f_k\rangle \otimes |s_\ell\rangle) = |s_\ell\rangle$ if $j = k$, or $= 0$ if $j \neq k$; and computing its action on any other vector of $\mathcal{H}$ by linearity.

The dynamics of an open quantum system can be described by a Lindblad master equation (see, e.g., [5]):

$$\frac{d\rho}{dt} = \mathfrak{L}(\rho) = -i[\mathbf{H},\rho] + \sum_\mu \mathbf{L}_\mu \rho \mathbf{L}_\mu^\dagger - \frac{\mathbf{L}_\mu^\dagger \mathbf{L}_\mu \rho + \rho \mathbf{L}_\mu^\dagger \mathbf{L}_\mu}{2} \,, \tag{1}$$

with any operators $\mathbf{L}_\mu$ and any Hermitian operator $\mathbf{H}$. In all the following, the cursive $\mathfrak{L}$ possibly with indexes denotes a super-operator, i.e. a linear function mapping operators-on-$\mathcal{H}$ to operators-on-$\mathcal{H}$, of this particular Lindbladian form.

In this paper, we consider a composite quantum system with two time scales:

$$\begin{aligned} \frac{d\rho}{dt} =& \mathfrak{L}_F(\rho) - i\epsilon[\mathbf{H}_{int},\rho] \;\text{ with} \qquad (2)\\ \mathfrak{L}_F(\rho) =& -i[\mathbf{H}_F \otimes 1_S, \rho] + \sum_\mu (\mathbf{L}_{F_\mu} \otimes 1_S)\rho(\mathbf{L}_{F_\mu}^\dagger \otimes 1_S) \\ & - \frac{((\mathbf{L}_{F_\mu}^\dagger \mathbf{L}_{F_\mu}) \otimes 1_S)\rho + \rho((\mathbf{L}_{F_\mu}^\dagger \mathbf{L}_{F_\mu}) \otimes 1_S)}{2} \,, \\ \mathbf{H}_{int} =& \sum_{j=1}^{n_{\mathrm{int}}} \mathbf{F}_j \otimes \mathbf{S}_j \end{aligned}$$

and where ϵ is a small positive parameter. Here 1_S denotes the identity operator on $\mathcal{H}_S$ and $\mathfrak{L}_F$ thus acts nontrivially only on $\mathcal{H}_F$. In other words, $\mathfrak{L}_F(\mathbf{X} \otimes \mathbf{Y}) = \mathfrak{L}_F(\mathbf{X}) \otimes \mathbf{Y}$ for any operators $\mathbf{X}$ on $\mathcal{H}_F$ and $\mathbf{Y}$ on $\mathcal{H}_S$. With a slight and common abuse of notation, we will thus sometimes refer to $\mathfrak{L}_F(\mathbf{X})$ with an operator $\mathbf{X}$ on $\mathcal{H}_F$ only. The $\mathbf{F}_j$ are some Hermitian operators on $\mathcal{H}_F$ and $\mathbf{S}_j$ are some Hermitian operators on $\mathcal{H}_S$. In particular, $\mathbf{H}_{int}$ may include an operator $1 \otimes \mathbf{H}_S$ consisting of all the Hamiltonian dynamics of the subsystem S.

For $\epsilon = 0$, the subsystem on $\mathcal{H}_F$ is completely decoupled from the subsystem on $\mathcal{H}_S$. The latter in fact does not move. For the former, we assume that for $\epsilon = 0$ it asymptotically converges towards a unique equilibrium state denoted by $\overline{\rho}_F = \sum_{m=1}^{n_F} r_m |\chi_m\rangle\langle\chi_m|$, where the second expression is a spectral decomposition with $r_m>0$ for all m and n_F is the rank of $\overline{\rho}_F$. Note that asymptotic convergence requires the presence of a dissipative (i.e., non Hamiltonian) term in the dynamics. Then on $\mathcal{H}$, for any initial state $\rho(0) = \rho_0$ the solution of $\frac{d\rho}{dt} = \mathfrak{L}_F(\rho)$ converges for t tending to $+\infty$ towards

$$\rho_\infty = e^{+\infty \mathfrak{L}_F}(\rho_0) = \overline{\rho}_F \otimes \mathrm{Tr}_F(\rho_0). \tag{3}$$

where $e^{t\mathfrak{L}_F}(\cdot)$ is the propagator. In the sequel, we will exploit the following key properties.

(a) $\mathrm{Tr}_F\left(\mathfrak{L}_F(\mathbf{A})\right) = 0$ for any operator $\mathbf{A}$ on $\mathcal{H}_F \otimes \mathcal{H}_S$.

(b) Several times we will have to find an operator $\mathbf{X}$ on $\mathcal{H}_F$ as the solution of $\mathfrak{L}_F(\mathbf{X}) = \mathbf{Y}$, with $\mathrm{Tr}(\mathbf{Y}) = 0$.

The asymptotic stability implies that $\mathfrak{L}_F$ is stable (eigenvalues with strictly negative parts) when considered as a linear map on the space $\mathcal{L}^0(\mathcal{H}_F)$ of linear operators on $\mathcal{H}_F$ with trace 0 (the fact that ρ is Hermitian and $\mathbf{X}$ possibly not, is easily covered by noting that $\mathfrak{L}_F(i\rho) = i\mathfrak{L}_F(\rho)$). The inverse is thus well defined for $\mathbf{Y}, \mathbf{X} \in \mathcal{L}^0$, and a formal expression is:

$$\mathfrak{L}_F^{-1}(\mathbf{Y}) = -\int_0^\infty e^{\tau \mathfrak{L}_F}(\mathbf{Y})\, d\tau \ , \tag{4}$$

which indeed converges since $t \mapsto e^{\tau \mathfrak{L}_F}(\mathbf{Y})$ converges exponentially towards 0.

(c) In the computation of item (b), if $\mathbf{Y}$ takes the form $\tilde{\mathbf{Y}}\overline{\rho}_F$ i.e. the states orthogonal to span$\{|\chi_1\rangle, ..., |\chi_{n_F}\rangle\}$ are in the kernel of $\mathbf{Y}$, then also $\mathbf{X} = \mathfrak{L}_F^{-1}(\mathbf{Y})$ takes the form $\tilde{\mathbf{X}}\overline{\rho}_F$ (see Lemma 3).

(d) Similarly, we will have to solve for operator $\mathbf{X}$ *on the composite Hilbert space* $\mathcal{H}$ in equations of the type $\mathfrak{L}_F(\mathbf{X}) = \mathbf{Y}$, where now $\mathbf{Y}$ as well acts on $\mathcal{H}$ instead of $\mathcal{H}_F$. Since $\mathfrak{L}_F$ acts nontrivially only on $\mathcal{H}_F$, the same formula (4) applies, provided that we assume that $\mathrm{Tr}_F(\mathbf{Y}) = 0$, and yields a unique solution $\mathbf{X}$ such that $\mathrm{Tr}_F(\mathbf{X}) = 0$.

(e) Standard computations show that for any operators $\mathbf{X}$ and $\mathbf{Y}$ on $\mathcal{H}_F$ we have the following identity

$$\begin{aligned}\mathfrak{L}_F(\mathbf{X}\overline{\rho}_F)\ \mathbf{Y}^\dagger + \mathbf{X}\ \mathfrak{L}_F(\overline{\rho}_F\mathbf{Y}^\dagger) = \\ \mathfrak{L}_F(\mathbf{X}\overline{\rho}_F\mathbf{Y}^\dagger) - \sum_\mu [\mathbf{L}_{F_\mu}, \mathbf{X}]\ \overline{\rho}_F\ [\mathbf{L}_{F_\mu}, \mathbf{Y}]^\dagger.\end{aligned} \tag{5}$$

Our goal, in system theoretic terms, is to obtain the expressions for the slow dynamics, i.e. the perturbation of the center manifold $\mathcal{D}_0 = \ker(\mathfrak{L}_F) = \{\rho = \overline{\rho}_F \otimes \rho_S, \forall \rho_S \in \mathcal{D}_S\}$, for $\epsilon \neq 0$ but small.

3 Asymptotic expansion

We now recall the asymptotic expansion method for obtaining such center manifold approximations, which leads to the recurrence relations which we must solve. We apply an adaptation of [7, Th.1 chapt.2] to construct an invariant submanifold of equation (2) to which the system converges quickly and on which it evolves slowly. This manifold is the image of $\mathcal{D}_0$ under some map $\mathfrak{K}$. Using [9], we can develop the map $\mathfrak{K}$ and the slow dynamics on $\mathfrak{K}$ in power series of ϵ. In order to preserve physical meaning, we will impose a specific quantum structure onto these standard techniques.

We parameterize the slow manifold, i.e. the perturbation of $\mathcal{D}_0$, by a density operator ρ_S of the same dimension of $\mathcal{D}_0$, and thus of the density operators on $\mathcal{H}_S$. We will then interpret ρ_S as approximately the state of the $\mathcal{H}_S$ system, although this is not exact: for $\epsilon \neq 0$ the subsystems of $\mathcal{H}_S$ and $\mathcal{H}_F$ get "hybridized" in the slow manifold, and we impose to describe this by a Kraus map:

$$\rho(t) = \mathfrak{K}(\rho_S(t)) = \sum_{k \geq 0} \epsilon^k \mathfrak{K}_k(\overline{\rho}_F \otimes \rho_S(t)) \, . \tag{6}$$

When approximating the infinite series by a finite sum up to $k = \bar{k}$, we impose to get an explicit Kraus map $\rho_S \mapsto \sum_j \mathbf{K}_j(\overline{\rho}_F \otimes \rho_S(t))\mathbf{K}_j^\dagger$ with $\mathrm{Tr}_F\left(\sum_j \mathbf{K}_j^\dagger \mathbf{K}_j(\overline{\rho}_F \otimes 1_S)\right) = 1_S$, up to terms of order $\epsilon^{\bar{k}+1}$.

Similarly, we impose that the evolution of $\rho_S(t)$, parameterizing the dynamics on the slow manifold, follows a Lindblad equation:

$$\frac{d\rho_S}{dt} = \sum_{k \geq 0} \epsilon^k \mathfrak{L}_{s,k}(\rho_S) \tag{7}$$

where, for any finite sum, we want to obtain on the right an expression of the type (1).

Using [9] and [7], we derive the following formal invariance condition binding the unknown maps $\mathfrak{L}_{s,k}$ and $\mathfrak{K}_k$:

$$\begin{aligned}\frac{d\rho}{dt} =& \mathfrak{L}_F\big(\sum_{k \geq 0} \epsilon^k \mathfrak{K}_k(\overline{\rho}_F \otimes \rho_S)\big) \\ & - i\epsilon[\mathbf{H}_{int}, \sum_{k \geq 0} \epsilon^k \mathfrak{K}_k(\overline{\rho}_F \otimes \rho_S)] \\ =& \mathfrak{K}\left(\frac{d\rho_S}{dt}\right) = \sum_{k \geq 0} \epsilon^k \mathfrak{K}_k \left(\sum_{j \geq 0} \epsilon^j \mathfrak{L}_{s,j}(\rho_S)\right)\end{aligned}$$

Identifying the terms of same order in ϵ yields recurrence relations to derive the $\mathfrak{K}_j$ and $\mathfrak{L}_{s,j}$.

To initialize this recurrence, it is natural to choose the zero order mapping

$$\mathfrak{K}_0(\overline{\rho}_F \otimes \rho_S) = \overline{\rho}_F \otimes \rho_S \, . \tag{8}$$

The zero order recurrence relation then yields $\overline{\rho}_F \otimes \mathfrak{L}_{s,0}(\rho_S) = \mathfrak{L}_F\big(\mathfrak{K}_0(\rho_S)\big) = 0$ so we get $\mathfrak{L}_{s,0}(\rho_S) = 0$ as expected. For $k \geq 1$, the term in ϵ^k gives the condition

$$\begin{aligned}\mathfrak{L}_F\big(\mathfrak{K}_k(\overline{\rho}_F \otimes \rho_S)\big) - i[\mathbf{H}_{int}, \mathfrak{K}_{k-1}(\overline{\rho}_F \otimes \rho_S)] \\ = \sum_{m=1}^{k} \mathfrak{K}_{k-m}\left(\overline{\rho}_F \otimes \mathfrak{L}_{s,m}(\rho_S)\right).\end{aligned} \tag{9}$$

The remaining difficulty is to solve (9) analytically, with the constraints on the structure of $\mathfrak{K}_j$ and $\mathfrak{L}_{s,j}$.

4 Main results

Our main result is an explicit solution for the first and second order approximation from the recurrence (9). In all the following, for an operator $\mathbf{X}$ on $\mathcal{H}_F$, we denote $\overline{\mathbf{X}} = \mathbf{X} - \mathrm{Tr}(\mathbf{X}\overline{\rho}_F)1_F$.

Theorem 1 *The first order approximation for the slow dynamics of* (2) *is given by the Hamiltonian evolution*

$$\frac{d}{dt}\rho_S = \epsilon\mathfrak{L}_{s,1}(\rho_S) = -i\left[\epsilon\mathbf{H}_{S_1}\ ,\ \rho_S\right] \qquad (10)$$

with $\mathbf{H}_{S_1} = \sum_{j=1}^{n_{int}} Tr(\mathbf{F}_j\overline{\rho}_F)\mathbf{S}_j$. *The parameterization of the slow manifold is given by:*

$$(\mathfrak{K}_0 + \epsilon\mathfrak{K}_1)(\overline{\rho}_F\otimes\rho_S) + O(\epsilon^2) = \big(1 - i\epsilon\sum_{j=1}^{n_{int}}\mathbf{Q}_j\otimes\mathbf{S}_j\big)\big(\overline{\rho}_F\otimes\rho_S\big)\big(1 + i\epsilon\sum_{j=1}^{n_{int}}\mathbf{Q}_j^\dagger\otimes\mathbf{S}_j\big) \qquad (11)$$

with $\mathbf{Q}_j$ *satisfying, thanks to property (c) from Section 2,*

$$\mathbf{Q}_j\overline{\rho}_F = -\mathfrak{L}_F^{-1}\left(\overline{\mathbf{F}}_j\ \overline{\rho}_F\right) \quad \text{with } Tr(\mathbf{Q}_j\overline{\rho}_F) = 0.$$

Proof 1 *Taking* (9) *for* $k = 1$ *and using* (8), *we must solve*

$$\mathfrak{L}_F(\mathfrak{K}_1(\overline{\rho}_F\otimes\rho_S)) - i[\mathbf{H}_{int}, \overline{\rho}_F\otimes\rho_S] = \overline{\rho}_F\otimes\mathfrak{L}_{s,1}(\rho_S)\,. \quad (12)$$

With a partial trace versus $\mathcal{H}_F$ *and using property (a) from Section 2, we get*

$$\mathfrak{L}_{s,1}(\rho_S) = -i\sum_{j=1}^{n_{int}}\left[Tr(\mathbf{F}_j\overline{\rho}_F)\mathbf{S}_j, \rho_S\right].$$

Plugging this into (12), *we compute* $\mathfrak{K}_1$ *by solving the equation* $\mathfrak{L}_F(\mathfrak{K}_1) = i\sum_{j=1}^{n_{int}}\left(\overline{\mathbf{F}}_j\overline{\rho}_F\otimes\mathbf{S}_j\rho_S - \overline{\rho}_F\overline{\mathbf{F}}_j\otimes\rho_S\mathbf{S}_j\right)$. *Inverting* $\mathfrak{L}_F$ *with properties (d),(c) from Section 2 and the associated operator spaces, we get*

$$\mathfrak{K}_1 = i\sum_{j=1}^{n_{int}}\left(\mathcal{L}_F^{-1}(\overline{\mathbf{F}}_j\overline{\rho}_F)\otimes\mathbf{S}_j\rho_S - \mathcal{L}_F^{-1}(\overline{\rho}_F\overline{\mathbf{F}}_j)\otimes\rho_S\mathbf{S}_j\right)$$

since $Tr\left(\overline{\mathbf{F}}_j\overline{\rho}_F\right) = 0$ *for each* j. $\square$

Theorem 1 shows that the slow evolution is coherent up to second order terms and driven by the Zeno Hamiltonian ϵH_{S_1}. However, the underlying state ρ of the bipartite system given by (11) admits in general entanglement terms of order one versus ϵ.

Lemma 1 *Consider the* $n_{int}\times n_{int}$ *matrix* Z *of entries*

$$Z_{j,j'} = -Tr\left(\mathbf{F}_j\mathfrak{L}_F^{-1}(\overline{\mathbf{F}}_{j'}\overline{\rho}_F)\right).$$

Then $X = Z + Z^\dagger$ *is always nonnegative and thus is factorized,* $X = N^\dagger N$, *where* N *is an* $n_{int}\times n_{int}$ *matrix with complex entries.*

Proof 2 *From* (5) *with* $\mathbf{X} = -\mathbf{Q}_{j'}$ *and* $\mathbf{Y} = -\mathbf{Q}_j$, *we have*

$$-\mathcal{L}_F(\mathbf{Q}_{j'}\overline{\rho}_F)\mathbf{Q}_j^\dagger - \mathbf{Q}_{j'}\mathcal{L}_F(\overline{\rho}_F\mathbf{Q}_j^\dagger) = \overline{\mathbf{F}}_{j'}\overline{\rho}_F\mathbf{Q}_j^\dagger + \mathbf{Q}_{j'}\overline{\rho}_F\overline{\mathbf{F}}_j = \sum_\mu[\mathbf{L}_{F_\mu}, \mathbf{Q}_{j'}]\ \overline{\rho}_F\ [\mathbf{L}_{F_\mu}, \mathbf{Q}_j]^\dagger - \mathcal{L}_F(\mathbf{Q}_{j'}\overline{\rho}_F\mathbf{Q}_j^\dagger).$$

Since $X_{j,j'} = Tr\left(\mathbf{F}_{j'}\overline{\rho}_F\mathbf{Q}_j^\dagger + \mathbf{Q}_{j'}\overline{\rho}_F\mathbf{F}_j\right) = Tr\left(\overline{\mathbf{F}}_{j'}\overline{\rho}_F\mathbf{Q}_j^\dagger + \mathbf{Q}_{j'}\overline{\rho}_F\overline{\mathbf{F}}_j\right)$, *we have*

$$X_{j,j'} = \sum_\mu Tr\left([\mathbf{L}_{F_\mu}, \mathbf{Q}_{j'}]\ \overline{\rho}_F\ [\mathbf{L}_{F_\mu}, \mathbf{Q}_j]^\dagger\right).$$

With the spectral decomposition of $\overline{\rho}_F = \sum_m r_m\,|\chi_m\rangle\langle\chi_m|$ *we have* $X^\top = \sum_{\mu,m} r_m G^{(\mu,m)}$ *where each matrix* $G^{(\mu,m)}$ *is the* $n_{int}\times n_{int}$ *Gram matrix associated with* n_{int} *vectors* $[\mathbf{L}_{F_\mu}, \mathbf{Q}_1]|\chi_m\rangle,\ \ldots,\ [\mathbf{L}_{F_\mu}, \mathbf{Q}_{n_{int}}]|\chi_m\rangle$. *Such Hermitian matrices are always nonnegative (see, e.g., [3, page 3]) and thus* X *is nonnegative because each* r_m *is positive.* $\square$

Theorem 2 *The* $n_{int}\times n_{int}$ *matrices* $Y = (Z - Z^\dagger)/2i$ *and* N *provided by lemma 1 yield the following second order approximation for the slow dynamics of* (2):

$$\frac{d}{dt}\rho_S = \epsilon\mathfrak{L}_{s,1}(\rho_S) + \epsilon^2\mathfrak{L}_{s,2}(\rho_S) = -i\left[\epsilon\mathbf{H}_{S_1} + \epsilon^2\mathbf{H}_{S_2}, \rho_S\right] + \epsilon^2\sum_{j=1}^{n_{int}}\left(\mathbf{C}_j\rho_S\mathbf{C}_j^\dagger - \tfrac{1}{2}\mathbf{C}_j^\dagger\mathbf{C}_j\rho_S - \tfrac{1}{2}\rho_S\mathbf{C}_j^\dagger\mathbf{C}_j\right) \quad (13)$$

where $\mathbf{H}_{S_1}$ *is given by* (10),

$$\mathbf{C}_j = \sum_{j'=1}^{n_{int}} N_{j,j'}\,\mathbf{S}_{j'} \quad \text{and} \quad \mathbf{H}_{S_2} = \sum_{j,j'=1}^{n_{int}} Y_{j,j'}\,\mathbf{S}_j\mathbf{S}_{j'}.$$

Proof 3 *Taking* (9) *for* $k = 2$, *we must solve*

$$\mathfrak{L}_F(\mathfrak{K}_2(\overline{\rho}_F\otimes\rho_S)) - i[\mathbf{H}_{int}, \mathfrak{K}_1(\overline{\rho}_F\otimes\rho_S)] = \mathfrak{K}_1(\overline{\rho}_F\otimes\mathfrak{L}_{s,1}(\rho_S)) + \overline{\rho}_F\otimes\mathfrak{L}_{s,2}(\rho_S)\,.$$

The two unknowns $\mathfrak{K}_2$ *and* $\mathfrak{L}_{s,2}$ *appear in exactly the same structure as for* $k = 1$, *only the independent term changes. With* $\mathfrak{L}_{s,1}$ *and* $\mathfrak{K}_1$ *provided by theorem 1, we get after some standard computations:*

$$\mathfrak{L}_F(\mathfrak{K}_2(\overline{\rho}_F\otimes\rho_S)) = \overline{\rho}_F\otimes\mathfrak{L}_{s,2}(\rho_S) + \sum_{j,j'=1}^{n_{int}}\left(\overline{\mathbf{F}}_j\mathbf{Q}_{j'}\overline{\rho}_F\otimes\mathbf{S}_j\mathbf{S}_{j'}\rho_S + \overline{\rho}_F\mathbf{Q}_{j'}^\dagger\overline{\mathbf{F}}_j\otimes\rho_S\mathbf{S}_{j'}\mathbf{S}_j\right) - \sum_{j,j'=1}^{n_{int}}\left(\left(\overline{\mathbf{F}}_j\overline{\rho}_F\mathbf{Q}_{j'}^\dagger + \mathbf{Q}_j\overline{\rho}_F\overline{\mathbf{F}}_{j'}\right)\otimes\mathbf{S}_j\rho_S\mathbf{S}_{j'}\right). \quad (14)$$

With a partial trace versus $\mathcal{H}_F$, *using property (a) from Section 2 and* $Tr(\mathbf{Q}_j\overline{\rho}_F) = 0$, *we get*

$$\mathfrak{L}_{s,2}(\rho_S) = \sum_{j,j'=1}^{n_{int}} Tr\left(\mathbf{F}_j\overline{\rho}_F\mathbf{Q}_{j'}^\dagger + \mathbf{Q}_j\overline{\rho}_F\mathbf{F}_{j'}\right)\mathbf{S}_j\rho_S\mathbf{S}_{j'} - \sum_{j,j'=1}^{n_{int}}\left(Tr\left(\mathbf{Q}_{j'}\overline{\rho}_F\mathbf{F}_j\right)\mathbf{S}_j\mathbf{S}_{j'}\rho_S + Tr\left(\mathbf{F}_j\overline{\rho}_F\mathbf{Q}_{j'}^\dagger\right)\rho_S\mathbf{S}_{j'}\mathbf{S}_j\right).$$

Since $Z_{j,j'} = Tr(\mathbf{Q}_{j'}\overline{\rho}_F\mathbf{F}_j)$ and $Z = X/2+iY$ with $X^\dagger = X$ and $Y^\dagger = Y$, we have after some simple computations

$$\mathfrak{L}_{s,2}(\rho_S) = \sum_{j,j'=1}^{n_{int}} \Big(X_{j,j'}\mathbf{S}_j\rho_S\mathbf{S}_{j'} - \tfrac{X_{j,j'}}{2}\mathbf{S}_j\mathbf{S}_{j'}\rho_S - \tfrac{X_{j',j}}{2}\rho_S\mathbf{S}_{j'}\mathbf{S}_j \Big) - i\left(\sum_{j,j'=1}^{n_{int}} Y_{j,j'}\mathbf{S}_j\mathbf{S}_{j'}\right)\rho_S + i\rho_S\left(\sum_{j,j'=1}^{n_{int}} Y_{j',j}\mathbf{S}_{j'}\mathbf{S}_j\right).$$

We get thus the Hermitian operator $\mathbf{H}_{S_2}$. Since $X_{j,j'} = \sum_{j''=1}^{n_{int}} N^*_{j,j''}N_{j'',j'}$, we have by definition of C_j,

$$\sum_{j=1}^{n_{int}} \Big(\mathbf{C}_j\rho_S\mathbf{C}_j^\dagger - \tfrac{1}{2}\mathbf{C}_j^\dagger\mathbf{C}_j\rho_S - \tfrac{1}{2}\rho_S\mathbf{C}_j^\dagger\mathbf{C}_j \Big) = \sum_{j,j'=1}^{n_{int}} \Big(X_{j,j'}\mathbf{S}_j\rho_S\mathbf{S}_{j'} - \tfrac{X_{j,j'}}{2}\mathbf{S}_j\mathbf{S}_{j'}\rho_S - \tfrac{X_{j',j}}{2}\rho_S\mathbf{S}_{j'}\mathbf{S}_j \Big).$$

We have proved (13). □

Theorem 2 shows that the Hamiltonian coupling to a dissipative bath leads to decoherence at second order on the slow system. The number of decoherence operators C_j does not exceed $n_{\rm int}$ the number of *Hamiltonian coupling terms*, independently of the dimension of $\mathcal{H}_F$ and of the number of decoherence operators $\mathbf{L}_{F_\mu}$ involved in the fast relaxation described by $\mathcal{L}_F$.

These formulas allow to compute the approximate evolution of the coupled system, provided we integrate or invert a superoperator $\mathcal{L}_F$ that acts only on the fast subsystem.

5 Illustrative example

We illustrate our results on a simple example: a system constituted of two connected qubits (two-level systems). We denote by $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z$ the standard Pauli matrices, and by $\boldsymbol{\sigma}_- = (\boldsymbol{\sigma}_x - i\boldsymbol{\sigma}_y)/2$ the energy loss operator (jump from the higher to the lower of the two levels). An upper letter denotes the qubit on which the operator is acting non-trivially, e.g. $\boldsymbol{\sigma}_a^F = \boldsymbol{\sigma}_a \otimes 1_S$.

The first qubit is open to the outside world and thereby subject to significant energy loss. It can also be driven by a resonant electromagnetic field to stabilize it, on the fast timescale, to a unique stationary state of nonzero energy. This fast subsystem is coupled to another qubit, which is well-protected i.e. which would keep its state in absence of that coupling. For the sake of concreteness we consider a dispersive coupling Hamiltonian. The dynamics of the system is then given by the following master equation:

$$\frac{d\rho}{dt} = u[\boldsymbol{\sigma}_+^F - \boldsymbol{\sigma}_-^F\ ,\ \rho] + \gamma(\boldsymbol{\sigma}_-^F\rho\boldsymbol{\sigma}_+^F - \tfrac{\boldsymbol{\sigma}_+^F\boldsymbol{\sigma}_-^F\rho + \rho\boldsymbol{\sigma}_+^F\boldsymbol{\sigma}_-^F}{2}) - i\kappa[\boldsymbol{\sigma}_z \otimes \boldsymbol{\sigma}_z, \rho]\ .$$

Assuming $\gamma \gg \kappa$, the first line corresponds to the fast dynamics $\mathfrak{L}_F(\rho)$ and the second line to its perturbation. To make an easy correspondence with the general theory, we can choose the time-scale such that $\gamma = O(1)$ and then $\kappa = \epsilon \ll 1$.

For $\kappa = 0$, the steady state of the fast qubit is given by :

$$\overline{\rho}_F = \frac{1 + x_\infty\boldsymbol{\sigma}_x + z_\infty\boldsymbol{\sigma}_z}{2}$$
$$\text{with } x_\infty = \frac{4\gamma u}{\gamma^2 + 8u^2} \quad \text{and} \quad z_\infty = -\frac{\gamma^2}{\gamma^2 + 8u^2}.$$

For the first order approximation, we compute $\mathrm{Tr}(\boldsymbol{\sigma}_z\overline{\rho}_F) = z_\infty$ and formula (10) leads to :

$$\frac{d\rho_S}{dt} = -i\kappa z_\infty[\boldsymbol{\sigma}_z, \rho_s]$$

for the dynamics. For the parameterization of the slow manifold by

$$(\mathfrak{K}_0 + \kappa\mathfrak{K}_1)(\overline{\rho}_F \otimes \rho_S) + O(\kappa^2) = \big(1 - i\kappa\mathbf{Q}\otimes\boldsymbol{\sigma}_z\big)\big(\overline{\rho}_F \otimes \rho_S\big)\big(1 + i\kappa\mathbf{Q}^\dagger \otimes \boldsymbol{\sigma}_z\big)\ ,$$

we first write $\overline{\mathbf{F}} = \boldsymbol{\sigma}_z - z_\infty 1_F$ and to get $\mathbf{Q}$ we must solve

$$\boldsymbol{\sigma}_-(\mathbf{Q}\overline{\rho}_F)\boldsymbol{\sigma}_+ - \tfrac{\boldsymbol{\sigma}_+\boldsymbol{\sigma}_-(\mathbf{Q}\overline{\rho}_F) + (\mathbf{Q}\overline{\rho}_F)\boldsymbol{\sigma}_+\boldsymbol{\sigma}_-}{2} = z_\infty\overline{\rho}_F - \sigma_z\overline{\rho}_F\ ,$$

with the constraint $\mathrm{Tr}(\mathbf{Q}\overline{\rho}_F) = 0$. Such operators can be parameterized by $\mathbf{Q} = q_x\boldsymbol{\sigma}_x + q_y\boldsymbol{\sigma}_y + q_z\boldsymbol{\sigma}_z + q_I 1_S$ with $q_x, q_y, q_z, q_I \in \mathbb{C}$. Analytically solving the linear system in 4 variables is a more direct alternative than using the formula (4) by integrating the dynamics; anyways, with both methods we get

$$q_x = -\tfrac{2\gamma^2+4u^2}{\gamma^2 u + 8u^3} \quad ; \quad q_y = \tfrac{2i\gamma^2+8iu^2}{\gamma^2 u+8u^3}$$
$$q_z = -\tfrac{3\gamma}{\gamma^2+8u^2} \quad ; \quad q_I = \tfrac{5\gamma^3+16\gamma u^2}{(\gamma^2+8u^2)^2}\ .$$

Note that although $\mathbf{Q}$ appears to be large for $u \ll 1$, in fact the products $\mathbf{Q}\overline{\rho}_F$ and $\mathbf{Q}\overline{\rho}_F\mathbf{Q}^\dagger$ remain small.

For the second order approximation, the matrix Z of Lemma 1 reduces to a scalar. Furthermore, with the values just computed, $\mathrm{Tr}(\mathbf{F}\mathbf{Q}\overline{\rho}_F)$ turns out to be purely real, so there is no second-order correction to the Zeno Hamiltonian, and the slow dynamics is given by

$$\frac{d\rho_S}{dt} = -i\kappa z_\infty[\boldsymbol{\sigma}_z, \rho_s] + \kappa^2 64\gamma u^2\,\frac{(\gamma^2 + 2u^2)}{(\gamma^2 + 8u^2)^3}\,\big(\boldsymbol{\sigma}_z\rho_S\boldsymbol{\sigma}_z - \rho_S\big)\,.$$

The dissipative part expresses how the purity of the slow qubit is affected by its coupling to the strongly dissipative one: the uncertain energy level of the fast qubit induces a growing phase uncertainty (dissipation with the σ_z operator). For u very small, $\overline{\rho}_F$ is close to a pure state (lowest energy level) and ρ_S undergoes almost no phase uncertainty – there remains essentially the deterministic phase shift induced by the coupling to the lower energy level in $\mathcal{H}_F$. For u very large, $\overline{\rho}_F$ is close to $1_F/2$ which seems to indicate large uncertainty. But in fact, the system is undergoing a sort of quantum dynamical decoupling (see [16]): the value of $\boldsymbol{\sigma}_z$ on the fast subsystem gets averaged out by our particular dynamics stabilizing $1_F/2$, such that there remains no significant net coupling between slow and fast qubits. This explains physically why with $u \gg 1$, we indeed have $d\rho_S/dt = 0$ up to second order.

6 Conclusion

We have studied adiabatic elimination in composite open quantum systems, assuming that one of the subsystems converges on a fast time scale towards a unique steady state. The other subsystem admits Hamiltonian dynamics on a slow time scale and as is common in quantum dynamics, the subsystems are connected via Hamiltonian interaction. We treat this slow dynamics as a perturbation, with respect to the trivial dynamics where the slow subsystem would just stay at its initial condition. Hence, using geometric singular perturbation theory and center manifold theory, we have derived an asymptotic expansion for the perturbation dynamics and for the slow manifold, in terms of powers of the time-scale separation. The quantum specificity is to impose, at all orders in this expansion, dynamics in the form of a Lindblad master equation and a slow manifold parameterization in terms of a completely positive trace preserving map (Kraus form). We have solved the expansion with these constraints up to second order, giving explicit expressions which allow physical interpretations. The goal of our future work is to provide a systematic procedure for computing explicit Lindblad and Kraus forms at arbitrary orders. This would allow (i) to get accurate results for less significant time-scale separations – e.g. allowing some quantum experiments to engineer stronger coupling and faster operation of the slow system – and (ii) to bound more precisely the effects of detrimental ancillary couplings, as a function of different classes of perturbations.

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A Inversion Lemmas on $\mathfrak{L}_F$

Lemma 2 *Let $\mathfrak{L}_F(\overline{\rho}_F) = 0$, with $\mathfrak{L}_F$ of the form (1). Then for any $|\nu\rangle \in \ker(\overline{\rho}_F)$ we have $\sqrt{\overline{\rho}_F}\mathbf{L}_k^\dagger|\nu\rangle = 0$, for all k.*

Proof 4 *For $|\nu\rangle \in \ker(\overline{\rho}_F)$ we have $\langle\nu|\mathfrak{L}_F(\overline{\rho}_F)|\nu\rangle = \sum_k\langle\nu|\mathbf{L}_k\overline{\rho}_F\mathbf{L}_k^\dagger|\nu\rangle$. Since $\mathfrak{L}_F(\overline{\rho}_F) = 0$ each term of this positive sum must annihilate.* □

Lemma 3 *Denote by $\rho = \overline{\rho}_F$ the unique solution of $\mathfrak{L}_F(\rho) = 0$. For a traceless operator $\mathbf{Y}$ such that $\ker(\overline{\rho}_F) \subseteq \ker(\mathbf{Y})$, the traceless solution to $\mathbf{X} = \mathfrak{L}_F^{-1}(\mathbf{Y})$ also satisfies $\ker(\overline{\rho}_F) \subseteq \ker(\mathbf{X})$.*

Proof 5 *Note that the operators have such kernels if and only if they can be written $\mathbf{X} = \tilde{\mathbf{X}}\overline{\rho}_F$, $\mathbf{Y} = \tilde{\mathbf{Y}}\overline{\rho}_F$. Since $\mathfrak{L}_F$ is a bijection on the space of traceless operators, the property is equivalent to showing that $\mathbf{Y}|\nu\rangle = \mathfrak{L}_F(\tilde{\mathbf{X}}\overline{\rho}_F)|\nu\rangle = 0$ for all $|\nu\rangle \in \ker(\overline{\rho}_F)$. By using $\overline{\rho}_F|\nu\rangle = 0$ and Lemma 2, we directly get*

$$\mathfrak{L}_F(\tilde{\mathbf{X}}\overline{\rho}_F)|\nu\rangle = \mathbf{X}\left(i\overline{\rho}_F\mathbf{H}_F - \tfrac{1}{2}\overline{\rho}_F\sum_\mu \mathbf{L}_{F_\mu}^\dagger\mathbf{L}_{F_\mu}\right)|\nu\rangle\,.$$

Subtracting $0 = \mathfrak{L}_F(\overline{\rho}_F)$ inside the bracket, applying $\overline{\rho}_F|\nu\rangle = 0$ and Lemma 2 once again, we do get 0. □