The least-curvature principle of Gauss and Hertz and geometric dynamics

Constantin Udriste, Massimiliano Ferrara, Ionel Tevy, Oltin Dogaru, Armando Ciancio

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CONSTANTIN UDRISTE
University Politehnica of Bucharest
Faculty of Applied Sciences
Department of Mathematics
Splaiul Independentei 313
060042, BUCHAREST, ROMANIA

MASSIMILIANO FERRARA
University of Reggio Calabria
Faculty of Law
Via dei Bianchi, 2-Palazzo Zani
89100, REGGIO CALABRIA, ITALY

IONEL TEVY
University Politehnica of Bucharest
Faculty of Applied Sciences
Department of Mathematics
Splaiul Indpendentei 313
060042, BUCHAREST, ROMANIA

OLTIN DOGARU
University Politehnica of Bucharest
Faculty of Applied Sciences
Department of Mathematics
Splaiul Indpendentei 313
060042, BUCHAREST, ROMANIA

ARMANDO CIANCIO
University of Messina
Department of Mathematics
ITALY

Abstract: *This paper shows how we can study real life problems in economics, biology and engineering with tools from differential geometry. Section 1 emphasizes the S-shaped time evolutions and underlines that the torse forming vector field reflects economical phenomena. Section 2 analyzes the geometric dynamics on infinite dimensional Riemannian manifolds produced by first order ODEs and the Euclidean metric. Section 3 introduces and studies a least-curvature principle. Section 4 defines and studies the geometric dynamics on infinite dimensional Riemannian manifolds produced by second order ODEs and by the Euclidean metric. Section 5 analyzes the least-curvature principle of Gauss and Hertz in a general setting. Section 6 explores the controllability of the neoclassical growth geometric dynamics and underlines that the theory can be extended to infinite dimensional manifolds. Section 7 contains conclusions.*

Key–Words: Kinematic curvature, Gauss-Hertz curvature, S-shaped curves, geometric dynamics, economic growth, Riemannian manifolds.

1 S-shaped time evolutions

S-*shaped (or sigmoid) time evolutions* [1] are frequently observed in some dynamic economic phenomena (product life cycles, the gradual diffusion of technological innovations or long-term fluctuations in income, productivity growth etc). It is also useful in biology and demography dynamics. These S-shaped evolutions are usually incorporated into formal models using autonomous ODE systems such as $\dot{x}(t) = X(x(t))$, where a solution $x(t)$ has an *inflection point* $x_0 = x(t_0)$. In other words $x(t)$ must satisfy

$$\ddot{x}(t_0) = \mu\dot{x}(t_0)$$

or

$$D_X X(x(t_0)) = \mu X(x(t_0))$$

and $x'(t_0),\ x'''(t_0)$ are linearly independent vectors. This means that $X(x(t_0))$ is a proper vector of the Jacobian matrix $DX(x(t_0))$ with respect to the proper value μ.

The S-shaped evolutions degenerate in two ways:

(i) $D_X X(x(t)) = \mu(t)X(x(t)), \forall t \in I \subset R$, where $x(t)$ is a solution of (1); in this case the solution $x(t)$ of (1) is a straight line;

(ii) $D_X X(x(t)) = \mu(x(t))X(x(t))$, for any $t \in I \subset R$ and for any solution $x(t)$ of (1); in this case any solution $x(t)$ of (1) is a straight line.

This remark suggests to introduce a vector field X characterized by the relation $D_X X(x) = \mu(x)X(x), \forall x \in R^n$. Note that a *torse forming vector field* has this property. Indeed, such a field is defined by [2]

$$D_Z X = aZ + (Y, Z)X,$$

where Z is an arbitrary C^∞ vector field, a is a given C^∞ real function and Y is a given C^∞ vector field. On the other hand, in the case of torse forming vector field, the relation

$$D_X X = aX + (Y, X)X$$

is true everywhere, showing the collinearity between $D_X X$ and X. In this way, a torse forming vector field used till now only in differential geometry may reflect phenomena from economics, biology, demographics etc (*Open Problems*).

2 Geometric dynamics on infinite dimensional Riemannian manifolds produced by first order ODEs

The *geometric dynamics* [2]-[4] induced by a flow

$$\dot{x}(t) = X(x(t), t), \tag{1}$$

and by the Euclidean metric is described by the first order Lagrangian

$$L^1(x, \dot{x}, t) = \frac{1}{2}||\dot{x} - X(x, t)||^2.$$

Let

$$F = \frac{1}{2}||X||^2$$

be the *potential energy* associated to X. The second order Euler-Lagrange ODE system associated to L^1, i.e.,

$$\frac{\partial L^1}{\partial x} - \frac{d}{dt}\frac{\partial L^1}{\partial \dot{x}} = 0$$

or

$$\ddot{x}(t) = \left(DX(x(t)) - DX^T(x(t))\right)\dot{x}(t)$$

$$+\nabla F(x(t)) + \frac{\partial X}{\partial t}, \tag{2}$$

is an Euler-Lagrange prolongation of the system (1). Introducing the *gyroscopic vector field*

$$Y(x(t), \dot{x}(t), t)$$

$$= \left(DX(x(t)) - DX^T(x(t))\right)\dot{x}(t) + \nabla F(x(t)) + \frac{\partial X}{\partial t},$$

the ODE (2) can be written in the general form of a second order dynamical system.

A trajectory in the geometric dynamics (2) is fixed either by initial conditions

$$x(0) = x_0,\ \dot{x}(0) = v_0$$

or by boundary conditions

$$x(0) = x_0,\ x(t_0) = x_{t_0}.$$

If $v_0 = X(x(0))$, then the trajectory in geometric dynamics coincides with a field line of the vector field X, i.e., a solution of (1). If $v_0 \neq X(x(0))$, then the corresponding trajectory is transversal to the field lines. In [3, pag. 5], one asks what is the practical significance of these transversal trajectories; in the Sections 3 and 5, possible answers will be given.

We admit that the theory in [3, pag. 137] can be applied also to the first order differential systems consisting from an infinity of n-vector equations. For that we use the Riemannian (Euclidean) manifold $(R^n \times l_2, \delta_{ij} + \delta^{\alpha\beta})$ to transform a local flow in a geodesic motion in a gyroscopic field of forces. We start from an arbitrary local flow described by

$$\dot{x}_\alpha(t) = X_\alpha(x_\alpha(t), t),\ x_\alpha(t) \in R^n,\ \alpha \in N, \tag{3}$$

on the Riemannian (Euclidean) manifold $(R^n \times l_2, \delta_{ij} + \delta^{\alpha\beta})$. We introduce the least squares Lagrangian

$$L^1_\infty(x_\alpha, \dot{x}_\alpha, t)$$

$$= \frac{1}{2}\delta_{ij}\delta^{\alpha\beta}(\dot{x}^i_\alpha - X^i_\alpha(x_\alpha, t))(\dot{x}^j_\beta - X^j_\beta(x_\beta, t))$$

$$= \frac{1}{2}\sum_\alpha ||\dot{x}_\alpha - X_\alpha(x_\alpha, t)||^2$$

and the associated action on the Sobolev space $W^{1,2}([a, b])$. By the Euler-Lagrange ODEs, we obtain an Euler-Lagrange prolongation of the system (3),

$$\frac{d^2 x_\alpha}{dt^2} = \delta^{\beta\gamma}\left(\frac{\partial X_\alpha}{\partial x_\beta} - \frac{\partial X_\beta}{\partial x_\alpha}\right)\frac{dx_\gamma}{dt} + \frac{\partial f}{\partial x_\alpha} + \frac{\partial X_\alpha}{\partial t},$$

where $f = \frac{1}{2}\delta_{ij}\delta^{\alpha\beta}X^i_\alpha X^j_\beta$ is a density of energy. This prolongation determines a geometric dynamics, i.e., a geodesic motion in a gyroscopic field of forces. Automatically, the least squares Lagrangian determines the Hamiltonian

$$H^1_\infty(x_\alpha, \dot{x}_\alpha, t)$$

$$= \frac{1}{2}\delta_{ij}\delta^{\alpha\beta}(\dot{x}^i_\alpha - X^i_\alpha(x_\alpha, t))(\dot{x}^j_\beta + X^j_\beta(x_\beta, t)).$$

Of course, L^1_∞ and H^1_∞ are sums of series of functions. They ask the uniform convergence, and hence

$$\lim_{\alpha\to\infty}(\dot{x}_\alpha - X_\alpha(x_\alpha, t)) = 0$$

$$\lim_{\alpha\to\infty}(\dot{x}_\beta + X_\beta(x_\beta, t)) = 0$$

along the trajectories in geometric dynamics.

3 The least-curvature principle

Suppose that, for each index $\alpha \in N$, the point $x_\alpha(t)$ represents a particle involved in a first order dynamical system at time t and that we have constraints described by

$$A_k(x) = 0,\ x = (x_\alpha), \alpha \in N,\ k = 1, ..., m,$$

where A_k are C^1 functions. Admit that the particle is "sensitive" to the velocity vector field $X_\alpha(x_\alpha(t), t)$, in the sense that without the constraints action, the particle would follow a field line. The velocity vector $\dot{x}(t) = (\dot{x}_\alpha(t))$ refers to any kinematically possible path, i.e. solution of the constraints system, for which coordinates at the instant considered are the same as in actual trajectory.

Definition. *The least squares Lagrangian (sum of series of functions, uniform convergence)*

$$L^1_\infty = \frac{1}{2}\sum_\alpha ||\dot{x}_\alpha(t) - X_\alpha(x_\alpha(t), t)||^2$$

is called the kinematic curvature of the path $x(t) = (x_\alpha(t))$ *on the Riemannian (Euclidean) space* $(R^n \times l_2, \delta_{ij} + \delta^{\alpha\beta})$.

Theorem. *From all paths consistent with the constraints, the actual trajectory is that which has the least kinematic curvature*.

Proof. Differentiating the constraints we respect to t, we obtain

$$\frac{\partial A_k}{\partial x^i_\alpha}(x(t))\dot{x}^i_\alpha(t) = 0.$$

Let $\dot{x}^i_\alpha$ be a typical component of the velocity in the path considered and $\dot{x}^i_{\alpha 0}$ be the corresponding component of the velocity in the actual trajectory. Then, by the previous equality,

$$\frac{\partial A_k}{\partial x^i_\alpha}(x(t))(\dot{x}^i_\alpha(t) - \dot{x}^i_{\alpha 0}(t)) = 0,$$

i.e., a small displacement $\delta x^i_\alpha(t)$ of the system is proportional to $(\dot{x}^i_\alpha(t) - \dot{x}^i_{\alpha 0}(t))$. The components of the velocities exercised by the curvature are typified by $(\dot{x}^i_{\alpha 0}(t) - X^i_\alpha(x_{\alpha 0}(t), t))$. Since these components must be normal to the submanifold defined by the constraints, we have (orthogonality condition)

$$\delta_{ij}\delta^{\alpha\beta}(\dot{x}^i_{\alpha 0}(t) - X^i_\alpha(x_{\alpha 0}(t), t))(\dot{x}^j_\beta(t) - \dot{x}^j_{\beta 0}(t)) = 0.$$

This relation can be written as follows

$$\sum_\alpha ||\dot{x}_\alpha(t) - X_\alpha(x_{\alpha 0}(t), t)||^2$$
$$= \sum_\alpha ||\dot{x}_{\alpha 0}(t) - X_\alpha(x_{\alpha 0}(t), t)||^2$$
$$+ \sum_\alpha ||\dot{x}_\alpha(t) - \dot{x}_{\alpha 0}(t)||^2.$$

Consequently

$$\sum_\alpha ||\dot{x}_\alpha(t) - X_\alpha(x_{\alpha 0}(t), t)||^2$$
$$> \sum_\alpha ||\dot{x}_{\alpha 0}(t) - X_\alpha(x_{\alpha 0}(t), t)||^2.$$

Now, the result from the statement of the theorem follows.

This result shows that the trajectories of the Euler-Lagrange prolongation (2), which are not field lines of (1), can be trajectories of the dynamical system $\dot{x}(t) = X(x(t), t)$ subject to some constraints.

4 Geometric dynamics on infinite dimensional Riemannian manifolds produced by second order ODEs

The general form of a second order dynamical system (Newton Law) is

$$\ddot{x}(t) = Y(x(t), \dot{x}(t), t). \tag{4}$$

A trajectory of this second order dynamical system is fixed either by initial conditions

$$x(0) = x_0,\ \dot{x}(0) = v_0$$

or by boundary conditions

$$x(0) = x_0,\ x(t_0) = x_{t_0}.$$

The second order ODEs (4) suggests that giving a vector field $Y(x(t), \dot{x}(t), t)$, we can introduce a second order least squares Lagrangian

$$L^2 = \frac{1}{2}||\ddot{x}(t) - Y(x(t), \dot{x}(t), t)||^2,$$

using the Euclidean metric. The fourth order Euler-Lagrange equations produced by L^2, i.e.,

$$\frac{\partial L^2}{\partial x} - \frac{d}{dt}\frac{\partial L^2}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L^2}{\partial \ddot{x}} = 0$$

are Euler-Lagrange prolongations of a system of type (4). To simplify, we accept

$$Y(x(t), \dot{x}(t), t) = Y(x(t))$$

and we denote $F = \frac{1}{2}||Y||^2$. Then the Euler-Lagrange prolongation of the system (4) is

$$x^{(4)}(t) = \Big(DX(x(t)) - DX^T(x(t))\Big)\,\ddot{x}(t)$$

$$+\dot{x}^T(t)\,(\text{Hess}(Y)(x(t)))\,\dot{x}(t) - \nabla F(x(t)).$$

This prolongation determines a geometric dynamics.

We admit that the theory in [3, page. 137] can be applied also to the second order differential systems consisting from an infinity of n-vector equations. For that we use the Riemannian (Euclidean) manifold $(R^n \times l_2, \delta_{ij} + \delta^{\alpha\beta})$ to transform a Newton law in an Euler-Lagrange prolongation of fourth order. We start from an arbitrary local second order dynamical system described by

$$\ddot{x}_\alpha(t) = Y_\alpha(x_\alpha(t), \dot{x}_\alpha(t), t), \tag{5}$$

$$x_\alpha(t) \in R^n,\ \alpha \in N,$$

on the Riemannian (Euclidean) manifold $(R^n \times l_2, \delta_{ij} + \delta^{\alpha\beta})$. We introduce the second order least squares Lagrangian

$$L^2_\infty(x_\alpha, \dot{x}_\alpha, \ddot{x}_\alpha, t)$$

$$= \frac{1}{2}\delta_{ij}\delta^{\alpha\beta}(\ddot{x}^i_\alpha - Y^i_\alpha(x_\alpha, \dot{x}_\alpha, t))(\ddot{x}^j_\beta - Y^j_\beta(x_\beta, \dot{x}_\beta, t))$$

$$= \frac{1}{2}\sum_\alpha ||\ddot{x}_\alpha - Y_\alpha(x_\alpha, \dot{x}_\alpha, t)||^2$$

and the associated action on the Sobolev space $W^{1,2}([a,b])$. By the Euler-Lagrange ODEs, we obtain an Euler-Lagrange prolongation of the system (3), which is a fourth order system. This prolongation determines a geometric dynamics. Automatically, the geometric dynamics determines the Hamiltonian

$$H^2_\infty(x_\alpha, \dot{x}_\alpha, \ddot{x}_\alpha, t)$$

$$= \frac{1}{2}\delta_{ij}\delta^{\alpha\beta}(\ddot{x}^i_\alpha - Y^i_\alpha(x_\alpha, \dot{x}_\alpha, t))(\ddot{x}^j_\beta + Y^j_\beta(x_\beta, \dot{x}_\beta, t)).$$

Of course, L^2_∞ and H^2_∞ are sums of series of functions. They ask the uniform convergence, and hence

$$\lim_{\alpha\to\infty}(\ddot{x}_\alpha - Y_\alpha(x_\alpha, \dot{x}_\alpha, t)) = 0$$

$$\lim_{\alpha\to\infty}(\ddot{x}_\beta + Y_\beta(x_\beta, \dot{x}_\alpha, t)) = 0$$

along the trajectories in geometric dynamics.

5 The least-curvature principle of Gauss and Hertz

Suppose that, for each index $\alpha \in N$, the point $x_\alpha(t)$ represents a particle involved in a second order dynamical system at time t. Admit that the constraints are described by Pfaff equations

$$\delta^{\alpha\beta}A_{\alpha ki}(x)dx^i_\beta = 0,\ x = (x_\alpha),$$

$$\alpha,\, \beta \in N,\ k = 1, ..., m,\ i = 1, ..., n,$$

where the coefficients $A_{\alpha ki}$ are C^1 functions. We add the external force $Y_\alpha(x_\alpha(t), \dot{x}_\alpha(t), t)$ which acts on the particle. The acceleration vector $\ddot{x}(t) = (\ddot{x}_\alpha(t))$ refers to any kinematically possible path, i.e. solution of the previous Pfaff system, for which coordinates and velocities at the instant considered are the same as in actual trajectory.

Definition. *The least squares Lagrangian (sum of series of functions, uniform convergence)*

$$L^2_\infty = \frac{1}{2}\sum_\alpha ||\ddot{x}_\alpha(t) - Y_\alpha(x_\alpha(t), \dot{x}(t), t)||^2$$

is called the Gauss-Hertz curvature [5] of the path $x(t) = (x_\alpha(t))$ on the Riemannian (Euclidean) space $(R^n \times l_2, \delta_{ij} + \delta^{\alpha\beta})$.

Theorem. *From all paths consistent with the constraints, the actual trajectory is that which has the least Gauss-Hertz curvature.*

Proof. Differentiating the constraints

$$\delta^{\alpha\beta}A_{\alpha ki}(x(t))\dot{x}^i_\beta = 0$$

we respect to t, we obtain

$$\delta^{\alpha\beta}\left(A_{\alpha ki}(x(t))\ddot{x}^i_\beta(t) + \frac{\partial A_{\alpha ki}}{\partial x^j_\gamma}(x(t))\dot{x}^i_\beta(t)\dot{x}^j_\gamma(t)\right) = 0.$$

Let $\ddot{x}^i_\alpha$ be a typical component of the acceleration in the path considered and $\ddot{x}^i_{\alpha 0}$ be the corresponding component of the acceleration in the actual trajectory. Then, by the previous equality,

$$\delta^{\alpha\beta}A_{\alpha ki}(x(t))(\ddot{x}^i_\beta(t) - \ddot{x}^i_{\beta 0}(t)) = 0,$$

i.e., a small displacement $\delta x^i_\alpha(t)$ of the system is proportional to $(\ddot{x}^i_\alpha(t) - \ddot{x}^i_{\alpha 0}(t))$. The components of the forces exercised by the Gauss-Hertz curvature are typified by $(\ddot{x}^i_{\alpha 0}(t) - Y^i_\alpha(x_{\alpha 0}(t), \dot{x}_{\alpha 0}(t), t))$. Since these forces do no work, we must have (orthogonality condition)

$$\delta_{ij}\delta^{\alpha\beta}(\ddot{x}^i_{\alpha 0}(t) - Y^i_\alpha(x_{\alpha 0}(t), \dot{x}_{\alpha 0}(t), t))$$

$$(\ddot{x}^j_\beta(t) - \ddot{x}^j_{\beta 0}(t)) = 0.$$

This relation can be written as follows

$$\sum_{\alpha} ||\ddot{x}_\alpha(t) - Y_\alpha(x_{\alpha 0}(t), \dot{x}_{\alpha 0}(t), t)||^2$$

$$= \sum_{\alpha} ||\ddot{x}_{\alpha 0}(t) - Y_\alpha(x_{\alpha 0}(t), \dot{x}_{\alpha 0}(t), t)||^2$$

$$+ \sum_{\alpha} ||\ddot{x}_\alpha(t) - \ddot{x}_{\alpha 0}(t)||^2.$$

Consequently

$$\sum_{\alpha} ||\ddot{x}_\alpha(t) - Y_\alpha(x_{\alpha 0}(t), \dot{x}_{\alpha 0}(t), t)||^2$$

$$> \sum_{\alpha} ||\ddot{x}_{\alpha 0}(t) - Y_\alpha(x_{\alpha 0}(t), \dot{x}_{\alpha 0}(t), t)||^2.$$

Now, the result from the statement of the theorem follows.

6 Neoclassical growth geometric dynamics

We start with the *neoclassical growth flow*

$$\dot{k}(t) = X(k(t)) - \delta k(t) - c(t),\ k(0) = k_0 > 0,$$

where $k = (k^1, ..., k^n)$ is the *capital intensity vector*, $c = (c^1, ..., c^n)$ is the *control vector* and δ is a real number. The Jacobian matrix of the growth vector field is $DX - \delta I$. Therefore, the geometric dynamics induced by the growth vector field and the Euclidean metric is described by the second order ODE system

$$\ddot{k}(t) = \Big(DX(k(t)) - DX^T(k(t))\Big)\, \dot{k}(t)$$

$$+\nabla F(k(t)) - \dot{c}(t),$$

where

$$F = \frac{1}{2}||X(k) - \delta k - c||^2.$$

Obviously, the geometric dynamics is controlled by the command $c(t)$ and its derivative $\dot{c}(t)$.

Now, we underline that the growth flow can be analyzed as in Sections 2-5 (see also [3]), but the sense of index α is still an open problem.

7 Conclusions

The S-shaped evolutions, the geometric dynamics on infinite dimensional Riemannian manifolds, the least-curvature principles and the neoclassical growth flow are suitable mathematical objects and models for describing the real life problems in economics, biology and engineering with tools from differential geometry. As example, the least-curvature principles, in fact the distance between a point and a subset, can describe the evolution of a dynamical system subject to some constraints.

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References:

[1] G. Jarne, J. Sanchez-Choliz, F. Fatas-Villafranca, *S-shaped Economic Dynamics, The Logistic and Gompertz Curves Generalized*, The Electronic Journal of Evolutionary Modeling and Economic Dynamics, 1048 (2005), 1-37.

[2] C. Udrişte, *Geometric Dynamics*, Kluwer Academic Publishers, 2000.

[3] C. Udrişte, M. Ferrara, D. Opriş, *Economic Geometric Dynamics*, Geometry Balkan Press, Bucharest, 2004.

[4] C. Udrişte, Sorin Udrişte, *Queue geometric dynamics*, Proceedings of 2005 Annual Hawaii International Conference on Statistics, Mathematics and Related Fields, January 9-11, 2005, Honolulu, Hawaii (versiune electronica); Economic Geometric Dynamics, Geometry Balkan Press, Bucharest, 2004, pp. 121-140.

[5] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge Mathematical Library, 1989, pp. 254-256.