Generalized Virasoro algebra: left-symmetry and related algebraic and hydrodynamic properties

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Motivated by the work of Kupershmidt (J. Nonlin. Math. Phys. **6** (1998), 222 –245) we discuss the occurrence of left symmetry in a generalized Virasoro algebra. The multiplication rule is defined, which is necessary and sufficient for this algebra to be quasi-associative. Its link to geometry and nonlinear systems of hydrodynamic type is also recalled. Further, the criteria of skew-symmetry, derivation and Jacobi identity making this algebra into a Lie algebra are derived. The coboundary operators are defined and discussed. We deduce the hereditary operator and its generalization to the corresponding $3-$ary bracket. Further, we derive the so-called $\rho-$compatibility equation and perform a phase-space extension. Finally, concrete relevant particular cases are investigated.

Keywords: Virasoro algebra; Left-symmetric algebras; Quasi-associativity; Coboundary operators; Nonlinear systems of hydrodynamic type.

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1. Introduction

The Virasoro algebra, also known as centrally extended Witt algebra, is probably one of the most important algebra studied by physicists and mathematicians in last few decades. It has a profound impact on mathematical and physical sciences. It appears naturally in problem with conformal symmetry and where the essential space-time is one or two dimensional and space is compactified to a circle. For more details, see [1]– [6], [13], [14], [18], [20], [23], [26]– [31], [37], [38] but also references therein.

We deal with one of the most important infinite dimensional Lie algebra, the Witt algebra $\mathscr{W}$ and its universal central extension. The Witt algebra is defined as the complex Lie algebra of derivations of the algebra $\mathbb{C}[\theta,\theta^{-1}]$ of complex Laurent polynomials. The elements of Witt algebra $\mathscr{W}$ are defined as $d_n = ie^{in\theta}\frac{d}{d\theta},\ \ n \in \mathbb{Z}$, so

$$\mathscr{W} = \oplus_{n\in\mathbb{Z}}\mathbb{C}d_n.$$

The Lie-bracket of elements of $\mathscr{W}$ yields $[d_m, d_n] = (m-n)d_{m+n}$.

The Virasoro algebra is constructed from the Witt algebra $\mathscr{W}$ by non-trivial central extension, called the Gelfand-Fuchs cocycle.

Recently, Kuperschmidt [27] investigated the Virasoro algebra with the multiplication

$$\begin{aligned}
[e_p, e_q] &:= e_p \star e_q - e_q \star e_p = (p-q)e_{p+q} + \theta(p^3 - p)\delta_{p+q},\ \ p,q \in \mathbb{Z},\\
[\theta, e_p] &= 0,
\end{aligned} \tag{1.1}$$

in a quasiassociative algebra endowed with the product

$$\begin{aligned}
e_p \star e_q &= -\frac{q(1+\varepsilon q)}{1+\varepsilon(p+q)}e_{p+q} + \frac{1}{2}\theta[p^3 - p + (\varepsilon - \varepsilon^{-1})p^2]\delta^0_{p+q},\\
e_p \star \theta &= \theta \star e_p = 0.
\end{aligned} \tag{1.2}$$

He focussed his analysis on the centerless quasiassociative multiplication

$$e_p \star e_q = -\frac{q(1+\varepsilon q)}{1+\varepsilon(p+q)}e_{p+q}. \tag{1.3}$$

He verified that this multiplication satisfies the quasiassociativity property

$$e_p \star (e_q \star e_r) - (e_p \star e_q) \star e_r = e_q \star (e_p \star e_r) - (e_q \star e_p) \star e_r,\ \ p,q,r \in \mathbb{Z}, \tag{1.4}$$

and re-interpreted, in the language of $2-$cocycle, the property of a bilinear form to provide a central extension of a quasiassociative algebra. His study led to a complex on the space of cochains and its generalization. Besides, Kuperschmidt discussed the homology and performed the differential-variational versions of the main results for the case when the centerless Virasoro algebra is replaced by the Lie algebra of vector fields on the circle.

This paper addresses a generalization of the algebra (1.1), denoted by $(\mathscr{A}, [.,.])$, endowed with the multiplication

$$[e_{x_i}, e_{x_j}] = g(x_i, x_j)e_{x_i+x_j}, \tag{1.5}$$

coming from the commutator

$$[e_{x_i}, e_{x_j}] = ae_{x_i} \star e_{x_j} - be_{x_j} \star e_{x_i}, \tag{1.6}$$

where $(a,b) \in \mathbb{R}\times\mathbb{R}^+$, $(x_i, x_j) \in \mathbb{Z}^2$. We give the necessary and sufficient condition for this algebra to be a quasiassociative algebra with the multiplication

$$e_{x_i} \star e_{x_j} = f(x_i, x_j)e_{x_i+x_j}. \tag{1.7}$$

Let us immediately mention that such a generalization of the algebra (1.1) can lead to various classes of nonassociative algebras [33] such as alternative algebras, Jordan algebras, and so on, as well as to their various extensions, depending on the defining functions f and g, but also on the real constants a and b. However, without loss of generality, in the sequel, the functions f and g are assumed to be defined as follows:

$$f,g:\mathbb{Z}\times\mathbb{Z}\longrightarrow\mathbb{Z},\ \ (x_i,x_j)\mapsto f(x_i,x_j),\, g(x_i,x_j). \tag{1.8}$$

Moreover, in this work, we are only interested in the class of left symmetric algebras, also called quasi-associative algebras.

The main results obtained in this work can be summarized in the four following theorems:

Theorem 1.1. *For the multiplication $e_{x_i}\star e_{x_j}:=R^k_{ij}e_{x_k}$, the quasi-associativity condition*

$$(e_{x_i}\star e_{x_j})\star e_{x_l}-e_{x_i}\star(e_{x_j}\star e_{x_l})=(e_{x_j}\star e_{x_i})\star e_{x_l}-e_{x_j}\star(e_{x_i}\star e_{x_l}), \tag{1.9}$$

is expressed by the Nijenhuis-torsion free relation:

$$\left(R^k_{ij}-R^k_{ji}\right)R^m_{kl}-R^k_{jl}R^m_{ik}+R^k_{il}R^m_{jk}=0. \tag{1.10}$$

Theorem 1.2. *The hereditary condition for a linear map $\Phi:\mathscr{A}\longrightarrow\mathscr{A}$ associated with the generalized Virasoro algebra (1.5) makes into:*

- *For $\Phi:e_{x_i}\mapsto e_{x_i+x_0}$ for some fixed $x_0\in\mathbb{Z}$, $\forall x_i\in\mathbb{Z}$,*

$$g(x_i+x_0,x_j)+g(x_i,x_j+x_0)-g(x_i,x_j)=g(x_i+x_0,x_j+x_0), \tag{1.11}$$

  or, equivalently,

$$\left(\mathscr{T}_{x_0}+\mathscr{L}_{x_0}-\mathscr{T}_{x_0}\mathscr{L}_{x_0}-1\right)g(x_i,x_j)=0, \tag{1.12}$$

  where the right and left translation operators $\mathscr{T}_{x_0}$ and $\mathscr{L}_{x_0}$ are defined, respectively, by

$$\mathscr{T}_{x_0}g(x_i,x_j):=g(x_i,x_j+x_0),\ \ \mathscr{L}_{x_0}g(x_i,x_j):=g(x_i+x_0,x_j); \tag{1.13}$$

- *For $\Phi:e_{x_i}\mapsto R^k_i e_{x_k}$,*

$$\left[g(x_m,x_j)R^m_iR^s_{m+j}+g(x_i,x_n)R^n_jR^s_{n+i}-g(x_i,x_j)R^t_{i+j}R^s_t\right]e_{x_s}=g(x_m,x_n)R^m_iR^n_je_{x_m+x_n}; \tag{1.14}$$

- *For $\Phi:e_{x_i}\mapsto e_{x_i+1}$, $[e_{x_i},e_{x_j}]:=R^k_{ij}e_{x_k}$,*

$$R^k_{(i+1)j}+R^k_{(j+1)i}-R^k_{ij}-R^k_{(i+1)(j+1)}=0. \tag{1.15}$$

Theorem 1.3. *For a map $\rho:\mathscr{A}\longrightarrow\mathscr{A}$, $e_{x_i}\longmapsto e_{x_i+x_0}$ with some fixed $x_0\in\mathbb{Z}$, $\forall x_i\in\mathbb{Z}$, the $\rho-$compatibility equation is equivalent to the invariance of the operator $(1-\mathscr{E})$ under the action* *of the right translation operator* $\mathscr{T}_{x_0}$, *i.e.,*

$$\mathscr{T}_{x_0}(1-\mathscr{E}) = (1-\mathscr{E}), \tag{1.16}$$

where the right translation and exchange operators $\mathscr{T}_{x_0}$ *and* $\mathscr{E}$ *are defined, respectively, by*

$$\mathscr{T}_{x_0} f(x_i,x_j) = f(x_i,x_j+x_0),\ \mathscr{E} f(x_i,x_j) = f(x_j,x_i). \tag{1.17}$$

Theorem 1.4. *The universal identity, known for nonassociative algebras [35], turns out to be in the following form for the generalized algebra introduced with the multiplication (1.7):*

$$\begin{aligned}
&f(x_k,x_j)\Big(-\mathscr{L}_{x_i}(x_j,x_k+x_j)(1-\mathscr{E})(f(x_i,x_j)+[[f(x_j,x_k+x_s),f(x_i,x_k+x_j)]]\Big)\\
&-\mathscr{L}_{x_i+x_j} f(x_k,x_s)\Big(-\mathscr{L}_{x_j}(x_i,x_k)(1-\mathscr{E})(f(x_i,x_j)+[[f(x_j,x_k),f(x_i,x_k)]]\Big)\\
&-\mathscr{T}_{x_i+x_j} f(x_k,x_j)\Big(-\mathscr{L}_{x_j}(x_i,x_j)(1-\mathscr{E})(f(x_i,x_j)+[[f(x_j,x_s),f(x_i,x_s)]]\Big)=0
\end{aligned} \tag{1.18}$$

where

$$[[f(x_j,x_l),f(x_i,x_l)]] = f(x_j,x_l)\mathscr{T}_{x_s} f(x_i,x_l) - f(x_i,x_l)\mathscr{T}_{x_i} f(x_j,x_l), \tag{1.19}$$

and $\mathscr{L}_u$, $\mathscr{T}_v$ *are the usual left and right translation operators, respectively.*

Finding a general solution f satisfying this universal identity remains an open issue. There exist, however, known particular solutions such as those investigated by Kupershmidt in [27], i. e., (1.3) and $f(x_i,x_j) = \lambda - x_j$, $\lambda = \text{const}$, valid only in the case $a = b = 1$ and $g(x_i,x_j) = x_i - x_j$.

In the sequel, we give a full characterization of this generalized Virasoro algebra for which the quasi-associativity condition and the criteria making it a Lie algebra are discussed. We deduce the hereditary operator and its generalization to the corresponding $3-$ary bracket. Further, we deduce the so-called $\rho-$compatibility equation, and investigate a phase-space extension. Finally, concrete relevant particular situations are analyzed.

The paper is organized as follows. Section 2 deals with some preliminaries on left symmetric algebras (also called quasi-associative algebras). In section 3, we discuss the main properties of the generalized Virasoro algebra. The case of left-alternative algebra structure and its link to some classes of nonlinear systems of differential equations are also recalled. Coboundary operators and a $3-$ary bracket are defined and discussed. We define the hereditary operator, and generalize it to the case of $3-$ary bracket. Then, we derive the associated $\rho-$ compatibility equation. Phase-space extension is also discussed. In section 4, we investigate the full centrally extended Virasoro algebra in the framework of the considered formalism. In section 5, we analyze the case of an infinite dimensional Lie algebra of polynomial vector fields on the real line $\mathbb{R}^1$ and deduce some remarkable identities. We end, in section 6, by some concluding remarks.

2. Preliminaries on left symmetric algebras

The ordinary centerless Virasoro-Witt algebra belongs to the class of quasi-associative algebras, known, in the literature [26] (and references therein), under the name of left-symmetric algebras (LSAs), arising in many areas of mathematics and physics. The LSAs were initially introduced by Caley in 1896 in the context of rooted tree algebras and in recent years Vinberg and Koszul re-introduced them in the context of convex homogeneous cones. LSAs have also independently appeared in the works by Gerstenhaber. As a consequence, perhaps, LSAs are known under different names. They are also called Vinberg algebras, Koszul algebras, or quasi-associative algebras, Gerstenhaber algebras, or pre-Lie algebras. See [6], [38] and [39] (and references therein).

Geometrically, they are also connected to the theory of affine manifolds and affine structures on Lie groups. See [1], [30], [6] and references therein. Recall a smooth manifold which admits a linear connection $\triangledown$ whose torsion and curvature tensor vanish, is called an affinely flat (or simply affine in short) manifold. By a well known theorem of differential geometry, such a manifold is locally equivalent to an open subset of Euclidean space with the standard connection, i. e., for each point of the manifold, there are a neighborhood and a coordinate map into the Euclidean space which is an affine equivalence. In fact, the torsion and curvature are exactly the obstructions to the existence of such a map. In general, a connection $\triangledown$ on a Lie group is completely determined by the action on the left invariant vector fields, i.e., by $\triangledown_X Y$ for $X,Y \in \mathfrak{g}$ using the Leibniz rule. $\triangledown$ is left-invariant if and only if $\triangledown_X Y \in \mathfrak{g}$ whenever $X,Y \in \mathfrak{g}$. To perceive the problem algebraically, denote $\triangledown_X Y$ by $X.Y$ for a left-invariant connection $\triangledown$ and vector fields $X,Y \in \mathfrak{g}$. Then having a left-invariant connection on G is the same as having an algebra structure on $\mathfrak{g}$. In this way, the geometric problems involving left-invariant connection become algebraic ones.

A left-invariant connection $\triangledown$ on G is said to be bi-invariant if it is also right-invariant. As usual, this holds if and only if $\triangledown$ is adjoint invariant. We can characterize bi-invariant connections using the associated algebra structure $X.Y = \triangledown_X Y$ as follows [25]:

Proposition 2.1. *The following statements are equivalent:*

(i) A left-invariant connection $\triangledown$ on G is bi-invariant.
(ii) Ad_g is an algebra automorphism on $(\mathfrak{g},.)$ for all $g \in G$.
(iii) Ad_X is an algebra derivation on $(\mathfrak{g},.)$ for all $X \in \mathfrak{g}$, i.e.,

$$\begin{aligned} &Ad_X(Y.Z) = Ad_X(Y).Z + Y.Ad_X(Z) \\ \textit{or}\ &[X,Y.Z] = [X,Y].Z + Y.[X,Z], X,Y,Z \in \mathfrak{g}. \end{aligned} \tag{2.1}$$

Proof. See [25]. □

If furthermore the connection $\triangledown$ is torsion free, then the algebra automorphism becomes a Lie algebra automorphism since $[X,Y] = X.Y - Y.X, \quad X,Y \in \mathfrak{g}$. Suppose further that $\triangledown$ is affinely flat so that it has vanishing torsion and curvature tensor. Then, using small letters for elements of $\mathfrak{g}$ and xy for $X.Y = \triangledown_X Y$, the torsion-free condition and the flatness of $\triangledown$ become algebraically:

$$xy - yx = [x,y] \tag{2.2}$$

$$x(yz) - y(xz) - [x,y]z = 0, \tag{2.3}$$

respectively, for all $x,y,z \in \mathfrak{g}$. This leads to the following definition.

Definition 2.1. Let $\mathscr{A}$ be a vector space over a field $\mathbb{K}$ equipped with a bilinear product $(x,y) \mapsto xy$. $\mathscr{A}$ is called a *left symmetric* algebra, or, equivalently, a *quasi-associative* algebra, if for all $x,y,z \in \mathscr{A}$

the associator $\left(x,y,z\right)=x(yz)-(xy)z$ is symmetric in x,y, i.e.,

$$\left(x,y,z\right)=\left(y,x,z\right), \qquad \text{or} \qquad (xy)z-x(yz)=(yx)z-y(xz). \tag{2.4}$$

Hence finding a left invariant affinely flat connection on G is the same as finding a left-symmetric algebra structure on $\mathfrak{g}$ which is compatible with Lie algebra structure of $\mathfrak{g}$ in the sense of (2.2).

Left symmetric algebras are Lie-admissible algebras (cf. [32]). Let $\mathscr{A}$ be a LSA, then for any $x\in\mathscr{A}$, denote L_x the left multiplication operator, $L_x(y)=xy$ for all $y\in\mathscr{A}$. By setting $[x,y]:=xy-yx$ a Lie bracket defines a Lie algebra $\mathscr{G}(\mathscr{A})$, known as the sub-adjacent Lie algebra of $\mathscr{A}$. Thus $\mathscr{A}$ is called a compatible left symmetric algebra structure on the Lie algebra $\mathscr{G}(\mathscr{A})$.

Let $\mathscr{G}(\mathscr{A})\to gl(\mathscr{A})$ with $x\mapsto L_x$. Then $(L,\mathscr{A})$ gives a representation of the Lie algebra $\mathscr{G}(\mathscr{A})$, i.e., $[L_x,L_y]=L_{[x,y]}$ for all $x,y\in\mathscr{A}$. But this is neither sufficient nor necessary condition to give compatible LSA on any Lie algebra. These are given as follows. Let $\mathscr{G}$ be a Lie algebra with a representation $\rho:\mathscr{G}\to gl(V)$, then a one-cocycle $q:\mathscr{G}\to V$ be a linear map associated to (ρ,q) such that $q[x,y]=\rho(x)q(y)-\rho(y)q(x)$ for all $x,y\in\mathscr{G}$. It has been shown in [26] that there is a compatible LSA structure if and only if there exists a bijective one-cocycle of $\mathscr{G}$.

If (ρ,q) is a bijective one-cocycle of $\mathscr{G}$ then $x*y=q^{-1}\rho(x)q(y)$ defines a LSA structure on $\mathscr{G}$, where as for a LSA $\mathscr{A}$ the identity transformation is a one-cocycle of $\mathscr{G}(\mathscr{A})$ associated with the regular representation L.

It is also worth noticing that left-symmetric structures appear of natural way in the theory of integrable systems of hydrodynamic type (see [28], [10], [36] (and references therein)) and the generalized Burgers equation. Indeed, it was proved by Sokolov *and co-workers* [22], (see also [35] and references therein), the following theorem giving the link between LSAs and multicomponent generalizations of Burgers equations.

Theorem 2.1. *If C^i_{jk} are structure constants of any LSA, then the system of Burgers equations,*

$$u^i_t=u^i_{xx}+2C^i_{jk}u^ku^j_x+A^i_{jkm}u^ku^ju^m, \ \ \textit{where} \ \ i,j,k,m=1,\ldots,N \tag{2.5}$$

is integrable iff the following relations hold:

$$\begin{aligned} A^i_{jkm} &= \frac{1}{3}\left(C^i_{jr}C^r_{km}+C^i_{kr}C^r_{mj}+C^i_{mr}C^r_{jk}-C^i_{rj}C^r_{km}-C^i_{rk}C^r_{mj}-C^i_{rm}C^r_{jk}\right)\\ & C^i_{jr}C^r_{km}-C^i_{kr}C^r_{jm}=C^r_{jk}C^i_{rm}-C^r_{kj}C^i_{rm}. \end{aligned} \tag{2.6}$$

In this case, let $e_1,\ldots,e_N$ be a basis of a LSA $\mathscr{A}$, and $u=u^ie_i$. Then, the integrable system can be written as

$$u_t=u_{xx}+2u\circ u_x+u\circ(u\circ u)-(u\circ u)\circ u, \tag{2.7}$$

where $\circ$ denotes the multiplication in $\mathscr{A}$.

3. Generalized Virasoro algebra: quasi-associativity, hereditary operator and $\rho-$compatibility equation

In this section, we discuss the main properties of the generalized Virasoro algebra. The case of left-alternative algebra structure and its link to some classes of nonlinear systems of differential equations are also recalled. A $3-$ary bracket is defined and the relation between the functions f and g is given. We define the hereditary operator, and generalize it to the case of $3-$ary bracket. Then, we derive the associated $\rho-$ compatibility equation.

3.1. *Skew-symmetry, Jacobi identity, coboundary operators and derivation property*

- Skew-symmetry

  The skew-symmetry property

$$\begin{aligned}[e_{x_i}, e_{x_j}] &= ae_{x_i} \star e_{x_j} - be_{x_j} \star e_{x_i} = g(x_i, x_j) e_{x_i + x_j} \\ &= -[e_{x_j}, e_{x_i}] = -\left(ae_{x_j} \star e_{x_i} - be_{x_i} \star e_{x_j}\right) = -g(x_j, x_i) e_{x_i + x_j}\end{aligned} \tag{3.1}$$

  induces the conditions

$$a = b \text{ or } g(x_i, x_j) = -g(x_j, x_i). \tag{3.2}$$

- Jacobi identity criterion

  The Jacobi identity

$$\Big[[e_{x_i}, e_{x_j}], e_{x_k}\Big] + \Big[[e_{x_j}, e_{x_k}], e_{x_i}\Big] + \Big[[e_{x_k}, e_{x_i}], e_{x_j}\Big] = 0 \tag{3.3}$$

  reduces to a condition similar to the *Bianchi's identity*

$$\mathbb{J}_{ij}^{k} + \mathbb{J}_{jk}^{i} + \mathbb{J}_{ki}^{j} = 0 \tag{3.4}$$

  where

$$\mathbb{J}_{ij}^{k} := g(x_i, x_j) g(x_i + x_j, x_k). \tag{3.5}$$

  Exploiting the relation between the functions f and g, i.e.,

$$g(x_i, x_j) = af(x_i, x_j) - bf(x_j, x_i). \tag{3.6}$$

  we can re-express the criterion for the Jacobi identity by the following result:

$$\begin{aligned}&a^2\left(\mathbb{T}_{ij}^{k} + \mathbb{T}_{jk}^{i} + \mathbb{T}_{ki}^{j}\right) + b^2\left(\mathbb{G}_{ji}^{k} + \mathbb{G}_{ik}^{j} + \mathbb{G}_{kj}^{i}\right) \\ &- ab\left(\mathbb{G}_{ij}^{k} + \mathbb{G}_{jk}^{i} + \mathbb{G}_{ki}^{j} + \mathbb{T}_{kj}^{i} + \mathbb{T}_{ji}^{k} + \mathbb{T}_{ik}^{j}\right) = 0\end{aligned} \tag{3.7}$$

  where

$$\mathbb{T}_{ij}^{k} := f(x_i, x_j) f(x_i + x_j, x_k), \quad \mathbb{G}_{ij}^{k} =: f(x_i, x_j) f(x_k, x_i + x_j). \tag{3.8}$$

  The criteria (3.2) and (3.4) or (3.7) confer the Lie algebra structure to the algebra $(\mathscr{A}, [.,.])$.

  For $a = b$, skew-symmetric functions f, (i. e. $f(x_i, x_j) = -f(x_j, x_i)$ for all $x_i, x_j \in \mathbb{Z}$), are solutions of (3.7), which is consistent with the above skew-symmetry conditions (3.2). Note, however, that the search for a general solution to this functional equation remains an open issue.

- Coboundary operators, 2−cocycle and second cohomology

  In addition to the above criteria, we consider an associated Lie module $\mathscr{M}$ over $\mathscr{A}$, and a $k-$cochain, i.e. an alternating $\mathbb{K}-$multilinear map

$$\psi : \mathscr{A} \times \mathscr{A} \times \ldots \times \mathscr{A} \,(k\,\text{copies of}\,\mathscr{A}) \longrightarrow \mathscr{M}.$$

The most important $\mathscr{A}$-modules are the trivial module $\mathbb{K}$, i.e., the action reads $e_{x_n} \cdot \lambda = 0$ for all $\lambda \in \mathbb{K}$ and all $e_{x_n} \in \mathscr{A}$, and adjoint module $\mathscr{A}$, i.e., $\mathscr{A}$ acts on $\mathscr{A}$ by the adjoint action.

Denote the vector space of $k-$cochains by $\mathscr{C}^k(\mathscr{A},\mathscr{A})$ and define the coboundary operators

$$\delta_k : \mathscr{C}^k(\mathscr{A},\mathscr{A}) \longrightarrow \mathscr{C}^{k+1}(\mathscr{A},\mathscr{A}),\, k \in \mathbb{N},\, \text{with}\, \delta_{k+1} \circ \delta_k = 0. \tag{3.9}$$

Definition 3.1. A $k-$cochain ψ is called a $k-$cocycle if it lies in the kernel of the coboundary operator δ_k. It is called a $k-$coboundary if it lies in the image of the (k-1) coboundary operator.

A skew-symmetric map $\psi : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is a Lie algebra $2-$cocycle with values in the adjoint module if

$$\begin{aligned}\delta_2 \psi(e_{x_i}, e_{x_j}, e_{x_k}) :=\ & \psi([e_{x_i}, e_{x_j}], e_{x_k}) + \psi([e_{x_j}, e_{x_k}], e_{x_i}) + \psi([e_{x_k}, e_{x_i}], e_{x_j}) \\ & - [e_{x_i}, \psi(e_{x_j}, e_{x_k})] + [e_{x_j}, \psi(e_{x_i}, e_{x_k})] - [e_{x_k}, \psi(e_{x_i}, e_{x_j})] = 0\end{aligned} \tag{3.10}$$

and a coboundary if there exits a linear map $\phi : \mathscr{A} \to \mathscr{A}$ with

$$\psi(e_{x_i}, e_{x_j}) = (\delta_1 \phi)(e_{x_i}, e_{x_j}) := \phi([e_{x_i}, e_{x_j}]) - [e_{x_i}, \phi(e_{x_j})] + [e_{x_j}, \phi(e_{x_i})]. \tag{3.11}$$

The second cohomology of $\mathscr{A}$ with values in the adjoint representation is

$$H^2(\mathscr{A},\mathscr{A}) = Ker\, \delta_2 / Im\, \delta_1,$$

whereas $H^2(\mathscr{A},\mathbb{K})$ with values in the trivial module is related to the central extension of $\mathscr{A}$. It is worth saying that deformations of the Lie algebra $\mathscr{A}$ are related to the Lie algebra cohomology and $H^2(\mathscr{A},\mathscr{A})$ classifies infinitesimal deformations [17]. If $H^2(\mathscr{A},\mathscr{A}) = 0$, then $\mathscr{A}$ is infinitesimally formally rigid.

An elementary and direct calculation of the vanishing second Lie algebra cohomology of the Witt and Virasoro algebras with values in the adjoint module is given by Schlichenmaier [34]. In 1989, A. Fialowski showed by explicit calculations the vanishing of the second Lie algebra cohomology of the Witt algebra (in an unpublished manuscript). She also gave statements of the rigidity of the Witt and Virasoro algebras [11] without proof.

Write the $0-$cocycle as

$$\psi(e_{x_i}, e_{x_j}) = \psi_{x_i, x_j} e_{x_i + x_j}. \tag{3.12}$$

If it is a coboundary, then it can be given as a coboundary of a linear form of degree 0, i.e.,

$$\phi(e_{x_i}) = \phi_{x_i} e_{x_i}. \tag{3.13}$$

Then the following result holds.

Proposition 3.1. *The 2−cocycle ψ defined by*

$$\delta_2\psi(e_{x_i},e_{x_j},e_{x_k}) := 0 \tag{3.14}$$

leads to the functional relation:

$$\begin{aligned} & g(x_i,x_j)\psi_{x_i+x_j,x_k} + g(x_j,x_k)\psi_{x_j+x_k,x_i} + g(x_k,x_i)\psi_{x_i+x_k,x_j} \\ - & \, g(x_i,x_j+x_k)\psi_{x_j,x_k} + g(x_j,x_i+x_k)\psi_{x_i,x_k} - g(x_k,x_i+x_j)\psi_{x_i,x_j} = 0. \end{aligned} \tag{3.15}$$

Besides, there results from the expression (3.11):

$$(\delta_1\phi)(x_i,x_j) = g(x_i,x_j)\left(\phi_{x_i+x_j} - \phi_{x_i} - \phi_{x_j}\right) e_{x_i+x_j}. \tag{3.16}$$

Hence, ψ is a coboundary if and only if there exists a system of $\phi_{x_k} \in \mathbb{C}$, $x_k \in \mathbb{Z}$, such that

$$\psi_{x_i,x_j} = g(x_i,x_j)\left(\phi_{x_i+x_j} - \phi_{x_i} - \phi_{x_j}\right). \tag{3.17}$$

- Derivation property

  This property expressed as

$$[e_{x_i}, e_{x_j} \star e_{x_k}] := e_{x_j} \star [e_{x_i}, e_{x_k}] + [e_{x_i}, e_{x_j}] \star e_{x_k} \tag{3.18}$$

leads to

$$f(x_j,x_k)g(x_i,x_j+x_k) = g(x_i,x_k)f(x_j,x_i+x_k) + g(x_i,x_j)f(x_i+x_j,x_k) \tag{3.19}$$

or, equivalently, using the relation (3.6) between f and g, to

$$\begin{aligned} & f(x_j,x_k)\Big[af(x_i,x_j+x_k) - bf(x_j+x_k,x_i)\Big] \\ - & \, f(x_j,x_i+x_k)\Big[af(x_i,x_k) - bf(x_k,x_i)\Big] - f(x_i+x_j,x_k)\Big[af(x_i,x_j) - bf(x_j,x_i)\Big] = 0 \end{aligned} \tag{3.20}$$

which can be simply rewritten as

$$a\left(\mathbb{G}^i_{jk} - \mathbb{G}^j_{ik} - \mathbb{T}^k_{ij}\right) + b\left(\mathbb{G}^j_{ki} + \mathbb{T}^k_{ji} - \mathbb{T}^i_{jk}\right) = 0. \tag{3.21}$$

Let us extend now the notion of skew-symmetry and Jacobi identity to the vector space $\mathscr{B} = \mathscr{A} \oplus \mathscr{A}$ with the multiplication

$$\Big[(e_{x_p},e_{x_r}),(e_{x_q},e_{x_s})\Big] := \Big([e_{x_p},e_{x_q}], e_{x_p}\star e_{x_s} - e_{x_q}\star e_{x_r}\Big) \tag{3.22}$$

where $(.,.)$ is the ordinary dot product. Explicitly, this new bracket gives

$$\Big[(e_{x_p},e_{x_r}),(e_{x_q},e_{x_s})\Big] = g(x_p,x_q)\Big(e_{x_p+x_q}, f(x_p,x_s)e_{x_p+x_s} - f(x_q,x_r)e_{x_q+x_r}\Big), \tag{3.23}$$

or, equivalently,

$$\Big[(e_{x_p},e_{x_r}),(e_{x_q},e_{x_s})\Big] = g(x_p,x_q)\Big[f(x_p,x_s)\Big(e_{x_p+x_q},e_{x_p+x_s}\Big) - f(x_q,x_r)\Big(e_{x_p+x_q},e_{x_q+x_r}\Big)\Big]. \tag{3.24}$$

Therefore, the skew-symmetry criterion reads

$$\Big[g(x_p,x_q) - g(x_q,x_p)\Big]\Big[f(x_p,x_s)\Big(e_{x_p+x_q},e_{x_p+x_s}\Big) - f(x_q,x_r)\Big(e_{x_p+x_q},e_{x_q+x_r}\Big)\Big] = 0. \tag{3.25}$$

The Jacobi identity property

$$\begin{aligned}\Big[[(e_{x_p},e_{x_r}),(e_{x_q},e_{x_s})],(e_{x_t},e_{x_u})\Big] &+ \Big[[(e_{x_q},e_{x_s}),(e_{x_t},e_{x_u})],(e_{x_p},e_{x_r})\Big] \\ &+ \Big[[(e_{x_t},e_{x_u}),(e_{x_p},e_{x_r})],(e_{x_q},e_{x_s})\Big] = 0\end{aligned} \tag{3.26}$$

gives here the following functional relation:

$$\begin{aligned}&\Big[g(x_p,x_q)g(x_p+x_q,x_t)f(x_p+x_q,x_u) - g(x_q,x_t)g(x_q+x_t,x_p)f(x_q,x_u)f(x_p,x_q+x_u) \\ &\qquad + g(x_t,x_p)g(x_t+x_p,x_q)f(x_p,x_u)f(x_q,x_p+x_u)\Big]\Big(e_{x_p+x_q+x_t},e_{x_p+x_q+x_u}\Big) \\ &- \Big[g(x_p,x_q)g(x_p+x_q,x_t)f(x_p,x_s)f(x_t,x_p+x_s) - g(x_q,x_t)g(x_q+x_t,x_p)f(x_t,x_s)f(x_p,x_t+x_s) \\ &\qquad - g(x_t,x_p)g(x_t+x_p,x_q)f(x_p+x_t,x_s)\Big]\Big(e_{x_p+x_q+x_t},e_{x_p+x_t+x_s}\Big) \\ &+ \Big[g(x_p,x_q)g(x_p+x_q,x_t)f(x_q,x_r)f(x_t,x_q+x_r) + g(x_q,x_t)g(x_q+x_t,x_p)f(x_q+x_t,x_r) \\ &\qquad - g(x_t,x_p)g(x_t+x_p,x_q)f(x_t,x_r)f(x_q,x_r+x_t)\Big]\Big(e_{x_p+x_q+x_t},e_{x_q+x_r+x_t}\Big) = 0,\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\Big[\mathbb{J}^t_{qp}\mathscr{L}_{x_q}f(x_p,x_u) - \mathbb{J}^p_{qt}\mathbb{G}^p_{qu} + \mathbb{J}^q_{tp}\mathbb{G}^q_{pu}\Big]\Big(e_{x_p+x_q+x_t},e_{x_p+x_q+x_u}\Big) \\ &+ \Big[\mathbb{J}^q_{tp}\mathscr{L}_{x_t}f(x_p,x_s) + \mathbb{J}^p_{qt}\mathbb{G}^p_{ts} - \mathbb{J}^t_{pq}\mathbb{G}^t_{ps}\Big]\Big(e_{x_p+x_q+x_t},e_{x_p+x_t+x_s}\Big) \\ &+ \Big[\mathbb{J}^t_{pq}\mathbb{G}^t_{qr} + \mathbb{J}^p_{qt}\mathscr{L}_{x_q}f(x_t,x_r) - \mathbb{J}^q_{tp}\mathbb{G}^q_{tr}\Big]\Big(e_{x_p+x_q+x_t},e_{x_q+x_r+x_t}\Big) = 0,\end{aligned} \tag{3.27}$$

where $\mathscr{L}_{x_t}$ stands for the left-translation operator acting as $\mathscr{L}_{x_t}f(x_q,x_r) = f(x_q+x_t,x_r)$.

3.2. *Quasi-associativity condition*

From the definition of the quasiassociativity w.r.t. the multiplication rule (1.7), we infer the following relation

$$\Big[f(x_i,x_j) - f(x_j,x_i)\Big]f(x_i+x_j,x_k) = f(x_j,x_k)f(x_i,x_j+x_k) - f(x_i,x_k)f(x_j,x_i+x_k) \tag{3.28}$$

expressing the necessary and sufficient property which confers a space phase structure to the subadjacent Lie algebra [27].

Setting the ansatz

$$f(x_i,x_j) := \phi_{x_i}(x_j) \tag{3.29}$$

for some linear map $\phi : \mathbb{Z} \to \mathrm{End}(\mathbb{Z})$, $\phi_x = \phi(x)$, then the quasiassociativity condition (3.28) reads

$$\Big[\phi_{x_i}(x_j) - \phi_{x_j}(x_i)\Big]\phi_{x_i+x_j}(x_k) = \phi_{x_j}(x_k)\phi_{x_i}(x_j+x_k) - \phi_{x_i}(x_k)\phi_{x_j}(x_i+x_k), \tag{3.30}$$

which can be further simplified into the expression

$$\Big[\mathscr{R}_i^j - \mathscr{R}_j^i\Big]\mathscr{R}_{i+j}^k - \mathscr{R}_j^k\mathscr{R}_i^{j+k} + \mathscr{R}_i^k\mathscr{R}_j^{i+k} = 0 \tag{3.31}$$

by defining $\phi_{x_i}(x_j) := \mathscr{R}_i^j$. The relation (3.31) can be given an interpretation in terms of the null value of the Nijenhuis-torsion. Indeed, we have the following result.

Proposition 3.2. *Let $e_{x_i} \star e_{x_j} := R_{ij}^k e_{x_k}$. Then the quasi-associativity condition, i.e.,*

$$(e_{x_i} \star e_{x_j}) \star e_{x_l} - e_{x_i} \star (e_{x_j} \star e_{x_l}) = (e_{x_j} \star e_{x_i}) \star e_{x_l} - e_{x_j} \star (e_{x_i} \star e_{x_l}), \tag{3.32}$$

is expressed by the Nijenhuis-torsion free relation:

$$\Big(R_{ij}^k - R_{ji}^k\Big)R_{kl}^m - R_{jl}^k R_{ik}^m + R_{il}^k R_{jk}^m = 0. \tag{3.33}$$

This well agrees with the result exposed in [39]. In this case, as well pointed out by Kuperschmidt [28] (see also references therein), all associated hydrodynamic systems are diagonalizable in Riemann invariants whenever they are hyperbolic. Moreover, a diagonal $N-$component hyperbolic hydrodynamic system whose Nijenhuis torsion is zero, is isomorphic to N noninteracting scalar equations.

Definition 3.2 (Alternative algebra). An alternative algebra $\mathscr{U}$ over a field $\mathbb{F}$ is an algebra defined with the two identities [33]

$$x^2y = x(xy) \text{ for all } x,y \in \mathscr{U} \tag{3.34}$$

and

$$yx^2 = (yx)x \text{ for all } x,y \in \mathscr{U}, \tag{3.35}$$

known, respectively, as left and right alternative laws.

In terms of associators, (3.34) and (3.35) are equivalent to

$$(x,x,y) = 0 \text{ for all } x,y \in \mathscr{U} \tag{3.36}$$

and

$$(y,x,x) = 0 \text{ for all } x,y \in \mathscr{U}, \tag{3.37}$$

respectively. Therefore, the relations (3.31) and (3.33) are trivially satisfied, $0 = 0$, for $i = j$, i.e., in the case of left-alternative algebra. In accordance with the modified Riccati scheme introduced by Kazakov [24], such a left-alternative algebra is associated with the following vector system of equations:

$$\mathbf{Y}_t = -\mathbf{Y}\star\mathbf{Y}+\varepsilon, \tag{3.38}$$

where the vector function $\varepsilon(t)$ takes value in the considered left alternative algebra (LAA). The nonlinear system (3.38) can be reduced to linear problems by means of this LAA. Notice that, in contrast to systems of hydrodynamic type, the nonlinear systems of equations generated by left-alternative algebras do not, in general, have integrals of the motion. For more details, see [24]. It is also worth mentioning the well-known Burgers vectorial equation [35]:

$$\mathbf{u}_t = \mathbf{u}_{xx}+2\mathbf{u}\star\mathbf{u}_x+\mathbf{u}\star(\mathbf{u}\star\mathbf{u})-(\mathbf{u}\star\mathbf{u})\star\mathbf{u}, \tag{3.39}$$

which is one of the important examples of equations associated with LSAs. In the case of the generalized Virasoro algebra examined in this work, the $\star-$ product is defined by the multiplication law (1.7). New examples of nonlinear systems may exist, but their full investigations remain totally open and may be the core of our forthcoming works.

3.3. $3-$*ary bracket*

Ternary algebra plays an important role in the construction of the world volume theories of multiple M2 branes [2]. The ternary bracket was introduced by Nambu [19] and developed by Filippov [12]. Several authors [9] studied Kac-Moody and centerless Virasoro (or Virasoro-Witt) $3-$algebras and demonstrated some of their applications to the Bagger-Lambert-Gustavsson theory. The su(1,1) enveloping algebra was used by Curtright *et al* [8] to construct ternary Virasoro-Witt algebra. Motivated by this work, we study the ternary algebra of the generalized Virasoro algebra defined by the following $3-$ary bracket:

$$[e_{x_i},e_{x_j},e_{x_k}] := e_{x_i}\star[e_{x_j},e_{x_k}]+e_{x_j}\star[e_{x_k},e_{x_i}]+e_{x_k}\star[e_{x_i},e_{x_j}]. \tag{3.40}$$

Putting

$$\mathbb{E}^k_{ij} := g(x_i,x_j)f(x_k,x_i+x_j)e_{x_i+x_j+x_k} \tag{3.41}$$

yields the following expression for the $3-$ary bracket:

$$[e_{x_i},e_{x_j},e_{x_k}] = \mathbb{E}^k_{ij}+\mathbb{E}^i_{jk}+\mathbb{E}^j_{ki}. \tag{3.42}$$

In the other hand, using the relations (1.6) and (1.7) the same bracket can also be evaluated in terms of the real a and b. In this case, denoting by

$$\mathbb{F}^k_{ij} := \Big[af(x_i,x_j)-bf(x_j,x_i)\Big]f(x_k,x_i+x_j)e_{x_i+x_j+x_k}, \tag{3.43}$$

we obtain

$$[e_{x_i},e_{x_j},e_{x_k}] = \mathbb{F}^k_{ij}+\mathbb{F}^i_{jk}+\mathbb{F}^j_{ki}. \tag{3.44}$$

The relations (3.42) and (3.44) can therefore be used to define a $3-$algebra generalization of the generalized Virasoro algebra proposed by Kupershmidt [27]. Indeed, such a formulation of the

3−ary bracket, cyclic in the indices i, j, k, reminiscent of the *Bianchi's identity* for curvature tensor in differential geometry, is quite relevant for detailed analysis of 3−algebras as will be developed in a forthcoming paper.

3.4. *Hereditary operator*

The hereditary operator is defined as follows [16]:

Definition 3.3 (Hereditary operator).
A linear map $\Phi : (\mathscr{A}, \circ) \to (\mathscr{A}, \circ)$ defined as

$$[a,b]_\Phi := (\Phi a)\circ b + a\circ(\Phi b) - \Phi(a\circ b), \tag{3.45}$$

where $\circ$ is some binary bilinear operator on $\mathscr{A}$, is called hereditary if

$$\Phi[a,b]_\Phi = (\Phi a)\circ(\Phi b), \tag{3.46}$$

or, equivalently,

$$\Phi^2(a\circ b) + (\Phi a)\circ(\Phi b) = \Phi\Big[(\Phi a)\circ b + a\circ(\Phi b)\Big]. \tag{3.47}$$

Next, let us introduce the map $\mathscr{L}_k$ such that

$$\mathscr{L}_k(\Phi)(b) := k\circ\Phi(b) - \Phi(b)\circ k - \Phi(k\circ b) + \Phi(b\circ k) \tag{3.48}$$

for all $b \in \mathscr{A}$. Then Φ is k−invariant iff $\mathscr{L}_k\Phi = 0$.
Formally, let us write

$$\mathscr{L}_a b = a\circ b - b\circ a. \tag{3.49}$$

Then, the Leibniz rule reads

$$\mathscr{L}_a(\Phi\circ b) = \mathscr{L}_a(\Phi)\circ b + \Phi(\mathscr{L}_a b). \tag{3.50}$$

Therefore, the map Φ is hereditary iff

$$\Phi(\mathscr{L}_a\Phi) = \mathscr{L}_{\Phi(a)}\Phi. \tag{3.51}$$

Three particular cases give rise to interesting simpler conditions as follows.

Proposition 3.3. *The hereditary condition for the generalized Virasoro algebra (1.5) makes into:*

(i) For $\Phi : e_{x_i} \mapsto e_{x_i+x_0}$ for some fixed x_0,

$$g(x_i+x_0, x_j) + g(x_i, x_j+x_0) - g(x_i, x_j) = g(x_i+x_0, x_j+x_0), \tag{3.52}$$

or, equivalently,

$$\Big(\mathscr{T}_{x_0} + \mathscr{L}_{x_0} - \mathscr{T}_{x_0}\mathscr{L}_{x_0} - 1\Big)g(x_i, x_j) = 0, \tag{3.53}$$

where the right and left translation operators $\mathscr{T}_{x_0}$ and $\mathscr{L}_{x_0}$ are defined by

$$\mathscr{T}_{x0}g(x_i,x_j) := g(x_i,x_j+x_0),\ \ \mathscr{L}_{x0}g(x_i,x_j) := g(x_i+x_0,x_j); \tag{3.54}$$

(ii) For $\Phi : e_{x_i} \mapsto R_i^k e_{x_k}$,

$$\Big[g(x_m,x_j)R_i^m R_{m+j}^s + g(x_i,x_n)R_j^n R_{n+i}^s - g(x_i,x_j)R_{i+j}^t R_t^s\Big] e_{x_s} = g(x_m,x_n)R_i^m R_j^n e_{x_m+x_n}; \tag{3.55}$$

(iii) For $\Phi : e_{x_i} \mapsto e_{x_i+1}$, $[e_{x_i}, e_{x_j}] := R_{ij}^k e_{x_k}$,

$$R^k_{(i+1)j} + R^k_{(j+1)i} - R^k_{ij} - R^k_{(i+1)(j+1)} = 0. \tag{3.56}$$

Besides, defining the action of Φ on a $3-$ary bracket in the simple case of $\Phi : e_{x_i} \mapsto e_{x_i+x_0}$ as:

$$\Phi\Big[e_{x_i}, e_{x_j}, e_{x_k}\Big]_{\Phi} := \Big[\Phi(e_{x_i}), \Phi(e_{x_j}), \Phi(e_{x_k})\Big] \tag{3.57}$$

yields a Bianchi like identity

$$\mathbb{P}^k_{ij} + \mathbb{P}^i_{jk} + \mathbb{P}^j_{ki} = 0 \tag{3.58}$$

where

$$\mathbb{P}^k_{ij} := \Big[g(x_i+x_0,x_j) + g(x_i,x_j+x_0) - g(x_i,x_j) - g(x_i+x_0,x_j+x_0)\Big] f(x_k+x_0, x_i+x_j+2x_0) \tag{3.59}$$

or, equivalently, in operator form,

$$\mathbb{P}^k_{ij} := \Big(\mathscr{T}_{x_0} + \mathscr{L}_{x_0} - \mathscr{T}_{x_0}\mathscr{L}_{x_0} - 1\Big) g(x_i,x_j)\,\mathscr{T}_{2x_0+x_j}\mathscr{L}_{x_0} f(x_k,x_i). \tag{3.60}$$

Hereditary operators play an important role in the field of nonlinear evolution equations. Indeed, as showed in [15], they generate on a systematic level many new classes of nonlinear dynamical systems which possess infinite dimensional abelian groups of symmetry transformations. Their so-called permanence properties, given by Fuchssteiner [15], allow to construct new hereditary operators out of given ones.

3.5. $\rho-$*compatibility equation*

Following [27], define a new multiplication

$$e_{x_i} \star' e_{x_j} = e_{x_i} \star e_{x_j} + \varepsilon h_{e_{x_i}, e_{x_j}},\ \ \varepsilon^2 = 0 \tag{3.61}$$

with some (bi-)linear map $h : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$, such that it makes $\mathscr{A}$ into a left-symmetric algebra as well. The new associator $(e_{x_i}, e_{x_j}, e_{x_k})_{\star'}$ can be expressed in terms of old one as follows:

$$(e_{x_i}, e_{x_j}, e_{x_k})_{\star'} = (e_{x_i}, e_{x_j}, e_{x_k})_{\star} + \varepsilon T'_{e_{x_i}, e_{x_j}, e_{x_k}} \tag{3.62}$$

where

$$T'_{e_{x_i}, e_{x_j}, e_{x_k}} = h_{e_{x_i}, e_{x_j}\star e_{x_k}} - h_{e_{x_i}\star e_{x_j}, e_{x_k}} + e_{x_i} \star h_{e_{x_j}, e_{x_k}} - h_{e_{x_i}, e_{x_j}} \star e_{x_k}. \tag{3.63}$$

Therefore, the left symmetric condition for $(\mathscr{A}, \star')$ is provided by the relation

$$
\begin{aligned}
T_{e_{x_i},e_{x_j},e_{x_k}} &= T'_{e_{x_i},e_{x_j},e_{x_k}} - T'_{e_{x_j},e_{x_i},e_{x_k}} \\
&= h_{e_{x_i},e_{x_j}\star e_{x_k}} - h_{e_{x_j},e_{x_i}\star e_{x_k}} - h_{[e_{x_i},e_{x_j}],e_{x_k}} \\
&+ e_{x_i}\star h_{e_{x_j},e_{x_k}} - e_{x_j}\star h_{e_{x_i},e_{x_k}} \\
&- \left(h_{e_{x_i},e_{x_j}} - h_{e_{x_j},e_{x_i}}\right)\star e_{x_k} = 0.
\end{aligned} \tag{3.64}
$$

Write here also

$$
h_{e_{x_i},e_{x_j}} := \phi_{e_{x_i}}(e_{x_j}) \tag{3.65}
$$

for some linear map $\phi : \mathscr{A} \to \mathscr{A}, \ \phi_{e_{x_i}} = \phi(e_{x_i})$.

Then the quasiassociativity condition (3.62) takes the following form:

$$
\begin{aligned}
&\phi_{e_{x_i}}(e_{x_j}\star e_{x_k}) - \phi_{e_{x_j}}(e_{x_i}\star e_{x_k}) - \phi_{[e_{x_i},e_{x_j}]}(e_{x_k}) + e_{x_i}\star\phi_{e_{x_j}}(e_{x_k}) \\
&- e_{x_j}\star\phi_{e_{x_i}}(e_{x_k}) - [\phi_{e_{x_i}}(e_{x_j}) - \phi_{e_{x_j}}(e_{x_i})]\star e_{x_k} = 0.
\end{aligned} \tag{3.66}
$$

The relation (3.66) can be rewritten in an operator form as:

$$
\left[\phi_{e_{x_i}}, L_{e_{x_j}}\right] - \left[\phi_{e_{x_j}}, L_{e_{x_i}}\right] = L_{\phi_{e_{x_i}}(e_{x_j}) - \phi_{e_{x_j}}(e_{x_i}) + \phi_{[e_{x_i},e_{x_j}]}}. \tag{3.67}
$$

Suppose now that $\phi_{e_{x_i}}$ is of a special form:

$$
\phi_{e_{x_i}} = L_{\rho(e_{x_i})}, \ \ \rho \in \mathrm{End}(\mathscr{A}), \tag{3.68}
$$

with some operator $\rho : \mathscr{A} \to \mathscr{A}$. Since for LSAs,

$$
[L_u, L_v] = L_{[u,v]}, \ \ \forall u, v \in \mathscr{A} \tag{3.69}
$$

the equation (3.66) becomes

$$
L_{E(e_{x_i},e_{x_j})} = 0 \tag{3.70}
$$

where

$$
\begin{aligned}
E(e_{x_i},e_{x_j}) &= [\rho(e_{x_i}), e_{x_j}] - [\rho(e_{x_j}), e_{x_i}] - \left(\rho(e_{x_i})\star e_{x_j} - \rho(e_{x_j})\star e_{x_i}\right) - \rho\left([e_{x_i},e_{x_j}]\right) \\
&= -e_{x_j}\star\rho(e_{x_i}) + e_{x_i}\star\rho(e_{x_j}) - \rho\left(e_{x_i}\star e_{x_j} - e_{x_j}\star e_{x_i}\right)
\end{aligned} \tag{3.71}
$$

The $\rho-$compatibility equation then reads

$$
\begin{aligned}
E(e_{x_i},e_{x_j}) = 0 \Leftrightarrow e_{x_i}\star\rho(e_{x_j}) - e_{x_j}\star\rho(e_{x_i}) &= \rho\left(e_{x_i}\star e_{x_j} - e_{x_j}\star e_{x_i}\right) \\
&:= \rho\left[f(x_i,x_j)e_{x_i+x_j} - f(x_j,x_i)e_{x_i+x_j}\right],
\end{aligned} \tag{3.72}
$$

instead of the weaker deformation condition (3.70). The operator ρ is called a strong deformation.

For the particular case, when $\rho : e_{x_i} \longmapsto e_{x_i+x_0}$ for fixed $x_0 \in \mathbb{Z}$, the $\rho-$compatibility equation (3.72) turns out to be a simpler difference equation

$$\Big[f(x_i,x_j+x_0)-f(x_i,x_j)\Big]-\Big[f(x_j,x_i+x_0)-f(x_j,x_i)\Big]=0. \tag{3.73}$$

Define the right translation and exchange operators $\mathscr{T}_{x_0}$ and $\mathscr{E}$, respectively, by

$$\mathscr{T}_{x_0}f(x_i,x_j)=f(x_i,x_j+x_0),\ \mathscr{E}f(x_i,x_j)=f(x_j,x_i). \tag{3.74}$$

Then, the relation (3.73) reads

$$\mathscr{T}_{x_0}(1-\mathscr{E})f(x_i,x_j)=(1-\mathscr{E})f(x_i,x_j). \tag{3.75}$$

Therefore the following result holds.

Proposition 3.4. *Let $\rho : e_{x_i} \longmapsto e_{x_i+x_0}$ for some fixed $x_0 \in \mathbb{Z}$. Then the $\rho-$compatibility equation (3.72) is equivalent to the invariance of the operator $(1-\mathscr{E})$ under the action of the right translation operator $\mathscr{T}_{x_0}$, i.e.,*

$$\mathscr{T}_{x_0}(1-\mathscr{E})=(1-\mathscr{E}). \tag{3.76}$$

3.6. *Universal identity*

All nonassociative algebras naturally arising in connection with integrable systems satisfy a universal identity [35], i.e.,

$$[[e_{x_i},e_{x_j},e_{x_k}\star e_{x_s}]]-[[e_{x_i},e_{x_j},e_{x_k}]]\star e_{x_s}-e_{x_k}\star[[e_{x_i},e_{x_j},e_{x_s}]]=0, \tag{3.77}$$

where

$$[[x,y,z]]=(x,y,z)-(y,x,z). \tag{3.78}$$

Here we get

$$\begin{aligned}[[e_{x_i},e_{x_j},e_{x_k}\star e_{x_s}]] = f(x_k,x_s)\Big(&\Big[f(x_j,x_k+x_s)f(x_i,x_j+x_k+x_s)\\ &- f(x_i,x_k+x_s)f(x_j,x_i+x_k+x_s)\Big]\\ &- f(x_i+x_j,x_k+x_s)\Big[f(x_i,x_j)-f(x_j,x_i)\Big]\Big)\end{aligned} \tag{3.79}$$

$$\begin{aligned}[[e_{x_i},e_{x_j},e_{x_k}]]\star e_{x_s} = f(x_i+x_j+x_k,x_s)\Big(&f(x_i+x_j,x_k)\Big[f(x_j,x_i)-f(x_i,x_j)\Big]\\ &+ f(x_j,x_k)f(x_i,x_j+x_k)-f(x_i,x_k)f(x_j,x_i+x_k)\Big)\end{aligned} \tag{3.80}$$

$$\begin{aligned}e_{x_k}\star[[e_{x_i},e_{x_j},e_{x_s}]] = f(x_k,x_i+x_j+x_s)\Big(&f(x_i+x_j,x_s)\Big[f(x_j,x_i)-f(x_i,x_j)\Big]\\ &+ f(x_j,x_s)f(x_i,x_j+x_s)-f(x_i,x_s)f(x_j,x_i+x_s)\Big)=0.\end{aligned} \tag{3.81}$$

Therefore,

Proposition 3.5. *The universal identity (3.77) turns out to be of the following form for the generalized quasiassociative algebra defined with the multiplication (1.7):*

$$\begin{aligned} & f(x_k,x_j)\Big(-\mathscr{L}_{x_i}(x_j,x_k+x_j)(1-\mathscr{E})(f(x_i,x_j)+[[f(x_j,x_k+x_s),f(x_i,x_k+x_j)]]\Big) \\ & -\mathscr{L}_{x_i+x_j}f(x_k,x_s)\Big(-\mathscr{L}_{x_j}(x_i,x_k)(1-\mathscr{E})(f(x_i,x_j)+[[f(x_j,x_k),f(x_i,x_k)]]\Big) \\ & -\mathscr{T}_{x_i+x_j}f(x_k,x_j)\Big(-\mathscr{L}_{x_j}(x_i,x_j)(1-\mathscr{E})(f(x_i,x_j)+[[f(x_j,x_s),f(x_i,x_s)]]\Big)=0 \end{aligned} \tag{3.82}$$

where

$$[[f(x_j,x_l),f(x_i,x_l)]]=f(x_j,x_l)\mathscr{T}_{x_s}f(x_i,x_l)-f(x_i,x_l)\mathscr{T}_{x_i}f(x_j,x_l), \tag{3.83}$$

and $\mathscr{L}_u$, $\mathscr{T}_v$ are the usual left and right translation operators, respectively.

From the definition of the algebra product (1.7), the functions f can be regarded as the algebra structure constants. Therefore, the form (3.82) of the universal identity (3.77) could be linked to the integrability condition of the differential equations associated with the considered generalized algebra. However, at this stage of our study, such an assertion deserves further investigations.

3.7. *Phase space extension*

As known from [28], the category of LSAs is closed with respect to the operation of phase-space extension, unlike the smaller category of associative algebras: if $\mathscr{A}$ is LSA then so is $T^\star\mathscr{A}=\mathscr{A}\oplus\mathscr{A}^\star$:

$$\begin{pmatrix} e_{x_i} \\ e'_{x_i}\end{pmatrix}\star'\begin{pmatrix} e_{x_j} \\ e'_{x_j}\end{pmatrix}=\begin{pmatrix} e_{x_i}\star e_{x_j} \\ e_{x_i}\star e'_{x_j}-e_{x_j}\star e'_{x_i}\end{pmatrix},\ e_{x_i},e_{x_j}\in\mathscr{A},\ e'_{x_i},e'_{x_j}\in\mathscr{A}^\star, \tag{3.84}$$

where,

$$\langle e_{x_i}\star e'_{x_j},e_{x_k}\rangle=-\langle e'_{x_j},e_{x_i}\star e_{x_k}\rangle,\ \ e_{x_i},e_{x_k}\in\mathscr{A},\ e'_{x_j}\in\mathscr{A}^\star. \tag{3.85}$$

The integrability of the hydrodynamical systems of the type [28]:

$$u_t=\rho u_{e_{x_i}}+u_{e_{x_i}}\star u \tag{3.86}$$

is preserved under such phase-space extensions. In (3.86), ρ is the operator whose matrix elements are $\rho^{x_i}_{x_j}$:

$$\rho(e_{x_j})=\sum_{x_i}\rho^{x_i}_{x_j}e_{x_i}, \tag{3.87}$$

$\{e_{x_i}\}$ being a basis of $\mathscr{A}$, whose structure constants are $C^{x_i}_{x_jx_k}$:

$$e_{x_j}\star e_{x_k}=\sum_{x_i}C^{x_i}_{x_jx_k}e_{x_i}. \tag{3.88}$$

Assume ρ satisfies the following $\rho-$ compatibility equation (3.72), i. e.,

$$
\begin{aligned}
e_{x_i}\star\rho(e_{x_j})-e_{x_j}\star\rho(e_{x_i}) &= \rho\Big(e_{x_i}\star e_{x_j}-e_{x_j}\star e_{x_i}\Big)\\
&:= \rho\Big[f(x_i,x_j)e_{x_i+x_j}-f(x_j,x_i)e_{x_i+x_j}\Big]
\end{aligned}
\tag{3.89}
$$

By formulae (3.61),(3.65),(3.68), we realize a new left-symmetric multiplication in the following way:

$$
e_{x_i}\star' e_{x_j} := \Big(e_{x_i}+\varepsilon\rho(e_{x_i})\Big)\star e_{x_j},\ \varepsilon^2=0 \tag{3.90}
$$

corresponding to the phase-space extension

$$
\begin{pmatrix} e_{x_i}\\ e'_{x_i}\end{pmatrix}\star'_1\begin{pmatrix} e_{x_j}\\ e'_{x_j}\end{pmatrix}=\begin{pmatrix} e_{x_i}\star e_{x_j}\\ e_{x_i}\star e'_{x_j}-e_{x_j}\star e'_{x_i}\end{pmatrix},\ e_{x_i},e_{x_j}\in\mathscr{A},\ e'_{x_i},e'_{x_j}\in\mathscr{A}^\star, \tag{3.91}
$$

where

$$
\begin{aligned}
\langle e_{x_i}\star' e'_{x_j},e_{x_k}\rangle &= -\langle e'_{x_j},e_{x_i}\star' e_{x_k}\rangle\\
&= -\langle e'_{x_j},\Big(e_{x_i}+\varepsilon\rho(e_{x_i})\Big)\star e_{x_k}\rangle\\
&= \langle\Big(e_{x_i}+\varepsilon\rho(e_{x_i})\Big)\star e'_{x_j},\star e_{x_k}\rangle
\end{aligned}
\tag{3.92}
$$

implying

$$
e_{x_i}\star' e'_{x_j}=\Big(e_{x_i}+\varepsilon\rho(e_{x_i})\Big)\star e'_{x_j}. \tag{3.93}
$$

Provided the former relation and taking into account the natural extension : $\mathscr{A}\to T^\star\mathscr{A}=\mathscr{A}\oplus\mathscr{A}^\star$: $\rho\mapsto\rho_1$ should satisfy:

$$
\begin{pmatrix} e_{x_i}\\ e'_{x_i}\end{pmatrix}\star'_1\begin{pmatrix} e_{x_j}\\ e'_{x_j}\end{pmatrix}=\left(\begin{pmatrix} e_{x_i}\\ e'_{x_i}\end{pmatrix}+\varepsilon\rho_1\begin{pmatrix} e_{x_i}\\ e'_{x_i}\end{pmatrix}\right)\star'_1\begin{pmatrix} e_{x_j}\\ e'_{x_j}\end{pmatrix}, \tag{3.94}
$$

we are in right to postulate the following choice:

Proposition 3.6.

$$
\rho_1:\begin{pmatrix} e_{x_i}\\ e'_{x_i}\end{pmatrix}\mapsto\begin{pmatrix}\rho(e_{x_i})\\ 0\end{pmatrix} \tag{3.95}
$$

satisfies the $\rho-$compatibility equation (3.72) in $T^\star\mathscr{A}$.

Proof. We have:

$$\rho_1\left(\begin{pmatrix} e_{x_i} \\ e'_{x_i} \end{pmatrix} \star_1 \begin{pmatrix} e_{x_j} \\ e'_{x_j} \end{pmatrix} - \begin{pmatrix} e_{x_j} \\ e'_{x_j} \end{pmatrix} \star_1 \begin{pmatrix} e_{x_i} \\ e'_{x_i} \end{pmatrix}\right) = \rho_1 \begin{pmatrix} e_{x_i} \star e_{x_j} - e_{x_j} \star e_{x_i} \\ e_{x_i} \star e'_{x_j} - e_{x_j} \star e'_{x_i} \end{pmatrix}$$
$$= \begin{pmatrix} \rho(e_{x_i} \star e_{x_j} - e_{x_j} \star e_{x_i}) \\ 0 \end{pmatrix} \tag{3.96}$$

$$\begin{pmatrix} e_{x_i} \\ e'_{x_i} \end{pmatrix} \star_1 \rho_1 \begin{pmatrix} e_{x_j} \\ e'_{x_j} \end{pmatrix} - \begin{pmatrix} e_{x_j} \\ e'_{x_j} \end{pmatrix} \star_1 \rho_1 \begin{pmatrix} e_{x_i} \\ e'_{x_i} \end{pmatrix} = \begin{pmatrix} e_{x_i} \\ e'_{x_i} \end{pmatrix} \star_1 \begin{pmatrix} \rho(e_{x_j}) \\ 0 \end{pmatrix}$$
$$- \begin{pmatrix} e_{x_j} \\ e'_{x_j} \end{pmatrix} \star_1 \begin{pmatrix} \rho(e_{x_i}) \\ 0 \end{pmatrix} = \begin{pmatrix} e_{x_i} \star \rho(e_{x_j}) - e_{x_j} \star \rho(e_{x_i}), \\ 0 \end{pmatrix}. \tag{3.97}$$

Therefore, the $\rho-$compatibility equation (3.72) also holds in $T^\star \mathscr{A}$. $\square$

When $\rho : e_{x_i} \mapsto e_{x_i+x_0}$ for some fixed x_0, the choice (3.95) reduces to a simpler relation:

$$\rho_1 : \begin{pmatrix} e_{x_i} \\ e'_{x_i} \end{pmatrix} \mapsto \begin{pmatrix} e_{x_i+x_0} \\ 0 \end{pmatrix}. \tag{3.98}$$

We infer the following result.

Proposition 3.7. *Let $\rho : e_{x_i} \mapsto e_{x_i+x_0}$. Then the relations (3.96) and (3.97) reduced to*

$$\begin{pmatrix} \Big[f(x_i,x_j) - f(x_j,x_i)\Big] e_{x_i+x_j+x_0} \\ 0 \end{pmatrix} = \begin{pmatrix} \Big[f(x_i,x_j+x_0) - f(x_j,x_i+x_0)\Big] e_{x_i+x_j+x_0} \\ 0 \end{pmatrix}, \tag{3.99}$$

which is equivalent to

$$\begin{pmatrix} \Big[1 - \mathscr{E} + \mathscr{T}_{x_0}(\mathscr{E} - 1)\Big] f(x_i,x_j) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.100}$$

giving

$$\mathscr{T}_{x_0}(1 - \mathscr{E}) = (1 - \mathscr{E}). \tag{3.101}$$

We thus recover the $\rho-$compatibility condition (3.76).
The phase-space extension of the system (3.86) reads:

$$\begin{pmatrix} u \\ u' \end{pmatrix}_t = \rho_1 \begin{pmatrix} u_{e_{x_i}} \\ u'_{e_{x_i}} \end{pmatrix} + \begin{pmatrix} u_{e_{x_i}} \\ u'_{e_{x_i}} \end{pmatrix} \star_1 \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} \rho(u_{e_{x_i}}) + u_{e_{x_i}} \star u \\ u_{e_{x_i}} u' \end{pmatrix}. \tag{3.102}$$

It is also worth noticing that, in the case of the left-symmetric double [28]: $\mathscr{A}^d = \mathscr{A} \oplus \mathscr{A}$, we get

$$\begin{pmatrix} e_{x_i} \\ e_{x_a} \end{pmatrix} \star_2 \begin{pmatrix} e_{x_j} \\ e_{x_b} \end{pmatrix} = \begin{pmatrix} e_{x_i} \star e_{x_j} \\ e_{x_i} \star e_{x_b} - e_{x_j} \star e_{x_a} \end{pmatrix} = \begin{pmatrix} f(x_i,x_j) e_{x_i+x_j} \\ f(x_i,x_b) e_{x_i+x_b} - f(x_j,x_a) e_{x_j+x_a} \end{pmatrix} \tag{3.103}$$

and if $\rho : \mathscr{A} \to \mathscr{A}$ is an operator of strong deformation, then so is $\rho_2 : \mathscr{A}^d \to \mathscr{A}^d$,

$$\rho_2\begin{pmatrix} e_{x_i} \\ e_{x_a} \end{pmatrix} = \begin{pmatrix} \rho(e_{x_i}) + e_{x_g} e_{x_a} \\ e_{x_\mu} e_{x_i} + e_{x_\nu} e_{x_a} \end{pmatrix}, \quad e_{x_g}, e_{x_\mu}, e_{x_\nu} = \text{constants} \tag{3.104}$$

yielding

$$\rho_2\begin{pmatrix} e_{x_i} \\ e_{x_a} \end{pmatrix} = \begin{pmatrix} e_{x_i+x_0} + e_{x_g} e_{x_a} \\ e_{x_\mu} e_{x_i} + e_{x_\nu} e_{x_a} \end{pmatrix}, \quad \text{when } \rho(e_{x_i}) := e_{x_i+x_0}. \tag{3.105}$$

Furthermore, as claimed in [28], the formula (3.90) shows that, in addition to the integrable hydrodynamic hierarchy starting with the equation (3.86), i.e.,

$$u_t = \rho u_{e_{x_i}} + u_{e_{x_i}} \star u \tag{3.106}$$

we have a second hierarchy, starting with the equation

$$u_t = u_{e_{x_i}} \star' u = u_{e_{x_i}} \star u + \varepsilon \rho(u_{e_{x_i}}) \star u, \;\; \varepsilon^2 = 0. \tag{3.107}$$

4. Centrally extended Virasoro algebra

In this section we aim at investigating the quasi-associativity condition, the 3−ary bracket and the fundamental identity for the above generalized algebra in the particular case when $a = b = 1$ and

$$f(x_i, x_j) = -\frac{x_j(1+\varepsilon x_j)}{1+\varepsilon(x_i+x_j)} + \frac{1}{2}\theta\left[{x_i}^3 - x_i + (\varepsilon - \varepsilon^{-1}){x_i}^2\right]\delta^0_{x_i+x_j}, \tag{4.1}$$

$$g(x_i, x_j) = (x_i - x_j) + \theta({x_i}^3 - x_i)\delta_{x_i+x_j} \tag{4.2}$$

This algebra corresponds to the Virasoro algebra, also called central extension of the Witt algebra with the multiplication:

$$\begin{aligned} [e_{x_i}, e_{x_j}] &= g(x_i, x_j) e_{x_i+x_j}, \\ e_{x_i} \star \theta &= \theta \star e_{x_i} = 0 \end{aligned} \tag{4.3}$$

coming from the commutator

$$\begin{aligned} [e_{x_i}, e_{x_j}] &= e_{x_i} \star e_{x_j} - e_{x_j} \star e_{x_i} \\ e_{x_i} \star e_{x_j} &= f(x_i, x_j) e_{x_i+x_j} \end{aligned} \tag{4.4}$$

where $(x_i, x_j) \in \mathbb{Z}^2$.

Proposition 4.1. *For the centrally extended Virasoro algebra,*

(i) The skew-symmetry condition (3.2) is equivalent to the system of equations

$$\begin{cases} x_i + x_j + \varepsilon\left(x_i^2 + x_j^2\right) = 0 \\ x_i^3 + x_j^3 - (x_i + x_j) + \left(\varepsilon - \varepsilon^{-1}\right)\left(x_i^2 + x_j^2\right) = 0; \end{cases} \tag{4.5}$$

(ii) The Jacobi identity

$$\Big[[e_{x_i}, e_{x_j}], e_{x_k}\Big] + \Big[[e_{x_j}, e_{x_k}], e_{x_i}\Big] + \Big[[e_{x_k}, e_{x_i}], e_{x_j}\Big] = 0 \tag{4.6}$$

is identified to the condition

$$\mathbb{J}^k_{ij} + \mathbb{J}^i_{jk} + \mathbb{J}^j_{ki} = 0 \tag{4.7}$$

where

$$\begin{aligned}\mathbb{J}^k_{ij} := \Big[(x_i - x_j) + \theta(x_i{}^3 - x_i)\delta_{x_i+x_j}\Big]\Big[&(x_i + x_j - x_k) \\ &+ \theta\Big((x_i + x_j)^3 - (x_i + x_j)\Big)\delta_{x_i+x_j+x_k}\Big];\end{aligned} \tag{4.8}$$

(iii) The derivation property, i.e.,

$$[e_{x_i}, e_{x_j} \star e_{x_k}] := e_{x_j} \star [e_{x_i}, e_{x_k}] + [e_{x_i}, e_{x_j}] \star e_{x_k} \tag{4.9}$$

leads to

$$\begin{aligned}&-\frac{x_k(1+\varepsilon x_k)}{1+\varepsilon(x_j+x_k)}(x_i - (x_j + x_k))e_{x_i+x_j+x_k} + \theta(x_i{}^3 - x_i)\delta^0_{x_i+x_j+x_k} = \frac{1}{1+\varepsilon(x_i+x_j+x_k)} \\ &\times \Big[-(x_i^2 - x_k^2) - x_k(x_i - x_j) - \varepsilon[(x_i^2 - x_k^2)(x_i + x_k) + x_k^2(x_i - x_j)]\Big]e_{x_i+x_j+x_k} \\ &+ \frac{1}{2}\theta\Big[(x_j{}^3 - x_j)(x_i - x_k) + (x_i - x_j)[(x_i + x_j)^3 - (x_i + x_j)] \\ &+ (\varepsilon - \varepsilon^{-1})[x_j{}^2(x_i - x_k) + (x_i - x_j)(x_i + x_j)^2]\Big]\delta^0_{x_i+x_j+x_k}.\end{aligned} \tag{4.10}$$

4.1. *Quasi-associativity condition*

We answer the question: Does it exist a necessary and sufficient condition for this algebra to be a quasi-associative algebra with the multiplication

$$\begin{aligned} e_{x_i} \star e_{x_j} &= -\frac{x_j(1+\varepsilon x_j)}{1+\varepsilon(x_i+x_j)}e_{x_i+x_j} + \frac{1}{2}\theta\Big[x_i{}^3 - x_i + (\varepsilon - \varepsilon^{-1})x_i{}^2\Big]\delta^0_{x_i+x_j} \\ e_{x_i} \star \theta &= \theta \star e_{x_i} = 0? \end{aligned} \tag{4.11}$$

Theorem 4.1. *The algebra defined with the multiplication rule (4.11) is neither associative, nor left-symmetric.*

Proof. By direct computation, we find:

$$\begin{aligned} &e_{x_i} \star (e_{x_j} \star e_{x_k}) - (e_{x_i} \star e_{x_j}) \star e_{x_k} \\ &= \frac{x_k(1+\varepsilon x_k)}{1+\varepsilon(x_i+x_j+x_k)}\Big(\frac{x_k + \varepsilon\Big[(x_i+x_j)(x_j+x_k) - x_j^2\Big]}{1+\varepsilon(x_i+x_j)}\Big)e_{x_i+x_j+x_k} \\ &- \frac{\theta}{2}\Big(\frac{x_k(1+\varepsilon x_k)}{1+\varepsilon(x_j+x_k)}\Big[x_i{}^3 - x_i + (\varepsilon - \varepsilon^{-1})x_i{}^2\Big] \\ &- \frac{x_j(1+\varepsilon x_j)}{1+\varepsilon(x_i+x_j)}\Big[(x_i+x_j)^3 - (x_i+x_j) + (\varepsilon - \varepsilon^{-1})(x_i+x_j)^2\Big]\Big)\delta^0_{x_i+x_j+x_k} \end{aligned} \tag{4.12}$$

while

$$
\begin{aligned}
&e_{x_j}\star(e_{x_i}\star e_{x_k})-(e_{x_j}\star e_{x_i})\star e_{x_k}\\
=&\ \frac{x_k(1+\varepsilon x_k)}{1+\varepsilon(x_i+x_j+x_k)}\Big(\frac{x_k+\varepsilon\Big[(x_i+x_k)(x_i+x_j)-x_i^2\Big]}{1+\varepsilon(x_i+x_j)}\Big)e_{x_i+x_j+x_k}\\
-&\ \frac{\theta}{2}\Big(\frac{x_k(1+\varepsilon x_k)}{1+\varepsilon(x_i+x_k)}\Big[{x_j}^3-x_j+(\varepsilon-\varepsilon^{-1}){x_j}^2\Big]\\
-&\ \frac{x_i(1+\varepsilon x_i)}{1+\varepsilon(x_i+x_j)}\Big[(x_i+x_j)^3-(x_i+x_j)+(\varepsilon-\varepsilon^{-1})(x_i+x_j)^2\Big]\Big)\delta^0_{x_i+x_j+x_k}\\
\neq&\ e_{x_i}\star(e_{x_j}\star e_{x_k})-(e_{x_i}\star e_{x_j})\star e_{x_k}.
\end{aligned}
\tag{4.13}
$$

□

It is worth noticing that for $i=j$, this algebra becomes a left-alternative algebra as required by the general formalism developed in the previous section.

4.2. $3-$*ary bracket and fundamental identity*

The $3-$ary bracket, defined by the relation (3.40), i.e.,

$$
[e_{x_i},e_{x_j},e_{x_k}]:=e_{x_i}\star[e_{x_j},e_{x_k}]+e_{x_j}\star[e_{x_k},e_{x_i}]+e_{x_k}\star[e_{x_i},e_{x_j}]
$$

leads to the expression

$$
\begin{aligned}
[e_{x_i},e_{x_j},e_{x_k}]=&-\frac{1}{1+\varepsilon(x_i+x_j+x_k)}\varepsilon\Big[(x_j^2-x_i^2)x_k+(x_i^2-x_k^2)x_j+(x_k^2-x_j^2)x_i\Big]e_{x_i+x_j+x_k}\\
&+\frac{\theta}{2}\Big[({x_i}^3+{x_j}^3+{x_k}^3)(1+\varepsilon-\varepsilon^{-1})-(x_i+x_j+x_k)\Big]\delta^0_{x_i+x_j+x_k}.
\end{aligned}
\tag{4.14}
$$

Defining the fundamental identity, (also called Filippov identity), in this case as:

$$
\begin{aligned}
\Big[x_i,e_{x_j},[e_{x_k},e_{x_l},e_{x_m}]\Big]:=&\Big[[x_i,e_{x_j},e_{x_k}],e_{x_l},e_{x_m}\Big]+\Big[x_k,[e_{x_i},e_{x_j},e_{x_l}],e_{x_m}\Big]\\
&+\Big[x_k,e_{x_l},[e_{x_i},e_{x_j},e_{x_m}]\Big],
\end{aligned}
\tag{4.15}
$$

implies cumbersome functional equations:

$$
\begin{aligned}
&\frac{\Big[(x_l^2-x_k^2)x_m+(x_k^2-x_m^2)x_l+(x_m^2-x_l^2)x_k\Big]}{1+\varepsilon(x_k+x_l+x_m)}\Big[(x_j^2-x_i^2)(x_k+x_l+x_m)\\
+&\Big(x_i^2-(x_k+x_l+x_m)^2\Big)x_j+\Big((x_k+x_l+x_m)^2-x_j^2\Big)x_i\Big]\\
=&\frac{\Big[(x_j^2-x_i^2)x_k+(x_i^2-x_k^2)x_j+(x_k^2-x_j^2)x_i\Big]}{1+\varepsilon(x_i+x_j+x_k)}\Big[(x_l^2-(x_i+x_j+x_k)^2)x_m\\
+&\Big((x_i+x_j+x_k)^2-x_m^2\Big)x_l+(x_m^2-x_l^2)(x_i+x_j+x_k)\Big]\\
+&\frac{\Big[(x_j^2-x_i^2)x_l+(x_i^2-x_l^2)x_j+(x_l^2-x_j^2)x_i\Big]}{1+\varepsilon(x_i+x_j+x_l)}\Big[\Big((x_i+x_j+x_l)^2-x_k^2\Big)x_m\\
+&(x_k^2-x_m^2)(x_i+x_j+x_l)+(x_m^2-(x_i+x_j+x_l)^2)x_k\Big]
\end{aligned}
$$

$$+\frac{\left[(x_j^2-x_i^2)x_m+(x_i^2-x_m^2)x_j+(x_m^2-x_j^2)x_i\right]}{1+\varepsilon(x_i+x_j+x_m)}\Big[(x_l^2-x_k^2)(x_i+x_j+x_m)$$
$$+(x_k^2-(x_i+x_j+x_m)^2)x_l+\Big((x_j+x_j+x_m)^2-x_l^2\Big)x_k\Big] \tag{4.16}$$

and

$$\Big[(x_i+x_j+x_k)^3+(x_j+x_k+x_l)^3+(x_i+x_j+x_m)^3-(x_k+x_l+x_m)^3$$
$$+2(x_k^3+x_l^3+x_m^3)-(x_i^3+x_j^3)\Big](1+\varepsilon+\varepsilon^{-1})-2(x_i+x_j+x_k+x_l+x_m)=0. \tag{4.17}$$

These functional equations reduce to the simpler relation (4.17) when $\varepsilon^2=0$.

5. $\mathbb{L}_k-$ infinite dimensional Lie algebra of polynomial vector fields on the real line $\mathbb{R}^1$

Consider the algebra $\mathbb{L}_k-$ as the infinite dimensional Lie algebra of polynomial vector fields on the real line $\mathbb{R}^1$. Let us define this algebra by the infinite basis $\{e_i\}$:

$$e_i=x^{i+1}\frac{d}{dx},\quad i\in\mathbb{N} \tag{5.1}$$

with the commutator

$$[e_i,e_j]=e_i\star e_j-e_j\star e_i=(j-i)e_{i+j} \tag{5.2}$$

with the multiplication

$$e_p\star e_q=(q+1)e_{p+q}+e_{p+q+1}\frac{d}{dx}. \tag{5.3}$$

Here the $\star-$ multiplication is nothing but the ordinary operators product.

One can easily prove by direct computation that this algebra endowed with the product (5.3) is an associative algebra, i.e., its associator is equal to zero. Indeed,

$$\begin{aligned}e_p\star(e_q\star e_r)&=(e_p\star e_q)\star e_r)\\&=(r+1)(q+r+1)e_{p+q+r}+(q+2r+3)e_{p+q+r+1}\frac{d}{dx}+e_{p+q+r+2}\frac{d^2}{dx^2}.\end{aligned} \tag{5.4}$$

Further the corresponding Nambu brackets are null, i.e.,

$$[e_p,e_q,e_r]:=e_p\star[e_q,e_r]+e_q\star[e_r,e_p]+e_r\star[e_p,e_q]=0, \tag{5.5}$$

that is the Jacobi identity is automatically satisfied. Thus we have a null $3-$ algebra for an infinite set of non-trivial noncommuting oscillator charges. The Filippov condition is trivially satisfied in this case, i.e.,

$$\Big[e_p, e_q, [e_r, e_s, e_t]\Big] := \Big[[e_p, e_q, e_r], e_s, e_t\Big] + \Big[e_r, [e_p, e_q, e_s], e_t\Big] + \Big[e_r, e_s, [e_p, e_q, e_t]\Big]. \tag{5.6}$$

The Bremner operator, also called the associative operator, of course, perfectly works, i.e.,

$$\Big[\Big[e_p, [e_q, e_r, e_s], e_t\Big], e_u, e_v\Big] := \Big[[e_p, e_q, e_r], [e_s, e_t, e_u], e_v\Big] \tag{5.7}$$

as a consequence of the associativity. Moreover, the skew-symmetry property is obeyed by the definition of the product, i.e.,

$$[e_i, e_j] = e_i \star e_j - e_j \star e_i = -[e_j, e_i] = -(j-i)e_{i+j}. \tag{5.8}$$

Provided the Jacobi identity and skew-symmetry properties are satisfied, this algebra is made into a Lie algebra structure. On the other hand,

$$[e_p, e_q \star e_r] = \Big[(r+1)(q+r+1) - (p+1)(p+r+1)\Big] e_{p+q+r} + (q+r-2p)e_{p+q+r+1}\frac{d}{dx}, \tag{5.9}$$

$$e_q \star [e_p, e_r] = (p+r+1)(r-p)e_{p+q+r} + (r-p)e_{p+q+r+1}\frac{d}{dx}, \tag{5.10}$$

$$[e_p, e_q] \star e_q = (r+1)(q-p)e_{p+q+r} + (q-p)e_{p+q+r+1}\frac{d}{dx}. \tag{5.11}$$

Summing the relations (5.10) and (5.11), we find that

$$[e_p, e_q \star e_r] = e_q \star [e_p, e_r] + [e_p, e_q] \star e_q, \tag{5.12}$$

showing that the derivation property is satisfied, what makes $(\mathbb{L}_k, [.,])$ into a Poisson structure. These properties induce the following consequences:

Proposition 5.1. *Let L_{e_q} and R_{e_q} be the left and right multiplication operators by e_q, (for some fixed $e_q \in \mathbb{L}_k$), defined, respectively, as:*

$$L_{e_q}(e_p) = e_q \star e_p, \;\; R_{e_q}(e_p) = e_p \star e_q, \forall e_p \in \mathbb{L}_k. \tag{5.13}$$

Then the following relations hold, for all $e_p, e_q \in \mathbb{L}_k$:

- $$[L_{e_p}, L_{e_q}] = L_{[e_p, e_q]} \tag{5.14}$$

- $$R_{[e_p, e_q]}(.) = (.) \star [e_p, e_q] = -(.) \star [R_{e_p}, R_{e_q}] \tag{5.15}$$

- $$[L_{e_p}, R_{e_q}](.) = e_p \star [(.), e_q] = 0 \tag{5.16}$$

- $$[R_{e_p}, e_q](.) = e_q \star [e_p, (.)] \tag{5.17}$$

- $$[R_{e_p} R_{e_q} + R_{e_p \star e_q}](.) = (.) \star [R_{e_p}(e_q) + R_{e_q}(e_p)]. \tag{5.18}$$

Let us mention an interesting identity of general interest:

$$\Big[e_p, [e_q, e_r]\Big] + \Big[e_r, [e_p, e_q]\Big] = e_q \star [e_p, e_r] - [e_p, e_r] \star e_q, \tag{5.19}$$

valid whatever the $\star-$product, with the usual commutator.

6. Concluding remarks

In this paper, we have discussed the appearence of left symmetric algebras in a generalized Virasoro algebra. We have provided the necessary and sufficient condition for this algebra to be a quasiassociative algebra. The criteria of skew-symmetry, derivation and Jacobi identity making this algebra a Lie algebra have been derived. Coboundary operators are defined, the $2-$cocycle and coboundary are discussed. We have deduced the hereditary operator and its generalization to the corresponding $3-$ary bracket. Further, we have derived the so-called $\rho-$compatibility equation, and performed a phase-space extension. Concrete relevant particular cases have also been investigated and discussed.

This study brings some interesting questions to light which merit a separate in-depth treatment. For instance, new examples of nonlinear systems associated with the considered generalization of the Virasoro algebra may exist, but their full investigation remains totally open. Besides, a detailed analysis of the main properties of $3-$algebras on the basis of definition (3.42) or (3.44) is also of great importance. These topics will be the core of our forthcoming works.

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