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Abstract

In this work we study the numerical approximation of the solutions of a class of abstract parabolic time optimal control problems with unbounded control operator. Our main results assert that, provided that the target is a closed ball centered at the origin and of positive radius, the optimal time and the optimal controls of the approximate time optimal problems converge (in appropriate norms) to the optimal time and to the optimal controls of the original problem. In order to prove our main theorem, we provide a nonsmooth data error estimate for abstract parabolic systems.

Keywords. distributed parameter systems, optimal control, numerical approximation
AMS subject classifications. 93C25; 93B07; 93C20; 35R30;

1 Introduction

Time optimal control of infinite dimensional systems is a subject of growing interest, motivated by numerous applications in domains such as guidance of complex systems or temperature regulation in large buildings. In recent year, using new tools from infinite dimensional systems theory, the literature devoted to this topic grew in a considerable manner (see Arada and Raymond [1], Barbu [4], [5], Fattorini [11], [12], [13], Kunisch and Wang [17], Kunisch and Wachsmuth [16], Li and Yong [21] and Tröltzsch [29]). The specific case of time optimal control for systems governed by parabolic PDE's has numerous applications, from which we quote optimization of building thermal storage (see, for instance [14] and references therein).

The aim of this paper, containing results partially announced in [30], is to study the approximation of the solutions of time optimal (internally or boundary) control problems for a class of infinite dimensional linear systems by projecting the original problem on an appropriate family of finite dimensional spaces. This is a delicate question since, as shown in the above mentioned references, time optimal controls are usually highly oscillating functions (due to the bang-bang property). As far as we know, the only papers having already investigated this issue are Knowles [15], Wang and Wang [32], Wang and Zheng [33] which investigated finite elements approximation for systems governed by the heat

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equation with internal controls. To our knowledge, the similar results obtained in these papers are unknown to boundary time optimal control problems. A similar topic dealing with the homogenization of parabolic system, where we study the asymptotic behavior of solutions and (time-)optimal controls has been developed in recent years (see for example, Carja [8], [9], Castro and Zuazua [10] and Tebou [27]).

To state our results we need some notation. Let X and U be real Hilbert spaces, and let $A_0 : \mathcal{D}(A_0) \to X$ be a strictly positive operator. It is known that $-A_0$ generates an exponentially stable analytic semigroup, denoted by $\mathbb{T}_t$. We denote by $\|.\|$ (resp. $\langle .,. \rangle$) the norm (resp. the inner product) on X. For $\gamma > 0$ we denote by X_γ the space $\mathcal{D}(A_0^\gamma)$, endowed with the graph norm. For $\gamma < 0$, X_γ stands for the dual of $X_{-\gamma}$ with respect to the pivot space X. We also introduce an operator $B \in \mathcal{L}(U, X_{-\alpha})$ with $0 \leqslant \alpha < \frac{1}{2}$, called *control operator*. In this paper we consider systems of the form

$$\begin{aligned} \dot{z}(t) + A_0 z(t) &= Bu(t) \qquad (t \geqslant 0), & (1.1)\\ z(0) &= z_0 \qquad (z_0 \in X), & (1.2) \end{aligned}$$

where $u \in L^\infty([0, +\infty); U)$ is the input function and z is the state trajectory. It is well known (this follows, for instance, by combining Theorem 4.4.3 and Proposition 5.1.3 in [31]) that if $z_0 \in X$ and $u \in L^\infty([0, +\infty); U)$, there exists a unique solution $z \in C([0, +\infty); X)$ of (1.1)-(1.2) and z satisfies

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u, \quad \text{where} \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) \mathrm{d}\sigma. \tag{1.3}$$

Given $\varepsilon > 0$, denote by $\bar{B}(0, \varepsilon)$ the closed ball centered in zero and of radius ε in X. We consider the time optimal control problem which states as follows
(**TP**) *Determine* $\tau_0^* > 0$ *such that*

$$\tau_0^* = \min\{\tau \geqslant 0 \ \mid \ \text{there exists } u \in L^\infty([0, +\infty); U) \text{ s.t. } \|u\|_{L^\infty([0,+\infty);U)} \leqslant 1 \ \text{ and } \ z(\tau) \in \bar{B}(0, \varepsilon)\},$$

and the corresponding optimal controls u_0^*.

Denote

$$\mathcal{U}_{\text{ad}} = \{u \in L^\infty([0, +\infty); U) \ \mid \ \|u\|_{L^\infty([0,+\infty);U)} \leqslant 1\}. \tag{1.4}$$

It is well-known that the above optimal time τ_0^* and optimal control u_0^* exist under additional assumptions (see, for instance, [23]).

Let $(V_h)_{h>0}$ (resp. $(U_h)_{h>0}$) be a family of finite dimensional subspaces of $X_{\frac{1}{2}}$ (resp. U), which are normed spaces when endowed with the restriction of the norm of $X_{\frac{1}{2}}$ (resp. U). We denote P_h (resp. Q_h) the orthogonal projector from X onto V_h (resp. U onto U_h). For each $h > 0$ we consider the following system:

$$\begin{aligned} \dot{z}_h(t) + A_h z_h(t) &= B_h u_h(t) \qquad (t \geqslant 0), & (1.5)\\ z_h(0) &= P_h z_0, & (1.6) \end{aligned}$$

where $u_h \in L^\infty([0, +\infty); U_h)$, $(A_h)_{h>0}$ is defined by

$$< A_h \varphi, \psi > \ = \ < A_0^{\frac{1}{2}} \varphi, A_0^{\frac{1}{2}} \psi > \qquad (\varphi, \psi \in V_h), \tag{1.7}$$

and $B_h \in \mathcal{L}(U, V_h)$ is defined by:

$$\langle B_h u, \varphi \rangle \ = \ \langle u, B^* \varphi \rangle_U \qquad (\varphi \in V_h, \ u \in U). \tag{1.8}$$

The above system is the Galerkin approximation of (1.1)-(1.2), and its solution z_h can be written as

$$z_h(t) = \mathbb{T}_{t,h}P_h z_0 + \Phi_{t,h}u_h, \quad \text{where} \quad \Phi_{t,h}u_h = \int_0^t \mathbb{T}_{t-\sigma,h}B_h u_h(\sigma)\mathrm{d}\sigma, \tag{1.9}$$

where $\mathbb{T}_{t,h} = \exp(-tA_h)$ is the semigroup generated by A_h. Denote by $\bar{B}_h(0,\varepsilon)$ the closed ball centered in zero in V_h with radius ε. For each $h > 0$, we consider the time optimal control problem for the above system (1.5)-(1.6) which states as follows:
($\mathbf{TP_h}$) *Determine* $\tau_h^* > 0$ *such that*

$$\tau_h^* = \min\{\tau \geqslant 0 \ \mid \ \text{there exists } u_h \in L^\infty([0,+\infty);U_h) \text{ s.t. } \|u_h\|_{L^\infty([0,+\infty);U_h)} \leqslant 1 \ \text{ and } \ z_h(\tau) \in \bar{B}_h(0,\varepsilon)\},$$

and the corresponding optimal controls u_h^* *in the admissible set* $\mathcal{U}_{\mathrm{ad},h}$*, where* $\mathcal{U}_{\mathrm{ad},h}$ *is defined by*

$$\mathcal{U}_{\mathrm{ad},h} = \{u \in L^\infty([0,+\infty);U_h) \ \mid \ \|u\|_{L^\infty([0,+\infty);U_h)} \leqslant 1\}. \tag{1.10}$$

The goal of this work is to study the convergence of τ_h^* to τ_0^* and of u_h^* to u_0^* when $h \to 0$. To this aim, we need appropriate assumptions on the approximation properties of the spaces $(V_h)_{h>0}$ and $(U_h)_{h>0}$. More precisely, we assume that there exist $\theta > 0$, $h_1 > 0$, $C > 0$ such that for every $h \in (0,h_1)$ and $0 \leqslant \gamma \leqslant 1$ we have:

$(C1)$ $\quad \|x - P_h x\| \leqslant Ch^{\theta\gamma}\|x\|_\gamma$ for every $x \in X_\gamma$,

$(C2)$ $\quad A_h$ is uniformly (with respect to h) analytic, in the sense of the Definition 3.2 below,

$(C3)$ $\quad \|(I - P_h)B\|_{\mathcal{L}(U,X)} \leqslant Ch^{\theta(1-\alpha)}$,

$(C4)$ $\quad \|P_h B\|_{\mathcal{L}(U,V_h)} \leqslant Ch^{-\theta\alpha}$,

$(C5)$ $\quad \lim_{h\to 0}\|Q_h u - u\|_U = 0$ for every $u \in U$,

where α represents the unboundedness degree of the operator B (introduced in the beginning of this section).

We are now in a position to state the main results of this paper:

Theorem 1.1. *With the notation and assumptions on operators* A_0*,* B*,* A_h*,* B_h*, assume that* $(C1)$-$(C5)$ *hold and that* $z_0 \in X$*,* $\|z_0\| > \varepsilon$*. Then* $\lim_{h\to 0}\tau_h^* = \tau_0^*$*.*

Theorem 1.2. *With the notation and assumptions on operators* A_0*,* B*,* A_h*,* B_h*, assume that* $(C1)$-$(C5)$ *hold and that the only* $\eta \in X$ *for which there exists an open non-empty interval* I *with* $B^*\mathbb{T}_t^*\eta = 0$ *for* $t \in I$ *is* $\eta = 0$*. Then we have*

$$u_h^* \to u_0^* \ \textit{strongly in} \ L^2([0,T];U),$$

where $T = 2\,\frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}$ *(*λ_1 *is the first eigenvalue of the operator* A_0*) and extending* $(u_h^*)_h$ *and* u_0^* *to time* T *by zero.*

It is worth mentioning that similar convergence results have been obtained in [15] and [33]. In [15], the control space takes only in finite dimensional subspaces of U and in [33] the initial data z_0 is taken in $X_{\frac{1}{2}}$ and with bounded operator B, which can not apply to boundary control problem. Our results

can be seen as generalizations of those obtained in [33], in the sense that we consider a class of abstract problems including the PDE systems studied there. The novelty of our results is that we weaken the regularity assumptions on the initial data with $z_0 \in X$ and also that we weaken the assumption on control operator B with $B \in \mathcal{L}(U, X_{-\alpha})$, $0 \leqslant \alpha < \frac{1}{2}$.

The outline of the remaining part of this paper is as follows. Section 2 contains some necessary background on time optimal control problems. In Section 3, we provide some error estimate results with smooth initial data which play an essential role to prove our Theorem 1.1. In Section 4, we provide the proof of our main theorems. In Section 5, we apply our abstract results to some equations. Throughout the paper, we denote by C a positive constant that may change from line to line.

2 Some background on time optimal control problem

This section, in which we continue to use the notation and assumptions introduced in Section 1 for X, U, A_0, B, is devoted to some background on the time optimal control problem (**TP**). We first introduce some basic notions on controllability (see for example Ch6. and Ch11. in [31]).

For $\tau > 0$, we denote $\Psi_\tau \in \mathcal{L}(X, L^2([0,\tau];U))$ the observation operator of (A_0^*, B^*) on $[0,\tau]$ defined by

$$(\Psi_\tau z_0)(\sigma) = B^* \mathbb{T}_\sigma^* z_0 \qquad (z_0 \in X,\ \sigma \in [0,\tau]).$$

It is clear that (see, for instance, [31, Ch. 4]),

$$\Phi_\tau^* = R_\tau \Psi_\tau \qquad (\tau > 0), \tag{2.11}$$

where R_τ is the reflection operator on $L^2([0,\tau];U)$ defined by $R_\tau u(t) = u(\tau - t)$.

We then recall the following existence result, which can be proved by using standard techniques providing the existence of time optimal controls for linear systems. We refer to [12] or Lions [22] for a detailed description of the methodology or to Micu *et al* [23, Proposition 2.6] for a slightly more general version of this result.

Proposition 2.1. *For every $z_0 \in X$ and $\varepsilon > 0$, the time optimal control problem* (**TP**) *admits at least one solution (τ^*, u^*).*

We need below a special version of the maximum principle. Although maximum principle is a classical tool in optimal control problems, we were unable to find the version we need (abstract setting with unbounded control operators) in the existing literature. Therefore we give the precise statement and a short proof below.

Theorem 2.2. *With the notation of Proposition 2.1, assume that $\tau^* > 0$. Then the time optimal control u^* satisfies the maximum principle, i.e., there exists $\xi \in X$, $\xi \neq 0$ such that,*

$$\left\langle u^*(\sigma), \left(\Psi_{\tau^*}\xi\right)(\tau^* - \sigma)\right\rangle_U = \max_{v \in U,\ \|v\| \leqslant 1} \left\langle v, \left(\Psi_{\tau^*}\xi\right)(\tau^* - \sigma)\right\rangle_U, \qquad (\sigma \in (0,\tau^*) \text{ a.e.}). \tag{2.12}$$

Moreover, ξ satisfies the transversality condition, i.e.:

$$\left\langle \xi, z - \mathbb{T}_{\tau^*} z_0 - \Phi_{\tau^*} u^* \right\rangle \geqslant 0 \qquad (z \in \bar{B}(0,\varepsilon)). \tag{2.13}$$

Proof. Let (τ^*, u^*) be a solution of the time optimal control in Proposition 2.1 and let

$$z_1 = \mathbb{T}_{\tau^*} z_0 + \Phi_{\tau^*} u^*.$$

Moreover, for each $z_0 \in X$ and $\tau > 0$ denote

$$U_\tau := \{u|_{[0,\tau]} \mid u \in \mathcal{U}_{\text{ad}}\} \text{ and } \mathcal{R}_\tau z_0 = \mathbb{T}_\tau z_0 + \Phi_\tau U_\tau.$$

We claim that

$$z_1 \in \partial \mathcal{R}_{\tau^*} z_0 \cap \partial B(0, \varepsilon). \tag{2.14}$$

Indeed, it can be easily checked that we necessarily have $\|z_1\| = \varepsilon$. If we admit, by contradiction, that z_1 is an interior point of $\mathcal{R}_{\tau^*} z_0$, this implies that there exists $z_2 \in \mathcal{R}_{\tau^*} z_0$ with $\|z_2\| < \varepsilon$. Consequently, there exists $u_2 \in U_{\tau^*}$ such that

$$\mathbb{T}_{\tau^*} z_0 + \Phi_{\tau^*} u_2 = z_2.$$

Since the map $t \mapsto \|\mathbb{T}_t z_0 + \Phi_t u_2\|$ is continuous and that $\tau^* > 0$, we obtain that there exists $\tau \in (0, \tau^*)$ such that

$$\|\mathbb{T}_\tau z_0 + \Phi_\tau u_2\| = \varepsilon.$$

This contradicts the fact that (τ^*, u^*) is a solution of our time optimal control problem.We have thus proved our claim (2.14).

Thus we clearly have that the sets $\mathcal{R}_{\tau^*} z_0$ and $B(0, \varepsilon)$ are non-empty, convex and they have no common point. Moreover, since $B(0, \varepsilon)$ is open, we can apply the geometric version of the Hahn Banach theorem (see, for instance [7, Theorem 1.6]) to obtain that there exists a hyperplane separating $\mathcal{R}_{\tau^*} z_0$ and $B(0, \varepsilon)$. This means that there exists $\alpha \in \mathbb{R}$ and $\xi \in X$, $\xi \neq 0$ such that

$$\langle \xi, \eta \rangle \leqslant \alpha \qquad (\eta \in \mathcal{R}_{\tau^*} z_0), \tag{2.15}$$

$$\langle \xi, \eta \rangle \geqslant \alpha \qquad (\eta \in B(0, \varepsilon)). \tag{2.16}$$

The two above inequalities and (2.14) imply that

$$\langle \xi, z_1 \rangle = \alpha. \tag{2.17}$$

From (2.15) and (2.17) we deduce that

$$\langle \xi, z_1 \rangle \geqslant \langle \xi, \eta \rangle \qquad (\eta \in \mathcal{R}_{\tau^*} z_0),$$

which combined with the duality (2.11), clearly implies the maximum principle (2.12).

Moreover, we easily deduce the transversality condition (2.13) from (2.16) and (2.17).

□

Remark 2.3. *To be more precise, we can deduce from the transversality condition (2.13) that:*

$$\xi = -k \left(\mathbb{T}_{\tau^*} z_0 + \Phi_{\tau^*} u^* \right), \qquad \textit{for some positive constant } k. \tag{2.18}$$

Corollary 2.4. *With the notation of Theorem 2.2, assume that $\tau^* > 0$ and that the assumption of Theorem 1.2 holds, i.e., the only $\eta \in X$ for which there exists an open non-empty interval I with $B^* \mathbb{T}_t^* \eta = 0$ for $t \in I$ is $\eta = 0$. Then the time optimal control u^* is unique and it has the bang-bang property:*

$$\|u^*(t)\| = 1 \qquad (t \in (0, \tau^*) \text{ a.e.}).$$

Proof. We first remark that the following statement holds: If $\eta \in X$ such that there exists a subset $e \subset [0,\tau]$ of positive measure with $B^*\mathbb{T}_t^*\eta = 0$ for $t \in e$, then $\eta = 0$. Indeed, the facts that the map $t \mapsto B^*\mathbb{T}_t^*\eta$ is analytic (which is due to the analyticity of $\mathbb{T}_t$) and that it vanishes in e imply that this map vanishes for t in some non-empty open interval I. Therefore, according to the hypothesis, we have $\eta = 0$.

With the above property, it is clear that $(\Psi_{\tau^*}\xi)(\tau^* - t) \neq 0$ for almost every $t \in (0,\tau^*)$. Indeed, if it is not the case, we deduce from the above property that $\xi = 0$, which contradicts Theorem 2.2. We can deduce from the maximum principle (2.12) that $u^*(t) = \frac{(\Psi_{\tau^*}\xi)(\tau^*-t)}{\|(\Psi_{\tau^*}\xi)(\tau^*-t)\|}$. Thus, the bang-bang property of the time-optimal control holds.

The uniqueness of the time-optimal control can be deduced from the strict convexity of U. This ends the proof. □

3 Nonsmooth data error estimates for abstract parabolic equations

In this section, we gather, for easy reference, some results providing error estimates for the semidiscrete Galerkin approximation of some abstract parabolic systems. Our results are strongly inspired by Lasiecka and Triggiani [20, Ch.4] and Badra [3, Ch.5], where they deal with a more general framework. We continue to use the same notation and assumption introduced in Section 1 for X, U, A_0, B, A_h, B_h and V_h and we recall that $\mathbb{T}_t$ (resp. $\mathbb{T}_{t,h}$) is the semigroup generated by $-A_0$ (resp. $-A_h$). We then consider the following linear equation and its approximated scheme (where we do not approximate u by u_h):

$$\begin{aligned} \dot{z}(t) + A_0 z(t) &= Bu(t) \qquad (t \geqslant 0), && (3.1)\\ z(0) &= z_0 \in X, && (3.2) \end{aligned}$$

$$\begin{aligned} \dot{z}_h(t) + A_h z_h(t) &= B_h u(t) \qquad (t \geqslant 0), && (3.3)\\ z_h(0) &= P_h z_0, && (3.4) \end{aligned}$$

with $u \in L^\infty([0,+\infty);U)$. We first recall a classical characterization of analytic semigroups:

Proposition 3.1. *$\mathbb{T}_t$ is an analytic semigroup if and only if there exist $C, \delta > 0$ such that*

$$\Sigma \subset \rho(A_0) \text{ and } \|(\lambda I - A_0)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C}{|\lambda|} \qquad (\lambda \in \Sigma), \tag{3.5}$$

where Σ is defined by $\Sigma := \left\{\lambda \in \mathbb{C} \mid |arg(\lambda)| \leqslant \frac{\pi}{2} + \delta\right\}$.

It is well known (see, for instance, [24, Ch. 2.6]) that if $\mathbb{T}_t$ is analytic then

$$\|A_0^\gamma \mathbb{T}_t\|_{\mathcal{L}(X)} \leqslant \frac{C}{t^\gamma} \qquad (t \geqslant 0,\ \gamma \in (0,1)). \tag{3.6}$$

We also need the concept of uniform analyticity of the family of semigroups $(\mathbb{T}_{t,h})_{h\geqslant 0}$.

Definition 3.2. *$\mathbb{T}_{t,h}$ is an uniformly analytic semigroup if there exist C independant of h, $\delta > 0$ such that for every $h > 0$ we have*

$$\Sigma \subset \rho(A_h) \text{ and } \|(\lambda I - A_h)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C}{|\lambda|} \qquad (\lambda \in \Sigma),$$

where Σ is defined in Proposition 3.1.

Similarly, the above definition implies the following property:

$$\|A_h^\gamma \mathbb{T}_{t,h}\|_{\mathcal{L}(X)} \leqslant \frac{C}{t^\gamma} \qquad (t>0,\ \gamma\in[0,1]). \tag{3.7}$$

It is also clear that

$$B_h u = (P_h B)\, u \quad (u\in U),$$

and (see assumption $(C4)$)

$$\|B_h\|_{\mathcal{L}(U,V_h)} \leqslant C h^{-\theta\alpha}.$$

We recall in the following some well-known results:

Lemma 3.3. *Assume that $\mathbb{T}_t$ is analytic. Then, for any $0\leqslant\gamma\leqslant 1$, there exists $C>0$ such that the following estimates hold:*
1. $\|A_0^\gamma(\lambda I - A_0)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C}{|\lambda|^{1-\gamma}} \qquad (\lambda\in\Sigma)$.
2. $\mathbb{T}_t z = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,(\lambda I - A_0)^{-1} z\,\mathrm{d}\lambda \qquad (z\in X)$, *where Γ is the path composed from the two rays $\rho e^{i\psi}$ and $\rho e^{-i\psi}$, $0<\rho<\infty$ and $\pi/2<\psi<\pi/2+\delta$, Γ being oriented so that $\Im\lambda$ increases along Γ ($\Im\lambda$ denotes the imaginary part of λ).*

Proof. The proof of the first estimate can be found in [24, Ch. 2.6] and the proof of the last assertion can be found in [24, Th. 7.7, Ch. 1]. □

Similar to Lemma 3.3, it is clear that we have the following lemma:

Lemma 3.4. *Assume that $\mathbb{T}_{t,h}$ is uniformly analytic. Then, for any $0\leqslant\gamma\leqslant 1$, there exists $C>0$ such that the following statement hold:*
1. $\|A_h^\gamma(\lambda I - A_h)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C}{|\lambda|^{1-\gamma}} \qquad (\lambda\in\Sigma)$.
2. $\mathbb{T}_{t,h} z = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,(\lambda I - A_h)^{-1} z\,\mathrm{d}\lambda \qquad (z\in V_h)$.

We gather in the proposition below, with no claim of originality, several error estimates which play a central role in the proof of our main result. Some of the proofs are very similar to those in [3, Ch.5], so that they are detailed.

Proposition 3.5. *Assume that the assumption $(C1)-(C4)$ hold. Then, there exists $C>0$, such that for every $h>0$, we have the following properties:*
1. $\|A_0^{-1} - A_h^{-1}P_h\|_{\mathcal{L}(X)} \leqslant C h^\theta$.
2. $\|A_0^{-1}(B - B_h)\|_{\mathcal{L}(U,X)} \leqslant C h^{\theta(1-\alpha)}$.
3. $\|(\lambda I - A_0)^{-1} - (\lambda I - A_h)^{-1}P_h\|_{\mathcal{L}(X)} \leqslant\ C h^\theta \qquad (\lambda\in\Sigma)$.
4. $\|(\lambda I - A_0)^{-1}B - (\lambda I - A_h)^{-1}B_h\|_{\mathcal{L}(U,X)} \leqslant\ C h^{\theta(1-\alpha)} \qquad (\lambda\in\Sigma)$.
5. $\|\mathbb{T}_{t,h}P_h - \mathbb{T}_t\|_{\mathcal{L}(X)} \leqslant C t^{-1} h^\theta$.
6. $\|\mathbb{T}_{t,h}B_h - \mathbb{T}_t B\|_{\mathcal{L}(U,X)} \leqslant C t^{-1} h^{\theta(1-\alpha)}$.
7. $\|\mathbb{T}_{\cdot,h}B_h - \mathbb{T}_\cdot B\|_{L^1([0,t];\mathcal{L}(U,X))} \leqslant C h^{\theta(1-\alpha)}\left(|\ln h| + t + 1\right)$.

Proof.

1. We skip this proof which is based on Proposition 5.1.1 and Proposition 5.1.3 in [20].

2. We skip this proof as it is similar to the proof of previous assertion by using assumption $(C3)$.

3. Notice at first that by simple calculation, it is clear that we have

$$(\lambda I - A_0)^{-1} = -A_0^{-1} + \lambda(\lambda I - A_0)^{-1}A_0^{-1}$$

and

$$(\lambda I - A_h)^{-1}P_h = -A_h^{-1}P_h + \lambda(\lambda I - A_h)^{-1}A_h^{-1}P_h.$$

Denote $M_h(\lambda) = (\lambda I - A_0)^{-1} - (\lambda I - A_h)^{-1}P_h$, we then have:

$$\begin{aligned} M_h(\lambda) &= -A_0^{-1} + A_h^{-1}P_h + \lambda(\lambda I - A_0)^{-1}A_0^{-1} - \lambda(\lambda I - A_h)^{-1}A_h^{-1}P_h \\ &= -\left(A_0^{-1} - A_h^{-1}P_h\right) + \lambda(\lambda I - A_0)^{-1}\left(A_0^{-1} - A_h^{-1}P_h\right) + \lambda(\lambda I - A_0)^{-1}A_h^{-1}P_h \\ &\quad - \lambda(\lambda I - A_h)^{-1}A_h^{-1}P_h \\ &= -\left(I - \lambda(\lambda I - A_0)^{-1}\right)\left(A_0^{-1} - A_h^{-1}P_h\right) + \lambda M_h(\lambda)A_h^{-1}P_h. \end{aligned}$$

This leads to $M_h(\lambda) = -\left(I - \lambda(\lambda I - A_0)^{-1}\right)(A_0^{-1} - A_h^{-1}P_h)(I - \lambda A_h^{-1}P_h)^{-1}$.

Notice that

$$(I - \lambda A_h^{-1}P_h)^{-1} = I - P_h - A_h(\lambda I - A_h)^{-1}P_h. \tag{3.8}$$

Indeed, we have

$$\begin{aligned} &(I - \lambda A_h^{-1}P_h)\left(I - P_h - A_h(\lambda I - A_h)^{-1}P_h\right) \\ &= I - P_h - A_h(\lambda I - A_h)^{-1}P_h - \lambda A_h^{-1}P_h + \lambda A_h^{-1}P_h + \lambda A_h^{-1}A_h(\lambda I - A_h)^{-1}P_h \\ &= I - P_h + (\lambda I - A_h)(\lambda I - A_h)^{-1}P_h = I. \end{aligned}$$

In the same manner, $\left(I - P_h - A_h(\lambda I - A_h)^{-1}P_h\right)(I - \lambda A_h^{-1}P_h) = I$. Thus, (3.8) holds.

By analyticity assumption (3.5) and assertion 1. in Lemma 3.4, we know that there exists a constant C independent of λ and h such that,

$$\|\lambda(\lambda I - A_0)^{-1}\|_{\mathcal{L}(X)} \leqslant C\frac{|\lambda|}{|\lambda|} = C \quad \text{and} \quad \|A_h(\lambda I - A_h)^{-1}\| \leqslant C \quad (\lambda \in \Sigma),$$

which leads to, using assertion 1. of Proposition 3.5

$$\|M_h(\lambda)\|_{\mathcal{L}(X)} \leqslant \|I - \lambda(\lambda I - A_0)^{-1}\|_{\mathcal{L}(X)}Ch^{\theta}\|I - P_h - A_h(\lambda I - A_h)^{-1}P_h\|_{\mathcal{L}(X)} \leqslant Ch^{\theta}.$$

This ends the proof of assertion 3..

4. Denote $M_{h,B}(\lambda) = (\lambda I - A_0)^{-1}B - (\lambda I - A_h)^{-1}B_h$. Notice that $(\lambda I - A_0)^{-1} = A_0(\lambda I - A_0)^{-1}A_0^{-1}$. Then, we have:

$$\begin{aligned} M_{h,B}(\lambda) &= (\lambda I - A_0)^{-1}(B - B_h) + \left((\lambda I - A_0)^{-1} - (\lambda I - A_h)^{-1}\right)B_h \\ &= A_0(\lambda I - A_0)^{-1}A_0^{-1}(B - B_h) + \left((\lambda I - A_0)^{-1} - (\lambda I - A_h)^{-1}\right)B_h. \end{aligned}$$

Since $A_0(\lambda I - A_0)^{-1}$ is a bounded operator (by assertion 1. in Lemma 3.3), it is clear that by using assertions 2. and 3. of Proposition 3.5 and assumption $(C4)$, we have:

$$\begin{aligned} \|M_{h,B}(\lambda)\|_{\mathcal{L}(U,X)} &\leqslant C\|A_0^{-1}(B - B_h)\|_{\mathcal{L}(U,X)} + \|(\lambda I - A_0)^{-1} - (\lambda I - A_h)^{-1}\|_{\mathcal{L}(X)}\|B_h\|_{\mathcal{L}(U,V_h)} \\ &\leqslant Ch^{\theta(1-\alpha)} + Ch^{\theta}\|B_h\|_{\mathcal{L}(U,V_h)} \leqslant Ch^{\theta(1-\alpha)}. \end{aligned}$$

5. It is clear that, by using assertion 2. in Lemma 3.3 and assertion 2. in Lemma 3.4, we have:

$$\mathbb{T}_t - \mathbb{T}_{t,h}P_h = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\left((\lambda I - A_0)^{-1} - (\lambda I - A_h)^{-1}P_h\right)\mathrm{d}\lambda.$$

Then, by using assertion 3. in Proposition 3.5, we have:

$$\|\mathbb{T}_t - \mathbb{T}_{t,h}P_h\|_{\mathcal{L}(X)} = C\left(\int_\Gamma |e^{\lambda t}|\mathrm{d}\lambda\right)h^\theta \leqslant Ct^{-1}h^\theta.$$

This ends the proof.

6. We skip this proof as it is similar to the proof of the previous assertion by using assertion 4. in Proposition 3.5.

7. We calculate:

$$\|\mathbb{T}_{\cdot,h}B_h - \mathbb{T}_\cdot B\|_{L^1([0,t];\mathcal{L}(U,X))} = \int_0^{h^\theta}\|\mathbb{T}_{\sigma,h}B_h - \mathbb{T}_\sigma B\|_{\mathcal{L}(U,X)}\mathrm{d}\sigma + \int_{h^\theta}^t\|\mathbb{T}_{\sigma,h}B_h - \mathbb{T}_\sigma B\|_{\mathcal{L}(U,X)}\mathrm{d}\sigma = I_1 + I_2.$$

By using assertion 6. of Proposition 3.5, we have:

$$I_2 \leqslant Ch^{\theta(1-\alpha)}\left(\int_{h^\theta}^t \sigma^{-1}\mathrm{d}\sigma\right) \leqslant Ch^{\theta(1-\alpha)}\left(\int_{h^\theta}^1 \sigma^{-1}\mathrm{d}\sigma + \int_1^t 1\mathrm{d}\sigma\right) \leqslant Ch^{\theta(1-\alpha)}\left(|\ln h| + t + 1\right).$$

Note that the assumption $B \in \mathcal{L}(U, X_{-\alpha})$ means that $A_0^{-\alpha}B$ is a bounded operator in $\mathcal{L}(U,X)$. Then, in order to deal with I_1, using also the analyticity of $\mathbb{T}$ (see (3.5)), for h small enough, we have, for every $0 \leqslant \alpha < \frac{1}{2}$,

$$\|\mathbb{T}_\sigma B\|_{\mathcal{L}(U,X)} \leqslant \|\mathbb{T}_\sigma A_0^\alpha\|_{\mathcal{L}(X)}\|A_0^{-\alpha}B\|_{\mathcal{L}(U,X)} \leqslant C\sigma^{-\alpha}.$$

Similarly, by using again the assumption $B \in \mathcal{L}(U, X_{-\alpha})$, it is clear that $A_h^{-\alpha}B_h$ is a bounded operator in $\mathcal{L}(U,X)$. Thus, we have

$$\|\mathbb{T}_{\sigma,h}B_h\|_{\mathcal{L}(U,X)} \leqslant \|\mathbb{T}_{\sigma,h}A_h^\alpha\|_{\mathcal{L}(U,X)}\|A_h^{-\alpha}B_h\|_{\mathcal{L}(U,X)} \leqslant C\sigma^{-\alpha},$$

using (3.7).

Finally, by the two previous inequalities, we have :

$$I_1 \leqslant C\int_0^{h^\theta}\sigma^{-\alpha}\mathrm{d}\sigma \leqslant Ch^{\theta(1-\alpha)}.$$

Thus, we have

$$\|\mathbb{T}_{\cdot,h}B_h - \mathbb{T}_\cdot B\|_{L^1([0,t];\mathcal{L}(U,X))} \leqslant Ch^{\theta(1-\alpha)}\left(|\ln h| + t + 1\right).$$

This ends the proof.

□

We will now present the main result of this section which gives us a nonsmooth initial data error estimate.

Theorem 3.6. *Let $z_0 \in X$, let z be the solution of (3.1),(3.2) and let z_h be the solution of (3.3), (3.4). Moreover, assume that $(C1)-(C4)$ hold. Then, there exist $C > 0$, $\tilde{h} > 0$ such that for $h \in (0, \tilde{h})$ we have the following error estimate:*

$$\|z(t) - z_h(t)\|_X \leqslant Ch^{\theta}t^{-1}\|z_0\|_X + Ch^{\theta(1-\alpha)}\left(|\ln h| + t + 1\right)\ \|u\|_{L^{\infty}([0,+\infty);U)}, \qquad \forall t > 0. \tag{3.9}$$

Proof. Denote $K_h(t) = \mathbb{T}_t B - \mathbb{T}_{t,h}B_h$.

We recall that z, solution of system (1.1)-(1.2), satisfies (1.3) and z_h, solution of (1.5)-(1.6), satisfies (1.9). It is clear that

$$\|z(t) - z_h(t)\|_X \leqslant \|\mathbb{T}_t z_0 - \mathbb{T}_{t,h}P_h z_0\|_X + \left\|\int_0^t K_h(t-\sigma)u(\sigma)\mathrm{d}\sigma\right\|_X.$$

By using assertions *5.* and *7.* in Proposition 3.5, we have:

$$\begin{aligned}\|z(t) - z_h(t)\|_X &\leqslant Ch^{\theta}t^{-1}\|z_0\|_X + \|K_h * u\|_{L^1([0,t];X)}\\ &\leqslant Ch^{\theta}t^{-1}\|z_0\|_X + \|K_h\|_{L^1([0,t];\mathcal{L}(U,X))}\|u\|_{L^{\infty}([0,t];U)}\\ &\leqslant Ch^{\theta}t^{-1}\|z_0\|_X + Ch^{\theta(1-\alpha)}\left(|\ln h| + t + 1\right)\ \|u\|_{L^{\infty}([0,t];U)}.\end{aligned}$$

This ends the proof. □

4 Proof of the main results

We denote $\tau_0^*(z_0)$ (resp.$\tau_h^*(z_{0,h})$) the optimal time associated with the initial state $z_0 \in X$ (resp. $z_{0,h} \in V_h$) and with final state in $\bar{B}(0,\epsilon)$ (resp. $\overline{B_h}(0,\varepsilon)$). We use also the notation $z(t, z_0, u)$ for the solution of the system (1.1)-(1.2) (resp. $z_h(t, z_{0,h}, u_h)$ for the system (1.5)-(1.6)) associated with the initial state z_0 (resp. $z_{0,h}$) and with the control u (resp. u_h) at time t.

We first recall a generalized Aubin-Lions theorem (Theorem 4.1) and a standard energy estimate (Lemma 4.2).

Theorem 4.1. *Let $T > 0$ and let Y_0, Y_1, Y_2 be Banach spaces such that $Y_0 \subset Y_1 \subset Y_2$. Assume that Y_i is reflexive for $i = 0, 1, 2$ and that the embedding of Y_0 into Y_1 is compact. Let $1 < r < \infty$.*

Then, the space

$$W = L^{\infty}([0,T];Y_0) \cap W^{1,r}([0,T];Y_2)$$

is compactly embedded in $C([0,T];Y_1)$.

Proof. See for example [26, Cor. 4, p.85]. □

Lemma 4.2. *Assume that $z_0 \in X_{\frac{1}{2}-\alpha}$. Then, there exists $C > 0$ such that*

$$\|z(\tau)\|^2_{\frac{1}{2}-\alpha} + \int_0^{\tau}\left(\|\dot{z}(s)\|^2_{-\alpha} + \|z(s)\|^2_{1-\alpha}\right)\mathrm{d}s \leqslant C\left(\int_0^{\tau}\|Bu(s)\|^2_{-\alpha}\mathrm{d}s + \|z_0\|^2_{\frac{1}{2}-\alpha}\right) \quad (\tau > 0), \tag{4.1}$$

where z is the solution of (1.1)-(1.2).

Proof. Using the facts that

$$\langle \dot{z}(t), A_0^{1-2\alpha}z(t)\rangle_{-\alpha,\alpha} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_0^{\frac{1}{2}-\alpha}z(t)\|^2, \qquad \langle A_0 z(t), A_0^{1-2\alpha}z(t)\rangle_{-\alpha,\alpha} = \|A_0^{1-\alpha}z(t)\|^2,$$

it follows that if z satisfies the equation (1.1)-(1.2), then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_0^{\frac{1}{2}-\alpha}z(t)\|^2 + \|A_0^{1-\alpha}z(t)\|^2 = \langle Bu(t), A_0^{1-2\alpha}z(t)\rangle_{-\alpha,\alpha}.$$

Integrating the above estimate from 0 to τ and using the fact that

$$\left|\langle Bu(t), A_0^{1-2\alpha}z(t)\rangle_{-\alpha,\alpha}\right| \leqslant \|Bu(t)\|_{-\alpha}\left\|A_0^{1-\alpha}z(t)\right\|,$$

it follows that

$$\|A_0^{\frac{1}{2}-\alpha}z(\tau)\|^2 + \int_0^\tau \|A_0^{1-\alpha}z(t)\|^2\,\mathrm{d}t \leqslant \|A_0^{\frac{1}{2}-\alpha}z_0\|^2 + \|Bu\|^2_{L^2([0,\tau];X_{-\alpha})}.$$

This leads to:

$$\|z(\tau)\|^2_{\frac{1}{2}-\alpha} + \int_0^\tau \|z(s)\|^2_{1-\alpha}\mathrm{d}s \leqslant \int_0^\tau \|Bu(s)\|^2_{-\alpha}\mathrm{d}s + \|z_0\|^2_{\frac{1}{2}-\alpha}. \tag{4.2}$$

Similarly, we have:

$$\langle \dot{z}(t), A_0^{-2\alpha}\dot{z}(t)\rangle_{-\alpha,\alpha} = \|A_0^{-\alpha}\dot{z}(t)\|^2, \quad \langle A_0 z(t), A_0^{-2\alpha}\dot{z}(t)\rangle_{-\alpha,\alpha} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_0^{\frac{1}{2}-\alpha}z(t)\|^2$$

and then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_0^{\frac{1}{2}-\alpha}z(t)\|^2 + \|A_0^{-\alpha}\dot{z}(t)\|^2 = \left\langle Bu(t), A_0^{-2\alpha}\dot{z}(t)\right\rangle_{-\alpha,\alpha}.$$

Using again the fact that

$$\left|\langle Bu(t), A_0^{-2\alpha}\dot{z}(t)\rangle_{-\alpha,\alpha}\right| \leqslant \|Bu(t)\|_{-\alpha}\left\|A_0^{-\alpha}\dot{z}(t)\right\|,$$

and integrating by time from 0 to τ, we obtain the following inequality:

$$\|A_0^{\frac{1}{2}-\alpha}z(\tau)\|^2 + \int_0^\tau \|A_0^{-\alpha}\dot{z}(t)\|^2\,\mathrm{d}t \leqslant \|A_0^{\frac{1}{2}-\alpha}z_0\|^2 + \|Bu\|^2_{L^2([0,\tau];X_{-\alpha})}.$$

This leads to:

$$\|z(\tau)\|^2_{\frac{1}{2}-\alpha} + \int_0^\tau \|\dot{z}(s)\|^2_{-\alpha}\mathrm{d}s \leqslant \int_0^\tau \|Bu(s)\|^2_{-\alpha}\mathrm{d}s + \|z_0\|^2_{\frac{1}{2}-\alpha}. \tag{4.3}$$

Then (4.1) is deduced from (4.2) and (4.3). This ends the proof.

□

We give here some convergence results which play an important rule in the proof of our main results.

Lemma 4.3. *Let $z_0 \in X$, $T > 0$, $\bar{u} \in L^\infty([0,T];U)$ and $\bar{\tau} \in [0,T]$. Let $(u_n)_n$ be a sequence of $L^\infty([0,T];U)$ and $(\tau_n)_n$ be a sequence of $[0,T]$ such that $u_n \in L^\infty([0,T];U_{h_n})$ for any $n \in \mathbb{N}$ and*

$$u_n \to \bar{u} \quad \textit{weakly* in} \quad L^\infty([0,T];U),$$

$$\tau_n \to \bar{\tau} \quad \textit{in } [0,T].$$

Then, the following convergence results hold

1. $\lim_{n\to+\infty}\|z(.,z_0,u_n) - z(.,z_0,\bar{u})\|_{C([0,T];X)} = 0,$

2. $\lim_{n\to+\infty}\|z(\tau_n,z_0,u_n) - z(\bar{\tau},z_0,u_n)\| = 0.$

Proof. 1. Denote $\psi(t) = z(t, z_0, \bar{u})$ and $\psi_n(t) = z(t, z_0, u_n)$. By the standard energy estimate (4.1), we know that $(\psi_n)_n$ is a bounded sequence in

$$W = C([0,T]; X_{\frac{1}{2}-\alpha}) \cap L^2([0,T]; X_{1-\alpha}) \cap W^{1,2}([0,T]; X_{-\alpha}). \tag{4.4}$$

Using Theorem 4.1 with $Y_0 = X_{\frac{1}{2}-\alpha}, Y_1 = X$ and $Y_2 = X_{-\alpha}$, we deduce that:

$$\exists\, \widetilde{\psi} \in C([0,T]; X) \; s.t. \; \psi_n \to \tilde{\psi} \text{ strongly in } C([0,T]; X) \text{ and } \psi_n \to \tilde{\psi} \text{ weakly in } W.$$

Now we prove that $\widetilde{\psi} = \psi$. We know that ψ_n satisfies:

$$\begin{aligned} \dot{\psi_n} + A_0\psi_n &= Bu_n, \\ \psi_n(0) &= z_0. \end{aligned}$$

Then it is clear that $\dot{\psi_n} \to \dot{\widetilde{\psi}}$ weakly in $L^2([0,T]; X_{-\alpha})$ and $A_0\psi_n \to A_0\widetilde{\psi}$ weakly in $L^2([0,T]; X_{-\alpha})$ since $\psi_n \to \tilde{\psi}$ weakly in W. Moreover, $u_n \overset{*}{\rightharpoonup} \bar{u}$ weakly* in $L^\infty([0,T]; U)$ implies that $u_n \rightharpoonup \bar{u}$ weakly in $L^2([0,T]; U)$. Thus, $Bu_n \rightharpoonup B\bar{u}$ weakly in $L^2([0,T]; X)$. Finally, $\tilde{\psi}(0) = z_0$. Indeed, we know that $z_0 = \psi_n(0) \to \tilde{\psi}(0)$, since $\psi_n \to \tilde{\psi}$ strongly in $C([0,T]; X)$.

Consequently, $\tilde{\psi}$ satisfies:

$$\begin{aligned} \dot{\tilde{\psi}} + A_0\tilde{\psi} &= B\bar{u}, \\ \tilde{\psi}(0) &= z_0, \end{aligned}$$

which implies that $\tilde{\psi} = \psi$. This leads to the first assertion.

2. We first notice that

$$\begin{aligned} &\|z(\tau_n, z_0, u_n) - z(\bar{\tau}, z_0, u_n)\| \\ &\leqslant \|\mathbb{T}_{\tau_n} z_0 - \mathbb{T}_{\bar{\tau}} z_0\| + \left\| \int_0^{\tau_n} \mathbb{T}_{\tau_n - \sigma} B u_n(\sigma) \mathrm{d}\sigma - \int_0^{\bar{\tau}} \mathbb{T}_{\bar{\tau} - \sigma} B u_n(\sigma) \mathrm{d}\sigma \right\| \\ &\leqslant \|\mathbb{T}_{\tau_n} z_0 - \mathbb{T}_{\bar{\tau}} z_0\| + \left\| \int_0^{\widetilde{\tau_n}} \mathbb{T}_{\tau_n - \sigma} B u_n(\sigma) \mathrm{d}\sigma - \int_0^{\widetilde{\tau_n}} \mathbb{T}_{\bar{\tau} - \sigma} B u_n(\sigma) \mathrm{d}\sigma \right\| \\ &\quad + \left\| \int_{\widetilde{\tau_n}}^{\tau_n} \mathbb{T}_{\tau_n - \sigma} B u_n(\sigma) \mathrm{d}\sigma \right\| + \left\| \int_{\widetilde{\tau_n}}^{\bar{\tau}} \mathbb{T}_{\bar{\tau} - \sigma} B u_n(\sigma) \mathrm{d}\sigma \right\| \\ &= L_1 + L_2 + L_3 + L_4, \end{aligned}$$

where $\widetilde{\tau_n} = \min\{\tau_n, \bar{\tau}\}$.

By using the continuity of the map $t \mapsto \mathbb{T}_t z_0$, it is clear that L_1 converges to zero when n tends to $+\infty$.

Moreover, since $(u_n)_n \subset L^\infty([0,T], U)$ and using (3.6), we have

$$\begin{aligned} L_3 &\leqslant \int_{\widetilde{\tau_n}}^{\tau_n} \|\mathbb{T}_{\tau_n - \sigma} B u_n(\sigma)\| \, \mathrm{d}\sigma \\ &\leqslant \int_{\widetilde{\tau_n}}^{\tau_n} \|\mathbb{T}_{\tau_n - \sigma} A_0^\alpha\|_{\mathcal{L}(X)} \left\|A_0^{-\alpha} B\right\|_{\mathcal{L}(U,X)} \|u_n(\sigma)\|_U \, \mathrm{d}\sigma \\ &\leqslant C \int_{\widetilde{\tau_n}}^{\tau_n} (\tau_n - \sigma)^{-\alpha} \mathrm{d}\sigma = \frac{C}{1-\alpha} (\tau_n - \widetilde{\tau_n})^{1-\alpha} \to 0, \end{aligned}$$

since $1-\alpha>0$. Similarly, L_4 converges to zero.

Now we prove that L_2 converges to zero. We have

$$\begin{aligned} L_2 &= \left\| \int_0^{\widetilde{\tau_n}} \left(\mathbb{T}_{\tau_n-\sigma} - \mathbb{T}_{\bar{\tau}-\sigma}\right) B\left(u_n(\sigma)-\bar{u}(\sigma)\right) \mathrm{d}\sigma \right\| + \left\| \int_0^{\widetilde{\tau_n}} \left(\mathbb{T}_{\tau_n-\sigma} - \mathbb{T}_{\bar{\tau}-\sigma}\right) B\bar{u}(\sigma) \mathrm{d}\sigma \right\| \\ &= L_{2,1} + L_{2,2}. \end{aligned}$$

By using the continuity of the map $t \mapsto \mathbb{T}_t z_0$, it is clear that $L_{2,2}$ converges to zero. Now we prove that $L_{2,1}$ tends to 0 when n tends to $+\infty$.

We have

$$\begin{aligned} L_{2,1} &\leqslant \left\| \int_0^{\widetilde{\tau_n}} \mathbb{T}_{\tau_n-\sigma} B\left(u_n(\sigma)-\bar{u}(\sigma)\right) \mathrm{d}\sigma \right\| + \left\| \int_0^{\widetilde{\tau_n}} \mathbb{T}_{\bar{\tau}-\sigma} B\left(u_n(\sigma)-\bar{u}(\sigma)\right) \mathrm{d}\sigma \right\| \\ &= \|\mathbb{T}_{\tau_n-\widetilde{\tau_n}}\| \|z(\widetilde{\tau_n}, z_0, u_n) - z(\widetilde{\tau_n}, z_0, \bar{u})\| + \|\mathbb{T}_{\bar{\tau}-\widetilde{\tau_n}}\| \|z(\widetilde{\tau_n}, z_0, u_n) - z(\widetilde{\tau_n}, z_0, \bar{u})\|. \end{aligned}$$

Since $\|\mathbb{T}_{\tau_n-\widetilde{\tau_n}}\|$ and $\|\mathbb{T}_{\bar{\tau}-\widetilde{\tau_n}}\|$ are bounded, by using the first assertion of this lemma, we can deduce that $L_{2,1}$ converge to zero when n tends to $+\infty$. This ends the proof of the second assertion.

□

We then give some properties which will be used to prove the convergence results:

Lemma 4.4.
Denote λ_1 (resp. $\lambda_{1,h}$) the first eigenvalue of the operator A_0 (resp. $A_{0,h}$). For every $z_0 \in X$ and $z_{0,h} \in V_h$ the following properties hold
1. $\tau_0^*(z_0) \leqslant \frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}$.
2. $\tau_h^*(z_{0,h}) \leqslant \frac{ln(\|z_{0,h}\|/\varepsilon)}{\lambda_{1,h}}$.
3. $\lambda_1 \leqslant \lambda_{1,h}$.

Proof. To prove assertion 1. we notice that $\|z(t,z_0,0)\| \leqslant e^{-\lambda_1 t}\|z_0\|$ and taking $t = \frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}$ we obtain that $\|z(t,z_0,0)\| \leqslant \varepsilon$. This proves 1. By a similar argument, it is clear that assertion 2. holds.

We end by proving the last assertion. In fact, this inequality is easily deduced by the min-max formula: $\lambda_1 = \min_{z\in X_{\frac{1}{2}}} \frac{\|A_0^{\frac{1}{2}} z\|^2}{\|z\|^2}$ and $\lambda_{1,h} = \min_{z\in V_h} \frac{\|A_0^{\frac{1}{2}} z\|^2}{\|z\|^2}$. This ends the proof. □

Given $\varepsilon > 0$ and $z_0 \in X$ such that $\|z_0\| > \varepsilon$, the lemma below shows that τ_h^* has a strictly positive lower and upper bounds, which are independent of h.

Lemma 4.5. *Assume that $(C1)$ holds. For every $z_0 \in X$ such that $\|z_0\| > \varepsilon$, there exist $c, C > 0, \tilde{h} > 0$ such that for any $h \in (0,\tilde{h})$, we have $c \leqslant \tau_h^*(P_h z_0) \leqslant C$, where $C = 2\ \frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}$.*

Proof. We begin by proving that $\tau_h^*(P_h z_0)$ is bounded from below. Assume by contradiction that $\lim_{h\to 0} \tau_h^*(P_h z_0) = 0$.

We first notice that

$$\begin{aligned} \|z_h(\tau_h^*, P_h z_0, u_h^*) - z_h(0, P_h z_0, u_h^*)\| &= \left\| \mathbb{T}_{\tau_h^*,h} P_h z_0 + \int_0^{\tau_h^*} \mathbb{T}_{\tau_h^*-\sigma,h} B u_h^*(\sigma) \mathrm{d}\sigma - P_h z_0 \right\| \\ &\leqslant \left\| \left(\mathbb{T}_{\tau_h^*,h} - Id\right) P_h z_0 \right\| + \left\| \int_0^{\tau_h^*} \mathbb{T}_{\tau_h^*-\sigma,h} B u_h^*(\sigma) \mathrm{d}\sigma \right\|. \end{aligned}$$

We can apply the Trotter-Kato theorem (see, for instance, [2, Theorem 3.6.1, Proposition 3.6.2]), to get that the first term of the right hand side of the previous inequality tends to zero when h tends to zero. Moreover, concerning the left hand side, using (3.7), we have

$$\begin{aligned}\left\|\int_0^{\tau_h^*}\mathbb{T}_{\tau_h^*-\sigma,h}Bu_h^*(\sigma)\mathrm{d}\sigma\right\| &\leqslant \int_0^{\tau_h^*}\left\|\mathbb{T}_{\tau_h^*-\sigma,h}A_h^\alpha\right\|_{\mathcal{L}(X)}\left\|A_h^{-\alpha}B\right\|_{\mathcal{L}(U,X)}\|u_h(\sigma)\|_U\,\mathrm{d}\sigma\\ &\leqslant C\int_0^{\tau_h^*}(\tau_h^*-\sigma)^{-\alpha}\mathrm{d}\sigma = \frac{C}{1-\alpha}(\tau_h^*)^{1-\alpha}\to 0,\end{aligned}$$

since $1-\alpha>0$.

Consequently, we have

$$\lim_{h\to 0}\|z_h(\tau_h^*,P_hz_0,u_h^*)-z_h(0,P_hz_0,u_h^*)\| = \lim_{h\to 0}\|z_h(\tau_h^*,P_hz_0,u_h^*)-P_hz_0\| = 0.$$

Using the fact that $\|z_h(\tau_h^*,P_hz_0,u_h^*)\|\leqslant\varepsilon$, it is clear that $\lim_{h\to 0}\|P_hz_0\|\leqslant\varepsilon$. However, note that $\lim_{h\to 0}\|P_hz_0-z_0\|=0$, which comes from $(C1)$ since X_γ is dense in X for any $\gamma>0$ and P_h is a projection. This leads to the contradiction with the fact that $\|z_0\|>\varepsilon$.

We prove now that $\tau_h^*(P_hz_0)$ is bounded from above. This is obvious by using Lemma 4.4, since

$$\tau_h^*(P_hz_0)\leqslant\frac{ln(\|P_hz_0\|/\varepsilon)}{\lambda_{1,h}}\leqslant 2\frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}<+\infty.$$

□

Proof of Theorem 1.1. It suffices to prove the following two inequalities:

$$\liminf_{h\to 0}\tau_h^*(P_hz_0)\geqslant\tau_0^*(z_0), \tag{4.5}$$

$$\limsup_{h\to 0}\tau_h^*(P_hz_0)\leqslant\tau_0^*(z_0). \tag{4.6}$$

We begin by proving (4.5). In the following, we denote simply $\tau_0^*(z_0)$ by τ_0^* and $\tau_h^*(P_hz_0)$ by τ_h^* when there is no ambiguity. We notice at first the following property:

$$\forall\ T>0,\ \forall\ u\in\mathcal{U}_{ad},\ \tau_0^*(z_0)\leqslant T+\tau_0^*(z(T,z_0,u)). \tag{4.7}$$

We deduce from Theorem 3.6 that

$$\|z(\tau_h^*,z_0,u_h^*)-z_h(\tau_h^*,P_hz_0,u_h^*)\|\leqslant Ch^\theta{\tau_h^*}^{-1}\|z_0\|_X+Ch^{\theta(1-\alpha)}(|\ln h|+1+\tau_h^*)\ \|u_h^*\|_{L^\infty([0,+\infty);U_h)},$$

which leads to:

$$\begin{aligned}\|z(\tau_h^*,z_0,u_h^*)\| &\leqslant \varepsilon+Ch^\theta{\tau_h^*}^{-1}\|z_0\|_X+Ch^{\theta(1-\alpha)}(|\ln h|+1+\tau_h^*)\ \|u_h^*\|_{L^\infty([0,+\infty);U_h)}\\ &\leqslant \varepsilon+Ch^\theta{\tau_h^*}^{-1}+Ch^{\theta(1-\alpha)}(|\ln h|+1+\tau_h^*).\end{aligned}$$

We denote $\widetilde{z_0}=z(\tau_h^*,z_0,u_h^*)$. According to (4.7) with $T=\tau_h^*(P_hz_0)$, we have:

$$\tau_0^*(z_0)\leqslant\tau_h^*(P_hz_0)+\tau_0^*(\widetilde{z_0}).$$

Since $u_h^* \in L^\infty([0,+\infty);U_h) \subset L^\infty([0,+\infty);U)$ and $\|u_h^*(t)\| \leqslant 1$, u_h^* is then an admissible control for the continuous system. Then, according to Lemma 4.4, we have:

$$\begin{aligned}\tau_0^* &\leqslant \tau_h^* + \frac{ln((\varepsilon + Ch^\theta {\tau_h^*}^{-1} + Ch^{\theta(1-\alpha)}(|\ln h| + 1 + \tau_h^*))/\varepsilon)}{\lambda_1} \\ &\leqslant \tau_h^* + \frac{Ch^\theta}{\lambda_1\varepsilon}{\tau_h^*}^{-1} + C\frac{h^{\theta(1-\alpha)}}{\lambda_1\varepsilon}(1+\tau_h^*) + C\frac{h^{\theta(1-\alpha)}|\ln h|}{\lambda_1\varepsilon}. \end{aligned} \tag{4.8}$$

Thus, (4.5) can be deduced by taking h to zero and by the fact that $C > \lim_{h\to 0}\tau_h^*(P_h z_0) > c > 0$ (see Lemma 4.5).

We now prove the second inequality (4.6). We have:

$$\begin{aligned}&\|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z(\tau_0^*, z_0, u^*)\| \\ &\leqslant \|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z_h(\tau_0^*, P_h z_0, u^*)\| + \|z_h(\tau_0^*, P_h z_0, u^*) - z(\tau_0^*, z_0, u^*)\| \\ &\leqslant \|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z_h(\tau_0^*, P_h z_0, u^*)\| + Ch^\theta {\tau_0^*}^{-1} + Ch^{\theta(1-\alpha)}(|\ln h| + 1 + \tau_0^*),\end{aligned}$$

using (3.9). Denote $f(h) = \|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z_h(\tau_0^*, P_h z_0, u^*)\|$. Notice that $\lim_{h\to 0} f(h) = 0$. Indeed,

$$\lim_{h\to 0} f(h) = \lim_{h\to 0}\|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z_h(\tau_0^*, P_h z_0, u^*)\| = \lim_{h\to 0}\|\Phi_{\tau^*,h}(u^* - Q_h u^*)\|.$$

Moreover, we have

$$\|\Phi_{\tau^*,h}(u^* - Q_h u^*)\| \leqslant \|\left(\Phi_{\tau^*,h} - \Phi_{\tau^*}\right)(u^* - Q_h u^*)\| + \|\Phi_{\tau^*}(u^* - Q_h u^*)\|.$$

Since $B \in \mathcal{L}(U, X_{-\alpha})$, with $0 \leqslant \alpha < \frac{1}{2}$, it is known that $\Phi_{\tau^*} \in \mathcal{L}(L^2([0,\tau^*];U), X)$ (see for instance [31, Proposition 5.1.3]). Combining this fact with assertion 7. in Proposition 3.5 and with assumption $(C5)$, it is clear that

$$\lim_{h\to 0} f(h) = 0.$$

Thus, we have:

$$\|z_h(\tau_0^*, P_h z_0, Q_h u^*)\| \leqslant \varepsilon + f(h) + Ch^\theta {\tau_0^*}^{-1} + Ch^{\theta(1-\alpha)}(|\ln h| + 1 + \tau_0^*).$$

By the similar argument as in (4.8), we have:

$$\begin{aligned}\tau_h^*(P_h z_0) &\leqslant \tau_0^* + \frac{\ln((\varepsilon + f(h) + Ch^\theta {\tau_0^*}^{-1} + Ch^{\theta(1-\alpha)}(|\ln h| + 1 + \tau_0^*))/\varepsilon)}{\lambda_{1,h}} \\ &\leqslant \tau_0^* + \frac{f(h) + Ch^\theta {\tau_0^*}^{-1} + C(1+\tau_0^*)h^{\theta(1-\alpha)} + Ch^{\theta(1-\alpha)}|\ln h|}{\lambda_1\varepsilon}.\end{aligned}$$

This leads to inequality (4.6) by taking h to zero. □

Proof of Theorem 1.2. Denote $T = 2\,\frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}$. It is clear that $\tau_h^* \leqslant T$ for all $h > 0$ and $\tau_0^* \leqslant T$ (see Lemma 4.4 and Lemma 4.5). We extend $(u_h^*)_h$ and u_0^* to time T by zero.

Since $\|u_h^*\|_{L^\infty([0,T];U_h)} \leqslant 1$, there exist a control $\bar{u} \in L^\infty([0,T];U)$ and a subsequence $(h_n)_n \to 0$, such that:

$$u_{h_n}^* \to \bar{u} \quad \text{weakly* in} \quad L^\infty([0,T];U).$$

Now we prove that $\bar{u} = u_0^*$.

The main step here is to prove the following convergence property:

$$\|z_{h_n}(\tau^*_{h_n}, P_{h_n} z_0, u^*_{h_n}) - z(\tau^*_0, z_0, \bar{u})\| \to 0. \tag{4.9}$$

Indeed, since $\bar{B}(0,\varepsilon)$ is complete (notice that $\overline{B_h}(0,\varepsilon) \subset \bar{B}(0,\varepsilon)$), (4.9) leads to $z(\tau^*_0, z_0, \bar{u}) \in \bar{B}(0,\varepsilon)$. Then, by the uniqueness of the time optimal control and Theorem 1.1, we deduce that $\bar{u} = u^*_0$.

We are then reduced to prove (4.9). We have:

$$\begin{aligned}\|z_{h_n}(\tau^*_{h_n}, P_{h_n} z_0, u^*_{h_n}) - z(\tau^*_0, z_0, \bar{u})\| \leqslant & \|z_{h_n}(\tau^*_{h_n}, P_{h_n} z_0, u^*_{h_n}) - z(\tau^*_{h_n}, z_0, u^*_{h_n})\| \\ & + \|z(\tau^*_{h_n}, z_0, u^*_{h_n}) - z(\tau^*_0, z_0, u^*_{h_n})\| \\ & + \|z(\tau^*_0, z_0, u^*_{h_n}) - z(\tau^*_0, z_0, \bar{u})\|. \end{aligned} \tag{4.10}$$

The first term of the right hand side of (4.10) converges to zero using the error estimate (3.9). Moreover, by using the first and the second assertions in Lemma 4.3, it is clear that the second and the third terms of the right hand of (4.10) converges to zero. We then deduce (4.9).

Thus, we have:

$$u^*_{h_n} \to u^*_0 \text{ weakly}^* \text{ in } L^\infty([0,T];U). \tag{4.11}$$

We deduce immediately that:

$$u^*_{h_n} \to u^*_0 \text{ weakly in } L^2([0,T];U).$$

At last, since both $u^*_{h_n}$ and u^*_0 are bang-bang controls (see Corollary 2.4), we have

$$\lim_{n\to+\infty} \|u^*_{h_n}\|_{L^2([0,T];U)} = \|u^*_0\|_{L^2([0,T];U)}.$$

According to the Radon-Riesz property, this leads to the strong convergence in $L^2([0,T];U)$. This ends the proof.

□

5 Applications

Example 5.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a boundary $\partial\Omega$ of class C^2. For each $z_0 \in L^2(\Omega)$ and $u \in L^\infty([0,+\infty);L^2(\Omega))$, we consider the following initial and boundary value problem

$$\begin{aligned} \dot{z}(x,t) - \triangle z(x,t) &= \chi_\omega(x) u(x,t) && (x \in \Omega, \quad t \geqslant 0), && (5.1)\\ z(x,t) &= 0 && (x \in \partial\Omega, \quad t \geqslant 0), && (5.2)\\ z(x,0) &= z_0(x) && (x \in \Omega), && (5.3) \end{aligned}$$

where $\omega \subset \Omega$ is an open and non-empty subset with its characteristic function χ_ω. The system (5.1)-(5.3) can be formulated in the abstract form (1.1)-(1.2) by taking $A_0 = -\triangle$ with Dirichlet boundary conditions, of domain $D(A_0) = H^1_0(\Omega) \cap H^2(\Omega)$ on $X = L^2(\Omega)$. The control operator $B \in \mathcal{L}(U,X)$ (here $\alpha = 0$) is defined by

$$B\varphi = \chi_\omega \tilde{\varphi},$$

where $U = L^2(\omega)$ and where $\tilde{\varphi}$ is the extension by 0 of φ on Ω outside ω.

We then consider the standard P_1 finite element method with regular triangulation $\mathcal{T}_h$ of Ω (see for example, Thomée [28], Raviart and Thomas [25]) and we build the finite element space $V_h \subset H_0^1(\Omega)$ of X, defined by

$$V_h = \left\{\varphi \in C(\bar{\Omega}) \ |\varphi_{|\mathcal{T}} \in P_1(\mathcal{T}) \quad \text{for every} \quad \mathcal{T} \in \mathcal{T}_h,\ \varphi_{|\partial\Omega} = 0\right\}$$

where $P_1(\mathcal{T})$ is the set of affine functions on $\mathcal{T}$. We also define U_h a finite dimensional subspace of U by

$$U_h := \{\chi_\omega \varphi_h \mid \varphi_h \in V_h\}.$$

Denote P_h the L^2-projection from $L^2(\Omega)$ to V_h, and we consider the following space semi-discrete scheme formulated in the form of (1.5)-(1.6):

$$\begin{aligned} \dot{z}_h(t) + A_h z_h(t) &= B_h u_h(t) \qquad (t \geqslant 0), & (5.4)\\ z_h(0) &= P_h z_0 \qquad (z_0 \in X), & (5.5) \end{aligned}$$

where u_h is taken in $L^\infty([0,+\infty); U_h)$ and the operator A_h and B_h are defined by

$$< A_h\varphi, \psi > \ = \int_\Omega \nabla\varphi \nabla\psi \ \mathrm{d}x \qquad (\varphi, \psi \in V_h), \tag{5.6}$$

and $B_h \in \mathcal{L}(U, V_h)$ is defined by :

$$\langle B_h u, \varphi\rangle \quad = \quad \langle u, B^*\varphi\rangle_U \qquad (\varphi \in V_h,\ u \in U). \tag{5.7}$$

Given $\varepsilon > 0$ with the same notation as in Section 1, we consider the time optimal control problem (**TP**) associated to (5.1)-(5.3) (resp. ($\mathbf{TP_h}$) associated to (5.4)-(5.5)) and denote (τ_0^*, u_0^*) the corresponding optimal time and time-optimal control (resp. (τ_h^*, u_h^*)).

We can now state the following convergence results:

Proposition 5.2. *With the notation above, let, for every $\varepsilon > 0$, $z_0 \in X$ such that $\|z_0\| > \varepsilon$. Then we have that $\tau_h^* \to \tau_0^*$, and $u_h^* \to u_0^*$ in $L^2([0,T]; L^2(\Omega))$.*

Proof. In order to apply Theorem 1.1 and Theorem 1.2, we verify that conditions $(C1) - (C5)$ are satisfied with $\alpha = 0$ and $\theta = 2$. Indeed, $(C1)$ is a standard error estimate when applying the P_1 finite element scheme (see for example [28]). $(C2)$ is proved in Bramble *et al* [6] and in Lasiecka [18]. Moreover, $(C3) - (C5)$ clearly hold true because here the control operator B is bounded. Furthermore, it is known that the assumption made in Theorem 1.2 holds true (see for example [23]). Thus, we can apply Theorem 1.1 and Theorem 1.2 to conclude. □

Example 5.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a boundary $\partial\Omega$ of class C^2. We consider the heat equation

$$\dot{z}(x,t) - \triangle z(x,t) = 0 \qquad (x \in \Omega, \quad t \geqslant 0), \tag{5.8}$$

with the initial and boundary conditions

$$\begin{aligned} \frac{\partial z}{\partial \nu}|_{\partial\Omega}(x,t) &= u(x,t) \qquad (x \in \partial\Omega, \quad t \geqslant 0), & (5.9)\\ z(x,0) &= z_0(x) \in L^2(\Omega) \qquad (x \in \Omega). & (5.10) \end{aligned}$$

It is known that the system (5.8)-(5.10) can be written in the form of (1.1)-(1.2) by taking $X = L^2(\Omega)$, $U = L^2(\partial\Omega)$, $D(A_0) = \left\{z \in H^2(\Omega) \mid \frac{\partial z}{\partial \nu}|_{\partial\Omega} = 0\right\}$, $A_0 = -\triangle$ and the control operator $B \in \mathcal{L}(U, X_{-\alpha})$ (here $\alpha = \frac{1}{4} + \varepsilon$, $\varepsilon > 0$) defined by

$$Bu = A_0 Nu \qquad (u \in U),$$

where N is the Neumann map. This map is defined by $Nv = z$, where $z \in L^2(\Omega)$ is the unique solution of the nonhomogeneous elliptic equation

$$\begin{cases} \triangle z = 0 & \text{in } \Omega \\ \frac{\partial z}{\partial \nu}|_{\partial\Omega} = v & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

We refer to [19, Par. 5] which shows that the Galerkin semi-discrete approximation of system (5.8)-(5.10) satisfies conditions $(C1) - (C5)$ (and for other numerical approximation method).

We can obtain the following result:

Proposition 5.4. *With the notation above, let, for every $\varepsilon > 0$, $z_0 \in X$ such that $\|z_0\| > \varepsilon$. Then we have that $\tau_h^* \to \tau_0^*$, and $u_h^* \to u_0^*$ in $L^2([0,T]; L^2(\partial\Omega))$.*

Proof. In order to apply Theorem 1.1 and Theorem 1.2, we verify that conditions $(C1) - (C5)$ are satisfied with $\alpha = \frac{1}{4} + \varepsilon$, $\varepsilon > 0$ and $\theta = 2$ (see for example [19, Ch. 5]). Moreover, it is known that the assumption made in Theorem 1.2 holds true (see for example [20, Section 3.3, Ch. 5.3]). Thus, we can apply Theorem 1.1 and Theorem 1.2 to conclude. □

We illustrate in the follows some numerical results for the 1d-heat equation with Neumann boundary condition. More precisely, we take $\Omega = (0,1)$, $\epsilon = 1$ (the radius of the target ball), control constraints $|u(t,0)| \leqslant 1$ and $|u(t,1)| \leqslant 1$ and the initial data $z_0(x) = 5sin(\pi x)$. We choose to discretize totally the system, with the implicit Euler method for time and finite difference scheme for space. We also use the optimization solver IPOPT in Matlab.

We obtain the following result:

Number of discretisation	10	20	40	60	80	100
time optimal	1.3442	1.3469	1.3464	1.3459	1.3456	1.3454

Moreover, we find that $u_h(t,0) = u_h(t,1) = -1$ for any $t > 0$ and for any number of discretization, that validates the fact that the time optimal control for the discrete system converges towards a bang-bang time optimal control.

Remark 5.5. *The error estimates of the time and control convergence remain to study.*

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