Principles of Least Action and of Least Constraint

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Dedicated to Professor Erwin Stein on the occasion of his 80th birthday with special gratitude as my mentor and a leading scholar in computational mechanics.

The present contribution describes the evolution of two major extremum principles in me-chanics proposed in the 18th and the first half of the 19th century, namely the Principle of Least Action associated with the name Maupertuis and Gauss's Principle of Least Constraint. We will briefly mention the d'Alembert Principle strongly related to the Principle of Gauss.

Preliminary Remark

Like the other contributions in this special issue the present paper is also based on a lecture given in a session on the history of mechanics at the GAMM annual meeting in 2010. The author not being a historian apologizes for a rather short and partly superficial essay on both principles, based mainly on secondary sources. It can by far not replace any of the excellent rigorous and detailed descriptions on the subject, some of them mentioned in the list of references. This holds in particular for the description of the scientific controversy on the Principle of Least Action, probably one of the most documented affairs in the history of sciences.

1 Introduction

The search for the existence of extremum principles in nature and technology can be traced back to the ancient times. The scholars were not only driven by scientific observations; often their objective was based on metaphysical arguments. In Figure 1 major figures involved in the development of extremum principles in mechanics are arranged in the respective time course. In the following essay the era of the mid 18th and the early 19th century is selected; in particular we will concentrate on Pierre-Louis Moreau de Maupertuis related to the *Principle of Least Action* in 1744/46 and on Carl Friedrich Gauss who stated the *Principle of Least Constraint* in 1829.

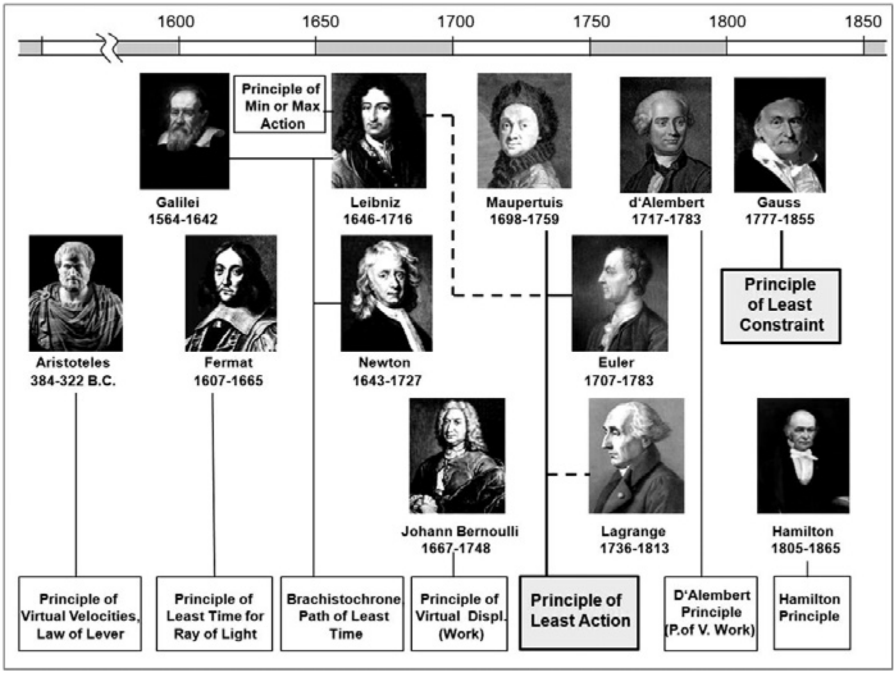


Fig. 1 Evolution of Extremum Principles.

2 Principle of Least Action

2.1 Maupertuis and his Principle

The discussion on the Principle of Least Action is one of the most extensive examinations of a scientific controversy. The protagonists (Figure 2) are Maupertuis and Euler on one side and

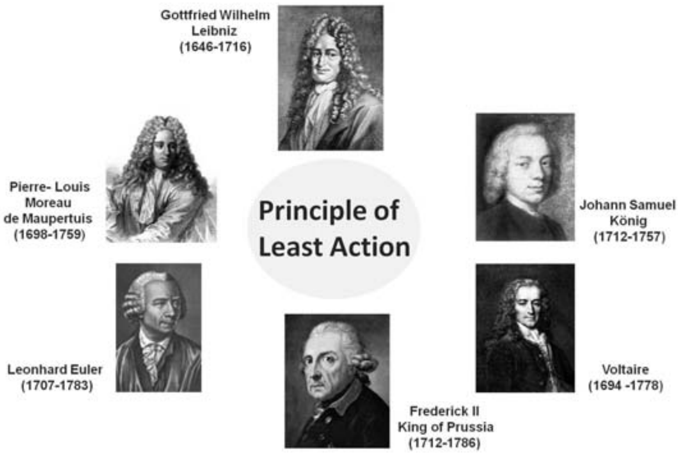
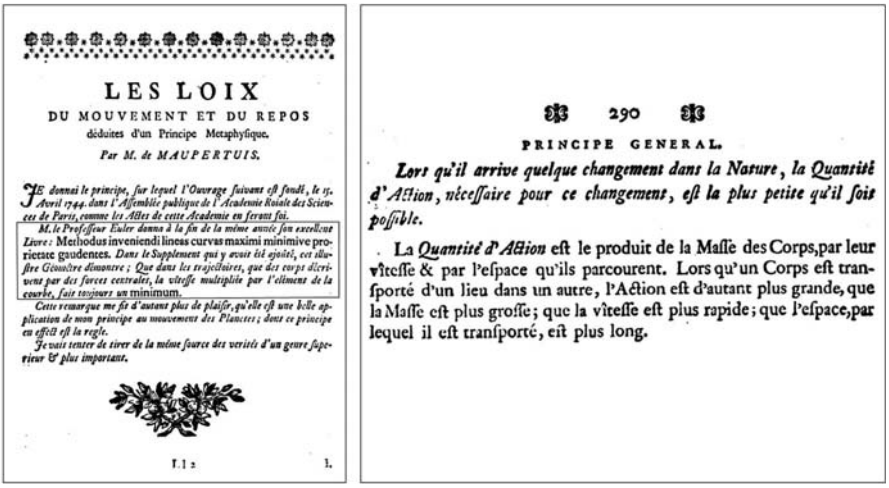


Fig. 2 The Protagonists.

Fig. 3 Pierre-Louis Moreau de Maupertuis.

König and Voltaire on the other side. Leibniz' work also played a role in the fight along with Frederick the Great, King of Prussia, as a leading authority at that time in Berlin.

The French mathematician Pierre-Louis Moreau de Maupertuis (Figure 3) was very much influenced by the work of Newton. Already in papers to the French Academy of Sciences in 1741 and 1744 he mentioned a principle minimizing a quantity which he called action. After he became the President of the Prussian Academy of Sciences in Berlin, on invitation of King Frederick the Great in 1746, he presented the book "*Les Loix du Movement et du Repos*" (The Laws of Movement and of Rest derived from a Metaphysical Principle) [1], see Figure 4, left. In the introduction he points to his previous work at the Paris Academy in 1744



LES LOIX

DU MOUVEMENT ET DU REPOS

déduites d'un Principe Metaphysique.

Par M. de MAUPERTUIS.

JE donnai le principe, sur lequel l'Ouvrage suivant est fondé, le 15. Avril 1744. dans l'Assemblée publique de l'Academie Roiale des Sciences de Paris, comme les Actes de cette Academie en feront foi.

M. le Professeur Euler donna à la fin de la même année son excellent Livre: Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes. *Dans le Supplement qui y avoit été ajoûté, cet illustre Géomètre démontre; Que dans les trajectoires, que des corps décrivent par des forces centrales, la vitesse multipliée par l'élément de la courbe, fait toujours un* minimum.

Cette remarque me fit d'autant plus de plaisir, qu'elle est une belle application de mon principe au mouvement des Planetes; dont ce principe en effet est la regle.

Je vais tenter de tirer de la même source des verités d'un genre superieur & plus important.

I. I 2 1.

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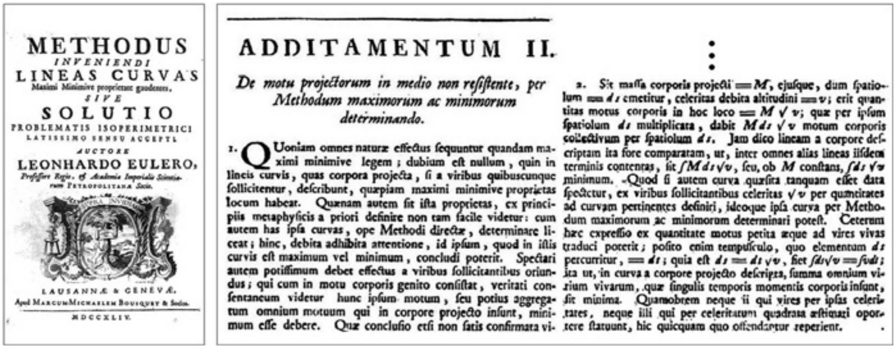
PRINCIPE GENERAL.

Lors qu'il arrive quelque changement dans la Nature, la Quantité d'Action, nécessaire pour ce changement, est la plus petite qu'il soit possible.

La *Quantité d'Action* est le produit de la Masse des Corps, par leur vîtesse & par l'espace qu'ils parcourent. Lors qu'un Corps est transporté d'un lieu dans un autre, l'Action est d'autant plus grande, que la Masse est plus grosse; que la vîtesse est plus rapide; que l'espace, par lequel il est transporté, est plus long.

Fig. 4 Maupertuis' Les Loix (1746) [1] and the Principe de la Moindre Action (Principle of Least Action).



METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum Petropolitanæ Socio.

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

ADDITAMENTUM II.

De motu projectorum in medio non resistente, per Methodum maximorum ac minimorum determinando.

1. QUoniam omnes naturæ effectus sequuntur quandam maximi minimive legem; dubium est nullum, quin in lineis curvis, quas corpora projecta, si a viribus quibuscunque sollicitentur, describunt, quæpiam maximi minimive proprietas locum habeat. Quænam autem sit ista proprietas, ex principiis metaphysicis a priori definire non tam facile videtur: cum autem has ipsa curvas, ope Methodi directæ, determinare liceat; hinc, debita adhibita attentione, id ipsum, quod in istis curvis est maximum vel minimum, concludi poterit. Spectari autem potissimum debet effectus a viribus sollicitantibus oriundus; qui cum in motu corporis genito consistat, veritati consentaneum videtur hunc ipsum motum, seu potius aggregatum omnium motuum qui in corpore projecto insunt, minimum esse debere. Quæ conclusio etsi non satis confirmata vi-

⋮

2. Sit massa corporis projecti $= M$, ejusque, dum spatiolum $= ds$ emetitur, celeritas debita altitudini $= v$; erit quantitas motus corporis in hoc loco $= M\sqrt{v}$; quæ per ipsum spatiolum ds multiplicata, dabit $M ds \sqrt{v}$ motum corporis collectivum per spatiolum ds. Jam dico lineam a corpore descriptam ita fore comparatam, ut, inter omnes alias lineas iisdem terminis contentas, sit $\int M ds \sqrt{v}$, seu, ob M constans, $\int ds \sqrt{v}$ minimum. Quod si autem curva quæsita tanquam esset data spectetur, ex viribus sollicitantibus celeritas $\sqrt{v}$ per quantitates ad curvam pertinentes definiri, ideoque ipsa curva per Methodum maximorum ac minimorum determinari potest. Ceterum hæc expressio ex quantitate motus petita æque ad vires vivas traduci poterit; posito enim tempusculo, quo elementum ds percurritur, $= dt$; quia est $ds = dt\sqrt{v}$, fiet $\int ds\sqrt{v} = \int v dt$; ita ut, in curva a corpore projecto descripta, summa omnium virium vivarum, quæ singulis temporis momentis corporis insunt, sit minima. Quamobrem neque ii qui vires per ipsas celeritates, neque illi qui per celeritatum quadrata æstimari oportere statuunt, hic quicquam quo offendantur reperient.

Fig. 5 Euler's Methodus Inveniendi Lineas Curvas (1744) [2] and Supplement II.

and adds the remark "*At the end of the same year, Professor Euler published his excellent book Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*" (A method for finding curved lines enjoying properties of maximum or minimum) [2], Figure 5. He then continues "*In a Supplement to his book, this illustrious Geometer showed that, in the trajectory of a particle acted on by a central force, the velocity multiplied by the line element of the trajectory is always a minimum*". In other words Maupertuis was aware of maximum and minimum properties however he reduced his further considerations to a minimum principle.

In his "*Les Loix…*" Maupertuis first critized the usual proofs of the existence of God and refers to the fundamental laws of nature. His self-confident statements are remarkable when he writes "*After so many great men have worked on this subject, I almost do not dare to say that I have discovered the universal principle upon which all these laws are based. This is the principle of least action, a principle so wise and so worthy of the Supreme Being, and intrinsic to all natural phenomena; one observes it at work not only in every change, but also in every constancy that Nature exhibits*".

Finally he states the general principle as follows (Figure 4, right): "*When a change occurs in Nature, the Quantity of Action necessary for that change is as small as possible*", and defines "*The Quantity of Action is the product of the Mass of Bodies times their velocity and the distance they travel. When a Body is transported from one place to another, the Action is proportional to the Mass of the Body, to its velocity and to the distance over which it is transported*". Expressing the Action A in a formula yields

$$A \sim M \cdot v \cdot \mathrm{d}s \tag{1}$$

With the mass of a particle M, the velocity v and the transported distance $\mathrm{d}s$. The relation of the Action to the kinetic energy is apparent if the distance $\mathrm{d}s$ is replaced by velocity times the time increment $\mathrm{d}t$.

In the sequel of this definition Maupertuis adds three examples as a proof for the generality of his principle, namely on the laws of motion for inelastic as well as elastic bodies and the law of mechanical equilibrium. In the mentioned Supplement II "*De motu projectorum…*" (On a motion of particles in a non-resistant medium, determined by a Method of maxima and minima), Figure 5, right, Leonard Euler (Figure 6) says "*1. Since all natural phenomena obey a certain maximum or minimum law, there is no doubt that some property must be maximized*

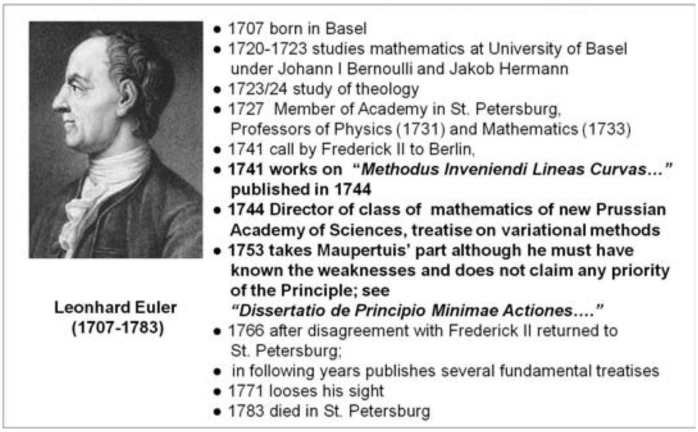


Fig. 6 Leonard Euler.

or minimized in the trajectories of particles acted upon by external forces. However, it does not seem easy to determine which property is minimized from metaphysical principles known a priori. Yet if the trajectories can be determined by a direct method, the property being minimized or maximized by these trajectories can be determined, provided that sufficient care is taken. After considering the effects of external forces and the movements they generate, it seems most consistent with experience to assert that the integrated momentum, i.e. the sum of all momenta contained in the particle's movement, is the minimized quantity".

In section 2 of the Additamentum Euler defines mass M, velocity $\sqrt{v}$ (he used $\sqrt{v}$ instead of v), infinitesimal distance $\mathrm{d}s$, the momentum integrated over the distance $\mathrm{d}s$ as $M \cdot \mathrm{d}s \cdot \sqrt{v}$ and writes: "*Now I assert that the true trajectory of the moving particle is the trajectory to be described, from among all possible trajectories connecting the same end point, that minimizes $\int M\mathrm{d}s\sqrt{v}$ or since M is constant $\int \mathrm{d}s\sqrt{v}$. Since the velocity $\sqrt{v}$ resulting from the external forces can be calculated a posteriori from the trajectory itself, a method of maxima and minima should suffice to determine the trajectory a priori. The minimized integral can be expressed in terms of the momentum (as above), but also in terms of the living forces (*kinetic energies*). For, given an infinitesimal time $\mathrm{d}t$ during which the element $\mathrm{d}s$ is traversed, we have $\mathrm{d}s = \mathrm{d}t\sqrt{v}$. Hence $\int \mathrm{d}s\sqrt{v} = \int v\mathrm{d}t$, i.e. the true trajectory of a moving particle minimizes the integral over time of its instantaneous living forces (*kinetic energies*). Thus, this minimum principle should appeal both to those who favor momentum for mechanics calculations and to those who favor living forces*".

Euler realized that for the entire path the action has to be a sum along all segments $\mathrm{d}s$ and essentially defined as quantity (velocity defined as v as usual)

$$A = \int Mv\mathrm{d}s = \int Mv^2\mathrm{d}t \tag{2}$$

which is equal to the kinetic energy up to the factor $\frac{1}{2}$.

Thus it turned out that Maupertuis' pretension on universality of his principle was wrong, confer [3, 4]; the examples were not well chosen and partially not correct. His claim that the

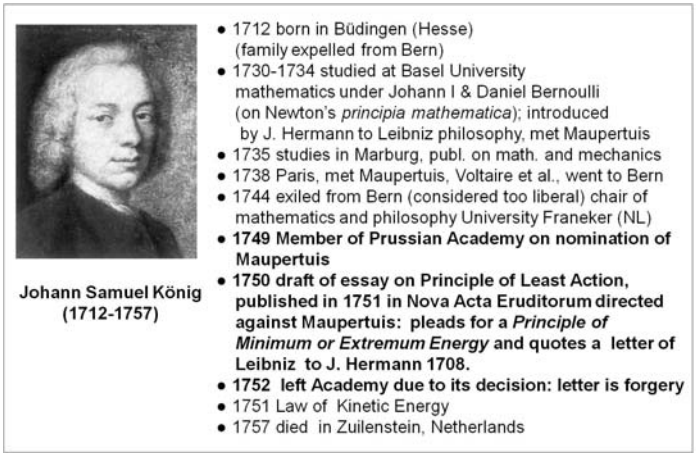


Fig. 7 Johann Samuel König.

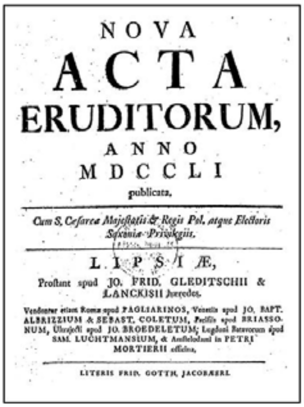
minimization is an economy principle did not hold because some problems lead to a maximum. Chevalier P. D'Arcy reproached Maupertuis for this error; for example the refraction of a concave mirror is based on a maximum property, [3]. Consequently his interpretation of Euler's statements was not correct.

2.2 The critics

It was his fellow student and friend Johann Samuel König (Figure 7) who became Maupertuis' strongest critic. Two years after he was appointed member of the Berlin Academy, strongly supported by the president, the Swiss lawyer and later mathematician König (1712–1757) passed an essay to Maupertuis who approved it for publication without reading. In his article König pled for a principle of extremal energy from which an extremum for the action can be derived. The essay appeared in Nova Acta Eruditorum 1751 [5]. When Maupertuis read it he became upset because not only the limitation of his principle was questioned but it also claimed that G.W. Leibniz (1646–1716) has given a more precise formulation already in a letter to the Swiss mathematician and theologian Jakob Hermann (1678–1733) in 1707. At the end of the paper [5] he quotes from Leibniz' letter to Hermann [6] (see Figure 8) "*But action is in no way what you think, there the consideration of the time enters; it is the product of the mass and the time*[1]*; or of the time by the living force. I have pointed out that in the modifications of movements, it is usually derived as a Maximum, or a Minimum. From this can be deduced several important propositions; it can be used to determine the curves describing the bodies that are attached to one or several centres*".

According to König the "force vive" (vis viva) has been already introduced by Leibniz expressing $M \cdot v^2$, i.e. it is two times the kinetic energy. It was not his intention though to accuse Maupertuis being a plagiarist or to disparage his merits; he simply wanted to point out the more general formulation of Leibniz (minimum or maximum). Nevertheless it was

[1] The quote contains a misprint; in the first definition of the action time must be replaced by distance and speed.



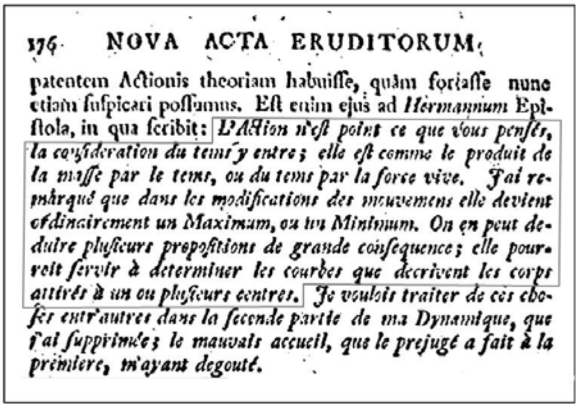
NOVA
ACTA
ERUDITORUM,
ANNO
MDCCLI
publicata.

Cum S. Cæsareæ Majestatis & Regis Pol. atque Electoris Saxoniæ Privilegiis.

LIPSIÆ,
Prostant apud JO. FRID. GLEDITSCHII & LANCKISII hæredes.

Venduntur etiam Romæ apud PAGLIARINOS, Venetiis apud JO. BAPT. ALBRIZZIUM & SEBAST. COLETUM, Parisiis apud BRIASSONUM, Ultrajecti apud JO. BROEDELETUM; Lugduni Batavorum apud SAM. LUCHTMANSIUM, & Amstelodami in PETRI MORTIERII officina.

LITERIS FRID. GOTTH. JACOBAEERI.



176 NOVA ACTA ERUDITORUM.

patentem Actionis theoriam habuisse, quam forsasse nunc etiam suspicari possumus. Est enim ejus ad *Hermannum* Epistola, in qua scribit: *L'Action n'est point ce que vous pensés, la consideration du temps y entre; elle est comme le produit de la masse par le tems, ou du tems par la force vive. J'ai remarqué que dans les modifications des mouvemens elle devient ordinairement un Maximum, ou un Minimum. On en peut deduire plusieurs propositions de grande consequence; elle pourroit servir à determiner les courbes que decrivent les corps attirés à un ou plusieurs centres. Je voulois traiter de ces choses entr'autres dans la seconde partie de ma Dynamique, que j'ai supprimée; le mauvais accueil, que le prejugé a fait à la premiere, m'ayant degouté.*

Fig. 8 König's reference letter of Leibniz to Hermann (1708) in Nova Acta Eruditorum (1751) [5].

the beginning of intensive mutual disparagements ending in one of the ugliest of all scientific disputes [7].

The letter König presented was a copy, a fact which would play a key role in the subsequent vehement quarrel. All attempts to find the original letter ended only in further copies; König claimed that he got his copy from the Swiss poet and politician Samuel Henzi in Bern who collected letters of Leibniz among other things. Henzi was decapitated in 1749 because of conspiration and all his original papers were supposed to be burned. The Berlin Academy met in spring 1752 charging König of forgery. It is noteworthy that the meeting was headed by Euler who from the very beginning was a strong supporter of Maupertuis despite his own more rigorous work on the same principle in 1744 (Supplement II, Figure 5); the reason for Euler's reaction is still a matter of conjectures; see e.g. [3, 4, 8]. König's appeal was rejected; he resigned as member of the Academy in summer 1752.

King Frederick used all his authority to support Maupertuis, the President of "his" Academy. It was the time when another figure entered the scene, namely Voltaire (Figure 9), the famous French writer. Voltaire liked satires and polemics and was known for his sharp, malicious, sometimes also witty remarks. Although he was invited by Frederick to come to Berlin he took part of König; he published an anonymous letter "*A reply from an Academician of Berlin to an Academician of Paris*" in fall 1752 defending König (Figure 13, right); he refers to Monsieur Moreau de Maupertuis and says: "*He asserts that in all possible cases, Action is always a Minimum, which has been demonstrated false; and he says he discovered this law of Minimum, what is not less false. Mr. Koenig, as well as other Mathematicians, wrote against this strange assertion, & he cited among other things, a fragment of Leibnitz in which the great man has remarked, that in modifications of movement, the action usually becomes either a Maximum, or a Minimum*". König once more referred to copies of Leibniz' letter in a further pamphlet. Frederick was provoked and defended Maupertuis in a "*Letter of an Academician of Berlin to an Academician in Paris*" (Figure 13, right).

Voltaire published further polemics, the most famous being the "*Diatribe due Docteur Akakia, Médicin du Pape*" (Figure 10, left), which was a defamatory piece of writing. The Greek word Akakia means "without guile". The *Diatribe* printed in Potsdam was burned by

- 1694 born in Paris
- 1711-13 studies Law, Paris, Caen, The Hague
- 1717 returns to Paris, writing essays and poems, several imprisonments (Bastille), due to attacks of government and church
- 1727/28 exile in GB, **inspired by Newton's theories**
- 1729-1739 Paris, Rouen, Brussels, Château de Cirey
- 1741 at court of Frederick II, 1742 detained for treason
- 1743 Acadèmie Française, mission to Berlin
- 1743-1747 at court of Louis XV in Versailles
- **1750-1753 at court of Frederick II in Sanssouci**
- **1752/53 takes König's side against Maupertuis who is protected by Frederick II, publishes several polemics, e.g. *Diatribe du Docteur Akakia, Médizin du Pape***
- 1754-1760 lives in and nearby Geneva
- 1756 cooperates with Diderot and d'Alembert on *Encyclopédie*
- 1778 dies in Paris leaving over 700 texts

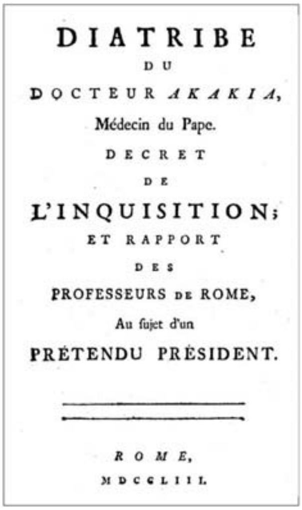
François-Marie Arouet (1694 – 1778), Voltaire (since 1717)

Anagram of „AROUET LI" Arouet, Le Jeune (The Younger)

Fig. 9 François-Marie Arouet, called himself Voltaire.

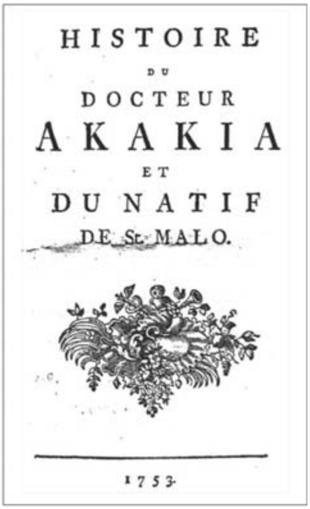
order of King Frederick. However since it was also published in Holland, some extra copies appeared in Berlin and were again burned in front of Voltaire's apartment. This was the final break-up with the king; Voltaire left for Leipzig in spring 1753 where he continued attacking Maupertuis.

Several letters of Voltaire followed compiled in "*Histoire du Docteur Akakia et du Natif de St. Malo*" in April 1753 (Figure 10, right) again making snide remarks because St. Malo is the birthplace of Maupertuis. Like the *Diatribe* this pamphlet was a sham presented as a defence but in reality being a backhanded compliment. In the *Histoire* a fictitious peace contract between the President (Maupertuis) and the Professor (König) *Traité de Paix conclu entre M.*



DIATRIBE
DU
DOCTEUR *AKAKIA*,
Médecin du Pape.
DECRET
DE
L'INQUISITION;
ET RAPPORT
DES
PROFESSEURS DE ROME,
Au fujet d'un
PRÉTENDU PRÉSIDENT.

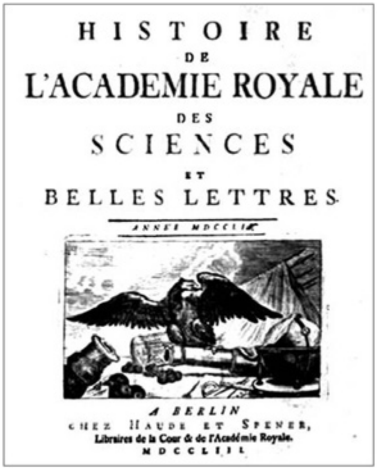
ROME,
MDCCLIII.



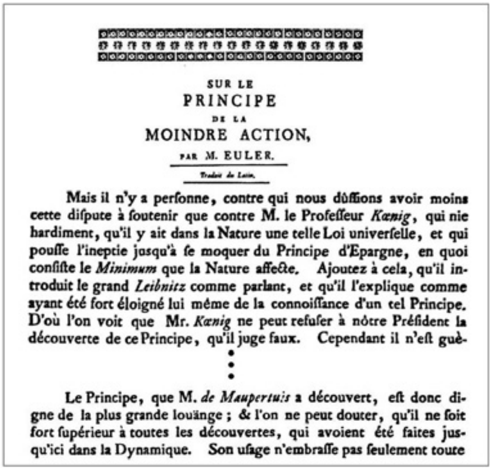
HISTOIRE
DU
DOCTEUR
AKAKIA
ET
DU NATIF
DE St. MALO.

1753.

Fig. 10 Voltaire's Diatribe [9] and Histoire du Docteur Akakia [10] (1753).


HISTOIRE
DE
L'ACADEMIE ROYALE
DES
SCIENCES
ET
BELLES LETTRES.

A BERLIN
CHEZ HAUDE ET SPENER,
Libraires de la Cour & de l'Académie Royale.
MDCCLIII.


SUR LE
PRINCIPE
DE LA
MOINDRE ACTION,
PAR M. EULER.

Mais il n'y a perſonne, contre qui nous dûſſions avoir moins cette diſpute à ſoutenir que contre M. le Profeſſeur *Kœnig*, qui nie hardiment, qu'il y ait dans la Nature une telle Loi univerſelle, et qui pouſſe l'ineptie jusqu'à ſe moquer du Principe d'Epargne, en quoi conſiſte le *Minimum* que la Nature affecte. Ajoutez à cela, qu'il introduit le grand *Leibnitz* comme parlant, et qu'il l'explique comme ayant été fort éloigné lui même de la connoiſſance d'un tel Principe. D'où l'on voit que Mr. *Kœnig* ne peut refuſer à nôtre Préſident la découverte de ce Principe, qu'il juge faux. Cependant il n'eſt guè-

⋮

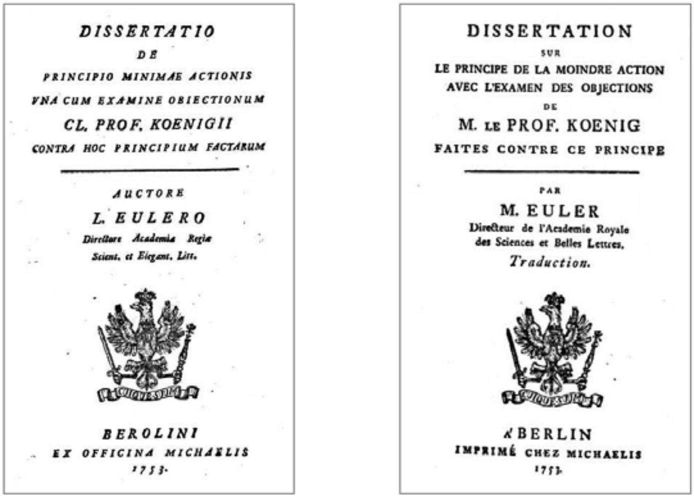
Le Principe, que M. *de Maupertuis* a découvert, eſt donc digne de la plus grande louänge ; & l'on ne peut douter, qu'il ne ſoit fort ſupérieur à toutes les découvertes, qui avoient été faites jusqu'ici dans la Dynamique. Son uſage n'embraſſe pas ſeulement toute

Fig. 11 Euler's Sur le Principe de la Moindre Action 1751/53 [12].

Le Président et M. Le Professeur, 1st January 1753 [10] was included saying among other things: "*In future we promise, not to put the Germans down and admit that the Copernicus's, the Kepler's, the Leibnitz's..., are something, and that we have studied under the Bernoulli's, and shall study again; and that, finally, Professor Euler, who was very anxious to serve us as a lieutenant, is a very great geometer who has supported our principle with formulae which we have been quite unable to understand, but which those who do understand have assured us they are full of genius, like the published works of the professor referred to, our lieutenant*", [11].

Thus Voltaire also included Euler in this dispute and holds him up to ridicule. Euler was indeed a vehement advocate of the Academy President despite his own more rigorous exposition of the subject. Dugas refers in [11] to an essay of Euler and quotes: "*This great geometer has not only established the principle more firmly than I had done but his method, more ubiquitous and penetrating than mine, has discovered consequences that I had not obtained. After so many vested interests in the principle itself, he has shown, with the same evidence, that I was the only one to whom the discovery could be attributed*".

Euler discussed his view of the development of the principle in [12], presented already in 1751 but not printed in the Histoires of the Prussian Academy until February 1753 (Figure 11). He referred to König's role and made a strong statement: "*But there is no one with whom we would be less likely to have dispute than Professor Koenig, who boldly denies that there is in nature such a universal law and pushes this absurdity to the point of mocking the Principle of Conservation, which constitutes the Minimum that nature appoints. In addition, he introduces the great Leibnitz as claiming and explaining that he was far from knowing such a principle. From this we see that Mr. Koenig cannot deny our President the discovery of the principle that he himself considers to be false*". Finally he said: "*The principle that Mr. de Maupertuis discovered is therefore worthy of the greatest of praise; and without a doubt it is far superior to all discoveries that have been made in dynamics up until now*" (Figure 11, right). This essay was included in the "*Dissertatio de Principio Minimae Actionis...*", a bilingual edition in



DISSERTATIO
DE
PRINCIPIO MINIMAE ACTIONIS
VNA CUM EXAMINE OBIECTIONUM
CL. PROF. KOENIGII
CONTRA HOC PRINCIPIUM FACTARUM

AUCTORE
L. EULERO
Directore Academiae Regiae
Scient. et Elegant. Litt.

BEROLINI
EX OFFICINA MICHAELIS
1753.

DISSERTATION
SUR
LE PRINCIPE DE LA MOINDRE ACTION
AVEC L'EXAMEN DES OBJECTIONS
DE
M. LE PROF. KOENIG
FAITES CONTRE CE PRINCIPE

PAR
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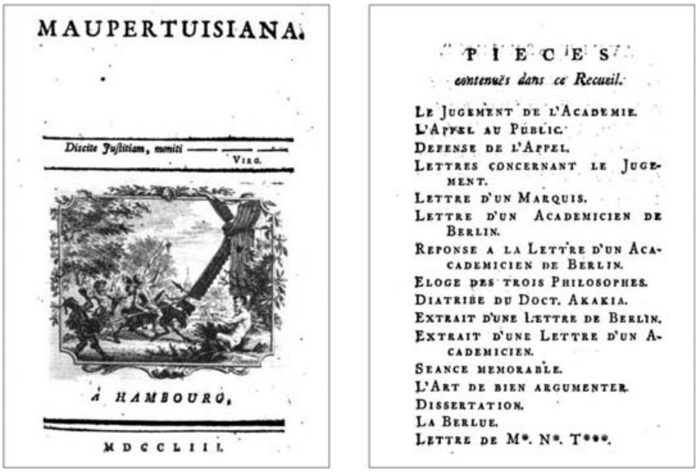
des Sciences et Belles Lettres.
Traduction.

A BERLIN
IMPRIMÉ CHEZ MICHAELIS
1753.

Fig. 12 Bilingual edition of Euler's Dissertatio [13].



MAUPERTUISIANA.

Discite Justitiam, moniti ——
VIRG.

A HAMBOURG,

MDCCLIII.

PIECES
contenuës dans ce Recueil.

LE JUGEMENT DE L'ACADEMIE.
L'APPEL AU PUBLIC.
DEFENSE DE L'APPEL.
LETTRES CONCERNANT LE JUGEMENT.
LETTRE D'UN MARQUIS.
LETTRE D'UN ACADEMICIEN DE BERLIN.
REPONSE A LA LETTRE D'UN ACADEMICIEN DE BERLIN.
ELOGE DES TROIS PHILOSOPHES.
DIATRIBE DU DOCT. AKAKIA.
EXTRAIT D'UNE LETTRE DE BERLIN.
EXTRAIT D'UNE LETTRE D'UN ACADEMICIEN.
SEANCE MEMORABLE.
L'ART DE BIEN ARGUMENTER.
DISSERTATION.
LA BERLUE.
LETTRE DE M*. N*. T***.

Fig. 13 Maupertuisiana (1753), List of Essays (figure from [3]).

Latin and French [13] together with an examination of the objections of M. Professor Koenig made against the principle (Figure 12), see also [14].

Shortly thereafter in April 1753 sixteen polemics and letters in dispute were put together in a publication initiated by Voltaire [14]; it was entitled "*Maupertuisiana*" (Figure 13) and printed in a fictitious location Hambourg; among them are the aforementioned "*Reply*" and "*Letter*". The figure on the title page shows Don Quixote (Maupertuis) fighting the windmills with a broken lance shouting "*Tremblez*" (tremble); behind him Sancha Panza (Euler) is riding a donkey. On the right side a satyr is depicted saying "*sic itur ad astra*" (thus one reaches the stars), meaning "thus one is blamed forever". Above the picture a line of Virgil's "Aeneis" is added: "*Discite Justitiam, moniti*" (be warned and learn justice).

The "*Traité de Paix*" was also included in the *Maupertuisiana*; in the title page of that volume the first part of a quote from Horace's Satires was added: "*ridiculum acri forties ac melius (plerumque secat res)*" or "ridicule often settles matters of importance better (and with more effect than severity)".

With the publication of the *Maupertuisiana* the quarrel came more or less to an end. Maupertuis was ill, spent a year in Paris and St. Malo, returned to Berlin which he left in 1756. After another year in St. Malo he was accommodated by Johann II Bernoulli in Basel where he died in 1759. S. König being the moral victor in this battle died from a stroke in Holland two years earlier in 1757. Voltaire had still some active and eventful years; he left Berlin in 1753 for Paris, but was banned by Louis X. He then lived in Geneva and in Ferney across the French border. In 1778 he returned for the first time in 20 years to Paris where he died after three months. Euler as the fourth figure in this dispute stayed a couple of years in Berlin, but returned to St. Petersburg in 1766. He was extremely active and published several fundamental treatises. He passed away in 1783.

2.3 On Leibniz' Alleged Letter to Hermann

The claim for forgery was taken up again 140 years later when C.I. Gerhardt, editor of Leibniz' mathematical œvre at the Prussian Academy, reinvestigated the case. The Physicist H. von Helmholtz, who gave a speech on the history of the Principle of Least Action in the Academy in 1887 [15], refers to Gerhardt's remark, that the letter does not fit to the correspondence with Hermann. Leibniz may have postponed the publication because he planned a later application. In a detailed report [16] in 1898 (see minutes for a meeting of the Academy) Gerhardt claims that three of the four letters presented by König are genuine and concludes at the end of his paper: "*From the above explanations it should undoubtedly follow that the letter fragment, which König published, is not made up, and the entire letter has not been foisted. The letter is written by Leibniz. Also with a probability close to certainty it has been proven that it had been directed to Varignon* (Pierre de Varignon (1654–1722), French mathematicien and physicist). *The coincidental letters of the correspondence with Varignon in the Royal Library of Hannover have disappeared as the letters of Leibniz in the respective correspondence with Hermann did*". The assumption for the authenticity was underlined by another argument in 1913, when W. Kabitz [17] found a further copy of the letter in Gotha among a collection of other genuine letters of Leibniz. This was accepted as a "proof" for the originality of the letter at that time.

Recently Breger has re-examined the case in a remarkably thorough discussion [18] in which he elaborated on eight arguments against the authenticity of the letter. Among those is the argument that Leibniz never applied the notion "*limites*" in the sense as it is used in the letter. Leibniz also often referred to minimum and maximum properties and also mentioned the term action, however combining both these concepts would be a singular event; for further arguments see [18], a paper worth reading. Breger recomments that now it is time to inversely analyse which arguments are speaking in favor of the genuineness. He also mentions that in case of forgery the rather strange reaction of Euler is easier to comprehend; he understood the letter as an action against his own person. A further question namely with respect to the identity of the falsifier still remains open.

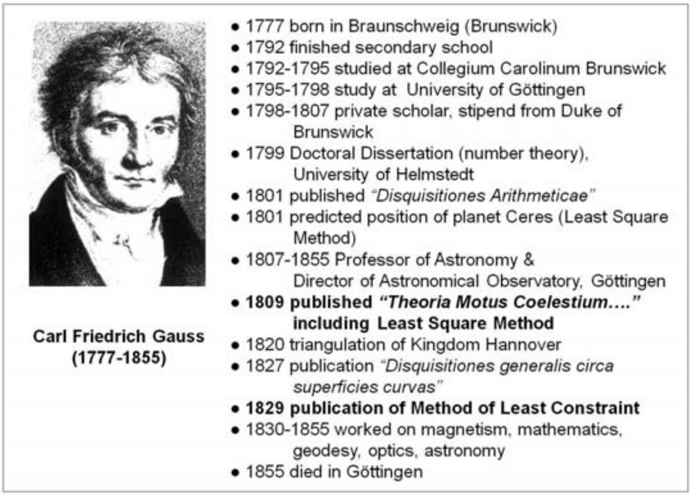
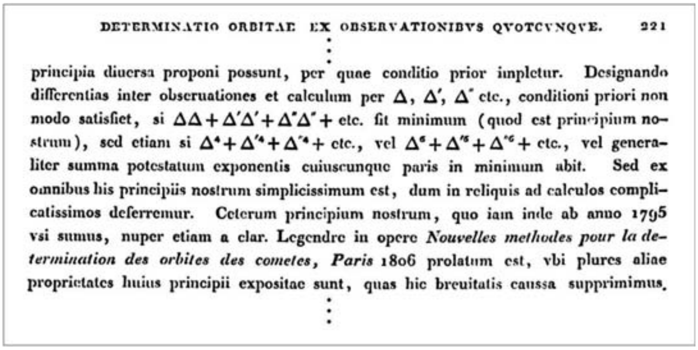


Fig. 14 Carl Friedrich Gauss.

2.4 Application of Principle of Least Action

As said above the Principle of Least Action has been defined in more rigorous form already by Euler in 1744 as an energy principle. It turned out to be rather a *Principle of Stationary Action*. Lagrange derived the equation of motion based on his newly developed calculus of variations (1760). In 1834/35 Hamilton introduced the Lagrangian function into the variational principle deriving what has been called later the Euler-Lagrange equations. Whereas Euler and Maupertuis concentrated in the Principle of Least Action on the kinetic energy (vis viva, living forces) tacitly assuming constant potential energy the difference between the kinetic and the potential energy entered the Lagrangian in Hamilton's principle. Despite this close relationship the Principle of Least Action has not found its way into the engineering community which mostly refers to the works of Lagrange, d'Alembert and Hamilton. In the 19th century the well known Principles of Minimum of Total Potential Energy (Dirichlet-Green) and Minimum of Total Complementary Energy (Menabrea-Castigliano) became basic theorems in mechanics. The Principles of Virtual Work and Complementary Work entered the scene in particular because of their generality being applicable also for problems where no potentials exist. They are nowadays together with mixed Variational Principles (Hellinger-Reissner, Fraeijs deVeubeke-Hu-Washizu, etc.) the foundations for the derivation of discretization methods.

Opposite to the engineering community the notion of Principle of Least (Stationary) Action is still present in modern physics. Entire chapters of treatises are devoted to the principle, e.g. [19, 20]; see also the famous Feynman Lectures on Physics [21]. Occasionally they are looked upon as generalization of Hamilton's principle rather than as its predecessor. The Principle has been applied for example in the theory of relativity or in quantum mechanics, see [22].



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principia diuersa proponi possunt, per quae conditio prior impletur. Designando differentias inter obseruationes et calculum per Δ, Δ', Δ'' etc., conditioni priori non modo satisfiet, si $\Delta\Delta + \Delta'\Delta' + \Delta''\Delta'' +$ etc. fit minimum (quod est principium nostrum), sed etiam si $\Delta^4 + \Delta'^4 + \Delta''^4 +$ etc., vel $\Delta^6 + \Delta'^6 + \Delta''^6 +$ etc., vel generaliter summa potestatum exponentis cuiuscunque paris in minimum abit. Sed ex omnibus his principiis nostrum simplicissimum est, dum in reliquis ad calculos complicatissimos deferremur. Ceterum principium nostrum, quo iam inde ab anno 1795 vsi sumus, nuper etiam a clar. Legendre in opere *Nouvelles methodes pour la determination des orbites des cometes, Paris* 1806 prolatum est, vbi plures aliae proprietates huius principii expositae sunt, quas hic breuitatis caussa supprimimus.

⋮

Fig. 15 C.F. Gauss's Method of Least Square (1809), extract from [23].

3 Principle of Least Constraint

3.1 Method of Least Squares

Carl Friedrich Gauss (1777–1855) (Figure 14) applied his method of least squares in 1801 when he determined the elliptical orbit of the astroid Ceres; however he had developed the basis of this method already in 1795 when he was 18 years old. It was not until 1809 that he published the method in the second volume of his book on the Theory of Celestial Bodies [23] (Figure 15). He said "*Our principle which we have made use of since the year 1795 has lately been published by Legendre in the work Nouvelles methodes…*". It was a further priority argument in the history of mechanics; however Gauss could present his correspondence with colleagues concerning its use much earlier.

3.2 Principle of Least Constraint

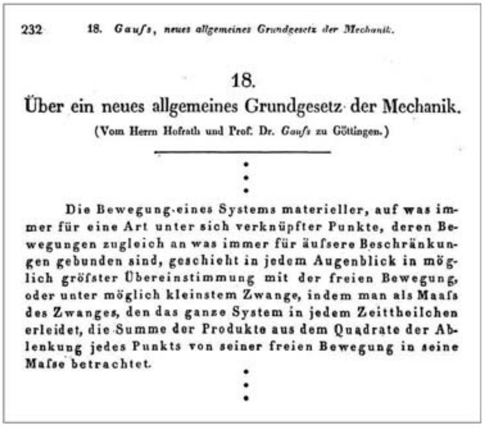
Twenty years later in 1829 Gauss wrote an essay on "*Über ein neues allgemeines Grundgesetz der Mechanik*" (On a new Fundamental Law of Mechanics) [24]. The paper was published in the 4th issue of *Journal für die reine und angewandte Mathematik (ed. A.L. Crelle)*, a journal still existing today. He describes his new law on four pages that became known as the Principle of Least Constraint (Figure 16), by the way with very little algebra. It says "*The motion of a system of material points… takes place in every moment in maximum accordance with the free movement or under least constraint;…*". He continues "*the measure of constraint, …, is considered as the sum of products of mass and the square of the deviation to the free motion*".

Applying the usual mathematical notation the acceleration a_i^{free} of the free unconstrained motion is defined by force F_i divided by mass m_i; r is the position vector.

$$\text{free motion} \qquad a_i^{\mathrm{free}} = \frac{F_i}{m_i} \qquad a_i = \ddot{r}_i \tag{3}$$

$$\text{constrained motion} \qquad a_i \neq a_i^{\mathrm{free}} \tag{4}$$

The acceleration of the constraint motion is called a_i where kinematic conditions constrain the corresponding material point i. For example a kinematic constraint may be given by a



232 18. *Gaufs, neues allgemeines Grundgesetz der Mechanik.*

18.

Über ein neues allgemeines Grundgesetz der Mechanik.

(Vom Herrn Hofrath und Prof. Dr. *Gaufs* zu Göttingen.)

⋮

Die Bewegung eines Systems materieller, auf was immer für eine Art unter sich verknüpfter Punkte, deren Bewegungen zugleich an was immer für äufsere Beschränkungen gebunden sind, geschieht in jedem Augenblick in möglich gröfster Übereinstimmung mit der freien Bewegung, oder unter möglich kleinstem Zwange, indem man als Maafs des Zwanges, den das ganze System in jedem Zeittheilchen erleidet, die Summe der Produkte aus dem Quadrate der Ablenkung jedes Punkts von seiner freien Bewegung in seine Maſse betrachtet.

⋮

Fig. 16 C.F. Gauss in 1828 and his paper on the Principle of Least Constraint (1829) [24].

prescribed trajectory. The measure Z for the "constraint" (Zwang) is proportional to the sum of squares of the differences between free and constrained accelerations. Each term in the sum is weighted by mass m_i.

$$Z \sim \sum_{i=1}^{N} \left(a_i - a_i^{\text{free}}\right)^2 \tag{5}$$

$$\text{or} \quad Z = \sum_{i=1}^{N} m_i \left(a_i - \frac{F_i}{m_i}\right)^2 = \text{MIN} \qquad \begin{array}{l} +\text{ kinematical conditions,} \\ \quad \text{e.g. prescribed displacements} \end{array} \tag{6}$$

According to the principle Z is supposed to be a Minimum. Thus from all possible motions (accelerations) the actual motion leads under given conditions to the least constraint.

Figure 17 applies the principle to the motion of a pendulum, the circle being the constrained trajectory. As expected the principle yields the equation of motion. This example elucidates that there is a strong relation with d'Alembert's Principle (Jean d'Alembert, 1717-1783). We start from the Principle of Least Constraint and its variation with respect to its free parameters a_i

$$\text{constraint} \qquad Z = \sum_{i=1}^{N} m_i \Big(a_i - \underbrace{\frac{F_i}{m_i}}_{\text{fix}}\Big)^2 = \text{MIN} \tag{7}$$

$$\text{variation} \qquad \delta Z = \sum_{i=1}^{N} 2m_i \left(a_i - \frac{F_i}{m_i}\right) \underbrace{\delta a_i}_{\hat{=}\delta r_i \hat{=} \delta d_i} = 0 \tag{8}$$

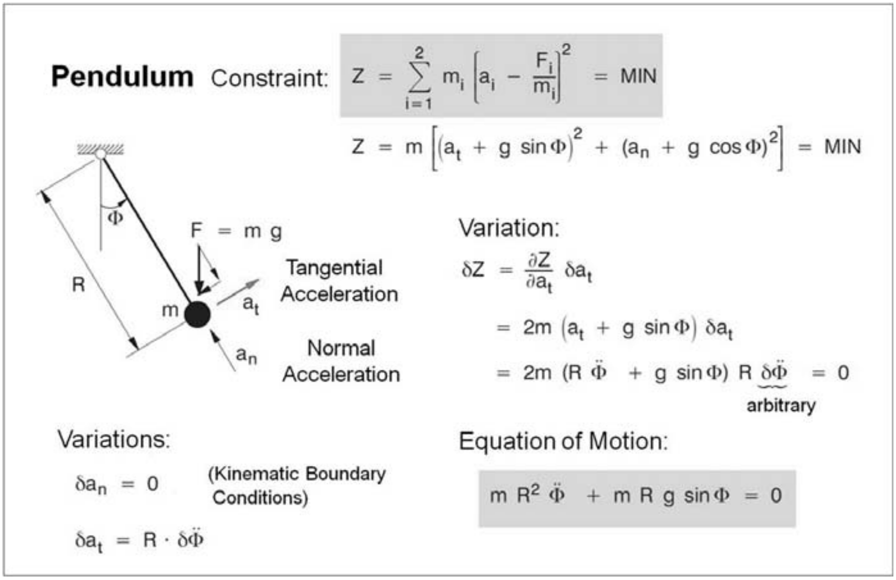


Fig. 17 Application of the Principle of Least Constraint for Pendulum.

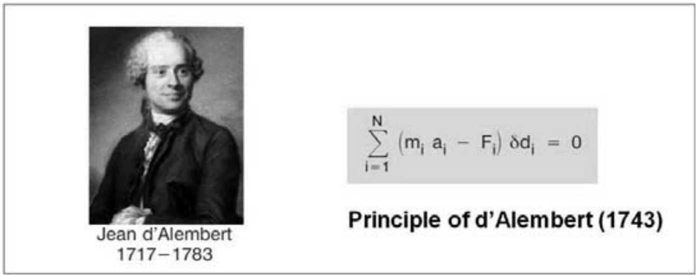


Fig. 18 Principle of d'Alembert.

where the position vector and its variation are

$$\text{position vector} \qquad r(t + \mathrm{d}t) = \underbrace{r(t) + v(t)\,\mathrm{d}t}_{\text{fix}} + 1/2\; a(t)\,\mathrm{d}t^2 \tag{9}$$

$$\text{variation} \qquad \delta r(t + \mathrm{d}t) = 1/2\; \delta a(t)\,\mathrm{d}t^2 \tag{10}$$

The variation or virtual displacements satisfy the essential boundary conditions. $\delta Z = 0$ directly yields the Principle of d'Alembert (1743) as a variational principle of the equation of motion (Figure 18).

Gauss finishes his paper with the remark (Figure 19) "*It is strange that the free movements, when they cannot withstand the necessary conditions, are modified in the same way as the analyzing mathematician, applying the method of least squares, balances experiences which are based on parameters depending on necessary interactions*", see the comparison in Figure 20.

> Es ist sehr merkwürdig, daſs die freien Bewegungen, wenn sie mit nothwendigen Bedingungen nicht bestehen können, von der Natur gerade auf dieselbe Art modificirt werden, wie der rechnende Mathematiker, nach der Methode der kleinsten Quadrate, Erfahrungen ausgleicht, die sich auf unter einander durch nothwendige Abhängigkeit verknüpfte Gröſsen beziehen. Diese Analogie lieſse sich noch weiter verfolgen, was jedoch gegenwärtig nicht zu meiner Absicht gehört.

Fig. 19 Conclusions of Paper [24].

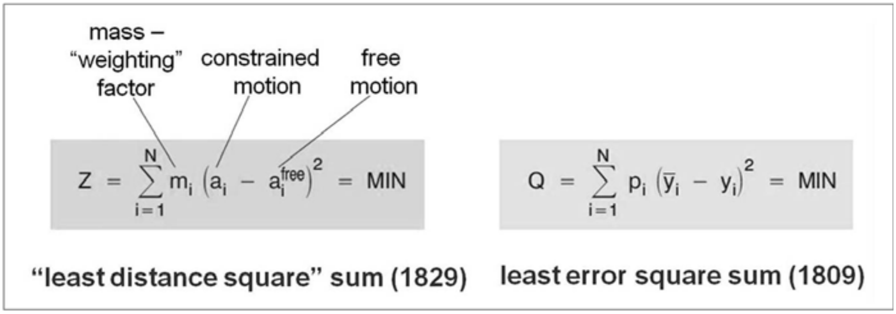


Fig. 20 Principle of Least Constraint and Least Square Method.

He adds a final sentence to his paper which says: "*This analogy could be further followed up, but this is currently not my intention*". However he never picked up this matter again.

3.3 Application of Principle of Least Constraint

Gauss's Principle is not very well known although it is mentioned as a fundamental principle in many treatises, e. g. [3, 25–27], see also [28]; correspondingly it has not been applied too often. Evans and Morriss [26] discuss in detail the application of the Principle for holonomic (constraints depend only on co-ordinates) and nonholonomic constraints (non-integrable constraints on velocity) and conclude "*The correct application of Gauss's principle is limited to arbitrary holonomic constraints and apparently, to nonholonomic constraint functions which are homogeneous functions of the momenta*". Evans et al. [29] applied the principle in the context of statistical mechanics, see also [30]. As another example the work of Glocker [31, 32] is mentioned where accelerations in rigid multibody systems under set-valued forces are evaluated applying Gauss's principle. Recently Udwadia and co-workers extended the Principle to underdetermined systems [33].

As a final remark the author would like to point out to the possibility of using Gauss's Principle in optimization and design of structures under dynamic excitation where the system ought to be tuned for selected accelerations.

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