Weighted local Weyl laws for elliptic operators

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Abstract

Let A be an elliptic pseudo-differential operator of order m on a closed manifold $\mathcal{X}$ of dimension $n > 0$, formally positive self-ajdoint with respect to some positive smooth density $d\mu_{\mathcal{X}}$. Then, the spectrum of A is made up of a sequence of eigenvalues $(\lambda_k)_{k\geq 1}$ whose corresponding eigenfunctions $(e_k)_{k\geq 1}$ are C^∞ smooth. Fix $s \in \mathbb{R}$ and define

$$K_L^s(x,y) = \sum_{0<\lambda_k\leq L} \lambda_k^{-s} e_k(x)\overline{e_k(y)}\,.$$

We derive asymptotic formulae near the diagonal for the kernels $K_L^s(x,y)$ when $L \to +\infty$ with fixed s. For $s = 0$, K_L^0 is the kernel of the spectral projector studied by Hörmander in [Hör68]. In the present work we build on Hörmander's result to study the kernels K_L^s. If $s < \frac{n}{m}$, K_L^s is of order $L^{-s+n/m}$ and near the diagonal, the rescaled leading term behaves like the Fourier transform of an explicit function of the symbol of A. If $s = \frac{n}{m}$, under some explicit generic condition on the principal symbol of A, which holds if A is a differential operator, the kernel has order $\ln(L)$ and the leading term has a logarithmic divergence smoothed at scale $L^{-1/m}$. Our results also hold for elliptic differential Dirichlet eigenvalue problems.

Contents

1 Introduction

The purpose of the present work is to compute pointwise asymptotics of the integral kernels of certain operators defined by functional calculus from either elliptic self-adjoint pseudo-differential operators on a closed manifold or an elliptic self-adjoint Dirichlet boundary problem. Stating the results in full generality requires some vocabulary from semi-classical analysis and some additional definitions. For this reason, we start by stating our results in the simpler case of elliptic self-adjoint differential operators on a closed manifold. The general case is presented in Section 2. This, of course, leads to some redundancy between different statements which we accept for the sake of accessibility and transparency of the main results.

Let $\mathcal{X}$ be a smooth compact manifold without boundary, of positive dimension $n > 0$ and equipped with a smooth positive density $d\mu_{\mathcal{X}}$. Let A be a positive elliptic differential operator on $\mathcal{X}$ of positive order m. By this we mean that in any local coordinate system $x = (x_1, \ldots, x_n)$ on $\mathcal{X}$ defined on $U \subset \mathbb{R}^n$, A, acts on $C_c^\infty(U)$ as

$$\sum_{0\leq|\alpha|\leq m} a_\alpha(x)\partial^\alpha$$

where $\alpha \in \mathbb{N}^n$ and $a_\alpha \in C^\infty(\mathbb{R}^n)$ and for each $\xi \in \mathbb{R}^n \setminus \{0\}$, we have

$$\sigma_A(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha > 0\,.$$

The function σ_A is called the **principal symbol** of A in these coordinates. It is well known (and easy to check) that the principal symbol of A read in different coordinates pieces together as a smooth function on the complement of the zero section in $T^*\mathcal{X}$. We assume that A is symmetric with respect to the L^2-scalar product on $(\mathcal{X}, d\mu_{\mathcal{X}})$. Then one can show (see Subsection 2.1) that A has a unique self-adjoint extension whose spectrum is made up of a non-decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive eigenvalues diverging to $+\infty$ with smooth L^2-normalized eigenfunctions $(e_k)_{k\in\mathbb{N}}$ forming a Hilbert basis for $L^2(\mathcal{X}, d\mu_{\mathcal{X}})$. For each $L \geq 0$, let Π_L be the L^2 orthogonal projector on the space spanned by the eigenfunctions e_k such that $\lambda_k \leq L$. Since this space is finite-dimensional, Π_L has a smooth integral kernel $E_L \in C^\infty(\mathcal{X}\times\mathcal{X})$. More explicitely,

$$\forall (x,y) \in \mathcal{X}\times\mathcal{X},\; E_L(x,y) = \sum_{\lambda_k\leq L} e_k(x)\overline{e_k(y)}\,.$$

In [Hör68], Hörmander studied the behavior of this kernel on a neighborhood of the diagonal as $L \to +\infty$. Integrating E_L over the diagonal he recovered the following estimate, also known as **Weyl's law**:

$$\mathrm{Card}\{k \in \mathbb{N} \mid \lambda_k \leq L\} \sim \frac{1}{(2\pi)^n}\int_{\mathcal{X}}\int_{\sigma_A(x,\xi)\leq 1} \widetilde{d_x\mu_{\mathcal{X}}}(\xi)d\mu_{\mathcal{X}}(x) \times L^{\frac{n}{m}}$$

where $\widetilde{d_x\mu_{\mathcal{X}}}(\xi)$ is the density induced on $T_x^*\mathcal{X}$ by $d\mu_{\mathcal{X}}$. Hörmander's result is stronger than the above estimates in two respects. First because the error term obtained is smaller than the ones known before and is sharp in all generality. Secondly, the result actually provides local information concerning the behavior of the kernel E_L near the diagonal, which is why is sometimes called the **local Weyl law**. We will state this theorem in Subsection 2.2 (see Theorem 2.2). In recent years, Hörmander's local Weyl law has received a lot of attention because E_L turns out to be the covariance of a certain Gaussian field on $\mathcal{X}$ defined as a random linear combination of eigenfunctions of A (see for instance [Bér85], [NS09], [Zel09], [GW14], [Let16], [GW16], [NR15], [HZZ15], [NS16],[SW16] and [CS17]). In [Riv17] we studied a natural variation of this random linear combination of eigenfunctions in dimension $n = 2$ and observed a very different asymptotic behavior of the covariance function. Following this work, we are interested in studying more general random linear combinations of these eigenfunctions. To this end, it is essential to gather some information about the corresponding covariance function. The purpose of this article is to provide an asymptotic for these kernels similar to the one we have for E_L. For each $s \in \mathbb{R}$ we consider the kernel

$$K_L^s(x,y) = \sum_{\lambda_k \le L} \lambda_k^{-s} e_k(x)\overline{e_k(y)}\,.$$

These kernels converge in distribution to the integral kernels of A^{-s} as $L \to +\infty$ but diverge on the diagonal for small or negative values of s. The pointwise behavior of the limiting kernel on the diagonal, which is well defined for large values of s, has been studied for instance in [See67] and [Sch86]. In [Sch86], the author proved that, as a function of s, the limit admitted a meromorphic extension to the whole complex plane. We focus instead on a fixed s for which the kernel diverges and study its pointwise divergence near the diagonal. We call these results **weighted local Weyl laws** by analogy with E_L (which is just K_L^0) because of the weights λ_n^{-s} on the terms of the sum defining K_L^s. As we shall see, the kernels K_L^s experience a sudden change in their asymptotic behavior between the phases $s < \frac{n}{m}$ and $s = \frac{n}{m}$. All our results will be local so we take the liberty of omitting with the composition with the chart when writing functions on $\mathcal{X}$ in local coordinates. Our first result provides information when $s < \frac{n}{m}$.

Theorem 1.1. *Assume that $s < \frac{n}{m}$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \dots, x_n)$ for $\mathcal{X}$ centered at x_0 and defined on an open subset $U \subset \mathbb{R}^n$ such that $d\mu_{\mathcal{X}}$ agrees with the Lebesgue measure in these coordinates. Then for each $\alpha, \beta \in \mathbb{N}^n$, there exists $V \subset U$ an open neighborhood of 0 such that, in these coordinates, we have the following estimates.*

1. *Uniformly for $w, x, y \in V$ and $L \ge 1$ we have $w + L^{-1/m}x, w + L^{-1/m}y \in U$ and*

$$L^{s-(n+|\alpha|+|\beta|)/m}\partial_x^\alpha \partial_y^\beta K_L^s\left(w + L^{-1/m}x, w + L^{-1/m}y\right) = \frac{1}{(2\pi)^n}\int_{\sigma_A(w,\xi)\le 1} e^{i\langle \xi, x-y\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{\sigma_A(w,\xi)^s}d\xi + O\left(L^{-1/m}\ln(L)^\eta\right)$$

   where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/m$ and 0 otherwise.

2. *Let $\varepsilon > 0$. Then, uniformly for $x, y \in V$ such that $|x - y| > \varepsilon$ and $L \ge 1$,*

$$L^{s-(n+|\alpha|+|\beta|)/m}\partial_x^\alpha \partial_y^\beta K_L^s(x,y) = O\left(L^{-1/m}\ln(L)^\eta\right)$$

   where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/m$ and 0 otherwise.

Here $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product.

Note that the case where $s = 0$ and $\alpha = \beta = 0$ is Theorem 5.1 of [Hör68] (see Theorem 2.2 and the discussion below for more details about this case). We prove Theorem 1.1 at the end of Section 2. Before stating the second result, we introduce the following notation. Firstly, $d\mu_{\mathcal{X}}$ defines a canonical dual density $\widetilde{d\mu_{\mathcal{X}}}$ on $T^*\mathcal{X}$. Around any $x \in \mathcal{X}$, there exist local coordinates in which $d\mu_{\mathcal{X}}$ corresponds to the Lebesgue measure. Then $\widetilde{d\mu_{\mathcal{X}}}$ is the unique density on $T^*\mathcal{X}$ who in these coordinates corresponds to the Lebesgue measure. For each $x \in \mathcal{X}$, let $S_x^* = \{\xi \in T_x^*\mathcal{X} \,|\, \sigma_A(x,\xi) = 1\}$. Since σ_A is m-homogeneous, S_x^* is a smooth compact hypersurface of $T_x^*\mathcal{X}$ strictly star-shaped[1] around the origin and there exists a smooth density $d_x\nu$ on S_x^* such that for each $u \in C_c^\infty(T_x^*\mathcal{X})$,

$$\int_{T_x^*\mathcal{X}} u(\xi)\widetilde{d_x\mu_{\mathcal{X}}}(\xi) = \int_0^{+\infty}\int_{S_x^*} u(t\xi)d_x\nu(\xi)t^{n-1}dt\,. \tag{1}$$

Our second result deals with the case where $s = \frac{n}{m}$. While Theorem 1.1 proves that the rate of growth of K_L^s does not depend on s for $s < n/m$, and that the main term depends continuously on s, the following result shows that this is not true for $s = n/m$. Indeed, while the first point is analogous to the results of Theorem 1.1 , the second point is quite different (and requires additional tools).

[1]More precisely, for each $\xi \in T_x^*\mathcal{X} \setminus \{0\}$, the ray $\{t\xi \,|\, t > 0\}$ intersects S^* exactly at $\sigma_A(x,\xi)^{-1/m}\xi$ and does so transversally.

Theorem 1.2. *Assume that* $s = \frac{n}{m}$*. Fix* $x_0 \in \mathcal{X}$ *and consider local coordinates* $x = (x_1, \dots, x_n)$ *for* $\mathcal{X}$ *centered at* x_0 *and defined on an open subset* $U \subset \mathbb{R}^n$ *such that* $d\mu_{\mathcal{X}}$ *agrees with the Lebesgue measure in these coordinates. Then, for each* $\alpha, \beta \in \mathbb{N}^n$*, there exists an open neighborhood* $V \subset U$ *of* 0 *such that the following holds.*

1. - *Assume that* $(\alpha, \beta) \neq (0,0)$*. In these coordinates, uniformly for* $w, x, y \in V$ *and* $L \geq 1$*,* $w + L^{-1/m}x, w + L^{-1/m}y \in U$ *and*

$$L^{-(|\alpha|+|\beta|)/m}\partial_x^\alpha \partial_y^\beta K_L^s\left(w + L^{-1/m}x, w + L^{-1/m}y\right) = \frac{1}{(2\pi)^n}\int_{\sigma_A(w,\xi)\leq 1} e^{i\langle \xi, x-y\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{\sigma_A(w,\xi)^{n/m}}d\xi + O\left(L^{-1/m}\ln(L)^\eta\right).$$

   where $\eta = 1$ *if* $1 = |\alpha| + |\beta|$ *and* 0 *otherwise.*

   - *Assume* $(\alpha, \beta) \neq (0,0)$ *and let* $\varepsilon > 0$*. Then, uniformly for* $x, y \in V$ *such that* $|x - y| > \varepsilon$*,*

$$L^{-(|\alpha|+|\beta|)/m}\partial_x^\alpha \partial_y^\beta K_L^s(x,y) = O\left(L^{-1/m}\ln(L)^\eta\right)$$

   where $\eta = 1$ *if* $1 = |\alpha| + |\beta|$ *and* 0 *otherwise.*

2. - *Uniformly for* $x, y \in V$ *and* $L \geq 1$*, in these coordinates,*

$$K_L^s(x,y) = g_A(x,y)\left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\right] + O(1)$$

   where

$$g_A(x,y) = \frac{1}{(2\pi)^n} \times \frac{\nu_x(S_x^*) + \nu_y(S_y^*)}{2}$$

   and $\ln_+(t) = \ln(t) \vee 0$*.*

   - *There exists a symmetric bounded function* $Q : U \times U \to \mathbb{R}$ *such that, uniformly for* $\kappa \geq 1$*,* $L \geq 1$ *and* $x, y \in V$ *such that* $|x - y| \geq \kappa L^{-1/m}$*, in these coordinates,*

$$K_L^s(x,y) = -g_A(x,y)\ln(|x-y|) + Q(x,y) + O\left(\kappa^{-1/k}\right)$$

   where, if $n = 1$ *then* $k = 1$ *and if* $n \geq 2$ *then* $k = m$*.*

Here $|\alpha| = \alpha_1 + \dots + \alpha_n$ *and* $\langle \cdot, \cdot \rangle$ *is the Euclidean scalar product.*

This theorem (especially the second point) generalizes Theorem 3 of [Riv17], which proved the second point in the case where $s = 1$, $\mathcal{X}$ was a closed surface (so $n = 2$) with a Riemmanian metric and A was the associated Laplacian (so $m = 2$). The main challenge in the extension comes from the need to apply a generalized stationary phase formula on the level sets of the symbol. In [Riv17], this is Proposition 23, where the traditional stationary phase formula applies directly. This general setting requires tools from singularity theory that are deployed in Section 7. The second point of Theorem 1.2 will follow from Theorem 2.5 below. As is apparent, in Figure 1, the proof of this result is more complex than that of the others. We prove Theorem 1.2 at the end of Section 2.

Corollary 1.3. *The Schwartz kernel* $K \in \mathcal{D}'(\mathcal{X} \times \mathcal{X})$ *of* $A^{-n/m}$ *belongs to* $L^1(\mathcal{X} \times \mathcal{X})$*. Moreover, for each smooth distance function* $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *on* $\mathcal{X}$ *there exists a bounded symmetric function* $Q_{A,d} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$*, smooth on the complement of the diagonal, such that, for any distinct* $x, y \in \mathcal{X}$*,*

$$K(x,y) = -g_A(x,y)\ln(d(x,y)) + Q_{A,d}(x,y).$$

We prove Corollary 1.3 at the end of Section 2.

1.1 An important example: the Laplacian

As explained above, this work is motivated by recent interest in the kernel K_L^0 as the covariance function of a Gaussian field. In further work, we wish to study certain Gaussian fields arising naturally in geometry and statistical mechanics with covariance K_L^s. One such field is the Gaussian Free Field, which is a central object in statistical mechanics today. In Corollaries 1.4 and 1.5 we detail our main results in this special case.

Let $(\mathcal{X}, g)$ be a closed Riemmanian manifold of dimension $n \geq 2$. Let $\Delta = -div \circ \nabla$ be the Laplace operator on $\mathcal{X}$ and let $|dV_g|$ be the Riemmanian volume density on $\mathcal{X}$. Then, Δ is an elliptic differential operator with principal symbol $\sigma(x,\xi) = g_x^{-1}(\xi,\xi)$ where g_x^{-1} is the scalar product induced on $T_x^*\mathcal{X}$ by g_x. Moreover, Δ is symmetric with respect to the L^2-scalar product induced by the density $|dV_g|$ on $\mathcal{X}$. Let $(\lambda_k)_{k\in\mathbb{N}}$ be the sequence of eigenvalues of Δ (counted with multiplicity) and arranged in increasing order. Let $(e_k)_{k\in\mathbb{N}}$ be a Hilbert basis of $L^2(\mathcal{X}, |dV_g|)$ made up of real valued functions, such that for each $k \in \mathbb{N}$, $\Delta e_k = \lambda_k e_k$. For each $L > 0$, each $s > 0$ and each $(x, y) \in \mathcal{X} \times \mathcal{X}$, let

$$K_L^s(x,y) = \sum_{0<\lambda_k\leq L} \lambda_k^{-1} e_k(x) e_k(y)\,.$$

Then, K_L^1 converges in distribution as $L \to +\infty$ to the Green function on $\mathcal{X}$ which is the (generalized) covariance function for the Gaussian Free Field (see for instance [She07]). We have the following results. In the case where $s < n/2$, K_L^s converges at scale $L^{-1/2}$ to a non-trivial function after rescaling by a polynomial factor.

Corollary 1.4. *Assume that $s < n/2$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \ldots, x_n)$ for $\mathcal{X}$ centered at x_0 such that $|dV_g|$ agrees with the Lebesgue measure in these coordinates. Then, for each $\alpha, \beta \in \mathbb{N}^n$ there exists $V \subset U$ an open neighborhood of 0 such that, in these coordinates, we have the following estimates.*

1. *Uniformly for $w, x, y \in V$ and $L \geq 1$ we have $w + L^{-1/2}x, w + L^{-1/2}y \in U$ and*

$$\partial_x^\alpha \partial_y^\beta K_L^s\left(w + L^{-1/2}x, w + L^{-1/2}y\right) = \frac{1}{(2\pi)^n}\int_{|\xi|_w^2\leq 1} e^{i\langle\xi,x-y\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{|\xi|_w^{2s}} d\xi L^{(n+|\alpha|+|\beta|-2s)/2} + O\left(L^{(n+|\alpha|+|\beta|-2s-1)/2}\ln(L)^\eta\right)$$

   where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/2$ and 0 otherwise. Here $|\xi|_w^2 = g_w^{-1}(\xi,\xi)$ and $d\xi$ is the Lebesgue measure.

2. *Let $\varepsilon > 0$. Then, uniformly for $x, y \in V$ such that $|x - y| > \varepsilon$ and for $L \geq 1$,*

$$\partial_x^\alpha \partial_y^\beta K_L^s\left(w + L^{-1/2}x, w + L^{-1/2}y\right) = O\left(L^{(n+|\alpha|+|\beta|-2s-1)/2}\ln(L)^\eta\right)$$

   where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/2$ and 0 otherwise.

Proof. This follows directly from Theorem 1.1 with $m = 2$, $s < n/2$, $A = \Delta$ and $\sigma_A(w,\xi) = |\xi|_w^2$. □

On the other hand, if $s = n/2$, although the derivatives of K_L^s also have non-trivial local limits at scale $L^{-1/2}$, K_L^s itself converges pointwise to a distribution with a logarithmic singularity on the diagonal. Note that when $s = 1$, the first part of the second point of Corollary 1.5 below yields Theorem 3 of [Riv17].

Corollary 1.5. *Assume that $s = n/2$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, x_2)$ for $\mathcal{X}$ centered at x_0 defined on an open subset $U \subset \mathbb{R}^2$ such that $|dV_g|$ agrees with the Lebesgue measure in these coordinates. Then, for each $\alpha, \beta \in \mathbb{N}^2$, there exists an open neighborhood $V \subset U$ of 0 such that the following holds.*

1. • *Assume that* $(\alpha,\beta)\neq(0,0)$. *In these coordinates, uniformly for* $w,x,y\in V$ *and* $L\geq 1$, *we have* $w+L^{-1/2}x, w+L^{-1/2}y\in U$ *and*

$$\partial_x^\alpha\partial_y^\beta K_L^s\left(w+L^{-1/2}x,w+L^{-1/2}y\right)=\frac{1}{(2\pi)^n}\int_{|\xi|_w^2\leq 1}e^{i\langle\xi,x-y\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{|\xi|_w^n}d\xi L^{(n+|\alpha|+|\beta|-2s)/2}$$
$$+O\left(L^{(n+|\alpha|+|\beta|-2s-1)/2}\ln(L)^\eta\right)$$

where $\eta=1$ *if* $1=|\alpha|+|\beta|$ *and* 0 *otherwise. Here* $|\xi|_w^2=g_w^{-1}(\xi,\xi)$ *and* $d\xi$ *is the Lebesgue measure.*

   • *Let* $\varepsilon>0$. *Then, uniformly for* $x,y\in V$ *such that* $|x-y|>\varepsilon$ *and for* $L\geq 1$,

$$\partial_x^\alpha\partial_y^\beta K_L^s\left(w+L^{-1/2}x,w+L^{-1/2}y\right)=O\left(L^{(n+|\alpha|+|\beta|-2s-1)/2}\ln\left(L\right)^\eta\right)$$

where $\eta=1$ *if* $1=|\alpha|+|\beta|$ *and* 0 *otherwise.*

2. • *Uniformly for* $x,y\in V$ *and* $L\geq 1$, *in these coordinates,*

$$G_L(x,y)=\frac{\left|S^{n-1}\right|}{(2\pi)^n}\left[\ln\left(L^{1/2}\right)-\ln_+\left(L^{1/2}|x-y|\right)\right]+O(1)$$

where $\ln_+(t)=\ln(t)\vee 0$.

   • *There exists a symmetric bounded function* $Q:U\times U\to\mathbb{R}$ *such that, uniformly for* $\kappa\geq 1$, $L\geq 1$ *and* $x,y\in V$ *such that* $|x-y|\geq\kappa L^{-1/2}$, *in these coordinates,*

$$G_L(x,y)=\frac{\left|S^{n-1}\right|}{(2\pi)^n}\ln(|x-y|)+Q(x,y)+O\left(\kappa^{-1/2}\right).$$

Proof. This follows directly from Theorem 1.2 with $m=2$, $s=n/2$, $A=\Delta$ and $\sigma_A(w,\xi)=|\xi|_w^2$. □

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2 Statement of the main results

In this section, we present the main objects of study and state our results in full generality. In Subsection 2.1 we present the general framework of the article. In Subsection 2.2 we state Hörmander's local Weyl law. In Subsection 2.3 we state the generalizations of the local Weyl law proved in this paper. We finish off by deducing Theorems 1.1 and 1.2 as well as Corollary 1.3.

2.1 General setting

In this article, we consider simultaneously two different elliptic eigenvalue problems. Since our arguments hold indifferently for the two cases, we present them in this section using the same notations. The first case is a closed eigenvalue problem. In this case we will follow [Hör68]. In the second case, we consider a Dirichlet eigenvalue problem, for which our main reference will be [Vas84].

1. In this setting we follow [Hör68]. Here $\mathcal{X}$ is a compact manifold without boundary. We consider a classical elliptic pseudo-differential operator A of positive order m acting on $C_c^\infty(\mathcal{X})$. We assume A is symmetric for the L^2-scalar product on $(\mathcal{X}, d\mu_\mathcal{X})$. This implies that the principal symbol σ_A of A is real valued and positive homogeneous of order m. Moreover, under these assumptions, A has a unique self-adjoint extension in $L^2(\mathcal{X}, d\mu_\mathcal{X})$ whose spectrum forms a discrete non-decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of real numbers diverging to $+\infty$ and whose corresponding eigenfunctions e_k are of class C^∞ (see for instance Section 29.1 of [Hör09]). For each $L > 0$, set

$$\forall x, y \in \overline{\mathcal{X}}, E_L(x,y) = \sum_{\lambda_j \leq L} e_j(x)\overline{e_j(y)}\,.$$

2. In this setting, we follow [Vas84]. Here $\mathcal{X}$ is the interior of a compact manifold $\overline{\mathcal{X}}$ with non-empty boundary $\partial\mathcal{X}$. We assume that $\mathcal{X}$ is equipped with a positive density $d\mu_\mathcal{X}$. We consider the Dirichlet eigenvalue problem

$$\begin{aligned} Au &= \lambda u \text{ on } \mathcal{X}\,; \\ \forall j = 1, \dots, j_0,\ B_j u &= 0 \text{ on } \partial\mathcal{X} \end{aligned}$$

where A is an elliptic differential operator of even order $m \geq 1$ with principal symbol σ_A, B_j are boundary differential operators (see Chapter 2, Section 1.4 of [LM72]) and $\lambda \in \mathbb{C}$. We assume that the problem is elliptic, formally self-adjoint with respect to $d\mu_\mathcal{X}$ and semi-bounded from below (see [Vas84] Section 1). As is well known, under these assumptions, the values of λ for which this problem has a non-trivial solution with sufficient regularity form an non-decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of real numbers diverging to $+\infty$ and the corresponding eigenfunctions e_k are smooth in $\mathcal{X}$ up to the boundary (the proof goes along the same lines as in the closed case treated in Section 29.1 of [Hör09] and is easily adapted using results from Chapter 20 of [Hör07]). For each $L > 0$, set

$$\forall x, y \in \overline{\mathcal{X}}, E_L(x,y) = \sum_{\lambda_j \leq L} e_j(x)\overline{e_j(y)}\,.$$

2.2 Hörmander's local Weyl law

Let us consider the sequence of real numbers $(\lambda_k)_k$ and the sequence of smooth functions $(e_k)_k$ from either of the two settings presented in Subsection 2.1. Recall that σ_A is the principal symbol of A, which we assumed to be positive homogeneous of order $m > 0$ in the second variable. We begin by stating Hörmander's local Weyl law, for which we need the following definition[2].

Definition 2.1. *Given an open subset $U \subset \mathbb{R}^n$, we will say that a function $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ is a* ***proper phase function*** *if it satisfies the following conditions.*

1. *The function ψ is a symbol of order one in its third variable.*
2. *For each $(x,y,\xi) \in U \times U \times \mathbb{R}^n$, $\langle x-y, \xi\rangle = 0$ implies that $\psi(x,y,\xi) = 0$.*
3. *For each $x \in U$ and $\xi \in \mathbb{R}^n$, $\partial_x\psi(x,y,\xi)|_{y=x} = \xi$.*
4. *There exists $\psi_\infty \in C^\infty(U \times U \times \mathbb{R}^n \setminus \{0\})$ satisfying all of the above properties and 1-homogeneous in ξ such that*

$$t^{-1}\psi(x,y,t\xi) \xrightarrow[t\to+\infty]{} \psi_\infty(x,y,\xi)$$

where the convergence takes place in $S^1(U \times U \times \mathbb{R}^n \setminus \{0\})$.

[2] This definition is inspired by Definition 2.3 of [Hör68]. However, our notion of proper phase function is more restrictive than Hörmander's notion of phase function.

An important example of proper phase function to have in mind is the phase function $\psi(x,y,\xi)=\langle x-y,\xi\rangle$. Hörmander's local Weyl law may be stated as follows.

Theorem 2.2 ([Hör68], Theorem 5.1 for $P=Id$)**.** *Let P be a differential operator of order d acting on $C^\infty(\mathcal{X}\times\mathcal{X})$. Fix a point in $\mathcal{X}$ and consider local coordinates $(x_1,\dots,x_n)$ around it. Suppose further that the density $d\mu_{\mathcal{X}}$ agrees with the Lebesgue measure in these coordinates. Let σ_A (resp. σ_P) be the principal symbol of A (resp. P) in these coordinates. Then, there exists an open neighborhood U of $0\in\mathbb{R}^n$, a proper phase function function $\psi\in C^\infty(U\times U\times\mathbb{R}^n)$ and a constant $C<+\infty$, such that, in these coordinates, for each $x,y\in U$ and $L>0$,*

$$\left|PE_L(x,y)-\frac{1}{(2\pi)^n}\int_{\sigma_A(x,\xi)\leq L}e^{i\psi(x,y,\xi)}\sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))d\xi\right|\leq C(1+L)^{(n+d-1)/m}\,.$$

Moreover, for each neighborhood $W\subset U\times U$ of the diagonal there exists $C>0$ such that in local coordinates, for each $(x,y)\in(U\times U)\setminus W$ and $L>0$,

$$\left|PE_L(x,y)\right|\leq C(1+L)^{(n+d-1)/m}\,.$$

Finally, there exists a symbol $\sigma\in S^1$ such that $\sigma_A^{1/m}-\sigma\in S^0$ and for each $x,y\in U$ and $\xi\in\mathbb{R}^n$,

$$\sigma(x,\partial_x\psi(x,y,\xi))=\sigma(y,\xi)\,. \tag{2}$$

Here $\partial_{x,y}\psi$ denotes the partial derivative of ψ with respect to the couple (x,y).

Remark 1. *The asymptotic provided by Theorem 2.5 is coordinate dependent since the notion of proper phase function is not invariant.*

Remark 2. *Equation (2) is called the* ***eikonal equation*** *and it has a unique solution with the boundary conditions imposed by the admissibility condition (see Section 3 of [Hör68]). The part concerning the eikonal equation is not usually stated as part of the local Weyl law but the function ψ provided by the theorem does satisfy this property and it will be useful in our proofs.*

Remark 3. *The case where $P=Id$ and was proved by Hörmander in [Hör68]. The case where $x=y$ and $\mathcal{X}$ is a closed manifold was treated in [SV97] with some restrictions on P. Finally, Gayet and Welschinger extended this result to a general P (see Theorem 2.3 of [GW14]) on a closed manifold. While in their statement, $x=y$, their proof yields the off-diagonal case with only minor modifications.*

Remark 4. *Hörmander manages to lift the compactness assumption using results on the local nature of the spectral projector Π_L. It is not clear that this approach could be applied for a general P.*

Remark 5. *Notice that in the boundary problem case (as in setting 2. form Subsection 2.1) we only get estimates in the interior of the domain.*

Remark 6. *One recent result closely related to this theorem is Canzani and Hanin's asymptotics for the monochromatic spectral projector of the Laplacian under some dynamical assumption on the geodesic flow (see [CH15] and [CH16]).*

For the convenience of the reader, in Appendix A we provide a proof of the full result relying on the wave kernel asymptotics provided in [Hör68].

2.3 Weighted local Weyl laws

In the present article, we generalize Theorem 2.2 in the following way. Consider A and P as in Theorem 2.2 and take U and ψ as provided by this theorem.

Theorem 2.3. *Fix $z = z_1 + iz_2 \in \mathbb{C}$. Let $f \in C^\infty(\mathbb{R})$ such that $f(t) = t^z$ for t large enough. Let K_L be the Schwartz kernel of $\Pi_L f(A)$. Suppose that $n + d + mz_1 > 0$. For each $x, y \in U$ and $L \geq 1$, let*

$$R_L(x,y) = L^{-z_1-(n+d)/m}\Big[PK_L(x,y) - \frac{1}{(2\pi)^n}\int_{\sigma_A(0,\xi)\leq 1} e^{i\langle \xi, x-y\rangle L^{1/m}}\sigma_A(0,\xi)^z\sigma_P(0,0,(\xi,-\xi))d\xi\Big].$$

Then, there exists an open neighborhood V of $0 \in U$ such that the following holds.

1. *Uniformly for $L \geq 1$ and $(w,x,y) \in V \times V \times V$, $w + L^{-1/m}x, w + L^{-1/m}y \in U$ and*

$$R_L\left(w + L^{-1/m}x, w + L^{-1/m}y\right) = O\left(L^{-1/m}\ln(L)^\eta\right)$$

   where $\eta = 1$ if $n + d + mz = 1$ and 0 otherwise.

2. *Uniformly for $L \geq 1$ and $(x,y) \in V \times V \setminus W$,*

$$PK_L(x,y) = O\left(L^{z_1+(n+d-1)/m}\ln(L)^\eta\right)$$

   where $\eta = 1$ if $n + d + mz = 1$ and 0 otherwise.

We prove Theorem 2.3 in Section 5. As we will see below, Theorem 1.1 and the first assertion of Theorem 1.2 are both direct consequences of Theorem 2.3. Before stating Theorem 2.5 below, we must introduce some more terminology. One key ingredient of the proof will be the decay of certain oscillatory integrals depending on the level sets of σ_A. To observe this behavior we must impose certain condition on σ_A. This is the object of Definition 2.4.

Definition 2.4. *Fix $m \in \mathbb{R}$, $m > 0$ and $k_0 \in \mathbb{N}$, $k_0 \geq 2$. We say that a positive m-homogeneous symbol σ on U is k_0-admissible there exists $k_0 \geq 2$ such that*

$$\forall (x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})\ \exists k \in \{2,\dots,k_0\},\ \sigma(x,\xi)^{k-1}\partial_\xi^k\sigma(x,\xi) \neq \frac{m(m-1)\dots(m-k+1)}{m^k}(\partial_\xi\sigma(x,\xi))^{\otimes k}. \quad (3)$$

This condition is invariant if we see σ as a function on $T^*\mathcal{X}$ because coordinate changes act linearly on the fibers of $T^*\mathcal{X}$. It is stable and generic for k_0 large enough, as explained in Proposition 7.5.

Theorem 2.5. *We use the same notations as in Theorem 2.3. Suppose that $n + d + mz = 0$ and that either $n = 1$ or σ_A is a k_0-admissible symbol for some $k_0 \geq 2$. For each $x, y \in U$ let*

$$Y_P(x,y) = \int_{S_y^*}\sigma_P\left(x,y,\partial_{x,y}(\partial_\xi\psi(x,y,0)\xi)\right)d_y\nu(\xi)\,.$$

Then, there exists $V \subset U$ an open neighborhood of 0 such that the following holds.

1. *Uniformly for $(x,y) \in V \times V$ and $L \geq 1$,*

$$PK_L(x,y) = \frac{1}{(2\pi)^n}Y_P(x,y)\Big[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\Big] + O(1)\,.$$

2. *There exists $Q \in L^\infty(V \times V)$ such that, uniformly for $\kappa \geq 1$, $L \geq 1$ and $(x, y) \in V \times V$ such that $|x - y| \geq \kappa L^{-1/m}$,*

$$PK_L(x,y) = -\frac{1}{(2\pi)^n} Y_P(x,y) \ln(|x-y|) + Q(x,y) + O\left(\kappa^{-1/k_0}\right).$$

Here, if $n = 1$ we set $k_0 = 1$.

We prove Theorem 2.5 in Section 6. As we will see below, the second point of Theorem 1.2 follows directly from this theorem.

Remark 7. *The admissibility condition on the symbol of A may appear to be unfamiliar. However, in practice, it is often satisfied. Here are two important examples of families of admissible symbols:*

- *If $n \geq 2$ and the level sets S_x^* are strictly convex, $\partial_\xi^2 \sigma_A$ is positive when restricted to their tangent spaces. Therefore, it cannot be a multiple of $(\partial_\xi \sigma)^{\otimes 2}$ so Theorem 2.5 applies with $k_0 = 2$.*
- *If σ_A is a positive homogeneous polynomial of degree $m \in \mathbb{N}$ in ξ, $m \geq 1$, then it is m-admissible. Indeed, otherwise, taking $k = k_0 = m$, we would have, for some $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$, $\sigma_A(x,\xi)^{m-1} \partial_\xi^m \sigma_A(x,\xi) = 0$. But since $\sigma_A(x,\xi) > 0$ we have $\partial_\xi^m \sigma_A(x,\xi) = 0$ which implies that all the coefficients of $\sigma_A(x,\cdot)$ vanish. This contradicts the fact that $\sigma_A(x,\xi) > 0$. In particular, Theorem 2.5 applies for all differential operators.*

In addition to the two examples of the last remark, we prove the following theorem.

Theorem 2.6. *Fix $n \in \mathbb{N}$, $n \geq 2$ and let $U \subset \mathbb{R}^n$ be an open subset. There exists $k_0 = k_0(n) \in \mathbb{N}$ such that for each $m > 0$, the set of k_0-admissible symbols is open and dense in the set of positive m-homogeneous symbols on U for the topology induced by the Whitney topology through the restriction map to the (Euclidean) unit sphere bundle on U.*

Theorem 2.6 follows immediately from Proposition 7.5, which is proved Subsection 7.2. The integer k_0 is explicit (see Proposition 7.5).

Finally, though we do not use this in the proof of Theorems 1.1 and 1.2, we prove the following result, which might be useful in further applications.

Theorem 2.7. *We use the same notations as in Theorem 2.3. Suppose that $n + d + mz_1 < 0$. Then, there exists and a function $K_\infty \in C^d(U \times U)$ such that the following holds. For each compact subset $\Omega \subset U \times U$, uniformly for $(x, y) \in \Omega$,*

$$PK_L(x,y) = PK_\infty(x,y) + O\left(L^{z_1 + (n+d)/m}\right).$$

Remark 8. *In Theorems 2.3, 2.5 and 2.7, the setting provided in Subection 2.1 only comes into play through Theorem 2.2. Therefore, if one could weaken the hypotheses for this theorem, one would automatically extend Theorems 2.3 and 2.5 as a corollary. In particular, since Hörmander proves Theorem 2.2 for $P = Id$ without any compactness assumption or boundary condition, both of these results remain valid in this case.*

Let us check that Theorems 2.3 and 2.5 imply the results presented in the introduction.

Proof of Theorem 1.1. Both results follow from Theorem 2.3 applied to the first setting of Subsection 2.1 with $z = -s$ by taking $P = \partial_x^\alpha \partial_y^\beta$ in a neighborhood of 0. In this case, the order of P is $d = |\alpha| + |\beta|$ and we have

$$\sigma_P(x,y,\xi) = (i\xi)^\alpha (-i\xi)^\beta$$

for any $\xi \in \mathbb{R}^n$ and $x, y \in V$ close enough to 0. □

Proof of Theorem 1.2. Set $z = -s = -n/m$. For the first part, set $P = \partial_x^\alpha \partial_y^\beta$ near 0 and proceed as in the proof of Theorem 1.1. Indeed, since $(\alpha, \beta) \neq (0,0)$, we have $n + d + mz_1 = |\alpha| + |\beta| > 0$. For the second part, since by Remark 7 σ_A is m-admissible, and since $n + d + mz_1 = 0$, we apply Theorem 2.5 instead. In our case, $P = Id$ so for each $x, y \in U$, $Y_P(x,y) = \nu_y\left(S_y^*\right)$ so

$$K_L^s(x,y) = \frac{1}{(2\pi)^n}\nu_y\left(S_y^*\right)\left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\right] + O(1)\,.$$

But since $K_L^s(x,y) = \overline{K_L^s(y,x)}$, we may replace $\nu_y\left(S_y^*\right)$ in the above expression by $\frac{\nu_x(S_x^*)+\nu_y\left(S_y^*\right)}{2}$. as announced. ☐

Proof of Corollary 1.3. We use the notations of Theorem 2.5. First of all, by definition, as $L \to +\infty$, $K_L^s \to K$ in distribution. Moreover, by Theorem 1.2, any point in $\mathcal{X}$ has a neighborhood V such that the sequence $(K_L^s)_{L\geq 1}$ is uniformly bounded on $V \times V$ by a locally integrable function and converge pointwise towards $-g_A(x,y)\ln(|x-y|) + Q(x,y)$ where $Q \in L^\infty(V \times V)$ on the complement of the diagonal in $V \times V$. In particular, they converge in distribution to this function. This implies that when restricted to $C^\infty(V \times V)$,

$$K(x,y) = -g_A(x,y)\ln(|x-y|) + Q(x,y)\,.$$

Now, given any smooth distance d on $\mathcal{X}$, for each x, y distinct,

$$\ln(|x-y|) = \ln(d(x,y)) + \ln\left(\frac{|x-y|}{d(x,y)}\right)$$

and the second term is bounded so, on $V \times V$,

$$K(x,y) = -g_A(x,y)\ln(d(x,y)) + Q_A(x,y)$$

for some $Q_A \in L^\infty(V \times V)$. But K is the integral kernel of A^s which is a self-adjoint pseudo-differential operator (see [See67] or Proposition 29.1.9 of [Hör09]). In particular, it is smooth and symmetric outside the diagonal (see for instance Theorem 18.1.16 of [Hör07]). Hence, Q_A must also be symmetric and smooth outside the diagonal. ☐

Remark 9. *These proofs work also in Setting 2 of Subsection 2.1 instead of that of Setting 1, so all of the results from the introduction hold also in Setting 2.*

3 Heuristics and proof outline

In this section we provide a heuristic justification for Theorems 2.3 and 2.5 and an outline of the skeleton of the proof. At the end of this section, we also provide a proof map to highlight the dependencies between intermediate results leading to the proofs of Theorems 2.3, 2.5 and 2.7, see Figure 1.

3.1 Heuristics

In order to get a sense of the kind of calculations we will carry out in the rest of the article, let us present a simple example, with few non-rigorous steps in order to shorten the argument. We assume that $\mathcal{X}$ is a closed Riemmanian manifold and that A denotes the associated Laplacian[3]. Then, A is indeed elliptic of order $m = 2$ and self-adjoint with respect to the riemmanian volume density $d\mu$. Moreover the symbol of A

[3]Here we use the convention that $\Delta = -\mathrm{div} \circ \nabla$ so that the operator is positive.

is $\sigma_A(x,\xi) = \|\xi\|_x^2$ where $\|\cdot\|_x$ is the norm induced by the riemmanian metric on $T_x^*\mathcal{X}$. Thus, in orthonormal coordinates $S_x^* = S^{n-1}$. Finally, we take $P = Id$. Now, if $s \geq \frac{n}{2}$,

$$K_L^s(x,y) = \int_1^L \lambda^{-s} E_\lambda'(x,y) d\lambda + O(1) = s\int_1^L \lambda^{-s-1} E_\lambda(x,y) d\lambda + O(1)\,. \tag{4}$$

Here, we artificially cut-off the first eigenvalues since they contribute a constant term to the sum defining K_L^s. By Theorem 2.2,

$$E_L(x,y) \simeq \frac{1}{(2\pi)^n}\int_{|\xi|^2\leq L} e^{i\psi(x,y,\xi)} d\xi \tag{5}$$

where

$$\psi(x,y,\xi) \simeq \langle x-y,\xi\rangle\,. \tag{6}$$

Replacing E_λ' in Equation (4) the expression given by Equations (5) and (6) we get

$$K_L^s(x,y) \simeq \frac{s}{(2\pi)^{n/2}}\int_1^L \lambda^{-s-1}\int_{\|\xi\|^2\leq\lambda} e^{i\langle x-y,\xi\rangle} d\xi d\lambda\,.$$

At this point, we make the additional assumption that $n \geq 2$. The one dimensional case is similar in spirit but follows a different argument. We then continue with a polar change of coordinates: $\xi = t\omega$.

$$K_L^s(x,y) \simeq \frac{s}{(2\pi)^n}\int_1^L \lambda^{-s-1}\int_0^{\lambda^{1/2}} J_{x,y}(t) t^{n-1} dt d\lambda \tag{7}$$

where

$$J_{x,y}(t) = \int_{S^{n-1}} e^{it\langle x-y,\omega\rangle} d\omega\,.$$

Let us first assume that $s > \frac{n}{2}$ and take $x - y = L^{-1/2}h$ where $h \in \mathbb{R}^n$ is fixed. Then,

$$\begin{aligned} K_L^s(x,y) &\simeq \frac{1}{(2\pi)^n}\int_{S^{n-1}}\int_1^L \lambda^{-s}\int_0^{\lambda^{1/2}} e^{iL^{-1/2}t\langle h,\omega\rangle} t^{n-1} dt d\lambda d\omega \\ &\simeq \frac{s}{(2\pi)^n} L^{n/2-s}\int_{S^{n-1}}\int_0^1 \lambda^{-s-1}\int_0^{\lambda^{1/2}} e^{it\langle h,\omega\rangle} t^{n-1} dt d\lambda d\omega \\ &\simeq \frac{1}{(2\pi)^n}\int_{\|\xi\|^2\leq 1} \|\xi\|^{-s} e^{i\langle h,\xi\rangle} d\xi L^{n/2-s}\,. \end{aligned}$$

This is the conclusion of Theorem 1.1.

Assume now that $s = \frac{n}{2}$. Starting off from Equation (7), we get

$$K_L^s(x,y) \simeq \frac{s}{(2\pi)^n}\int_{|x-y|^2}^{|x-y|^2 L} \lambda^{-s-1}\int_0^{\lambda^{1/2}} J_{x,y}(|x-y|^{-1}t) t^{n-1} dt d\lambda\,.$$

Note that, by the stationary phase method,

$$J_{x,y}(t) = O\left((|x-y|t)^{-1/2}\right)\,. \tag{8}$$

This crucial observation basically allows us to replace $J_{x,y}(|x-y|^{-1}t)$ with $|S^{n-1}|\mathbb{1}[|x-y|^{-1}t \le 1]$ and get

$$\begin{aligned} K_L^s(x,y) &\simeq \frac{s|S^{n-1}|}{(2\pi)^n}\int_{|x-y|^2}^{|x-y|^2L}\lambda^{-s-1}\int_0^{\lambda^{1/2}}\mathbb{1}[|x-y|^{-1}t\le 1]t^{n-1}dtd\lambda \\ &= \dots \\ &= \frac{|S^{n-1}|}{(2\pi)^n}\int_{|x-y|}^{|x-y|L^{1/2}}\mathbb{1}[u\le 1]\frac{du}{u}+O(1) \\ &= \frac{|S^{n-1}|}{(2\pi)^n}\ln\left(L^{1/2}\right)-\ln_+\left(L^{1/2}|x-y|\right)+O(1)\,. \end{aligned}$$

This is the essential statement of Theorem 1.2.

3.2 Proof strategy

There are two main obstacles to carry out the above calculation rigorously in the general case and Sections 4 and 7 are devoted to dealing with them. The first is to justify Equation (6). This is the role of Lemmas 4.1, 4.2 and 4.3 that roughly state that ψ behaves like $\langle x-y,\xi\rangle$. The second difficulty is to obtain an analog of Equation (8) when S^{n-1} is replaced by $S_x^*=\{\xi\,,\,\sigma_A(x,\xi)=1\}$ for a general symbol σ_A. Indeed, in this case, the standard stationary method need not apply and we must use more general results on oscillatory integrals. This requires the assumption that σ_A be admissible (see Definition 2.4). To make this point more precise, let us introduce some notation.

As in the previous section, we fix once and for all a point in $\mathcal{X}$ and consider a local chart centered at this point defined on $U\subset\mathbb{R}^n$ given by Theorem 2.2. We also take P with principal symbol σ_P, $W\subset U\times U$ and $\psi\in C^\infty(U\times U\times\mathbb{R}^n)$ as in this theorem. The following quantity will be central in our proofs. For any $t>0$, $x,y\in U$ and $\xi\in\mathbb{R}^n$ let

$$H_P(x,y,\xi,t)=e^{i\psi(x,y,t\xi)}\sigma_P\left(x,y,t^{-1}\partial_{x,y}\psi(x,y,t\xi)\right) \tag{9}$$

and

$$J_A(x,y,t)=\int_{S_y^*}H_P(x,y,\xi,t)d_y\nu(\xi). \tag{10}$$

Since σ_P is d-homogeneous in its third variable, H_P satisfies the following Equation. For any $s,t>0$, $x,y\in U$ and $\xi\in\mathbb{R}^n$,

$$H_P(x,y,s\xi,t)=s^dH_P(x,y,\xi,st). \tag{11}$$

We will prove the following proposition.

Proposition 3.1. *Assume that σ_A is k_0-admissible. Then, there exists $V\subset U$ an open neighborhood of 0 and $C<+\infty$ such that, uniformly for distinct $x,y\in V$ and $t>0$*

$$|J_A(x,y,t)|\le C\,(t|x-y|)^{-\frac{1}{k_0}}\,.$$

The proof of Proposition 3.1 is divided into two steps. First, we will prove that the admissibility condition on σ_A implies a property governing the decay of certain oscillatory integrals over the level sets of σ_A that we define below in Definition 3.2. Next, we prove that this property implies the required behavior of J_A. More precisely, we introduce the following terminology.

Definition 3.2. *Let $\varepsilon>0$, $m>0$, $E\subset\mathbb{R}^p$ a neighborhood of 0 and let $U\subset\mathbb{R}^n$ be an open subset. Let $\sigma\in C^\infty(U\times\mathbb{R}^n\setminus\{0\})$ be homogeneous of degree m in the second variable. For each $x\in U$ let $S_x^*=\{\xi\in\mathbb{R}^n\mid\sigma(x,\xi)=1\}$ and $d_x\mu$ be the area measure on S_x^*. Let $S^*U=\{(x,\xi)\in U\times\mathbb{R}^n\mid\xi\in S_x^*\}$.*

1. *Given a compact subset* $\Omega \subset U \times (\mathbb{R}^n \setminus \{0\})$ *let* $X = \{(x,\tau,\xi) \in \Omega \times \mathbb{R}^n \mid \xi \in S^*_x\}$. *We call a* ***deformation of the height function for*** σ ***over*** Ω *any family* $(f_\eta)_{\eta \in E}$ *of continuous, real-valued functions on* X*, smooth in the third variable* ξ*, with the following properties:*
   - *for each* $(x,\tau,\xi) \in \Omega \times \mathbb{R}^n$ *such that* $\xi \in S^*_x$*,* $f_0(x,\tau,\xi) = \langle \tau, \xi \rangle$
   - *for each* $\alpha \in \mathbb{N}^n$*, the map* $\eta \mapsto \partial^\alpha_\xi f_\eta$ *is continuous for the topology of uniform convergence on compact sets.*

2. *We say that* σ *has* ε***-non-degenerate level sets*** *if, for any compact subset* Ω *of* $U \times (\mathbb{R}^n \setminus \{0\})$ *and any deformation of the height function* $(f_\eta)_\eta$ *for* σ *over* Ω *there exists* $V \subset \mathbb{R}^p$ *a neighborhood of* 0 *depending only on* Ω *(and* ε*) such that for each* $\gamma \in C^\infty(S^*U)$ *and each continuous family of smooth functions on* $(u_\eta)_\eta \in (C^\infty(\mathbb{R}^n))^E$*, there exists* $C < +\infty$ *such that for each each* $\eta \in V$*, each* $(x,\tau) \in \Omega$ *and each* $\lambda > 0$*,*
$$\Big| \int_{S^*_x} e^{i\lambda f_\eta(x,\tau,\xi)} u_\eta(\xi)\gamma(x,\xi) d_x\mu(\xi) \Big| \leq C\lambda^{-\varepsilon} . \tag{12}$$
We say that σ *has* ***non-degenerate level sets*** *if it has* ε*-non-degenerate level sets for some* $\varepsilon > 0$*.*

3. *Let* $\varepsilon > 0$*. We say that a homogeneous symbol on a manifold has non-degenerate (resp.* ε*-non-degenerate) level sets if it has this property when written in any local coordinate system.*

Proposition 3.1 will then be a consequence of the following results. On the one hand, we will prove:

Proposition 3.3. *Fix* $n, k_0 \in \mathbb{N}$*,* $n \geq 1$*,* $k_0 \geq 2$*. Let* $U \subset \mathbb{R}^n$ *be an open subset and let* $\sigma \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\})$ *be positive and homogeneous of degree* $m > 0$ *in its second variable. If* σ *is* k_0*-admissible, then it has* $\frac{1}{k_0}$*-non-degenerate level sets.*

The proof of this proposition, which is presented in Subsection 7.1, is entirely independent of the rest of the present text and uses different techniques. It is followed by Subsection 7.2, in which we prove that the admissibility condition is generic in a suitable sense.

On the other hand, in Subsection 4.2, we will prove the following result.

Lemma 3.4. *Fix* $\varepsilon > 0$*. Suppose that the symbol* σ_A *has* ε*-non-degenerate level sets (see Definition 3.2). Then, there exists* $V \subset U$ *an open neighborhood of* 0 *and* $C < +\infty$ *such that, uniformly for distinct* $x, y \in V$ *and* $t > 0$
$$|J_A(x,y,t)| \leq C\,(t|x-y|)^{-\varepsilon} .$$

This corresponds to Proposition 23 of [Riv17] for $\varepsilon = \frac{1}{2}$ although, in that setting, the non-degeneracy condition was always satisfied.

In the one dimensional case, Proposition 3.1 is replaced by Lemma 4.4.

After proving all of these results, we carry out the calculation sketched in Subsection 3.1 in Sections 5 and 6 as we will now explain in more detail. We therefore suggest that the reader have Subsection 3.1 in mind for what follows. The integration by parts is of course valid in a general setting. This allows us to obtain an expression like Equation (4) where the map λ^{-s} is replaced by $f(\lambda)$ for some adequate function f. More explicitely, in Section 5, we derive the following result. We start by introducing a suitable function $f :]0,+\infty[\mapsto \mathbb{C}$ and studying the asymptotics of the following kernel:
$$K^f_L : (x,y) \mapsto \sum_{\lambda_k \leq L} f(\lambda_k) e_k(x)\overline{e_k}(y) .$$
This is again a smooth function. Since all of our results are local, we fix once and for all a point in $\mathcal{X}$ and consider $x = (x_1, \ldots, x_n)$ the local coordinate system at this point provided by Theorem 2.2, defined on an open neighborhood U of 0 in $\mathbb{R}^n$.

Proposition 3.5. *Take $f : \mathbb{R} \to \mathbb{C}$ with support in $]0, +\infty[$ differentiable almost everywhere. Then, in local coordinates, uniformly for each $x, y \in U$, for each $L > 0$,*

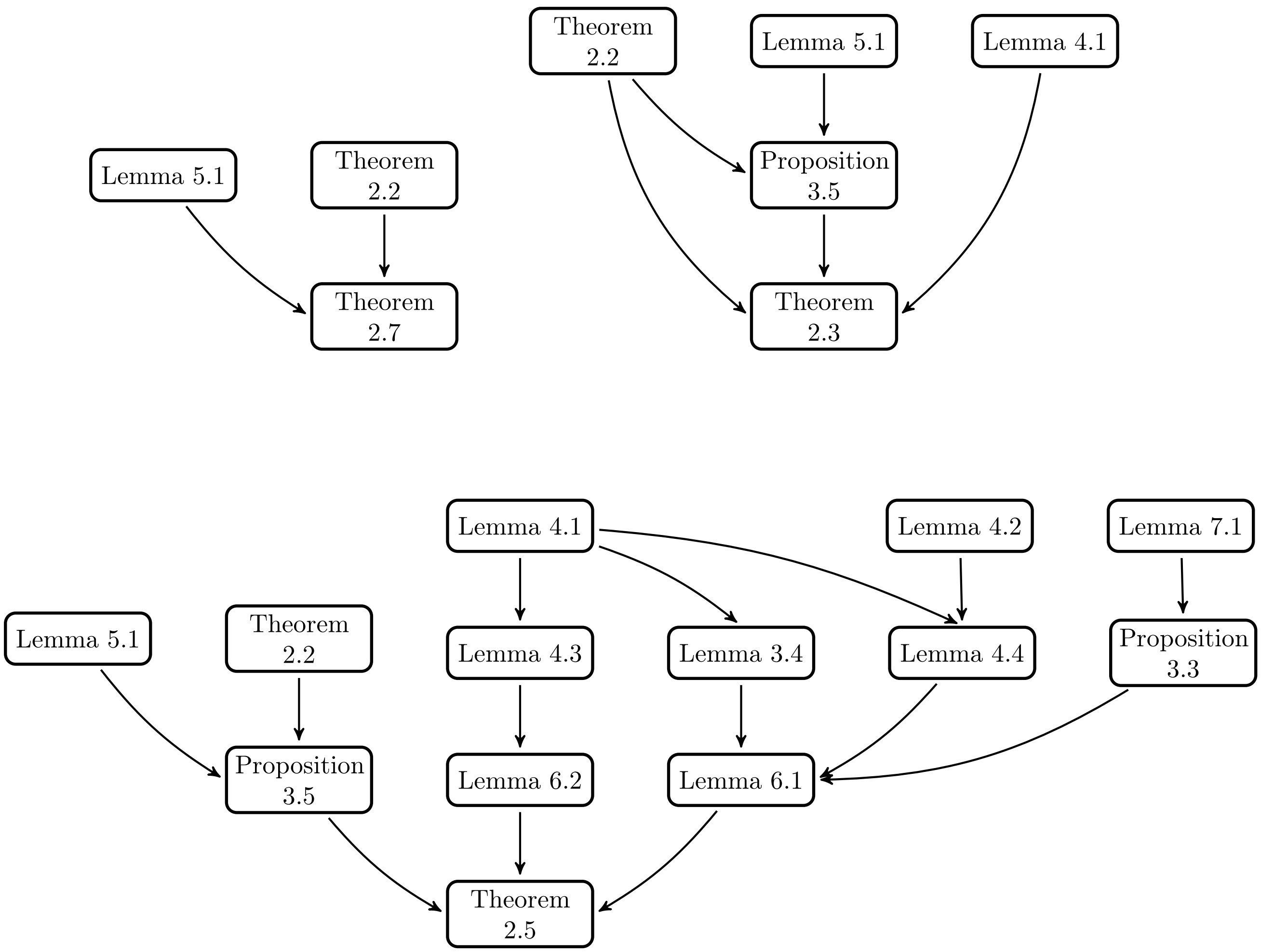
$$PK_L^f(x,y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi) \le L} e^{i\psi(x,y,\xi)} \sigma_P(x, y, \partial_{x,y}\psi(x,y,\xi)) f(\sigma_A(y,\xi)) d\xi$$
$$+ O\left(f(L) L^{(n+d-1)/m}\right) + O\left(\int_0^L f'(\lambda) \lambda^{(n+d-1)/m} d\lambda\right).$$

In addition, uniformly for any $(x, y) \in (U \times U) \setminus W$, for each $L \ge 1$,

$$PK_L^f(x,y) = O\left(f(L) L^{(n+d-1)/m}\right) + O\left(\int_0^L f'(\lambda) \lambda^{(n+d-1)/m} d\lambda\right).$$

Finally, the constants implied by the O's do not depend on f.

The condition on the support of f can be achieved by multiplying f by a suitable cut-off function when necessary since the spectrum of A is bounded from below. We prove Proposition 3.5 in Section 5. Then, we consider the case where f is of the form $f(t) = \chi(t) t^z$ where $z = z_1 + iz_2 \in \mathbb{C}$ and χ is some smooth function with support in $]0, +\infty[$ equal to 1 for t large enough. In Section 5, , we prove Theorem 2.7 using only a crude estimate from Theorem 2.2, and we also deduce Theorem 2.3 from Proposition 3.5 and results from Section 4. Next, in Section 6 we prove Theorem 2.5 using again Proposition 3.5 but also Proposition 3.1. We end this section with a diagram detailing the dependencies between different results involved in the proofs of Theorems 2.3, 2.5 and 2.7.



A map of the proofs of Theorems 2.3, 2.5 and 2.7. The result at the origin of each arrow is used in the proof of the result at its target.

4 Preliminary results

As before, in this section we fix once and for all a point in $\mathcal{X}$ and consider a local chart centered at this point defined on $U \subset \mathbb{R}^n$ given by Theorem 2.2. We also take P with principal symbol σ_P, $W \subset U \times U$ and $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ as in this theorem. The object of this section is to estimate the behavior of the phase ψ near the diagonal and to prove Lemma 4.3.

4.1 Basic properties of the phase ψ

The phase ψ from Theorem 2.2 will frequently appear in the calculations below. We begin by deducing a list of properties of ψ from those given in Definition 2.1. We gather these properties in Lemma 4.1. It is easy to check that all these properties are satisfied by the function $\psi(x, y, \xi) = \langle x - y, \xi \rangle$. Next, we present an additional lemma, Lemma 4.2, for the case $n = 1$. Finally, we use Lemma 4.1 to deduce some properties of the function H_P defined in Equation (9).

Lemma 4.1. *Let $U \subset \mathbb{R}^n$ and let $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ be a proper phase function. For each $t > 0$, let $\psi_t = t^{-1}\psi(\cdot, \cdot, t\cdot)$. Then,*

1. *For each $x, y \in U$ and each $t > 0$, $\psi_t(x, y, 0) = 0$.*
2. *For each $x \in U$, each $t > 0$ and each $\xi \in \mathbb{R}^n$, $\psi_t(x, x, \xi) = 0$.*
3. *For each $x \in U$, each $t > 0$ and each $\xi \in \mathbb{R}^n$, $\partial_{x,y}\psi_t(x, x, \xi) = (\xi, -\xi)$.*
4. *The sequence $(\psi_t)_{t>0}$ converges in $C^\infty(U \times U \times \mathbb{R}^n)$ as $t \to 0$ to the function ψ_0 defined by $\psi_0(x, y, \xi) = \partial_\xi \psi(x, y, 0)\xi$. In other words, for each compact subset $\Omega \subset U$, each $R < +\infty$ and each $\alpha, \beta, \gamma \in \mathbb{R}^n$,*
$$\lim_{t \to 0} \sup_{x,y \in \Omega,\ |\xi| \leq R} \left| \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_t(x, y, \xi) - \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_0(x, y, \xi) \right| = 0.$$
5. *The sequence $(\psi_t)_{t \geq 0}$ is bounded in $C^\infty(U \times U \times \mathbb{R}^n)$.*

Proof of Lemma 4.1. Let $t > 0$, $x, y \in U$ and $\xi \in \mathbb{R}^n$. Then, $\langle x - y, 0 \rangle = \langle x - x, t\xi \rangle = 0$ so $\psi_t(x, x, \xi) = \psi_t(x, y, 0) = 0$ by the second point of Definition 2.1. This proves the first two points of Lemma 4.1. By point 3 of Definition 2.1, for each $x \in U$ and $\xi \in \mathbb{R}^n$, $\partial_x \psi(x, x, \xi) = \xi$, so $\partial_x \psi_t(x, x, \xi) = t^{-1}(t\xi) = \xi$. Next, by differentiating the following equality

$$\psi_t(x + sv, x + sv, \xi) = 0$$

with respect to $s \in \mathbb{R}$, at $s = 0$, where $x \in U$, $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, we get

$$\partial_x \psi_t(x, x, \xi) + \partial_y \psi_t(x, x, \xi) = 0.$$

This proves the third point of Lemma 4.1.
To prove the fourth point, first, fix $\beta, \gamma \in \mathbb{N}^n$ and let $\Omega \subset U$ be a compact subset and $R < +\infty$. Then, for each $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$ such that $|\xi| \leq R$,

$$\partial_x^\beta \partial_y^\gamma \psi_t(x, y, \xi) = t^{-1} \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi).$$

By the first point, of Lemma 4.1, $\partial_x^\beta \partial_y^\gamma \psi(x, y, 0) = 0$. We apply Taylor's formula to $t \mapsto \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi)$ uniformly for $t \leq 1$, $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$ such that $|\xi| \leq R$ and get

$$t^{-1} \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi) = 0 + \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi(x, y, 0)\xi) + O(t).$$

In particular, as $t \to 0$, $\partial_x^\beta \partial_y^\gamma \psi_t \to \partial_x^\beta \partial_y^\gamma \psi_0$ uniformly for $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$, $|\xi| \le R$. Next, fix $\alpha \in \mathbb{N}$ and suppose $|\alpha| \ge 1$. Then, for each $x, y \in K$, $\xi \in \mathbb{R}^n$, $|\xi| \le R$ and $t > 0$,

$$\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_t(x,y,\xi) = t^{|\alpha|-1} \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi(x,y,t\xi).$$

If $|\alpha| = 1$, as $t \to 0$ the right hand side converges uniformly to $\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi(x,y,0) = \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi(x,y,0)\xi)$. On the other hand, if $|\alpha| > 1$, as $t \to 0$ it converges uniformly to $0 = \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi(x,y,0)\xi)$. This proves the fourth point of Lemma 4.1. Lastly, the family $(\psi_t)_{t>0}$ is obviously continuous into $C^\infty(U \times U \times \mathbb{R}^n)$ for $t > 0$. By the fourth point of Lemma 4.1 we may extend it by continuity to $t = 0$. On the other hand, by the fifth point of Definition 2.1, it also converges as $t \to \infty$. In particular, the family $(\psi_t)_{t \ge 0}$ is uniformly bounded in $C^\infty(U \times U \times \mathbb{R}^n)$. This proves the fifth point of Lemma 4.1. □

We use the following lemma to prove Lemma 4.4 below, which is the analog of Proposition 3.1 we use in dimension $n = 1$. It is the only place where we use the fact that ψ satisfies the eikonal equation (2).

Lemma 4.2. *Assume that $n = 1$. For each segment $I \subset U$ there exists $c \in]0, +\infty[$ such that for each $x, y \in I$ and $\xi \in \mathbb{R}$, $\frac{1}{c}|x-y| \le |\partial_\xi \psi(x,y,\xi)| \le c|x-y|$ and $|\partial_\xi^2 \psi(x,y,\xi)| \le c|x-y|(1+|\xi|)^{-1}$.*

Proof of Lemma 4.2. Let us fix $I \subset U$ a compact interval. Since the symbol σ_A is m-homogeneous and $dim(\mathcal{X}) = 1$ there exists a positive function $\varrho \in C^\infty(U)$ such that $\sigma_A(x,\xi) = \varrho(x)^m |\xi|^m$ for $\xi \neq 0$ and $x \in U$. By construction of ψ there exist $C_1 < +\infty$ and symbols $\tau \in S^0(U \times \mathbb{R})$ and $\sigma \in S^1(U \times \mathbb{R})$ such that $\sigma(x,\xi) = \varrho(x)|\xi| + \tau(x,\xi)$ for $|\xi| \ge C_1$ and $x \in U$ and such that

$$\forall \xi \in \mathbb{R} \setminus [-C_1, C_1],\ \forall x, y \in U,\ \sigma(x, \partial_x \psi(x,y,\xi)) = \sigma(y,\xi).$$

Since $\tau \in S^0$ and since ϱ, being positive and continuous, is bounded from below on I, there exists $C_2 \in [\max(C_1, 1), +\infty[$ such that for any $x \in I$ and $\xi \in \mathbb{R}$ such that $|\xi| \ge C_2$,

$$\frac{1}{2}\varrho(x)|\xi| \le \sigma(x,\xi) \le 2\varrho(x)|\xi|\,;$$
$$C_2^{-1} \le \mathrm{sign}(\xi)\partial_\xi \sigma(x,\xi) \le C_2.$$

Let $(\sigma^{-1})(x,\cdot)$ be the inverse of $\sigma(x,\cdot) : [C_2, +\infty[\to [\sigma(x,C_2) + \infty[$. Let us fix $x_0 \in I$. Then, for any $x \in I$,

$$\partial_x \psi(x, x_0, \xi) = (\sigma^{-1})(x, \sigma(x_0,\xi)). \tag{13}$$

Differentiating this Equation with respect to ξ we obtain the following expression for $\partial_\xi \partial_x \psi$.

$$\partial_\xi \partial_x \psi(x,x_0,\xi) = \partial_\xi(\sigma^{-1})(x, \sigma(x_0,\xi))\partial_\xi \sigma(x_0,\xi)\,.$$

Now, by definition of σ^{-1}, we have, for $x \in I$ and $\xi \in \mathbb{R}$ such that $\xi \ge C_3 = \max_{y \in I} \sigma(y, C_2)$,

$$\partial_\xi(\sigma^{-1})(x,\xi) = \left(\partial_\xi \sigma(x, \sigma^{-1}(x,\xi))\right)^{-1} = \left(\varrho(x) + \partial_\xi \tau(x, \sigma^{-1}(x,\xi))\right)^{-1},$$

where $\varrho(x)$ is bounded on I from above and below by positive constants and $\partial_\xi \tau(x, \sigma^{-1}(x,\xi))$ is $O(|\sigma^{-1}(x,\xi)|^{-1})$ uniformly for $x \in I$. Since $\sigma^{-1}(x,\xi) \xrightarrow[\xi \to +\infty]{} +\infty$ then there exists $C_4 > 0$ such that for any $x \in I$ and any $\xi \ge C_4 \ge \max(C_3, C_2)$,

$$C_4^{-1} \le \partial_\xi(\sigma^{-1})(x,\xi) \le C_4. \tag{14}$$

Therefore,

$$C_2^{-1}C_4^{-1} \le \partial_x \partial_\xi \psi(x,x_0,\xi) \le C_2 C_4\,.$$

Recall that, by the first point of Lemma 4.1, $\psi(x,x,\xi) = 0$ for any $x \in U$ and any $\xi \in \mathbb{R}$. Thus, for any $x \in I$, $\xi \ge C_4$,

$$|\partial_\xi \psi(x,x_0,\xi)| = \left|\int_{x_0}^x \partial_\xi \partial_x \psi(y,x_0,\xi)dy\right| \in \left[C_5^{-1}|x-x_0|, C_5|x-x_0|\right]$$

where $C_5 = C_2C_4$ is independent of the choice of x_0. The case where $\xi < 0$ is symmetric and this proves the first identity announced in the lemma. For the second identity, we start by differentiating Equation (13) with respect to ξ to obtain

$$\partial_\xi^2\partial_x\psi(x,x_0,\xi) = \partial_\xi^2(\sigma^{-1})(x,\sigma(x_0,\xi))(\partial_\xi\sigma(x_0,\xi))^2 + \partial_\xi(\sigma^{-1})(x,\sigma(x_0,\xi))\sigma_\xi^2(x_0,\xi)\,. \tag{15}$$

To deal with the second term of the right hand side, observe that, since σ is a symbol of order one and by Equation (14), there exists a constant $C_6 < +\infty$ such that for any $x, x_0 \in I$ and any $\xi \in \mathbb{R}$,

$$\left|\partial_\xi(\sigma^{-1})(x,\sigma(x_0,\xi))\sigma_\xi^2(x_0,\xi)\right| \leq C_6(1+|\xi|)^{-1}. \tag{16}$$

For the first term we proceed as follows. By definition of σ^{-1}, we have, for any $x \in I$ and $\xi \geq C_3$,

$$\partial_\xi^2\sigma(x,\sigma^{-1}(x,\xi))(\partial_\xi(\sigma^{-1})(x,\xi))^2 + \partial_\xi\sigma(x,\sigma^{-1}(x,\xi))\partial_\xi^2(\sigma^{-1})(x,\xi) = 0.$$

By Equation (14), since σ is a symbol of order one and since $\partial_\xi\sigma$ is bounded from below on $[C_2,+\infty[$, there exists $C_7 < +\infty$ such that for each $x, x_0 \in I$ and $\xi \geq C_3$,

$$\left|\partial_\xi^2(\sigma^{-1})(x,\sigma(x_0,\xi))(\partial_\xi\sigma(x_0,\xi))^2\right| \leq C_7(1+|\xi|)^{-1}\,. \tag{17}$$

We use Equations (16) and (17) on the right hand side of Equation (15) and get, for each $x, x_0 \in I$ and $\xi \geq C_3$,

$$\left|\partial_x\partial_\xi^2\psi(x,x_0,\xi)\right| \leq (C_6+C_7)(1+|\xi|)^{-1}\,.$$

As before, since for all $x \in I$ and $\xi \in \mathbb{R}$, $\psi(x,x,\xi) = 0$, we have

$$\left|\partial_\xi^2\psi(x,x_0,\xi)\right| \leq \int_{x_0}^{x}\left|\partial_x\partial_\xi^2\psi(y,x_0,\xi)\right|dy \leq C_8|x-x_0|(1+|\xi|)^{-1}$$

where $C_8 = C_6 + C_7$. The case $\xi < 0$ is symmetric. □

From Lemma 4.1, we deduce the following properties of the function H_P defined in Equation (9).

Lemma 4.3. *The function H_P satisfies the following properties.*

1. *The function $t \mapsto H_P(\cdot,\cdot,\cdot,t)$ extends continuously to $t = 0$ as a function from $\mathbb{R}_+$ to $C^\infty(U\times U\times\mathbb{R}^n)$ and*
$$H_P(x,y,\xi,0) = \sigma_P\left(x,y,\partial_{x,y}(\partial_\xi\psi(x,y,0)\xi)\right).$$
2. *Uniformly for $t \geq 0$ and x, y in compact subsets of U and $\xi \in \mathbb{R}^n$,*
$$H_P(x,y,\xi,t) - H_P(x,y,\xi,0) = O\left(t|x-y||\xi|^{d+1}\right)\,.$$

Note that the assertions are both easy to check for the prototype $H_P(x,y,t) = e^{it\langle x-y,\xi\rangle}\sigma_P(x,y,\xi)$.

Remark 10. *Lemma 4.3 implies that the function $t \mapsto J_A(\cdot,\cdot,t)$ extends continuously to $t = 0$ as a function from $\mathbb{R}_+$ to $C^\infty(U\times U)$ and*

$$J_A(x,y,0) = \int_{S_y^*}\sigma_P(x,y,\partial_{x,y}(\partial_\xi\psi(x,y,0)\xi))d_y\nu(\xi). \tag{18}$$

Proof. The first statement follows from the fourth point of Lemma 4.1. For the second statement, by Equation (11), we may therefore restrict our attention to the case where $\xi \in S_y^*$. Next, we observe that by the second point of Lemma 4.1, $H_P(y, y, \xi, t) = H_P(y, y, \xi, 0)$. The function H_P is clearly C^1 with respect to its first variable so that $|H_P(x, y, \xi, t) - H_P(x, y, \xi, 0)|$ is no greater than

$$|x-y| \sup_{s\in[0,1]} |\partial_x H_P(sx+(1-s)y, y, \xi, t) - \partial_x H_P(sx+(1-s)y, y, \xi, 0)| \,.$$

Let us fix $\Omega \subset U$ a compact set. Then by Taylor's inequality, there exists $C_1 < +\infty$ such that for each $x, y \in \Omega$, $\xi \in S_y^*$ and each $t > 0$,

$$|\partial_{x,y}\psi(x, y, t\xi) - \partial_{x,y}\psi(x, y, 0) - \partial_{x,y}(\partial_\xi\psi(x, y, 0)\xi)t| \leq C_1 t^2 \,.$$

By the first point of Lemma 4.1, $\partial_{x,y}\psi(x, y, 0) = 0$, so that

$$\partial_{x,y}\psi_t(x, y, \xi) = \partial_{x,y}(\partial_\xi\psi(x, y, 0)\xi) + O(t) \tag{19}$$

uniformly in $x, y \in \Omega$ and $\xi \in S_y^*$. On the other hand by the fifth point of Lemma 4.1, $(\psi_t)_{t>0}$ is bounded in C^∞. In particular, there exists a constant $C_2 < +\infty$ such that for each $t > 0$, each $x, y \in K$ and each $\xi \in S_y^*$, $|\psi_t(x, y, \xi)| \leq C_2$. In other words

$$\psi(x, y, t\xi) = O(t) \tag{20}$$

uniformly in $x, y \in \Omega$ and $\xi \in S_y^*$. Applying estimates (19) and (20) to each occurrence of ψ in H_P, we see that uniformly for x, y in compact subsets of U and $\xi \in S_y^*$,

$$\partial_x H_P(x, y, \xi, t) = \partial_x H_P(x, y, \xi, 0) + O(t) \,,$$

which completes the proof. □

4.2 Proof of Lemma 3.4 and its analogue in dimension one

In this subsection, we use the results of the previous subsection to prove Lemma 3.4. We will use this lemma in the proof of the multi-dimensional case of Theorem 2.5 (see Section 6). In the one dimensional case, we will use Lemma 4.4 presented below.

Proof of Lemma 3.4. To prove this lemma, we interpret J_A as an oscillatory integral whose phase is a deformation of $(\omega, \tau) \mapsto \langle \omega, \tau \rangle$. First, fix $\Omega \subset U$ a compact neighborhood of 0. Let $r_0 > 0$ be such that $\Omega_0 = \{x \in \mathbb{R}^n \mid \exists y \in \Omega,\ |y - x| \leq r_0\} \subset U$. By the fourth point of Lemma 4.1 the family $(\psi_t)_{t>0}$ extends by continuity to $t = 0$ in C^∞. For each $t \geq 0$, $y \in U$, $0 < r \leq r_0$ and $\xi, \tau \in \mathbb{R}^n$ such that $|\tau| \leq 1$, let

$$f_{t,r}(y, \xi, \tau) = r^{-1}\psi_t((y + r\tau), y, \xi).$$

Let $\alpha \in \mathbb{N}^n$. The Taylor expansion of $\partial_\xi^\alpha \psi_t(y + r\tau, y, \xi)$ along r yields, for each $y \in \Omega$, $|\tau| \leq 1$, $0 < r \leq r_0$, $t \geq 0$ and $\xi \in S_y^*$,

$$\left|\partial_\xi^\alpha \psi_t(y + r\tau, y, \xi) - \partial_\xi^\alpha \langle \xi, \tau \rangle\right| \leq \frac{1}{2} C_1 r$$

where

$$C_1 = \sup\{|\partial_x\partial_\xi\psi_s(w', w, \xi)| \mid w \in \Omega,\ w' \in \Omega_0,\ \xi \in S_w^*,\ s \geq 0\} \,.$$

The constant C_1 is finite by the fifth point of Lemma 4.1. In particular,

$$\lim_{r\to 0} f_{t,r}(y, \xi, \tau) = \langle \xi, \tau \rangle$$

smoothly in ξ, uniformly in $t \geq 0$, $y \in \Omega$ and $\tau \in \mathbb{R}^n$ such that $|\tau| \leq 1$. In particular, we have proved first that $f_{t,r}(y, \xi, \tau) \xrightarrow[t,r\to 0]{} \langle \xi, \tau \rangle$ in this same topology, and second that for each $\alpha \in \mathbb{N}^n$, the map $(t, r) \to \partial_\xi^\alpha f_{t,r}$

is continuous at $(t, 0)$ for any $t \geq 0$ for the topology of uniform convergence. Since this map is obviously continuous as long as $r > 0$ we have proved that the family $(f_{t,r})_{t,r}$ is a deformation of the height function in the sense of Definition 3.2. Now let $x \in U$ be such that $0 < r := |x - y| \leq r_0$ and let $\tau = \frac{x-y}{|x-y|}$. Then $|\tau| = 1$ and

$$\psi(x, y, t\xi) = t|x - y| f_{t,|x-y|}(y, \xi, \tau).$$

Moreover, by the fifth point of Lemma 4.1, the function

$$\xi \mapsto \sigma_P(x, y, \partial_{x,y}\psi_t(x, y, \xi))$$

is bounded in $C^\infty(\mathbb{R}^n)$ uniformly for $x, y \in \Omega$ and $t \geq 1$. Hence, the fact that the function σ_A has ε-non-degenerate level sets (see Definition 3.2) implies the existence an open neighborhood $V \subset U$ of 0 and a constant $C > 0$ such that, uniformly for $x, y \in V$ and $t > 0$,

$$\Big| \int_{S_y^*} e^{i\psi(x,y,t\xi)} \sigma_P(x, y, \partial_{x,y}\psi_t(x, y, \xi)) d_y\nu(\xi) \Big| \leq C(t|x - y|)^{-\varepsilon}.$$

Here we took $\omega = t|x - y|$ in Equation (12). $\square$

In dimension $n = 1$, the symbol will never have non-degenerate level sets (in fact they will be discrete). Instead of Lemma 3.4 we will use the following result.

Lemma 4.4. *Assume that $n = 1$. For each compact interval $I \subset U$, there exists $C < +\infty$ such that for each $0 < a \leq b$, each $\eta \in \{-1, +1\}$ and each $x, y \in I$*

$$\left| \int_{\eta a}^{\eta b} e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P(x, y, |x - y| \partial_{x,y}\psi(x, y, |x - y|^{-1}\eta)) \sigma_A(y, \xi)^{-(d+1)/m} d\eta \right| \leq C a^{-1} . \tag{21}$$

Proof of Lemma 4.4. Let $I \subset U$ be a compact interval. First of all, since σ_A is homogeneous of degree m and $n = 1$, there exists a positive function $\varrho \in C^\infty(U)$ such that $\sigma_A(x, \eta) = \varrho(x)|\eta|^m$. Thus, we may replace $\sigma_A(x, \eta)$ by $|\eta|^m$ in Equation (21). Observe that for each $t, \lambda > 0$, $x, y \in U$ and $\eta \in \mathbb{R}$,

$$\psi_t(x, y, \lambda\eta) = \lambda\psi_{\lambda t}(x, y, \eta) .$$

This Equation, combined with the fifth point of Lemma 4.1 implies that there exists $C < +\infty$ such that for each $x, y \in I$, each $t > 0$ and $\eta \in \mathbb{R}$

$$|\partial_{x,y}\psi_t(x, y, \eta)| \leq C|\eta| \text{ and } |\partial_{x,y}\partial_\xi\psi_t(x, y, \eta)| \leq C .$$

Since moreover σ_P is homogeneous of degree d in the third variable, we have, uniformly for $x, y \in I$ and for non-zero $\eta \in \mathbb{R} \setminus \{0\}$,

$$\sigma_P(x, y, |x - y|\partial_{x,y}\psi(x, y, |x - y|^{-1}\eta))|\eta|^{-d-1} = \sigma_P(x, y, \partial_{x,y}\psi_{|x-y|^{-1}}(x, y, \eta))|\eta|^{-d-1} = O(|\eta|^{-1})$$
$$\partial_\eta[\sigma_P(x, y, |x - y|\partial_{x,y}\psi(x, y, |x - y|^{-1}\eta))|\eta|^{-d-1}] = \partial_\eta[\sigma_P(x, y, \partial_{x,y}\psi_{|x-y|^{-1}}(x, y, \eta))|\eta|^{-d-1}] = O(|\eta|^{-2}).$$

In addition, again uniformly for $x, y \in I$ and non-zero $\eta \in \mathbb{R} \setminus \{0\}$, by Lemma 4.2, $\partial_\eta^2[\psi(x, y, |x - y|^{-1}\eta)] = O(|\eta|^{-1})$ and $\partial_\eta[\psi(x, y, |x - y|^{-1}\eta)]$ is bounded from above and below by a positive constant. Now, setting momentarily $u(\eta) := \psi(x, y, |x - y|^{-1}\eta)$ and $v(\eta) = \sigma_P(x, y, |x - y|\partial_{x,y}\psi(x, y, |x - y|^{-1}\eta))|\eta|^{-d-1}$, we have, for any $a, b > 0$ such that $a \leq b$,

$$\int_a^b e^{iu(\eta)} v(\eta) d\eta = \Big[\frac{1}{i} e^{iu(\eta)} \frac{v(\eta)}{u'(\eta)}\Big]_{\eta=a}^b - \int_a^b \frac{1}{i} e^{iu(\eta)} \left(\frac{v'(\eta)}{u'(\eta)} - \frac{v(\eta)u''(\eta)}{u'(\eta)^2}\right) d\eta.$$

The preceding observations show that, uniformly for $x, y \in I$, $0 < a \le b$ and $\eta \in [a, b]$, we have $\frac{v(a)}{u'(a)} = O(a^{-1})$, $\frac{v(b)}{u'(b)} = O(b^{-1})$, $\frac{v'(\eta)}{u'(\eta)} = O(\eta^{-2})$ and $\frac{v(\eta)u''(\eta)}{u'(\eta)^2} = O(\eta^{-2})$. Consequently, there exists $C < +\infty$ such that for any $x, y \in \Omega$ and any $0 < a \le b$,

$$\Big| \int_a^b e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P(x, y, |x-y|\partial_{x,y}\psi(x, y, |x-y|^{-1}\eta))|\eta|^{-d-1} d\eta \Big| \le Ca^{-1}.$$

The proof for $\int_{-b}^{-a}$ is identical. □

5 Proof of Theorem 2.7, Proposition 3.5 and Theorem 2.3

In this section, we prove Theorem 2.7, Proposition 3.5 and Theorem 2.3. We use only Theorem 2.2 and Lemma 4.1.

Let $f : \mathbb{R} \to \mathbb{R}$ with support in $]0, +\infty[$ differentiable almost everywhere. For each $L \ge 1$, let K_L^f be the integral kernel of $\Pi_L f(A)$. Later in the section, we will be interested in a special case of K_L^f. More precisely, we fix $z = z_1 + iz_2 \in \mathbb{C}$ and set $K_L = K_L^f$ where f is chosen so that $f(t) = t^z$ for $t > 0$ large enough. We begin by linking K_L^f with E_L.

Lemma 5.1. *For any $L \in \mathbb{R}$,*

$$K_L^f = f(L)E_L - \int_0^L f'(\lambda)E_\lambda d\lambda.$$

This lemma generalizes Proposition 21 of [Riv17].

Proof. The functions $L \mapsto E_L$ and $L \mapsto K_L^f$ are locally constant and define distributions on $\mathbb{R}$ with values in $C^\infty(\mathcal{X} \times \mathcal{X})$. We denote by $'$ the weak derivative with respect to L of these kernels. For each $u, v \in C^\infty(\mathcal{X})$ we let $u \boxtimes v \in C^\infty(\mathcal{X} \times \mathcal{X})$ be the function $(u \boxtimes v)(x, y) = u(x)v(y)$. For all $L > 0$,

$$E_L = \sum_{\lambda_k \le L} e_k \boxtimes \overline{e_k};\ K_L^f = \sum_{\lambda_k \le L} f(\lambda_k) e_k \boxtimes \overline{e_k}\,,$$

so that

$$\left(K_L^f\right)' = \sum_{k\in\mathbb{N}} \delta_{\lambda_k}(L) f(\lambda_k) e_k \boxtimes \overline{e_k} = f(L) \sum_{k\in\mathbb{N}} \delta_{\lambda_k}(L) e_k \boxtimes \overline{e_k} = f(L)E_L'$$

and

$$K_L^f = \int_0^L f(\lambda)E_\lambda' d\lambda.$$

By integration by parts,

$$K_L^f = f(L)E_L - f(0)E_0 - \int_0^L f'(\lambda)E_\lambda d\lambda = f(L)E_L - \int_0^L f'(\lambda)E_\lambda d\lambda$$

since $f(0) = 0$. □

We can now prove both Theorem 2.7 and Proposition 3.5 using Theorem 2.2. We start with Theorem 2.7.

Proof of Theorem 2.7. Let $L > 0$. Then, by Lemma 5.1, we have, for each $t \ge L$,

$$K_{L+t}^f = (L+t)^z E_{L+u} - \int_0^{L+t} f'(\lambda)E_\lambda d\lambda\,.$$

Now, if instead of K_L^f we consider the special case K_L, and if we apply the operator P, then, for all large enough values of $L > 0$ and all $t \geq 0$,

$$PK_{L+t} - PK_L = (L+t)^z PE_{L+t} - L^z PE_L - \int_L^{L+t} z\lambda^{z-1} PE_\lambda d\lambda\,.$$

By Theorem 2.2, we have, uniformly for $(x,y) \in U \times U$ and $t \geq 0$,

$$(L+t)^z PE_{L+t}(x,y) = O\left(L^{z_1+(n+d)/m}\right)$$

and

$$\int_L^{L+t} z\lambda^{z-1} E_\lambda d\lambda = O\left(\int_L^{+\infty} \lambda^{-1+z_1+(n+d)/m} d\lambda\right) = O\left(L^{z_1+(n+d)/m}\right)\,.$$

In particular, uniformly for $(x,y) \in U \times U$ and $t \geq 0$,

$$PK_{L+t}(x,y) - PK_L(x,y) = O\left(L^{z_1+(n+d)/m}\right)\,.$$

Since, $z_1 + (n+d)/m < 0$, this last estimate implies that the sequence $(PK_L)_{L>0}$ is a Cauchy sequence in $C^0(U \times U)$. Therefore, it converges uniformly on compact subsets of $U \times U$ to some function $K_\infty^P \in C^0(U \times U)$. Since this is actually true for any differential operator of order at most d (indeed, if $d' \leq d$, we still have $z_1 + (n+d')/m < 0$), all the derivatives of K_L, of order up to d, converge uniformly on compact sets. But this means that the limit K_∞ of $(K_L)_{L>0}$ is actually of class C^d and that the limits of the respective derivatives converge to the derivatives of the limit. In particular, $K_\infty^P = PK_\infty$. □

We now move on to Proposition 3.5.

Proof of Proposition 3.5. By Theorem 2.2, uniformly for $x, y \in U$ and $L \geq 1$,

$$\begin{aligned} PE_L(x,y) &= \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi) \leq L} e^{i\psi(x,y,\xi)} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi)) d\xi + O\left(L^{(n+d-1)/m}\right) \\ &= \frac{1}{(2\pi)^n} \int_0^{L^{1/m}} J_A(x,y,t) t^{n+d-1} dt + O\left(L^{(n+d-1)/m}\right). \end{aligned}$$

In the second equality we used the definition of $d\nu$ (see (1)) and J_A (see (10)) as well as the fact that σ_P is d-homogeneous along the fibers. Consequently, uniformly for any $x, y \in U$ and $L \geq 1$,

$$-\int_0^L f'(\lambda) PE_\lambda(x,y) d\lambda = -\frac{1}{(2\pi)^n} \int_0^L f'(\lambda) \int_0^{\lambda^{1/m}} J_A(x,y,t) t^{n+d-1} dt d\lambda + O\left(\int_{-\infty}^L f'(\lambda) \lambda^{(n+d-1)/m} d\lambda\right).$$

Integrating by parts along λ the first term in the right hand side, we get

$$-f(L) PE_L(x,y) + \frac{1}{(2\pi)^n} \int_0^L f(\lambda) \frac{1}{m} \lambda^{\frac{1}{m}-1} J_A(x,y,\lambda^{1/m}) \lambda^{(n+d-1)/m} d\lambda + O\left(f(L) L^{(n+d-1)/m}\right).$$

Setting $u = \lambda^{1/m}$ we get

$$\begin{aligned} \int_0^L f(\lambda) \frac{1}{m} \lambda^{\frac{1}{m}-1} J_A(x,y,\lambda^{1/m}) \lambda^{(n+d-1)/m} d\lambda &= \int_0^{L^{1/m}} f(u^m) J_A(x,y,u) u^{n+d-1} du \\ &= \int_{\sigma_A(y,\xi) \leq L} e^{i\psi(x,y,\xi)} f(\sigma_A(y,\xi)) \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi)) d\xi. \end{aligned}$$

By Lemma 5.1,

$$PK_L^f = f(L)PE_L - \int_0^L f'(\lambda)PE_\lambda d\lambda.$$

Replacing the integral term by the expression derived above, we see that the $f(L)PE_L$ terms cancel out, leaving the equation from the first result of Proposition 3.5. For the case where $(x,y) \in U \times U \setminus W$, we just apply the corresponding estimate from Theorem 2.2 and proceed accordingly. □

For the proof of Theorem 2.3, we remind teh reader that $K_L = K_L^f$ where $f(t) = t^z$ for $t > 0$ large enough.

Proof of Theorem 2.3. Throughout the proof, we let $\eta = 1$ if $n + d + mz = 1$ and 0 otherwise and set $g(L) = L^{(n+d-1)/m+z_1}\ln(L)^\eta$. Let Ω be a compact neighborhood of 0 in U such that for any $w, x \in \Omega$ and $L \geq 1$, $w + L^{-1/m}x$ belongs to U. Firstly, changing f on a compact set affects PK_L by adding a linear combination of smooth functions (independent of L). Thus, we may assume that $f(t) = t^z \mathbb{1}[t \geq 1]$. By Proposition 3.5, uniformly for $w, x, y \in \Omega$ and $L \geq 1$, $PK_L\left(w + L^{-1/m}x, w + L^{-1/m}y\right)$ equals

$$\begin{aligned}\frac{1}{(2\pi)^n}\int_{1\leq\sigma_A(w+L^{-1/m}y,\xi)\leq L} &\sigma_A(w + L^{-1/m}y, \xi)^z e^{i\psi(w+L^{-1/m}x, w+L^{-1/m}y,\xi)} \\ &\times \sigma_P\left(w + L^{-1/m}x, w + L^{-1/m}y, \partial_{x,y}\psi\left(w + L^{-1/m}x, w + L^{-1/m}y, \xi\right)\right) d\xi + O(g(L))\,.\end{aligned} \tag{22}$$

Indeed, since $n + d + z_1 > 0$, $O(g(L)) + O(1) = O(g(L))$. We need to check that replacing each occurrence of $w + L^{-1/m}y$ or $w + L^{-1/m}x$ by w in the integrand will produce an error of order $O(g(L))$. More precisely, we make the following claim.

Claim 1. *Uniformly for $w, x, y \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $L \geq 1$ such that $1 \leq \sigma_A(w + L^{-1/m}y, \xi) \leq L$, the quantity*

$$\begin{aligned}&\sigma_A(w + L^{-1/m}y, \xi)^z e^{i\psi(w+L^{-1/m}x, w+L^{-1/m}y,\xi)} \\ &\quad\times \sigma_P\left(w + L^{-1/m}x, w + L^{-1/m}y, \partial_{x,y}\psi\left(w + L^{-1/m}x, w + L^{-1/m}y, \xi\right)\right)\end{aligned} \tag{23}$$

equals

$$e^{iL^{-1/m}\langle\xi, x-y\rangle}\sigma_A(w,\xi)^z\sigma_P(w, w, (\xi, -\xi)) + O\left(|\xi|^{mz_1+d}L^{-1/m}\right)\,. \tag{24}$$

Proof. Throughout the proof we fix $w, x, y \in \Omega$, $L \geq 1$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\sigma_A(w + L^{-1/m}y, \xi) \leq L$. Unless otherwise stated, all the O estimates will be uniform with respect to these parameters. First of all, since σ_A is a positive m-homogeneous symbol in its second variable, σ_P is a symbol of order d in its third variable and $\partial_{x,y}\psi$ is a symbol of order 1 in its third variable, applying Taylor's inequality with respect to the L-dependent variables everywhere except the exponential in the quantity (23) shows that it equals

$$\sigma_A(w,\xi)^z e^{i\psi(w+L^{-1/m}x, w+L^{-1/m}y,\xi)}\sigma_P\left(w, w, \partial_{x,y}\psi\left(w, w, \xi\right)\right) + O\left(|\xi|^{mz_1+d}L^{-1/m}\right)\,. \tag{25}$$

Here the $|\xi|^{mz_1}$ appears regardless of the sign of z_1 because σ_A is positive homogeneous. Since ψ is a symbol of order one in ξ and $|\xi| = O\left(L^{1/m}\right)$,

$$\psi(w + L^{-1/m}x, w + L^{-1/m}y, \xi) = \psi(w, w, \xi) + \partial_x\psi(w, w, \xi)L^{-1/m}x + \partial_y\psi(w, w, \xi)L^{-1/m}y + O\left(L^{-1/m}\right)\,.$$

By points two and three of Lemma 4.1 we get

$$\psi(w + L^{-1/m}x, w + L^{-1/m}y, \xi) = L^{-1/m}\langle x - y, \xi\rangle + O\left(L^{-1/m}\right)\,.$$

Using this estimate in the exponential, together with the fact the rest of the integrand is $O\left(|\xi|^{mz_1+d}\right)$ we obtain that the quantity (25) equals

$$e^{iL^{-1/m}\langle\xi,x-y\rangle}\sigma_A(w,\xi)^z\sigma_P(w,w,(\xi,-\xi))+O\left(|\xi|^{mz_1+d}L^{-1/m}\right)$$

which is exactly (24). □

By Claim 1 and Equation (22) $PK_L\left(w+L^{-1/m}x,w+L^{-1/m}y\right)$ equals

$$PK_L\left(w+L^{-1/m}x,w+L^{-1/m}y\right)=\frac{1}{(2\pi)^n}\int_{1\le\sigma_A(w+L^{-1/m}y,\xi)\le L}e^{iL^{-1/m}\langle\xi,x-y\rangle}\sigma_A(w,\xi)^z\sigma_P(w,w,(\xi,-\xi))d\xi \tag{26}$$

$$+O\left(L^{-1/m}\int_{1\le\sigma_A(w+L^{-1/m}y,\xi)\le L}|\xi|^{mz_1+d}d\xi\right).$$

But since $mz_1+d+n>0$ and σ_A is m-homogeneous, the remainder is $O\left(L^{z_1+(n+d-1)/m}\right)=O(g(L))$. For each $y\in\Omega$ and each $L\ge 1$ let $\Delta(y,L)$ be the symmetric difference of the sets $\{\xi\in\mathbb{R}^n\mid 1\le\sigma_A(w,\xi)\le L\}$ and $\{\xi\in\mathbb{R}^n\mid 1\le\sigma_A(w+L^{-1/m}y,\xi)\le L\}$. Since σ_A is positive m-homogeneous in ξ and smooth in y, there exists $0<C<+\infty$ such that for each $L\ge 1$ and $w\in\Omega$, $Vol(\Delta(w,L))\le CL^{(n-1)/m}$ and for each $\xi\in\Delta(w,L)$, $C^{-1}L^{1/m}\le|\xi|\le CL^{1/m}$. Consequently, in Equation (26) we can replace the integration domain by $\{\xi\in\mathbb{R}^n\mid 1\le\sigma_A(w,\xi)\le L\}$ and produce an error of order $O\left(L^{z_1+(n+d-1)/m}\right)=O(g(L))$ uniformly for $y\in\Omega$ and $L\ge 1$. In other words,

$$PK_L\left(w+L^{-1/m}x,w+L^{-1/m}y\right)=\frac{1}{(2\pi)^n}\int_{1\le\sigma_A(w,\xi)\le L}e^{iL^{-1/m}\langle\xi,x-y\rangle}\sigma_A(w,\xi)^z\sigma_P(w,w,(\xi,-\xi))d\xi+O(g(L)).$$

Moreover, since $mz_1+d+n>0$ and the integrand scales like $|\xi|^{mz_1+d}$ near 0, adding the region $\sigma_A(w,\xi)\le 1$ to the integration domain creates a bounded error. Following this by the change of variable $\xi=L^{1/m}\zeta$ shows that uniformly for $w,x,y\in\Omega$ and $L\ge 1$

$$PK_L\left(w+L^{-1/m}x,w+L^{-1/m}y\right)=\frac{1}{(2\pi)^n}\int_{\sigma_A(w,\zeta)\le 1}e^{i\langle\zeta,x-y\rangle}\sigma_A(w,\zeta)^z\sigma_P(w,w,(\zeta,-\zeta))d\zeta L^{z+(n+d)/m}+O(g(L)).$$

This proves the first statement of the theorem for $V=\mathring{\Omega}$. To prove the second statement, observe that by Lemma 5.1, uniformly for $L\ge 1$ and $x,y\in\Omega$,

$$PK_L(x,y)=f(L)PE_L-\int_0^L f'(\lambda)PE_\lambda(x,y)d\lambda=L^zPE_L(x,y)-\int_1^L\lambda^{z-1}PE_\lambda(x,y)d\lambda+O(1)$$

Next, fix $W\subset V\times V$ a neighborhood of the diagonal. By Theorem 2.2, there exists $C'>0$ such that for any $(x,y)\in(V\times V)\setminus W$ and any $L\ge 1$, $|PE_L(x,y)|\le C'L^{(n+d-1)/m}$, which implies

$$|PK_L(x,y)|\le C'\left(L^{z_1+(n+d-1)/m}+\int_1^L\lambda^{z_1-1+(n+d-1)/m}d\lambda\right)=O(g(L)).$$

This proves the second statement of Theorem 2.3. □

6 Proof of Theorem 2.5

In this section we prove Theorem 2.5. We use the admissibility condition through Proposition 3.3. Suppose that $n+d+mz=0$, so that $z=-\frac{d+n}{m}$. By Proposition 3.5, uniformly for $x,y\in U$,

$$PK_L(x,y)=\frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)\le L}e^{i\psi(x,y,\xi)}\sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))f(\sigma_A(y,\xi))d\xi+O\left(L^{-1/m}\right).$$

Let $C < +\infty$ be such that $f(t) = t^z$ for $t > C$. Then,

$$PK_L(x,y) = \frac{1}{(2\pi)^n}\int_{C\le\sigma_A(y,\xi)\le L} e^{i\psi(x,y,\xi)}\sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(d+n)/m}d\xi$$
$$+Q_1(x,y) + O\left(L^{-1/m}\right)$$

where

$$Q_1(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)\le C} e^{i\psi(x,y,\xi)}\sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))f(\sigma_A(y,\xi))d\xi\,.$$

We will split the integral term in the last expression of PK_L as follows. For any $x, y \in U$, let

$$I_L(x,y) = \frac{1}{(2\pi)^n}\int_{C\le\sigma_A(y,\xi)\le L} \mathbb{1}[\sigma_A(y,\xi)|x-y|^m \ge 1]e^{i\psi(x,y,\xi)}\sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(d+n)/m}d\xi$$
$$II_L(x,y) = \frac{1}{(2\pi)^n}\int_{C\le\sigma_A(y,\xi)\le L} \mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1]e^{i\psi(x,y,\xi)}\sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(d+n)/m}d\xi.$$

Then, uniformly for $x, y \in U$,

$$PK_L(x,y) = I_L(x,y) + II_L(x,y) + Q_1(x,y) + O\left(L^{-1/m}\right). \tag{27}$$

Theorem 2.5 is an easy consequence of the following two lemmas.

Lemma 6.1. *Let $k_0 \in \mathbb{N}$, $k_0 \ge 2$. Suppose that either $n = 1$ or σ_A is $\frac{1}{k_0}$-admissible. There exist an open neighborhood $V \subset U$ of $0 \in \mathbb{R}^n$, a function $Q_2 \in L^\infty(V\times V)$ and a constant $C < +\infty$ such that for any $x, y \in V$ and $L \ge 1$,*

$$\Big|I_L(x,y) - Q_2(x,y)\Big| \le C\min\left(L^{-1/k_0 m}|x-y|^{-1/k_0}, 1\right)\,.$$

In dimension one, we prove the lemma using Lemma 4.4 while in the case of admissible symbols we use Proposition 3.1. This proof is the only place where we use these results.

Lemma 6.2. *There exist an open neighborhood $V \subset U$ of 0 and a constant $C < +\infty$ such that for all $x, y \in V$ and $L \ge 1$,*

$$\left|II_L(x,y) - \frac{1}{(2\pi)^n}Y_P(x,y)\left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\right]\right| \le C\,.$$

Moreover $II_L(x,y)$ is independent of L as long as $L \ge 1$ and $L|x-y|^m \ge 1$.

Let us first prove that these lemmas imply Theorem 2.5.

Proof of Theorem 2.5. Let V be the intersection of the V's appearing in Lemmas 6.1 and 6.2. Firstly, Lemma 6.1 implies that $I_L(x,y)$ is uniformly bounded for $x, y \in V$ and $L \ge 1$. Secondly, Lemma 6.2 implies that, uniformly for $x, y \in V$ and $L \ge 1$,

$$II_L(x,y) = \frac{1}{(2\pi)^n}Y_P(x,y)\left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\right] + O(1)\,.$$

Plugging these two estimates in Equation (27) we get the first point of Theorem 2.5. For the second point, we begin by observing that by Lemma 6.2, there exists a bounded function $Q_3 \in L^\infty(V\times V)$ such that for each $L \ge |x-y|^{-m}$,

$$II_L(x,y) = -\frac{1}{(2\pi)^n}Y_P(x,y)\ln\left(|x-y|\right) + Q_3(x,y)\,.$$

Moreover, if $L \geq |x-y|^{-m}$ then $L^{-1/k_0}|x-y|^{-1/k_0 m} \leq 1$ so by Lemma 6.1, uniformly for any such x, y and L,

$$I_L(x,y) = Q_2(x,y) + O\left(L^{-1/k_0}|x-y|^{-1/k_0 m}\right).$$

Applying these two estimates to Equation (27) we deduce that, uniformly for $x, y \in V$ and $L \geq 1$ such that $|x-y| \geq L^{-1/m}$,

$$PK_L(x,y) = -\frac{1}{(2\pi)^n} Y_P(x,y) \ln(|x-y|) + Q(x,y) + O\left(L^{-1/k_0}|x-y|^{-1/k_0 m}\right)$$

where $Q = Q_1 + Q_2 + Q_3 \in L^\infty(V \times V)$. This proves the estimate in the second point of Theorem 2.5. $\square$

Proof of Lemma 6.1. Suppose first that $\mathcal{X}$ has dimension $n = 1$ and fix $\Omega \subset U$ a compact neighborhood of 0. For $x \neq y$, setting $\eta = |x-y|\xi$, the integral $I_L(x,y)$ equals

$$\int_{a(x,y)}^{b(x,y,L)} e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P\left(x, y, |x-y|\partial_{x,y}\psi\left(x,y,|x-y|^{-1}\eta\right)\right) \sigma_A(y,\eta)^{-(d+1)/m} d\eta.$$

where $a(x,y)$ and $b(x,y,L)$ are the positive numbers defined by $\sigma_A(y, a(x,y)) = \max(C|x-y|^m, 1)$ and $\sigma_A(y, b(x,y,L)) = \max(|x-y|^m L, 1)$. Since σ_A is elliptic positive homogeneous of degree $m > 0$ there exists $C_1 > 0$ such that for each $x, y \in \Omega$ and $L \geq 1$,

$$b(x,y,L) \geq C_1 \min\left(|x-y|L^{-1/m}\right).$$

By Lemma 4.4, $I_L(x,y)$ converges to some limit $Q_2(x,y)$ as $L \to +\infty$ in such a way that the remainder term is $O\left(\min\left(|x-y|^{-1}L^{-1/m}, 1\right)\right)$. The case where $x = y$ follows by continuity and we have proved the lemma in the one-dimensional case with $V = \mathring{\Omega}$.

Suppose now that $n \geq 2$ and σ_A is $\frac{1}{k_0}$-admissible for some integer $k_0 \geq 2$. By Equations (1) and (10), for any $L \geq 1$ and $x, y \in U$,

$$\begin{aligned} I_L(x,y) &= \frac{1}{(2\pi)^n} \int_{C^{1/m}}^{L^{1/m}} \mathbb{1}[|x-y|t \geq 1] J_A(x,y,t) \frac{dt}{t} \\ &= \frac{1}{(2\pi)^n} \int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}[s \geq 1] J_A\left(x,y,|x-y|^{-1}s\right) \frac{ds}{s} \end{aligned}$$

By Proposition 3.1, there exist an open neighborhood $V \subset U$ of 0 and a constant $C_3 > 0$ such that, uniformly for distinct $x, y \in V$ and $t > 0$, $|J_A(x,y,t)| \leq C_3 (|x-y|t)^{-1/k_0}$. Therefore, for each $x, y \in V$ and $L > 0$,

$$\begin{aligned} \left|(2\pi)^n I_L(x,y) - \int_{C^{1/m}|x-y|}^{+\infty} \mathbb{1}[s \geq 1] J_A\left(x,y,|x-y|^{-1}s\right) \frac{ds}{s}\right| &\leq C_3 \int_{\max\left(|x-y|L^{\frac{1}{m}},1\right)}^{+\infty} s^{-1-1/k_0} ds \\ &= \frac{C_3}{k_0} \min\left(1, L^{-1/k_0 m}|x-y|^{-1/k_0}\right). \end{aligned}$$

By continuity, this stays true for $x = y$. This proves the lemma for σ_A admissible with

$$Q_2(x,y) = \int_{C^{1/m}|x-y|}^{+\infty} \mathbb{1}[s \geq 1] J_A\left(x,y,|x-y|^{-1}s\right) \frac{ds}{s}.$$

$\square$

Proof of Lemma 6.2. The proof of the second statement is obvious from the definition of II_L and the expression $\ln\left(L^{\frac{1}{m}}\right) - \ln_+\left(L^{\frac{1}{m}}|x-y|\right)$. We now prove the first statement. For each $y \in U$ and each $0 \le r_1 \le r_2$, we set

$$\mathcal{A}_y(r_1, r_2) = \{\xi \in \mathbb{R}^n \,|\, r_1 \le \sigma_A(y,\xi) \le r_2\}.$$

Recall that

$$II_L(x,y) = \frac{1}{(2\pi)^n}\int_{\mathcal{A}_y(C,L)} \mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1]e^{i\psi(x,y,\xi)}\sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(n+d)/m}d\xi..$$

By Equation (9), the integrand equals

$$\mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1]H_P\left(x,y,\sigma_A(y,\xi)^{-1/m}\xi, \sigma_A(y,\xi)^{1/m}\right)\sigma_A(y,\xi)^{-n/m}.$$

Since σ_A is positive homogeneous of degree m, $\sigma_A(y,\xi)^{-1/m}\xi$ is uniformly bounded for $y \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. By the second point of Lemma 4.3, uniformly for $x, y \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$H_P\left(x,y,\sigma_A(y,\xi)^{-1/m}\xi, \sigma_A(y,\xi)^{1/m}\right) = H_P\left(x,y,\sigma_A(y,\xi)^{-1/m}\xi, 0\right) + O\left(|x-y|\sigma_A(y,\xi)^{1/m}\right).$$

Again by m-homogeneity and positivity, $|x-y|\mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1]\sigma_A(y,\xi)^{(1-n)/m}$ is uniformly integrable in ξ for $x, y \in \Omega$ so

$$II_L(x,y) = \frac{1}{(2\pi)^n}\int_{\mathcal{A}_y(C,L)} \mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1]H_P\left(x,y,\sigma_A(y,\xi)^{-1/m}\xi, 0\right)\sigma_A(y,\eta)^{-n/m}d\xi + O(1).$$

Fix two distinct points $x, y \in U$. The change of variables $\eta = |x-y|\xi$ in the integral yields

$$\int_{|x-y|\mathcal{A}_y(C,L)} \mathbb{1}[\sigma_A(y,\eta) < 1]H_P(x,y,|x-y|\eta,0)\sigma_A(y,\eta)^{-n/m}d\eta$$

which, by definition of J_A (see Equation (10)), equals

$$J_A(x,y,0)\int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}[|x-y|s < 1]\frac{ds}{s}.$$

Observe that for any $0 < a \le b$,

$$\int_a^b \mathbb{1}[t<1]\frac{dt}{t} = \ln(b) - \ln_+(b) - \ln(a) + \ln_+(a)$$

where $\ln_+(s) = \max(\ln(s), 0)$. In our setting, uniformly for distinct $x, y \in \Omega$,

$$\int_{C^{1/m}}^{L^{1/m}} \mathbb{1}[|x-y|s < 1]\frac{ds}{s} = \int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}[t<1]\frac{dt}{t} = \ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right) + O(1).$$

Hence, uniformly for any $(x,y) \in \Omega \times \Omega$ and $L \ge 1$,

$$II_L(x,y) = \frac{1}{(2\pi)^n}J_A(x,y,0)\left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\right] + O(1).$$

Finally, by Equation (18) $J_A(x,y,0) = Y_P(x,y)$ so the lemma is proved with $V = \mathring{\Omega}$. $\square$

7 Admissible symbols

In this section, we deal with results concerning admissible symbols (see Definition 2.4). These results are useful in the proofs of Theorems 2.6 and 2.5. More precisely, in Subsection 7.1 we prove Proposition 3.3 which says that admissible symbols have non-degenerate level sets and is used in the proof of Theorem 2.5. Then, in Proposition 7.5 of Subsection 7.2 we prove that admissibility is both stable and generic in a suitable topology. Theorem 2.6 follows directly from Proposition 7.5.

7.1 Proof of Proposition 3.3

The object of this subsection is to prove Proposition 3.3. To prove this result, we will use partitions of unity and local charts to carry the integral onto $\mathbb{R}^n$ and then apply the following lemma, which we prove later in the section.

Lemma 7.1. *Let $n \in \mathbb{N}$, $n \geq 1$. Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 and $(f_\eta)_{\eta \in E}$ be a continuous family of smooth functions on U indexed by $E \subset \mathbb{R}^p$, an open neighborhood of 0. Fix $k \geq 1$ and assume that $d_0^k f_0 \neq 0$. Then, there exist $E' \subset E$ and $U' \subset U$ two open neighborhoods of the origin in $\mathbb{R}^p$ and $\mathbb{R}^n$ respectively, such that for each $u \in C_c^\infty(U')$ there exists $C(u) < +\infty$ such that for each $\lambda > 0$ and each $\eta \in E'$,*

$$\left| \int_{U'} e^{i\lambda f_\eta(x)} u(x) dx \right| \leq C(u) \lambda^{-\frac{1}{k}} \,.$$

Moreover, $C(u)$ depends continuously on u in the $C_c^\infty(U')$ topology.

We now begin the proof of Proposition 3.3.

Proof of Proposition 3.3. Take Ω, γ, $(f_\eta)_\eta$ and $(u_\eta)_\eta$ as in Definition 3.2. Recall that $d_x\mu$ is the area measure on S_x^*. By using partitions of unity on $\mathbb{R}^n$, we may fix $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and assume that the functions u_η are supported near ξ_0. Let $\xi_1, \dots, \xi_{n-1} \in \mathbb{R}^n$ be such that $(\xi_0, \xi_1, \dots, \xi_{n-1})$ forms a basis for $\mathbb{R}^n$. For any $x \in S^*U$, let

$$\beta_x : (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1} \mapsto \sigma(x, \xi_0 + t_1\xi_1 + \dots + t_{n-1}\xi_{n-1})^{-\frac{1}{m}} (\xi_0 + t_1\xi_1 + \dots + t_{n-1}\xi_{n-1}) \in S_x^* \,.$$

The map β_x defines a local coordinate system at $\sigma(x, \xi_0)^{-\frac{1}{m}} \xi_0 \in S_x^*$. Moreover, the map $x \mapsto \beta_x \in C^\infty(\mathbb{R}^{n-1})$ is continuous. The density $g_x = \frac{\beta_x^*(\gamma(x,\cdot)d_x\mu)}{dt} \in C^\infty(\mathbb{R}^{n-1})$ also depends continously on x in $C^\infty(\mathbb{R}^{n-1})$. Now, for any $\lambda > 0$, $\eta \in E$ and $(x, \tau) \in \Omega$, if u_η is supported close enough to ξ_0,

$$\int_{S_x^*} e^{i\lambda f_\eta(x,\tau,\xi)} u_\eta(\xi) \gamma(x, \xi) d_x\mu(\xi) = \int_{\mathbb{R}^{n-1}} e^{i\lambda f_\eta(x,\tau,\beta_x(t))} u_\eta(\beta_x(t)) g_x(t) dt \,.$$

We now set $\tilde{E} = U \times \mathbb{R}^n \times E$, for any $\tilde{\eta} = (x, \tau, \eta) \in \tilde{E}$, $\tilde{f}_{\tilde{\eta}} = f_\eta(x, \tau, \beta_x(\cdot)) \in C^\infty(\mathbb{R}^{n-1})$ and $\tilde{u}_{\tilde{\eta}} = u_\eta(\beta_x(\cdot)) g_x \in C^\infty(\mathbb{R}^{n-1})$. By compactness, it is enough to fix $(x_0, \tau_0) \in \Omega$ and prove estimate (12) for $\tilde{\eta} = (x, \eta, \tau)$ close enough to $\tilde{\eta}_0 = (x_0, 0, \tau_0)$. Also, without loss of generality, we may assume $x_0 = 0$. Our task is therefore to find $C > 0$ such that for each $\tilde{\eta}$ close enough to $\tilde{\eta}_0$ and each $\lambda > 0$,

$$\left| \int_{\mathbb{R}^{n-1}} e^{i\lambda \tilde{f}_{\tilde{\eta}}(t)} \tilde{u}_{\tilde{\eta}}(t) dt \right| \leq C \lambda^{-\frac{1}{k_0}} \,.$$

We wish to apply Lemma 7.1. The estimate is obvious for $\lambda \leq 1$ while, for $\lambda \geq 1$, replacing k_0 by some smaller integer would improve the estimate. Thus, we need only to check that there exists $k \in \{1, \dots, k_0\}$ such that

$$d_0^k \tilde{f}_{\tilde{\eta}_0} \neq 0 \,. \tag{28}$$

Let $g = \tilde{f}_{\tilde{\eta}_0}$. Since $f_0(x, \tau_0, \xi) = \langle \tau_0, \xi \rangle$, we have, for all $t \in \mathbb{R}^{n-1}$,

$$g(t) = (\langle \tau_0, \xi_1 \rangle t_1 + \dots + \langle \tau_0, \xi_{n-1} \rangle t_{n-1} + \langle \tau_0, \xi_0 \rangle) \sigma(0, \xi_0 + t_1\xi_1 + \dots + t_{n-1}\xi_{n-1})^{-\frac{1}{m}} \,.$$

We proceed by contradiction and assume that $d_0^j g = 0$ for each $j \in \{1, \dots, k\}$. To understand how this condition affects σ we use the following claim which we prove at the end.

Claim 2. *Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 and $f \in C^\infty(U)$ be positive valued. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{N}$ such that $k \geq 1$. Assume that there exist $b \in \mathbb{R}$ and $\tau \in \mathbb{R}^n$ such that $(\tau, b) \neq (0, 0)$ such that, writing $h : x \in \mathbb{R}^n \mapsto \langle \tau, x\rangle + b \in \mathbb{R}$ we have, for each $j \in \{1, \dots, k\}$,*

$$d_0^j[hf^\alpha] = 0\,. \tag{29}$$

Then,

$$f(0)^{k-1} d_0^k f = (\alpha+1)(2\alpha+1)\dots((k-1)\alpha+1)(d_0 f)^{\otimes k}\,. \tag{30}$$

We wish to use this claim with $\alpha = -\frac{1}{m}$, $h(t) = \langle \tau_0, \xi_1\rangle t_1 + \dots + \langle \tau_0, \xi_{n-1}\rangle t_{n-1} + \langle \tau_0, \xi_0\rangle$ and $f(t) = \sigma(0, \xi_0 + t_1\xi_1 + \dots + t_{n-1}\xi_{n-1})$. In order to apply it, the only thing to check is that h is not identically 0. But $h = 0$ would imply that $\langle \tau_0, \xi_0\rangle = \dots = \langle \tau_0, \xi_{n-1}\rangle = 0$. This cannot happen since $\tau_0 \neq 0$. Hence, by Claim 2 we have the following equality between (symmetric) k-forms on the hyperplane H spanned by $(\xi_1, \dots, \xi_{n-1})$,

$$\sigma(0, \xi_0)^{k-1} \partial_\xi^k \sigma(0, \xi) = C(m, k)(\partial_\xi \sigma(0, \xi_0))^{\otimes k} \tag{31}$$

where

$$C(m, k) = \left(-\frac{1}{m} + 1\right) \cdots \left(-\frac{k-1}{m} + 1\right) = \frac{m(m-1)\dots(m-k+1)}{m^k}\,.$$

Next, we make the following claim, which we prove at the end.

Claim 3. *Let m be a positive real number and let $f \in C^\infty(\mathbb{R}^p \setminus \{0\})$ be a real-valued m-homogeneous function. Then, for each $x \in \mathbb{R}^p \setminus \{0\}$, each hyperplane $H \subset \mathbb{R}^p$ not containing x and each $k_0 \geq 2$,*

$$\forall k \in \{2, \dots, k_0\},\ f(x)^{k-1} d_x^k f = \frac{m(m-1)\dots(m-k+1)}{m^k}(d_x f)^{\otimes k} \tag{32}$$

is equivalent to

$$\forall k \in \{2, \dots, k_0\},\ f(x)^{k-1} d_x^k f\big|_H = \frac{m(m-1)\dots(m-k+1)}{m^k}(d_x f)^{\otimes k}\big|_H\,. \tag{33}$$

This claim implies that σ actually satisfies Equation (31) on the whole of $T_\xi^*\mathbb{R}^n \simeq \mathbb{R}^n$. By the assumption on σ, this Equation cannot be satisfied for all $k \leq k_0$. Hence, $d_0^k g$ cannot vanish for each $k \in \{1, \dots, k_0\}$. In particular, there exists $k \in \{1, \dots, k_0\}$ for which $\tilde{f}_{\tilde{\eta}}$ satisfies Equation (28). Hence, Lemma 7.1 applies for this k and we are done.

Proof of Claim 2. Let f, τ, b, α, h and k be as in the statement of the claim. Let $g(x) = h(x)^{-\frac{1}{\alpha}}$. First of all, by Equation (29) with $j = 1$,

$$f(0)\tau = -\alpha b d_0 f$$

In particular, since $(\tau, b) \neq 0$ and $f(0) > 0$, we actually have $b \neq 0$. Thus, the function $g : x \mapsto h(x)^{-\frac{1}{\alpha}}$ is well defined and positive near the origin. Moreover, $hg^\alpha = 1$ so all of its derivatives vanish. Consequently, for each $j \in \{1, \dots, k\}$, $d_0^j(f^\alpha g^{-\alpha}) = 0$ which in turn gives, for each $j \in \{1, \dots, k\}$, $d_0^j(fg^{-1}) = 0$ (here we use the fact that $fg^{-1} = (f^\alpha g^{-\alpha})^{\frac{1}{\alpha}}$ which is well defined near 0). In particular, the Taylor expansions of f and g coincide to the kth order up to a multiplicative constant. By homogeneity of Equation (30) we may assume that they agree up to order k. But

$$\begin{aligned} d_0^k g &= \prod_{j=0}^{k-1}\left(-\frac{1}{\alpha} - j\right) \times b^{-\frac{1}{\alpha}-k}\tau^{\otimes k} \\ &= \left(b^{-\frac{1}{\alpha}}\right)^{1-k}(\alpha+1)(2\alpha+1)\dots((k-1)\alpha+1)\left(-\alpha b^{\frac{1}{\alpha}+1}\right)^{-k}\tau^{\otimes k} \end{aligned}$$

and $g(0) = b^{-\frac{1}{\alpha}}$ and $d_0 g = \left(-\alpha b^{\frac{1}{\alpha}+1}\right)^{-1} \tau$. Thus,

$$g(0)^{k-1} d_0^k g = (\alpha+1)(2\alpha+1)\dots((k-1)\alpha+1)(d_0 g)^{\otimes k}.$$

Since f agrees with g up to order k, f satisfies Equation (30). □

Proof of Claim 3. Equation (32) implies (33) by restriction to H. Let us assume (33) and prove the converse. Since $x \notin H$, $\mathbb{R}x \bigoplus H$ generate $\mathbb{R}^p$. By multilinearity, it is enough to prove (32) when the k forms are evaluated on families of the form $(x, \dots, x, y_1, \dots, y_h)$ where $y_1, \dots, y_h \in H$ and $h \in \{1, \dots, k\}$. Now, since f is homogeneous, by Euler's Equation, for any $h \le k$, and for any $y_1, \dots, y_h \in H$,

$$d_x^k f(x, \dots, x, y_1, \dots, y_h) = \underbrace{(m-h)\dots(m-k+1)}_{1 \text{ if } k=h} d_x^h f(y_1, \dots, y_h)$$
$$\text{and } (d_x f)^{\otimes k}(x, \dots, x, y_1, \dots, y_h) = m^{k-h} f^{k-h}(x) (d_x f)^{\otimes h}(y_1, \dots, y_h).$$

Applying (33) to compare the right hand sides of each line we get Equation (32). □

□

The proof of Lemma 7.1 will combine two theorems from singularity theory and oscillatory integral asymptotics which we state now.

The following theorem is a corollary of the Malgrange preparation theorem presented in [Hör03]. We give a slightly different formulation and add the continuity with respect to smooth perturbations, which actually follows from Hörmander's original proof.

Theorem 7.2 ([Hör03], Theorem 7.5.13)**.** *Let $U \subset \mathbb{R} \times \mathbb{R}^n$ (resp. $E \subset \mathbb{R}^p$) be an open neighborhood of $0 \in \mathbb{R} \times \mathbb{R}^n$ (resp. $0 \in \mathbb{R}^p$) and $(f_\eta)_{\eta \in E}$ be a continuous family of smooth functions on U. We denote by (t, x) the elements of U. Let $k \in \mathbb{N}$, $k \ge 2$. Assume that for each $\eta \in E$ and $j \in \{0, \dots, k-1\}$*

$$\partial_t^j f_\eta(0,0) = 0$$

and that $\partial_t^k f_\eta(0,0) > 0$. Then, there exist $W \subset \mathbb{R} \times \mathbb{R}^n$ (resp. $V \subset \mathbb{R}^n$) a neighbohood of $0 \in \mathbb{R} \times \mathbb{R}^n$ (resp. $0 \in \mathbb{R}^n$) with $U' \subset \mathbb{R} \times V$ such that for each $\eta \in E$, there exist $\phi_\eta \in C^\infty(W)$ as well as $a_\eta^1, \dots, a_\eta^{k-1} \in C^\infty(V)$, satisfying, for any $\eta \in E$, $(t, x) \in W$,

$$\phi_\eta(0,0) = 0,$$
$$\partial_t \phi_\eta(0,0) > 0,$$
$$a_\eta^1(0) = \dots = a_\eta^{k-1}(0) = 0,$$
$$\text{and } f_\eta(\phi_\eta(t,x), x) = t^k + \sum_{j=0}^{k-1} a_\eta^j(x) t^j.$$

Moreover, one can choose these functions such that the maps $\eta \mapsto \phi_\eta$ and $\eta \mapsto a_\eta^j$ are continuous into C^∞.

Proof. First, apply Theorem 7.5.13 of [Hör03] to each f_η and define $\tilde{\phi}_\eta(\cdot, x)$ as the inverse map of $T(\cdot, x)$ for the T corresponding to f_η. That the maps $\tilde{\phi}_\eta$ and $\tilde{a^j}_\eta$ depend continuously on η follows from the proof of the aforementioned result. Indeed, they are built as solutions of ODEs whose initial conditions depend continuously on f in C^∞. Finally, by rescaling the new variable t and thus replacing $\tilde{\phi}_\eta$ (resp. $\tilde{a^j}_\eta$) by ϕ_η (resp. a_η^j) we get rid of the $\frac{1}{k}$ factor appearing in front of T^k in Theorem 7.5.13 of [Hör03]. □

The following theorem is the special case of Theorem 4 of [Col77] (and the remarks 2.3 and 2.4 that follow it) of type A_n singularities.

Theorem 7.3 ([Col77], Theorem 4)**.** *Let $k \in \mathbb{N}$, $k \geq 2$. There exist $\delta = \delta(k) > 0$, $V \subset \mathbb{R}^k$ an open neighborhood of 0 such that for all $u \in C_c^\infty(]-\delta,\delta[)$ there exists $C(u) < +\infty$ such that for all $\lambda > 0$ and $(a_0,\dots,a_{k-1}) \in V$,*

$$\left|\int_{\mathbb{R}} e^{i\lambda(t^k+a_{k-1}t^{k-1}+\dots+a_0)}u(t)dt\right| \leq C(u)\lambda^{-\frac{1}{k}}\,.$$

Moreover $C(u)$ depends continuously on $u \in C_c^\infty(]-\delta,\delta[)$.

Proof. In the terminology of [Col77], the map $(t,a_0,\dots,a_{k-1}) \mapsto t^k + a_{k-1}t^{k-1} + \dots + a_0$ is the universal unfolding of the singularity type A_{k-1}. In the notations of [Col77] our k corresponds to their n while their k equals 1 in our setting. Moreover, as stated in the table preceding Theorem 4 of [Col77], in the case A_{k-1}, $\varepsilon(\sigma) = \frac{1}{2} - \frac{1}{k}$ so that the integral decays polynomially in λ at order $-\frac{1}{2} + \varepsilon(\sigma) = -\frac{1}{k}$. □

Proof of Lemma 7.1. Let $(f_\eta)_\eta$ and $k \geq 1$ be as in the statement of the lemma. To make use of the assumption $d_0^k f_0 \neq 0$ we use the following elementary result in multilinear algebra which we prove at the end.

Claim 4. *Let ω be a symmetric k-linear form on $\mathbb{R}^n$. Let $q : \mathbb{R}^n \to \mathbb{R}$ be defined as $q(x) = \omega(x,\dots,x)$. Then, $q = 0$ implies $\omega = 0$.*

By Claim 4 there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $d_0^k f_0(v,v,\dots,v) \neq 0$. Without loss of generality, we may assume that $v = e_n := (0,\dots,0,1)$. We write $x = (\tilde{x},x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$. Let $u \in C_c^\infty(U)$ be such that $u(\tilde{x},x_n) \neq 0$ implies that $\|\tilde{x}\|_\infty \leq 1$. Then, for each $\eta \in E$ and $\lambda > 0$,

$$\left|\int_U e^{i\lambda f_\eta(x)}u(x)dx\right| \leq \max_{\tilde{x}\in\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}} e^{i\lambda f_\eta(\tilde{x},x_n)}u(\tilde{x},x_n)dx_n\right|\,.$$

This way by replacing η by $(\eta,\tilde{x})$ and f_η by $f_\eta(\tilde{x},\cdot)$ we have reduced the problem to the one dimensional case. From now on, we assume that $n = 1$.

For each $\eta \in E$, each $x \in U$, and $q = (q_0,\dots,q_{k-1}) \in \mathbb{R}^k$, let

$$g_\eta(x,q) = f_\eta(x) - f_\eta(0) - f'_\eta(0)x - \dots - \frac{1}{(k-1)!}f_\eta^{(k-1)}(0)x^{k-1} + q_0 + a_1x + \dots + q_{k-1}x^{k-1}\,.$$

We will first prove the desired bound where we replace f_η by $g_\eta(\cdot,q)$, uniformly for q close enough to 0 and then deduce the result for f_η itself as a phase.

The map $\eta \mapsto g_\eta$ is continuous from E to $C^\infty(U\times\mathbb{R}^k)$. Moreover, for each $\eta \in E$ close enough to 0 we have

$$\forall j \in \{0,\dots,k-1\},\ \partial_x^j g_\eta(0,0) = 0$$
$$\partial_x^k g_\eta(0,0) \neq 0.$$

Replacing f_η by $-f_\eta$ does not change the estimate since it amounts to complex conjugation of the integrand. With this in mind, we may assume that $\partial_x^k g_\eta(0,0) > 0$. By Theorem 7.2, there exist $W \subset \mathbb{R}\times\mathbb{R}^k$ and continuous families of smooth functions $(a_\eta^1)_\eta \dots, (a_\eta^{k-1})_\eta$, as well as $(\phi_\eta)_\eta$ defined respectively in a neighborhood of $0 \in \mathbb{R}^k$ and a neighborhood of $(0,0)$ in $U\times\mathbb{R}^k$ such that for each $\eta \in E$ and (x,q) close enough to 0 and $(0,0)$ respectively,

$$\phi_\eta(0,0) = 0\,,$$
$$\partial_t\phi_\eta(0,0) > 0\,,$$
$$a_\eta^1(0) = \dots = a_\eta^{k-1}(0) = 0\,,$$
$$\text{and } g_\eta(\phi_\eta(x,q),q) = x^k + \sum_0^{k-1} a_\eta^j(q)x^j\,.$$

Hence, if $u \in C_c^\infty(\mathbb{R})$ is supported close enough to 0, we have, for all $\eta \in E$ close enough to 0 and all $q \in \mathbb{R}^k$,

$$\int_{\mathbb{R}} e^{i\lambda g_\eta(y,q)} u(y)dy = \int_{\mathbb{R}} e^{i\lambda(x^k + a_\eta^{k-1}(q)x^{k-1} + \cdots + a_\eta^0(q))} u(\phi_\eta(x,q)) \left(\phi_\eta^{-1}(\cdot,q)\right)'(x)dx\,.$$

By Theorem 7.3, there exist $W_1 \subset \mathbb{R}^k \times E$ a neighborhood of 0 such that $\delta > 0$ such that for each $(q,\eta) \in W_1$, for each $v \in C_c^\infty\left(]-\delta,\delta[\right)$, there exists $C'(v) < +\infty$ such that for each $\lambda > 0$ and each (q,η) close enough to $(0,0)$,

$$\left|\int_{\mathbb{R}} e^{i\lambda(x^k + a_\eta^{k-1}(q)x^{k-1} + \cdots + a_\eta^0(q))} v(x)dx\right| \leq C'(v)\lambda^{-\frac{1}{k}}\,.$$

Moreover, Theorem 7.3 specifies that the map $v \in C_c^\infty\left(]-\delta,\delta[\right) \to C'(v) \in \mathbb{R}$ is continuous. By continuity, there exist $\varepsilon > 0$ and $W_2 \subset W_1$ a compact neighborhood of 0 such that for any $(q,\eta) \in W_2$ and any $x \in \mathbb{R}$ with $|x| \geq \delta/2$, $|\phi_\eta(x,q)| \geq \varepsilon$. In particular, the map $(q,\eta,u) \in W_2 \times C_c^\infty\left(]-\varepsilon,\varepsilon[\right) \mapsto u(\phi_\eta(\cdot,q))\left(\phi_\eta(\cdot,q)^{-1}\right)' \in C_c^\infty\left(]-\delta,\delta[\right)$ is well defined and continuous. Consequently, so is the map

$$\begin{aligned} W_2 \times C_c^\infty\left(]-\varepsilon,\varepsilon[\right) &\to \mathbb{R} \\ (q,\eta,u) &\mapsto C_{q,\eta}(u) = C'\left(u(\phi_\eta(\cdot,q))\left(\phi_\eta(\cdot,q)^{-1}\right)'\right)\,. \end{aligned}$$

By compactness, $C(u) = \sup_{(q,\eta)\in W_2} C_{q,\eta}(u)$ is finite and continuous in u. We have proved that for any $(q,\eta) \in W_2$, any $\lambda > 0$ and any $u \in C_c^\infty(]-\varepsilon,\varepsilon[)$,

$$\left|e^{i\lambda g_\eta(y,q)}u(y)dy\right| \leq C(u)\lambda^{-\frac{1}{k}}\,.$$

To obtain the corresponding estimate with f_η instead of $g_\eta(\cdot,q)$, we make the following two observations. First, for each $\eta \in E$, and $x \in U$,

$$g_\eta(x, f_\eta(0), \ldots, f_\eta^{(k-1)}(0)) = f_\eta(x)\,.$$

Second, since $f_0(0) = \cdots = f_0^{(k-1)}(0) = 0$, there exists $E' \subset E$ a neighborhood of 0 such that for each $\eta \in E'$, $(f_\eta(0), \ldots, f_\eta^{(k-1)}(0), \eta) \in W_3$. Thus, for each $\eta \in E'$ each $u \in C_c^\infty(]-\varepsilon,\varepsilon[)$ and each $\lambda > 0$,

$$\left|e^{i\lambda f_\eta(y)}u(y)dy\right| \leq C(u)\lambda^{-\frac{1}{k}}$$

and the proof is over, save for the proof of Claim 4.

Proof of Claim 4. Let us prove the following formula.

$$\forall x_1,\ldots,x_k \in \mathbb{R}^n,\ \omega(x_1,\ldots,x_k) = \frac{1}{2^k}\sum_{\eta\in\{-1,1\}^k}\prod_{i=1}^k \eta_i \times q\left(\sum_{j=1}^k \eta_j x_j\right)\,.$$

For each $x_1,\ldots,x_k \in \mathbb{R}^n$ and $p = (p_1,\ldots,p_k) \in \mathbb{N}^k$ such that $p_1 + \cdots + p_k = k$, we denote by $\omega\left(x_1^{p_1}\ldots x_k^{p_k}\right)$ the form ω evaluated in any k-uple with exactly p_j occurrences of x_j ($\forall j \in \{1,\ldots,k\}$). Then, for each $x_1,\ldots,x_k \in \mathbb{R}^n$,

$$\begin{aligned}\sum_{\eta\in\{-1,1\}^k}\prod_i^k \eta_i \times q\left(\sum_{j=1}^k \eta_j x_j\right) &= \sum_{\eta\in\{-1,1\}^k}\prod_i^k \eta_i \times \sum_{p_1+\cdots+p_k=k}\binom{k}{p_1,\ldots,p_k}\omega\left((\eta_1x_1)^{p_1}\ldots(\eta_kx_k)^{p_k}\right) \\ &= \sum_{p_1+\cdots+p_k=k}\binom{k}{p_1,\ldots,p_k}\omega\left(x_1^{p_1}\ldots x_k^{p_k}\right)\sum_{\eta\in\{-1,1\}^k}\prod_i^k \eta_i^{p_i+1}\,.\end{aligned}$$

Given $j \in \{1, \ldots, j\}$ and $(p_1, \ldots, p_k)$ such that $p_j = 0$, applying the bijection

$$(\eta_1, \ldots, \eta_k) \mapsto (\eta_1, \ldots, -\eta_j, \ldots, \eta_k)$$

shows that $\sum_{\eta \in \{-1,1\}^k} \prod_j \eta_j^{p_j+1} = 0$. Thus, the only remaining term is the one corresponding to $p_1 = \cdots = p_k = 1$ for which the sum of products of the $\varepsilon_j^{p_j+1}$ equals 2^k. Therefore,

$$\sum_{\eta \in \{-1,1\}^k} \prod_i \eta_i q \left(\sum_i \eta_i v_i \right) = 2^k \omega(v_1, \ldots, v_k)$$

as announced. □

□

7.2 Genericity and stability of the non-degeneracy condition

The goal of this subsection is to prove Proposition 7.5 below, which says roughly that admissible symbols are stable and generic. To give a precise meaning to this statement, we first need to define a topology on the set of positive homogeneous symbols.

Definition 7.4. *Fix $n \in \mathbb{N}$, $n \geq 1$ and $m \in]0; +\infty[$. For each $U \subset \mathbb{R}^n$, let $S_h^m(U) \subset C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$ be the set of smooth functions m-homogeneous in the second variable. We write $S_{h,+}^m(U)$ for the set of positive valued functions in $S_h^m(U)$. The map*

$$S_h^m(U) \to C^\infty(U \times S^{n-1}),$$

restricting the second variable to the unit sphere, is a bijection. We endow $S_h^m(U)$ with the topology induced by the Whitney topology on $C^\infty(U \times S^{n-1})$ (see Definition 3.1 of Chapter II of [GG73]).

We have the following proposition.

Proposition 7.5. *For all $n \in \mathbb{N}$, $n \geq 2$ we define $k_0 = k_0(n) \in \mathbb{N}$ as follows. We set $k_0(2) = 5$, $k_0(3) = 3$, $k_0(4) = 3$ and $\forall n \geq 5$, $k_0(n) = 2$. Fix $n \geq 2$ and $m > 0$. Let $U \subset \mathbb{R}^n$ be an open subset. Then, the set of $\sigma \in S_{h,+}^m(U)$ such that for each $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ there exists $j \in \{2, \ldots, k_0\}$ such that*

$$\sigma^{j-1}(x,\xi)\partial_\xi^j \sigma(x,\xi) \neq \frac{m(m-1)\ldots(m-j+1)}{m^j} \left(\partial_\xi \sigma(x,\xi)\right)^{\otimes j} \tag{34}$$

is open and dense in $S_{h,+}^m(U)$.

To prove this proposition, we will apply Thom's transversality theorem (see Theorem 4.9 of Chapter II of [GG73]) to a well chosen submanifold of the jet bundle of $U \times S^{n-1}$ whose codimension grows with the degree of admissibility we consider. Lemmas 7.6, 7.7, 7.8, 7.9 and 7.10 below are devoted to the construction of this manifold. The proof of Proposition 7.5 is presented only after these are stated and proved. Throughout the rest of the section we fix $n \in \mathbb{N}$, $n \geq 2$, $U \subset \mathbb{R}^n$ an open subset and $m \in \mathbb{R}$, $m > 0$. We start by introducing some notation.

Notation:

1. For each $j, p \in \mathbb{N}$, $p \geq 1$, let Sym_p^j be the space of symmetric j-linear forms over $\mathbb{R}^p$. This is a vector space of dimension $\binom{p+j-1}{j}$. We adopt the convention that $\mathrm{Sym}_p^0 = \mathbb{R}$.

2. Let X be a smooth manifold. For each $k \geq 0$ we denote by $\mathcal{J}^k(X)$ the k-th jet space of mappings from X to $\mathbb{R}$, that is, the space $J^k(X, \mathbb{R})$ introduced in Definition 2.1 of Chapter II of [GG73]. For any $p \in \mathbb{N}$ and any open subset $V \subset \mathbb{R}^p$, the space $\mathcal{J}^k(V)$ is canonically isomorphic to $V \times \bigoplus_{j=0}^k \mathrm{Sym}_p^j$. We will denote its elements by (ξ, ω) where $\xi \in V$ and $\omega = (\omega_0, \ldots, \omega_k) \in \bigoplus_{j=0}^k \mathrm{Sym}_p^j$.

3. Let X be a smooth manifold and $k \in \mathbb{N}$. For each $f \in C^\infty(X)$, we write $j^k f$ for the section of $\mathcal{J}^k(X)$ whose value at each point is the k-jet of f at this point (see the paragraph below Definition 2.1 of Chapter II of [GG73]).

Since the jet bundle $\mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ is quite explicit, we will make most of our contructions inside it and them 'push them down' onto the sphere. In the following lemma, we build the map we need to 'push down' our constructions.

Lemma 7.6. *Let $\iota : S^{n-1} \to \mathbb{R}^n$ be the canonical injection. Then, there exists a bundle morphism*

$$\rho : \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \to \mathcal{J}^k\left(S^{n-1}\right)$$

such that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n \setminus \{0\}) & \xrightarrow{\iota^*} & C^\infty\left(S^{n-1}\right) \\ {\scriptstyle \iota^*(j^k \cdot)}\Big\downarrow & & \Big\downarrow{\scriptstyle j^k} \\ \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) & \xrightarrow{\rho} & \mathcal{J}^k\left(S^{n-1}\right) \end{array} .$$

Here the top arrow is the restriction map while the left arrow is the restriction of the k-jet to the sphere.

Proof. We construct ρ by defining its action on each fiber. Let $\xi \in S^{n-1}$ and let (V, ϕ) be a chart $\phi : V \to \mathbb{R}^{n-1}$ of S^{n-1} near ξ. Then, for each $f \in C^\infty(\mathbb{R}^n)$, the k-th order Taylor expansion of $f \circ \phi^{-1}$ at ξ depends only on the k-th order Taylor expansion of f at ξ and the dependence is linear. This defines a linear map $\rho|_\xi : \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})|_\xi \to \mathcal{J}^k\left(S^{n-1}\right)|_\xi$. The corresponding fiberwise map ρ is clearly smooth and defines a morphism of smooth vector bundles. Moreover, by construction, for each $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and each $\xi \in S^{n-1}$, $\rho|_\xi\left(j^k f(\xi)\right) = j^k(f \circ \iota)(\xi)$ so the diagram does indeed commute. $\square$

Notation:

For each $k \in \mathbb{N}$, each $\xi \in \mathbb{R}^n$ and each $\omega = (\omega_0, \dots, \omega_k) \in \bigoplus_{j=0}^k \mathrm{Sym}_n^j$ we introduce the following notation. For each $j \in \{0, \dots, k\}$, $\omega_j|_{\xi^\perp}$ is the restriction of ω_j to the orthogonal of ξ in $\mathbb{R}^n$. Moreover, we set $\omega|_{\xi^\perp} = (\omega_0|_{\xi^\perp}, \dots, \omega_k|_{\xi^\perp})$.

In the following lemma, we check that the set of jets of homogeneous maps is a smooth submanifold of $\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ and give an explicit description of it. Moreover, we show that the 'push down' map ρ maps it diffeomorphically on the space $\mathcal{J}^k\left(S^{n-1}\right)$.

Lemma 7.7. *Fix $k \in \mathbb{N}$. Let H_m^k be the subset of $\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ of jets of m-homogeneous functions. Then,*

1. *The set H_m^k is characterized by the following equations:*

$$H_m^k = \cap_{j=0}^{k-1} \left\{ (\xi, \omega) \in \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \,\middle|\, \omega_{j+1}(\xi, \dots) = (m-j)\omega_j \right\} .$$

2. *The set H_m^k is a submanifold of $\iota^* \mathcal{J}^k(U \times \mathbb{R}^n \setminus \{0\})$ of the same dimension as $\mathcal{J}^k\left(S^{n-1}\right)$.*

3. *The map $\rho|_{H_m^k} : H_m^k \to \mathcal{J}^k\left(U \times S^{n-1}\right)$ is a diffeomorphism.*

Proof. We set

$$\widetilde{H_m^k} = \cap_{j=0}^{k-1} \left\{ (\xi, \omega) \in \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \,\middle|\, \omega_{j+1}(\xi, \dots) = (m-j)\omega_j \right\} .$$

Firstly, each m-homogeneous $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$, satisfies Euler's equation. That is, for each $\xi \in \mathbb{R}^n \setminus \{0\}$, $d_\xi f(\xi) = m f(\xi)$. Next, notice that if f is m-homogeneous, then, for each $j \in \{1, \dots, k\}$, $\xi \mapsto d_\xi^j f$ is homogeneous of order $m-j$ so that for each $\xi \in \mathbb{R}^n \setminus \{0\}$, $d_\xi\left(d^j f\right)(\xi, \dots) = (m-j) d_\xi^j f$. Therefore, for each $\xi \in \mathbb{R}^n \setminus \{0\}$, $j^k f(\xi) \in \widetilde{H_m^k}$. We have shown that $H_m^k \subset \widetilde{H_m^k}$. Next, notice that for each $f \in C^\infty\left(S^{n-1}\right)$, the m-homogeneous function $\xi \mapsto |\xi|^m f\left(\frac{\xi}{|\xi|}\right)$ restricts back to f on S^{n-1}. Therefore, we have $\mathcal{J}^k\left(S^{n-1}\right) = \rho\left(H_m^k\right) \subset \rho\left(\widetilde{H_m^k}\right) \subset \mathcal{J}^k\left(S^{n-1}\right)$. So we have

$$\rho\left(H_m^k\right) = \rho\left(\widetilde{H_m^k}\right) = \mathcal{J}^k\left(S^{n-1}\right) . \tag{35}$$

Given this equation, in order to prove the lemma, it is enough to prove points 2 and 3 with H_m^k replaced by $\widetilde{H_m^k}$, which we call 2' and 3' respectively. Indeed, point 3' will imply that $\rho|_{\widetilde{H_m^k}}$ is one-to-one so by Equation (35), we will have $H_m^k = \widetilde{H_m^k}$ which is point 1. Moreover, since we will have already proved points 2 and 3 for $\widetilde{H_m^k}$ we will have them for H_m^k. Let us start by proving 2'. For each $j \in \{0, \dots, k-1\}$ set

$$F_m^j : (\xi, \omega) \mapsto \omega_{j+1}(\xi, \dots) - (m-j)\omega_j$$

so that $\widetilde{H_m^k} = \cap_{j=0}^{k-1} \left(F_m^j\right)^{-1}(0)$. Let us prove that the map

$$F_m = \left(F_m^0, \dots, F_m^{k-1}\right) : \iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right) \to \bigoplus_{j=0}^{k-1} \mathrm{Sym}_n^j$$

is a submersion. Fix $(\xi, \omega) \in \iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)$. Let $(\eta_0, \dots, \eta_{k-1}) \in \bigoplus_{j=0}^{k-1} \mathrm{Sym}_n^j \simeq T_{F_m(\xi,\omega)} \bigoplus_{j=0}^{k-1} \mathrm{Sym}_n^j$. Then, for each $j \in \{0, \dots, k-1\}$,

$$\partial_{\omega_{j+1}} F_m^j(\xi, \omega)\left(|\xi|^{-2}\langle \xi, \cdot\rangle \otimes \eta_j\right) = \eta_j .$$

In particular, $d_{(\xi,\omega)} F_m$ is surjective. Therefore $\widetilde{H_m^k}$ is a submanifold of $\iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)$ of codimension

$$codim_{\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})}\left(\widetilde{H_m^k}\right) = \sum_{j=0}^{k-1} dim\left(\mathrm{Sym}_n^j\right) = \sum_{j=0}^{k-1} \binom{n+j-1}{j} .$$

Indeed, recall that $dim\left(\mathrm{Sym}_n^j\right) = \binom{n+j-1}{j}$. Using this identity, we also have:

$$dim\left(\iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)\right) = (n-1) + \sum_{j=0}^{k} \binom{n+j-1}{j} ;$$

$$dim\left(\mathcal{J}^k\left(S^{n-1}\right)\right) = (n-1) + \sum_{j=0}^{k} \binom{n+j-2}{j} .$$

Therefore, firstly $dim\left(\widetilde{H_m^k}\right) = (n-1) + \binom{n+k-1}{k}$ and secondly

$$dim\left(\widetilde{H_m^k}\right) - dim\left(\mathcal{J}^k\left(S^{n-1}\right)\right) = \binom{n+k-1}{k} - \sum_{j=0}^{k} \binom{n+j-2}{j} = 0 . \tag{36}$$

In the last equality we use a well known binomial formula which is easily checked by induction on k. The conclusion here is that $\widetilde{H_m^k}$ has the same dimension as $\mathcal{J}^k\left(S^{n-1}\right)$ so we have proved 2'. To prove 3' observe that ρ is linear on each fiber of $\iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)$ so that its derivative $d\rho$ is constant on each fiber. Moreover, it is equivariant with respect to the automorphisms of the base space S^{n-1} so its derivative must have the same rank on different fibers. Since ρ is surjective (see Equation (35)) $d\rho$ must be of maximal rank. This proves that ρ is a local diffeomorphism. But since it is a morphism of vector bundles, it must be a diffeomorphism, which is the claim of 3'. This concludes the proof of the lemma. □

In the following lemma, we build a submanifold of H_m^k that describes the condition of non-admissibility and compute its codimension.

Lemma 7.8. *For each $k \in \mathbb{N}$, $k \geq 2$, define*

$$Y_m^k = \cap_{j=2}^k \left\{ (\xi, \omega) \in \iota^* \mathcal{J}^k \left(\mathbb{R}^n \setminus \{0\} \right) \,\middle|\, \omega_0 > 0,\, \omega_0^{j-1} \omega_j|_{\xi^\perp} = \frac{m(m-1)\dots(m-j+1)}{m^j} \left(\omega_1|_{\xi^\perp} \right)^{\otimes j} \right\} .$$

Then, $Y_m^k \cap H_m^k$ is a closed submanifold of H_m^k of codimension $\sum_{j=2}^k \binom{n+j-2}{j}$.

Proof. For each $j \in \{0, \dots, k-1\}$, each $l \in \{2, \dots, k\}$ and each $(\xi, \omega) \in \iota^* \mathcal{J}^k \left(\mathbb{R}^n \setminus \{0\} \right)$, let, as before, $F_m^j(\xi, \omega) = \omega_{j+1}(\xi, \dots) - (m-j)\omega_j \in Sym_n^{j-1}$. Moreover, let $\mathrm{Sym}_n^l|_{\xi^\perp}$ be the set of symmetric l-linear forms acting on the orthogonal of ξ in $\mathbb{R}^n$ and let $G_m^l(\xi, \omega) = \omega_l|_{\xi^\perp} - \frac{m(m-1)\dots(m-l+1)}{m^l} \left(\omega_1|_{\xi^\perp} \right)^{\otimes l} \in \mathrm{Sym}_n^l|_{\xi^\perp}$. Then, $Y_m^k \cap H_m^k$ is the intersection of the zero sets of the functions F_m^j and G_m^l for $j \in \{0, \dots, k-1\}$ and $l \in \{2, \dots, k\}$. In particular, it is closed. Note first that $\partial_{\omega_0} F_m^0 = m \neq 0 \in \mathrm{Hom}\left(\mathrm{Sym}_n^0, \mathrm{Sym}_n^0 \right) \simeq \mathbb{R}$. In particular this map is invertible. We will now prove that for each $l \in \{2, \dots, k\}$, the map $(\partial_{\omega_l} F_m^{l-1}, \partial_{\omega_l} G_m^l)$ is of maximal rank on Y_m^k. For any $(\xi, \omega) \in Y_m^k$ and any $l \in \{2, \dots, k\}$, $(\partial_{\omega_l} F_m^{l-1}(\xi, \omega), \partial_{\omega_l} G_m^l(\xi, \omega))$ acts as follows.

$$\begin{aligned} \mathrm{Sym}_n^l &\to \mathrm{Sym}_n^{l-1} \bigoplus \mathrm{Sym}_n^l|_{\xi^\perp} \\ \eta_l &\mapsto \left(\eta_l(\xi, \dots), \omega_0^{l-1} \eta_l|_{\xi^\perp} \right). \end{aligned}$$

But this map is invertible. To see this, let $pr_{\xi^\perp}^* : \mathrm{Sym}_n^l|_{\xi^\perp} \to \mathrm{Sym}_n^l$ be the pull-back map by the orthogonal projection onto the orthogonal of ξ. Also, recall that on Y_m^k, we have $\omega_0 > 0$. Then, the inverse of $(\partial_{\omega_l} F_m^{l-1}(\xi, \omega), \partial_{\omega_l} G_m^l(\xi, \omega))$ is

$$\begin{aligned} \mathrm{Sym}_n^{l-1} \bigoplus \mathrm{Sym}_n^l|_{\xi^\perp} &\to \mathrm{Sym}_n^l \\ (\eta_{l-1}, \eta|_\perp) &\mapsto |\xi|^{-2} \langle \xi, \cdot \rangle \otimes \eta_{l-1} + \omega_0^{1-l} pr_{\xi^\perp}^* \eta_\perp . \end{aligned}$$

All in all, we have shown so far that $\partial_{\omega_0} F_m^0$ is surjective and that for each $l \in \{2, \dots, k\}$, $(\partial_{\omega_l} F_m^{l-1}, \partial_{\omega_l} G_m^l)$ is of maximal rank. Therefore, $Y_m^k \cap H_m^k$ is a submanifold of H_m^k of codimension

$$\begin{aligned} codim_{H_m^k}(Y_m^k \cap H_m^k) &= codim_{\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})} \left(Y_m^k \cap H_m^k \right) - codim_{\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})} \left(H_m^k \right) \\ &= 1 + \sum_{l=2}^k \binom{n+l-1}{l} - \sum_{j=0}^{k-1} \binom{n+j-1}{j} \\ &= \binom{n+k-1}{k} - \binom{n+1-1}{1} \\ &= \sum_{j=2}^k \binom{n+j-2}{j} \end{aligned}$$

where in the last line we use the same binomial identity as in Equation (36). □

So far we have neglected the U coordinate in the product $U \times S^{n-1}$. To take this coordinate into account, in the following lemma, we introduce a submersion $pr_2 : \mathcal{J}^k \left(U \times S^{n-1} \right) \to \mathcal{J}^k \left(S^{n-1} \right)$ by which we will pull back the submanifold $\rho \left(Y_m^k \right)$.

Lemma 7.9. *Let $k \in \mathbb{N}$. Let $\pi : U \times S^{n-1} \to S^{n-1}$ be the map $(x, \xi) \mapsto \xi$. Also, for each $x \in U$, let $\iota_x : S^{n-1} \to U \times S^{n-1}$ be the map $\xi \mapsto (x, \xi)$. Then, there exists a surjective vector bundle morphism $pr_2 : \mathcal{J}^k\left(U \times S^{n-1}\right) \to \pi^*\mathcal{J}^k\left(S^{n-1}\right)$ such that for each $x \in U$, the following diagram commutes:*

$$\begin{array}{ccccc} C^\infty\left(U \times S^{n-1}\right) & & \xrightarrow{\quad\iota_x^*\quad} & & C^\infty\left(S^{n-1}\right) \\ \downarrow j^k & & & & \downarrow j^k \\ \mathcal{J}^k\left(U \times S^{n-1}\right) & \xrightarrow{pr_2} & \pi^*\mathcal{J}^k\left(S^{n-1}\right) & \xrightarrow{\iota_x^*} & \mathcal{J}^k\left(S^{n-1}\right). \end{array}$$

In particular, pr_2 is a submersion.

Proof. Given $f \in C^\infty\left(U \times S^{n-1}\right)$ and $x \in U$, the k-jet of $f(x, \cdot)$ at $\xi \in S^{n-1}$ depends only on the k-jet of f at (x, ξ). This allows us to define a map $pr_2|_{(x,\xi)} : \mathcal{J}^k\left(U \times S^{n-1}\right)|_{(x,\xi)} \to \pi^*\mathcal{J}^k\left(S^{n-1}\right)_{(x,\xi)}$. This defines a bundle morphism $pr_2 : \mathcal{J}^k\left(U \times S^{n-1}\right) \to \pi^*\mathcal{J}^k\left(\right)$. The fact that the diagram commutes follows by construction. Finally, since the composition of the top and right arrows : $j^k \circ \iota_x^*$ is onto, so is the composition of the left and bottom arrows. But this implies that the composition of bottom arrows is onto. Since $\pi^*\mathcal{J}^k\left(S^{n-1}\right)$ and $\mathcal{J}^k\left(S^{n-1}\right)$ have the same rank, then pr_2 must also be onto. In particular, it defines a submersion from the manifold $\mathcal{J}^k\left(U \times S^{n-1}\right)$ to the manifold $\pi^*\mathcal{J}^k\left(S^{n-1}\right)$. ☐

In this last lemma, we check that the previous construction does indeed characterize non-admissibility of a symbol by the intersection of the k-jet with the submanifold constructed in Lemma 7.8 and 'pushed down' by ρ.

Lemma 7.10. *Let $k \in \mathbb{N}$, $k \geq 2$. Let $\sigma \in S^m_{h,+}(U)$. Then, there exists $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ such that for each $j \in \{2, \dots, k\}$*

$$\sigma^{j-1}(x,\xi)\partial_\xi^j \sigma(x,\xi) = \frac{m(m-1)\dots(m-j+1)}{m^j}\left(\partial_\xi \sigma(x,\xi)\right)^{\otimes j} \tag{37}$$

if and only if $pr_2 \circ j^k\left(\sigma|_{U \times S^{n-1}}\right)\left(U \times S^{n-1}\right) \cap \rho\left(Y_m^k\right) \neq \emptyset$.

Proof. Firstly, Equation (37) is homogeneous in ξ so there exists a pair $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ satisfying it if and only if there exists such a pair in $U \times S^{n-1}$. Now, since σ is m-homogeneous, for each $x \in U$, $j^k(\sigma(x,\cdot))(S^{n-1}) \subset H_m^k$. Therefore, $(x, \xi) \in U \times S^{n-1}$ satisfy Equation (37) if and only if $j^k(\sigma(x,\cdot))(\xi) \in Y_m^k \cap H_m^k$ (here we use that the symbols are positive, as well as m-homogeneous). Since, moreover, by Lemma 7.8, $\rho|_{H_m^k}$ is bijective, this is equivalent to $\rho \circ j^k(\sigma(x,\cdot))(\xi) \in \rho(Y_m^k)$. But, by Lemmas 7.6 and 7.9, $\rho \circ j^k(\sigma(x,\cdot)) = j^k(\sigma(x,\cdot)|_{S^{n-1}}) = pr_2 \circ j^k\left(\sigma|_{U \times S^{n-1}}\right)(x,\cdot)$. To conclude, we have proved that for any $(x, \xi) \in U \times S^{n-1}$, (x, ξ) satisfies Equation (37) if and only if $pr_2 \circ j^k\left(\sigma|_{S^{n-1}}\right)(x, \xi) \in \rho\left(H_m^k\right)$. This concludes the proof of the lemma. ☐

We are now ready to prove Proposition 7.5.

Proof of Proposition 7.5. Firstly, by Lemma 7.10, Equation (34) has solutions in $U \times (\mathbb{R}^n \setminus \{0\})$ if and only if $j^k\left(\sigma|_{U \times S^{n-1}}\right)\left(U \times S^{n-1}\right) \cap pr_2^{-1}\left(\rho(Y_m^k)\right) \neq \emptyset$. Now, by Lemmas 7.7 and 7.8, $\rho\left(Y_m^k\right)$ is a closed submanifold of $\mathcal{J}^k\left(S^{n-1}\right)$ of codimension $\sum_{j=2}^k \binom{n+j-2}{j}$. Since moreover, by Lemma 7.9, pr_2 is a submersion, $Z_m^k = pr_2^{-1}\left(\rho(Y_m^k)\right)$ has the same codimension in $\mathcal{J}^k\left(U \times S^{n-1}\right)$. At this point, we apply Thom's transversality theorem (Corollary 4.10 of Chapter II of [GG73]). This theorem states that the functions $f \in C^\infty\left(U \times S^{n-1}\right)$ such that $j^k(f)(U \times S^{n-1})$ is transverse to Z_m^k is open and dense. But $j^k(f)\left(U \times S^{n-1}\right)$ has dimension at most $2n - 1$ so if k is such that

$$2n - 1 < \sum_{j=2}^k \binom{n+j-2}{j} \tag{38}$$

then such a transverse intersection must be empty. Inequality (38) is satisfied for instance for $n = 2$ and $k = 5$, for $n \in \{3, 4\}$ and $k = 3$ and for $n \geq 5$ and $k = 2$. This ends the proof of the proposition. ☐

A Proof of Theorem 2.2

In this section, we prove Theorem 2.2 by following closely the approach used in [Hör68] and in [GW14]. As explained above, [GW14] contains all the essential arguments for Theorem 2.2 despite the focus on the case where $x = y$ and $\mathcal{X}$ is closed. In this section we merely wish to confirm this by revisiting the proof. We consider A, σ_A and E_L indifferently as in any of the two settings presented in Subection 2.1.

A.1 Preliminaries

The following lemma summarizes the results proved in Section 4 of [Hör68] for the closed manifold setting. For the boundary problem, this was proved in Section 3 of [Vas84]. We introduce the following notation. For each $L > 0$, set $\tilde{E}_L = E_{L^m}$.

Lemma A.1. *Firstly, the spectral function $\tilde{E}_L(x, y)$ defines a tempered distribution of the L variable with values in $C^\infty(\mathcal{X} \times \mathcal{X})$. In addition, for each set of local coordinates in which $d\mu_{\mathcal{X}}$ coincides with the Lebesgue measure on $\mathbb{R}^n$, there is an open neighborhood U of $0 \in \mathbb{R}^n$ such that there exist $\varepsilon > 0$, a proper phase function $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$, a symbol $\sigma \in S^1(U, \mathbb{R}^n)$, a function $k \in C^\infty(U \times U \times]-\varepsilon, \varepsilon[)$ and a symbol $q \in S^0(U \times]-\varepsilon, \varepsilon[\times U, \mathbb{R}^n)$, for which*

$$\mathcal{F}_L[\tilde{E}'_L(x, y)](t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} q(x, t, y, \xi) e^{i(\psi(x,y,\xi) - t\sigma(y,\xi))} d\xi + k(x, y, t).$$

Here $\mathcal{F}_L$ (resp. $'$) denotes the Fourier transform (resp. the derivative) with respect to the variable L, in the sense of temperate distributions, and the integral is to be understood in the sense of Fourier integral operators (see Theorem 2.4 of [Hör68]). We have

1. *The function ψ satisfies the Equation*

$$\forall x, y \in U,\ \xi \in \mathbb{R}^n,\ \sigma(x, \partial_x \psi(x, y, \xi)) = \sigma(y, \xi).$$

2. *For each $t \in]-\varepsilon, \varepsilon[$ and $\xi \in \mathbb{R}^n$, the function $q(\cdot, t, \cdot, \xi)$ has compact support in $U \times U$ uniformly in (t, ξ) and $q(x, 0, y, \xi) - 1$ is a symbol of order -1 as long as x, y belong to some open neighborhood U_0 of 0 in U.*

3. *$\sigma - \sigma_A^{\frac{1}{m}} \in S^0$.*

We will also need the following classical lemma. Here and below, $\mathcal{S}(\mathbb{R})$ will denote the space of Schwartz functions.

Lemma A.2. *For each $\varepsilon > 0$ there is a function $\rho \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(\rho)$ has compact support contained in $]-\varepsilon, \varepsilon[$, $\rho > 0$ and $\mathcal{F}(\rho)(0) = 1$.*

Proof. Choose $f \in \mathcal{S}(\mathbb{R})$ whose Fourier transform has support in $]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. Then it is easy to see that $\rho = f^2 * f^2$ satisfies the required properties. □

Before we proceed, let us fix U, ψ, q, k and ρ as in Lemmas A.1 and A.2 as well as a differential operator P on $\mathcal{X} \times \mathcal{X}$ of order d with principal symbol σ_P. Let $\tilde{E}_{L,P} = P\tilde{E}_L$. In order to estimate this $\tilde{E}_{L,P}$, we will first convolve it with ρ in order to estimate it using Lemma A.1. Then, we will compare $\tilde{E}_{L,P}$ to its convolution with ρ which we denote - somewhat liberally - by

$$\rho * \tilde{E}_{L,P} = \int_{\mathbb{R}} \rho(\lambda) \tilde{E}_{L-\lambda,P} d\lambda\,.$$

The starting point of the following calculations will be the following Equation, which follows from Lemma A.1.

$$\frac{d}{d\lambda}(\rho * e_{\lambda,P}(x,y))|_{\lambda=L} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)(t)P\left(q(x,t,y,\xi)e^{i(\psi(x,t,y,\xi)-t\sigma(y,\xi))}\right)\Big](L)d\xi + \mathcal{F}_t^{-1}\big[\mathcal{F}(\rho)(t)Pk(x,t,y)\big](L). \tag{39}$$

A.2 Estimating the convolved kernel

In this section we provide the following expression for $\rho * \tilde{E}_{L,P}$ in the local coordinates chosen in Lemma A.1.

Lemma A.3. *There is an open set $V \subset U$ containing 0 such that, as $L \to \infty$ and uniformly for $(x,y) \in V \times V$,*

$$\rho * \tilde{E}_{L,P}(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\leq L} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))e^{i\psi(x,y,\xi)}d\xi + O(L^{n+d-1})\,.$$

In order to do so we use the three lemmas stated below, whose proofs are given at the end of the section. To begin with, we use the information of Lemma A.1 to give a first expression for $\rho * \tilde{E}_{L,P}$.

Lemma A.4. *The quantity*

$$\rho * \tilde{E}_{L,P}(x,y) - \int_{-\infty}^{L} \frac{1}{(2\pi)^n}\int_{T_y^*M} \mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)P\left(q(x,t,y,\xi)e^{i(\psi(x,y,t,\xi)-t\sigma(y,\xi))}\right)\Big](\lambda)d\xi d\lambda$$

is bounded uniformly for $(x,y) \in U \times U$.

Here and below $\mathcal{F}$ is the Fourier transform and the occasional subscript indicates the variable on which the transform is taken. Let us now investigate the effect of the differential operator P on the right hand side of this expression. By the Leibniz rule, there is a finite family of symbols $(\sigma_j)_{0\leq j\leq d} \in C^\infty(U \times]-\varepsilon,\varepsilon[\times U, \mathbb{R}^n)^{d+1}$ such that for each j, σ_j is homogeneous of degree j, such that

$$P\Big[q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\Big] = \Big[\sum_{j=0}^{d}\sigma_j(x,t,y,\xi)\Big]e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}$$

and such that

$$\sigma_d(x,t,y,\xi) = q(x,t,y,\xi)\sigma_P(x,y,\partial_{x,y}(\psi(x,y,\xi)-t\sigma(y,\xi)))\,.$$

Now, for each j, let

$$R_j(x,y,L,\xi) = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}} \mathcal{F}(\rho)(t)\sigma_j(x,t,y,\xi)e^{itL}dt$$

and

$$S_j(x,y,L) = \int_{-\infty}^{L}\int_{\mathbb{R}^n} R_j(x,y,\lambda-\sigma(y,\xi),\xi)e^{i\psi(x,y,\xi)}d\xi d\lambda\,.$$

Then,

$$\int_{-\infty}^{L}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)P\left(q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\right)\Big](\lambda)d\xi d\lambda = \sum_{j=0}^{d} S_j(x,y,L)\,.$$

Each S_j will grow at an order corresponding to the degree of the associated symbol. This is shown in the following lemma.

Lemma A.5. *There is an open set $V \subset U$ containing 0 such that, as $L \to \infty$ and uniformly for $(x,y) \in V \times V$,*

$$S_j(x,y,L) = \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\leq L}\sigma_j(x,0,y,\xi)e^{i\psi(x,y,\xi)}d\xi + O(L^{n+j-1})\,.$$

Similarly since $q(x,0,y,\xi)-1\in S^{-1}(U_0\times U_0,\mathbb{R}^n)$, from a computation analogous to the proof of Lemma A.5 and left to the reader, replacing σ_d by

$$(q(x,0,y,\xi)-1)\sigma_P(x,y,\partial_{x,y}(\psi(x,y,\xi)-t\sigma(y,\xi)))\in S^{d-1}$$

one can remove q from the main term, which results in the following.

Lemma A.6. *There is an open set $V\subset U$ containing 0 such that, as $L\to\infty$ and uniformly for $(x,y)\in V\times V$,*

$$S_d(x,y,L)=\frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\le L}\sigma_P(x,y,\partial_{x,y}(\psi(x,y,\xi)-t\sigma(y,\xi)))e^{i\psi(x,y,\xi)}d\xi+O(L^{n+d-1})\,.$$

The juxtaposition of these results yields Lemma A.3.

Proof of Lemma A.4. Since $k\in C^\infty(U\times U\times]-\varepsilon,\varepsilon[)$ and $\mathcal{F}(\rho)$ is supported in $]-\varepsilon,\varepsilon[$,

$$\mathcal{F}_t^{-1}\big[\mathcal{F}(\rho)(t)Pk(x,t,y)\big](L)\in\mathcal{S}(\mathbb{R})\,.$$

Therefore, by Equation (39),

$$\rho*\tilde{E}_{L,P}(x,y)-\int_{-\infty}^{L}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)P\left(q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\right)\Big](\lambda)d\xi d\lambda$$

is bounded. □

Proof of Lemma A.5. In this proof, all generic constants will be implicitly uniform with respect to $(x,y)\in V\times V$. Let us fix $y\in V$ and define the following three domains of integration.

$$\begin{aligned}D_1&=\{(\lambda,\xi)\in\mathbb{R}\times\mathbb{R}^n\mid\lambda\le L,\ \sigma(y,\xi)\le L\}\\D_2&=\{(\lambda,\xi)\in\mathbb{R}\times\mathbb{R}^n\mid\lambda\le L,\ \sigma(y,\xi)>L\}\\D_3&=\{(\lambda,\xi)\in\mathbb{R}\times\mathbb{R}^n\mid\lambda>L,\ \sigma(y,\xi)\le L\}\,.\end{aligned}$$

Moreover, for $l=1,2,3$, let $I_l=\int_{D_l}R_j(x,y,\lambda-\sigma(y,\xi),\xi)e^{i\psi(x,y,\xi)}d\xi d\lambda$. We will prove that I_2 and I_3 are $O(L^{n+j-1})$. The following calculation will then yield the desired identity. Here we use Fubini's theorem and the fact that $\mathcal{F}(\rho)(0)=\int_{\mathbb{R}}\rho(\lambda)d\lambda=1$.

$$\begin{aligned}S_j(x,y,L)&=I_1+I_2=I_1+I_3+O(L^{n+j-1})\\&=\int_{\sigma(y,\xi)\le L}\Big[\int_{\mathbb{R}}R_j(x,y,s,\xi)ds\Big]e^{i\psi(x,y,\xi)}d\xi+O(L^{n+j-1})\\&=\frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\le L}\sigma_j(x,0,y,\xi)e^{i\psi(x,y,\xi)}d\xi+O(L^{n+j-1}).\end{aligned}$$

First of all, R_j is rapidly decreasing in the third variable and, since σ is elliptic of degree 1, bounded by $\sigma(y,\xi)^j$ with respect to the last variable, ξ. Therefore, for each $N>0$ there is a constant $C>0$ such that

$$|R_j(x,y,\lambda,\xi)|\le\frac{C\sigma(y,\xi)^j}{(1+|\lambda|)^N}\,.$$

Since σ is elliptic of order 1, the hypersurface $L^{-1}\{\sigma(y,\xi)=L\}\subset\mathbb{R}^n$ converges smoothly for $L\to\infty$ uniformly in y to $S_y^*=\{\sigma_A(y,\xi)=1\}$ and the volume of $\{\sigma(x,\xi)=\beta\}\subset\mathbb{R}^n$ is $O(\beta^{n-1})$. Taking $N=$

$2n+j+1$, we deduce that

$$\begin{aligned}|I_2| &\le C\int_{-\infty}^{L}\int_{\sigma(y,\xi)>L}\frac{\sigma(y,\xi)^j}{(1+|\lambda-\sigma(y,\xi)|)^{2n+j+1}}d\xi d\lambda \le C\int_{-\infty}^{L}\int_{L}^{+\infty}\frac{\beta^{n+j-1}}{(1+|\lambda-\beta|)^{2n+j+1}}d\beta d\lambda\\ &\le C\int_{L}^{+\infty}\int_{-\infty}^{L-\beta}\frac{\beta^{n+j-1}}{(1+|s|)^{2n+j+1}}dsd\beta \le C\int_{L}^{\infty}\frac{\beta^{n+j-1}}{(1+\beta-L)^{2n+j}}d\beta\\ &\le C\int_{0}^{+\infty}\frac{(\gamma+L)^{n+j-1}}{(1+\gamma)^{2n+j}}d\gamma \le CL^{n+j-1}.\end{aligned}$$

Here we applied first the change of variables $s=\lambda-\beta$ and then $\gamma=\beta-L$. The case of I_3 is analogous and by a similar calculation we deduce that I_1 is well defined. □

A.3 Comparison of the kernel and its convolution

In this section we set about proving that $\tilde{E}_{L,P}$ is close enough to its convolution with ρ. This is encapsulated in the following lemma.

Lemma A.7. *There is an open set $V\subset U$ containing 0 such that, as $L\to\infty$ and uniformly for $(x,y)\in V\times V$,*

$$\rho*\tilde{E}_{L,P}(x,y)-\tilde{E}_{L,P}(x,y)=O(L^{n+d-1}).$$

As before, the proofs are relegated to the end of the section. In order to prove Lemma A.7 we first estimate the growth of the R_j as follows.

Lemma A.8. *There is an open set $V\subset U$ containing 0 such that, as $L\to\infty$ and uniformly for $(x,y)\in V\times V$,*

$$\int_{\mathbb{R}^n}R_j(x,y,L-\sigma(y,\xi),\xi)e^{i\psi(x,y,\xi)}d\xi=O(L^{n+j-1}).$$

This lemma follows from a computation analogous to the bound on I_2 and I_3 given in the proof of Lemma A.5 above and the details are left to the reader. It allows us to prove a second intermediate result from which we obtain Lemma A.7 directly.

Lemma A.9. *There is an open set $V\subset U$ containing 0 such that, as $L\to\infty$ and uniformly for $(x,y)\in V\times V$,*

$$\tilde{E}_{L+1,P}(x,y)-\tilde{E}_{L,P}(x,y)=O(L^{n+d-1}).$$

Proof of Lemma A.9. We begin with the case where $x=y$ and P is of the form $P_1\otimes P_1$. For brevity we define

$$u(L)=\tilde{E}_{L,P}(x,x)=\sum_{\lambda_k\le L}|(P_1e_k)(x)|^2.$$

Recall $\rho>0$ so it stays greater than some constant $a>0$ on the interval $[-1,0]$. Moreover u is increasing so by Equation (39) and Lemma A.8,

$$\begin{aligned}0\le u(L+1)-u(L)&=\int_{L}^{L+1}u'(\lambda)d\lambda\le\frac{1}{a}\int_{L}^{L+1}\rho(L-\lambda)u'(\lambda)d\lambda\\ &\le\frac{1}{a}\frac{d}{dL}(\rho*u)\le\frac{1}{a}\sum_{j=0}^{d}\int_{\mathbb{R}^n}R_j(x,y,L-\sigma(y,\xi))d\xi+O(L^{n+d-1})=O(L^{n+d-1}).\end{aligned}$$

Now if P is of the form $P_1 \otimes P_2$, and for any x and y, let $X = (P_1 e_k)_{L<\lambda_k \leq L+1}$ and $Y = (P_2 e_k)_{L<\lambda_k \leq L+1}$ be two vectors in some $\mathbb{C}^q$ which we equip with the standard hermitian product "$\star$". Then, $\tilde{E}_{L+1,P}(x,y) - \tilde{E}_{L,P}(x,y) = X \star \overline{Y}$ so

$$\begin{aligned}|\tilde{E}_{L+1,P}(x,y) - \tilde{E}_{L,P}(x,y)|^2 &\leq |X|^2|Y|^2 \\ &= |\tilde{E}_{L+1,P_1\otimes P_1}(x,y) - \tilde{E}_{L,P_1\otimes P_1}(x,y)||\tilde{E}_{L+1,P_2\otimes P_2}(x,y) - \tilde{E}_{L,P_2\otimes P_2}(x,y)| \\ &\leq \frac{1}{4}\Big(\tilde{E}_{L+1,P_1\otimes P_1}(x,x) - \tilde{E}_{L,P_1\otimes P_1}(x,x) + \tilde{E}_{L+1,P_1\otimes P_1}(y,y) - \tilde{E}_{L,P_1\otimes P_1}(y,y)\Big) \\ &\times \Big(\tilde{E}_{L+1,P_2\otimes P_2}(x,x) - \tilde{E}_{L,P_2\otimes P_2}(x,x) + \tilde{E}_{L+1,P_2\otimes P_2}(y,y) - \tilde{E}_{L,P_2\otimes P_2}(y,y)\Big) \\ &\leq CL^{2n+2d-2}.\end{aligned}$$

Here we used first the Cauchy-Schwarz inequality, then the mean value inequality, then on each factor,

$$2|P_1 e_k(x)\overline{P_1 e_k(y)}| \leq |P_1 e_k(x)|^2 + |P_1 e_k(y)|^2$$

and finally the above estimate. In general P is a locally finite sum of operators of the form $P_1 \otimes P_2$. $\square$

Proof of Lemma A.7. First of all, according to Lemma A.9 there is a constant C such that for all $L \geq 0$ and λ,

$$|\tilde{E}_{L+\lambda,P}(x,y) - \tilde{E}_{L,P}(x,y)| \leq C(1+|\lambda|+L)^{n+d-1}(1+|\lambda|).$$

Consequently

$$\begin{aligned}(\rho * \tilde{E}_{L,P}(x,y) - \tilde{E}_{L,P}(x,y)| &\leq |\int \rho(\lambda)\tilde{E}_{L+\lambda,P}(x,y)d\lambda - \tilde{E}_{L,P}(x,y)| \\ &\leq \int \rho(\lambda)\Big|\tilde{E}_{L+\lambda,P}(x,y) - \tilde{E}_{L,P}(x,y)\Big|d\lambda \\ &\leq C\int \rho(\lambda)(1+|\lambda|+L)^{n+d-1}(1+|\lambda|)d\lambda \\ &\leq C'L^{n+d-1}\end{aligned}$$

for some $C' > 0$. Here we used that $\rho > 0$, ρ is rapidly decreasing and $\int_{\mathbb{R}} \rho(\lambda)d\lambda = \mathcal{F}(\rho)(0) = 1$. $\square$

A.4 Conclusion

Combining Lemmas A.3 and A.7 we obtain the following:

$$\tilde{E}_{L,P}(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\leq L} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))e^{i\psi(x,y,\xi)}d\xi + O(L^{n+d-1}).$$

Since $\sigma - \sigma_A^{\frac{1}{m}} \in S^0$, replacing one by the other adds only a $O(L^{n+d-1})$ term. Therefore,

$$\tilde{E}_{L,P}(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)^{1/m}\leq L} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))e^{i\psi(x,y,\xi)}d\xi + O(L^{n+d-1}).$$

This estimate is valid and uniform for $x, y \in V$. To conclude, notice that $\sigma_A(x,\xi)^{1/m} \leq L$ is equivalent to $\sigma_A(x,\xi) \leq L^m$. Since $\tilde{E}_L = E_{L^m}$, replacing L by $L^{1/m}$ in the last estimate we get

$$E_{L,P}(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)\leq L} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))e^{i\psi(x,y,\xi)}d\xi + O(L^{(n+d-1)/m})$$

as announced.

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