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FACTORIZATION OF BIVARIATE SPARSE POLYNOMIALS

FRANCESCO AMOROSO AND MARTÍN SOMBRA

Abstract. We prove a function field analogue of a conjecture of Schinzel on the factorization of univariate polynomials over the rationals. We derive from it a finiteness theorem for the irreducible factorizations of the bivariate Laurent polynomials in families with fixed set of complex coefficients and varying exponents. Roughly speaking, this result shows that the truly bivariate irreducible factors of these sparse Laurent polynomials, are also sparse.

The proofs are based on a generalization of the toric Bertini's theorem due to Fuchs, Mantova and Zannier.

1. Introduction

A polynomial is said to be *sparse* (or *lacunary*) if it has few terms compared with its degree. The factorization problem for sparse polynomials can be vaguely stated as the question of whether the irreducible factors of a sparse polynomial are, apart from obvious exceptions, also sparse. Aspects of this problem have been studied in various settings and for different formalizations of the notion of sparsenness, see for instance [Len99, Sch00, KK06, AKS07, FGS08, Gre16, ASZ17]. Many times, these studies were based on tools from Diophantine geometry like lower bounds for the height of points and subvarieties, and unlikely intersections of subvarieties and subgroups of a torus.

In this text, we consider families of bivariate Laurent polynomials given as the pullback of a *fixed* regular function on a torus by a *varying* 2-parameter monomial map. Precisely, let t, z be variables, $\boldsymbol{x} = (x_1, \dots, x_n)$ a group of other n variables, and

$$F \in \mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, z^{\pm 1}]$$

a Laurent polynomial. For each vector $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we consider the bivariate Laurent polynomial given as the pullback of F by the monomial map $(t, z) \mapsto (t^{\boldsymbol{a}}, z) = (t^{a_1}, \dots, t^{a_n}, z)$, that is

$$F_{\boldsymbol{a}} = F(t^{\boldsymbol{a}}, z) \in \mathbb{C}[t^{\pm 1}, z^{\pm 1}]. \tag{1.1}$$

The number of coefficients of each $F_{\boldsymbol{a}}$ is bounded by those of F, and so these Laurent polynomials can be considered as sparse when F is fixed and $\boldsymbol{a}$ is large.

The following is our main result.

Theorem 1.1. *Let $F \in \mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}]$. There is a finite set of matrices $\Omega \subset \mathbb{Z}^{n \times n}$ satisfying the following property. For each $\boldsymbol{a} \in \mathbb{Z}^n$, there are $M \in \Omega$ and $\boldsymbol{b} \in \mathbb{Z}^n$ with $\boldsymbol{a} = M\boldsymbol{b}$ such that if P is an irreducible factor of $F(\boldsymbol{x}^M, z)$, then $P(t^{\boldsymbol{b}}, z)$ is, as an element of $\mathbb{C}(t)[z^{\pm 1}]$, either a unit or an irreducible factor of $F(t^{\boldsymbol{a}}, z)$.*

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This result shows that for each $\boldsymbol{a} \in \mathbb{Z}^n$, there is a matrix M within the finite set $\Omega \subset \mathbb{Z}^{n\times n}$ and a vector $\boldsymbol{b} \in \mathbb{Z}^n$ with $\boldsymbol{a} = M\boldsymbol{b}$ such that, unless $F\big(\boldsymbol{x}^M, z\big) = 0$, the irreducible factorization

$$F\big(\boldsymbol{x}^M, z\big) = \prod_P P(\boldsymbol{x}, z)^{e_P} \tag{1.2}$$

yields the irreducible factorization

$$F_{\boldsymbol{a}} = \gamma \prod_P{}' P(t^{\boldsymbol{b}}, z)^{e_P},$$

in the ring $\mathbb{C}(t)[z^{\pm 1}]$ for $F_{\boldsymbol{a}}$ as in (1.1), the product being over the irreducible factors P in (1.2) such that $P(t^{\boldsymbol{b}}, z)$ is not a unit, and with $\gamma \in \mathbb{C}(t)[z^{\pm 1}]^\times$.

Hence, the irreducible factorizations in $\mathbb{C}(t)[z^{\pm 1}]$ of the $F_{\boldsymbol{a}}$'s can be obtained by specializing the irreducible factorizations of the Laurent polynomials $F(\boldsymbol{x}^M, z)$ for a *finite* number of matrices M. These irreducible factors of the $F_{\boldsymbol{a}}$'s are sparse, in the sense that they are all represented as the pullback of a finite number of regular functions on the $(n+1)$-dimensional torus $\mathbb{G}_{\mathrm{m}}^{n+1}$ by 2-parameter monomial maps. In particular, both the number of these irreducible factors and of their coefficients is bounded above independently of $\boldsymbol{a}$.

For computations and concrete applications, it would be good to have explicit bounds for the set of matrices Ω in Theorem 1.1. In a more ambitious plan, one might even wonder if the set Ω in Theorem 1.1 could be chosen independently on the coefficients of F.

As explained in Section 2, Theorem 1.1 is a consequence of our function field analogue of a conjecture of Schinzel (Theorem 2.4).

The proof of Theorem 2.4 relies on a toric version of Bertini's theorem. The first toric Bertini's theorem was proved by Zannier [Zan10, Theorem 3], to be later precised and generalized by Fuchs, Mantova and Zannier [FMZ17, Theorem 1.5]. In this result, cosets of subtori replace the affine subspaces in the classical version of the theorem. Roughly speaking, it states that, for an irreducible quasiprojective variety W of dimension $n \geq 0$ and a dominant map

$$\pi\colon W \longrightarrow \mathbb{G}_{\mathrm{m}}^n$$

that is finite onto its image and satisfies the technical property PB (Definition 3.1), the fiber of a generic coset is irreducible.

Here we extend this result to the situation where π does not verify the property PB. In this more general situation, the conclusion of that theorem does not necessarily hold. Instead, this conclusion is replaced by an alternative that "explains" the possibility that a fiber is reducible by its factorization through a reducible pullback of the variety W by an isogeny of $\mathbb{G}_{\mathrm{m}}^n$ within a finite set.

Theorem 1.2. *Let W be an irreducible quasiprojective variety of dimension n and $\pi\colon W \to \mathbb{G}_{\mathrm{m}}^n$ a dominant map that is finite onto its image. There is a finite union $\mathcal{E}$ of proper subtori of $\mathbb{G}_{\mathrm{m}}^n$ and a finite set Λ of isogenies of $\mathbb{G}_{\mathrm{m}}^n$ such that, for a subtorus $T \subset \mathbb{G}_{\mathrm{m}}^n$ and a point $p \in \mathbb{G}_{\mathrm{m}}^n(\mathbb{C}) = (\mathbb{C}^\times)^n$, one of the next conditions holds:*

1. *$T \subseteq \mathcal{E}$;*
2. *there is $\lambda \in \Lambda$ with $\lambda^* W$ reducible and a subtorus $T' \subset \mathbb{G}_{\mathrm{m}}^n$ with λ inducing an isomorphism $T' \to T$;*
3. *$\pi^{-1}(p \cdot T)$ is irreducible.*

We prove this theorem by reducing it to the previous toric Bertini's theorem, through a variation (Proposition 3.8) of a factorization result for rational maps from [Zan10].

Back to the factorization problem for sparse polynomials, it is natural to consider the more general setting of pullbacks of regular functions on $\mathbb{G}_{\mathrm{m}}^n$ by *arbitrary* monomial maps, instead of only those appearing in (1.1). Let $\boldsymbol{y} = (y_1, \dots, y_n)$ and $\boldsymbol{t} = (t_1, \dots, t_k)$ be groups of n and k variables, respectively. For a Laurent polynomial $G \in \mathbb{C}[\boldsymbol{y}^{\pm 1}]$, consider the family of k-variate Laurent polynomials given by the pullback of G by the monomial map $\mathbb{G}_{\mathrm{m}}^k \to \mathbb{G}_{\mathrm{m}}^n$ defined by $\boldsymbol{t} \mapsto \boldsymbol{t}^A$ for a matrix $A \in \mathbb{Z}^{n\times k}$, that is

$$G_A = G(\boldsymbol{t}^A) \in \mathbb{C}[\boldsymbol{t}^{\pm 1}].$$

Denote by S the multiplicative subset of $\mathbb{C}[\boldsymbol{t}^{\pm 1}]$ generated by the Laurent polynomials of the form $f(\boldsymbol{t}^{\boldsymbol{d}})$ for $f \in \mathbb{C}[z^{\pm 1}]$ and $\boldsymbol{d} \in \mathbb{Z}^k$.

We propose the following conjecture which, as explained in Remark 4.2, partially generalizes Theorem 1.1.

Conjecture 1.3. *Let $G \in \mathbb{C}[\boldsymbol{y}^{\pm 1}]$ and $k \geq 2$. There is a finite set of matrices $\Omega \subset \mathbb{Z}^{n\times n}$ satisfying the following property. For each $A \in \mathbb{Z}^{n\times k}$, there are $N \in \Omega$ and $B \in \mathbb{Z}^{n\times k}$ with $A = NB$ such that if P is an irreducible factor of $G(\boldsymbol{y}^N)$, then $P(\boldsymbol{t}^B)$ is, as an element of $\mathbb{C}[\boldsymbol{t}^{\pm 1}]_S$, either a unit or an irreducible factor of $G(\boldsymbol{t}^A)$.*

The validity of this conjecture would imply that the irreducible factors of the G_A's that truly depend on more than one variable, are also the pullback of a finite number of regular functions on $\mathbb{G}_{\mathrm{m}}^n$ by k-parameter monomial maps. The possible univariate irreducible factors of the G_A's split completely, and so they cannot be accounted from a finite number of such regular functions.

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2. A conjecture of Schinzel and its function field analogue

In [Sch65], Schinzel proposed the conjecture below on the factorization of univariate polynomials over $\mathbb{Q}$.

An element of $\mathbb{Q}[\boldsymbol{x}^{\pm 1}]$ is *cyclotomic* if it is an irreducible factor of a binomial of the form $\boldsymbol{x}^{\boldsymbol{a}} - 1$ for a nonzero vector $\boldsymbol{a} \in \mathbb{Z}^n$. The *cyclotomic part* of a Laurent polynomial $F \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}] \setminus \{0\}$, denoted by $\mathrm{cyc}(F)$, is defined as a maximal factor of F that is a product of cyclotomic Laurent polynomials. This cyclotomic part is well-defined up to a unit of $\mathbb{Q}[\boldsymbol{x}^{\pm 1}]$.

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n$, we denote by $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i=1}^n a_i b_i$ their scalar product.

Conjecture 2.1. *Let $F \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}]$ be a non-cyclotomic irreducible Laurent polynomial. There are finite sets $\Omega^0 \subset \mathbb{Z}^{n\times n}$ of nonsingular matrices and $\Gamma \subset \mathbb{Z}^n$ of nonzero vectors satisfying the property that, for $\boldsymbol{a} \in \mathbb{Z}^n$, one of the next conditions holds:*

1. *there is $\boldsymbol{c} \in \Gamma$ verifying $\langle \boldsymbol{c}, \boldsymbol{a} \rangle = 0$;*
2. *there are $M \in \Omega^0$ and $\boldsymbol{b} \in \mathbb{Z}^n$ with $\boldsymbol{a} = M\boldsymbol{b}$ such that if*

$$F(\boldsymbol{x}^M) = \prod_P P^{e_P}$$

is the irreducible factorization of $F(\boldsymbol{x}^M)$*, then*

$$\frac{F(t^{\boldsymbol{a}})}{\mathrm{cyc}(F(t^{\boldsymbol{a}})} = \prod_P \Big(\frac{P(t^{\boldsymbol{b}})}{\mathrm{cyc}(P(t^{\boldsymbol{b}}))}\Big)^{e_P}$$

is the irreducible factorization of $F(t^{\boldsymbol{a}})/\mathrm{cyc}(F(t^{\boldsymbol{a}})$.

For the validity of this statement, in its condition (2) it is necessary to take out the cyclotomic part of $F(t^{\boldsymbol{a}})$ and of the $P(t^{\boldsymbol{b}})$'s, as shown by the example below.

Example 2.2. Set $F = x_1+x_2-2 \in \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}]$. Let $\boldsymbol{a} \in \mathbb{Z}^2$ and choose a nonsingular matrix $M \in \mathbb{Z}^{2\times 2}$ and a vector $\boldsymbol{b} \in \mathbb{Z}^2$ with $\boldsymbol{a} = M\boldsymbol{b}$. We have that

$$F(\boldsymbol{x}^M) = x_1^{m_{1,1}} x_2^{m_{1,2}} + x_1^{m_{2,1}} x_2^{m_{2,2}} - 2$$

is irreducible, and so $P := F(\boldsymbol{x}^M)$ is the only irreducible factor of this Laurent polynomial. However, $t-1$ divides $F(t^{\boldsymbol{a}}) = P(t^{\boldsymbol{b}})$, and so these univariate Laurent polynomials are not irreducible, unless we divide them by this cyclotomic factor.

Schinzel proved this conjecture when $n = 1$ in *loc. cit.* and, under a restrictive hypothesis (non-symmetry), when $n \geq 2$ [Sch70], see also [Sch00, §6.2]. The general case when $n \geq 2$ remains open. In this paper, we prove a function field analogue for Laurent polynomials over the field $\mathbb{C}(z)$, from which we deduce Theorem1.1.

An element of $\mathbb{C}(z)[\boldsymbol{x}^{\pm 1}]$ is *constant* if it lies in $\mathbb{C}[\boldsymbol{x}^{\pm 1}]$, up to a scalar factor in $\mathbb{C}(z)^\times$. The *constant part* of a Laurent polynomial $F \in \mathbb{C}(z)[\boldsymbol{x}^{\pm 1}] \setminus \{0\}$, denoted by $\mathrm{ct}(F)$, is defined as its maximal constant factor. This constant part is well-defined up to a unit of $\mathbb{C}(z)[\boldsymbol{x}^{\pm 1}]$.

Remark 2.3. The analogy between cyclotomic Laurent polynomials over $\mathbb{Q}$ and irreducible constant Laurent polynomials over $\mathbb{C}(z)$ stems from height theory. Let $\mathbb{K}$ denote either $\mathbb{Q}$ or $\mathbb{C}(z)$, and h the canonical height function on subvarieties of the torus $\mathbb{G}^n_{\mathrm{m},\mathbb{K}}$, induced by the standard inclusion $\mathbb{G}^n_{\mathrm{m},\mathbb{K}} \hookrightarrow \mathbb{P}^n_{\mathbb{K}}$.

Let $F \in \mathbb{K}[\boldsymbol{x}^{\pm 1}]$ be an irreducible Laurent polynomial defining a hypersurface $V(F)$ of $\mathbb{G}^n_{\mathrm{m}}$. Then the condition that $\mathrm{h}(V(F)) = 0$ is equivalent to the fact that F is cyclotomic when $\mathbb{K} = \mathbb{Q}$, and to the fact that F is constant when $\mathbb{K} = \mathbb{C}(z)$.

Theorem 2.4. *Let* $F \in \mathbb{C}(z)[\boldsymbol{x}^{\pm 1}]$ *be a non-constant irreducible Laurent polynomial. There are finite sets* $\Omega^0 \subset \mathbb{Z}^{n\times n}$ *of nonsingular matrices and* $\Gamma \subset \mathbb{Z}^n$ *of nonzero vectors satisfying the property that, for* $\boldsymbol{a} \in \mathbb{Z}^n$*, one of the next conditions holds:*

1. *there is* $\boldsymbol{c} \in \Gamma$ *verifying* $\langle \boldsymbol{c}, \boldsymbol{a}\rangle = 0$*;*
2. *there are* $M \in \Omega^0$ *and* $\boldsymbol{b} \in \mathbb{Z}^n$ *with* $\boldsymbol{a} = M\boldsymbol{b}$ *such that if*

$$F(\boldsymbol{x}^M) = \prod_P P^{e_P}$$

is the irreducible factorization of $F(\boldsymbol{x}^M)$*, then*

$$\frac{F(t^{\boldsymbol{a}})}{\mathrm{ct}(F(t^{\boldsymbol{a}}))} = \prod_P \Big(\frac{P(t^{\boldsymbol{b}})}{\mathrm{ct}(P(t^{\boldsymbol{b}}))}\Big)^{e_P};$$

the irreducible factorization of $F(t^{\boldsymbol{a}})/\mathrm{ct}(F(t^{\boldsymbol{a}})$.

Similarly as for Conjecture 2.1, for the validity this statement it is is necessary to take out in its condition (2) the constant part of $F(t^{\boldsymbol{a}})$ and of the $P(t^{\boldsymbol{b}})$'s.

Example 2.5. Set $F = x_1 + zx_2 - z - 1 \in \mathbb{C}(z)[x_1^{\pm 1}, x_2^{\pm 1}]$. Let $\boldsymbol{a} \in \mathbb{Z}^2$ and choose $M \in \mathbb{Z}^{2\times 2}$ nonsingular and $\boldsymbol{b} \in \mathbb{Z}^2$ with $\boldsymbol{a} = M\boldsymbol{b}$. We have that

$$F(\boldsymbol{x}^M) = x_1^{m_{1,1}} x_2^{m_{1,2}} + z\, x_1^{m_{2,1}} x_2^{m_{2,2}} - z - 1$$

is irreducible, and so $P := F(\boldsymbol{x}^M)$ is its only irreducible factor. Again, $t - 1$ divides $F(t^{\boldsymbol{a}}) = P(t^{\boldsymbol{b}})$, and so these univariate Laurent polynomials are not irreducible, unless we divide them by a suitable constant factor.

The proof of Theorem 2.4 relies on the toric Bertini's theorem 1.2 or, more precisely, on its polynomial version in Theorem 3.15. In Section 3 we prove these latter results, while in Section 4 we explain how to deduce Theorem 2.4 from Theorem 3.15, and how to deduce Theorem 1.1 from Theorem 2.4.

3. A variant of Zannier's toric Bertini's theorem

Building on [DZ07], Zannier proved in [Zan10] a toric version of Hilbert irreducibility theorem and then deduce from it an analogue of Bertini's theorem for covers, where the subtori of $\mathbb{G}_{\mathrm{m}}^n$ replaced the linear subspaces in the classical version of this theorem. This result was precised and generalized (with a completely different proof) by Fuchs, Mantova and Zannier to include fibers of arbitrary cosets of subtori [FMZ17, Theorem 1.5] and to obtain a more uniform result.

As before, let $\boldsymbol{x} = (x_1, \dots, x_n)$ be a group of n variables and denote by $\mathbb{G}_{\mathrm{m}}^n = \operatorname{Spec}(\mathbb{C}[\boldsymbol{x}^{\pm 1}])$ the n-dimensional torus over $\mathbb{C}$.

Let W be a variety, that is, a reduced separated scheme of finite type over $\mathbb{C}$. We assume that W is irreducible and quasiprojective of dimension $n \geq 0$, and equipped with a dominant map

$$\pi\colon W \longrightarrow \mathbb{G}_{\mathrm{m}}^n$$

of degree $e \geq 1$ that is finite onto its image. Given an isogeny λ of $\mathbb{G}_{\mathrm{m}}^n$, that is, an endomorphism of $\mathbb{G}_{\mathrm{m}}^n$ with finite kernel, we denote by $\lambda^* W$ the fibered product $\mathbb{G}_{\mathrm{m}}^n \times_{\lambda,\pi} W$, and by

$$\begin{array}{ccc} \lambda^* W & \xrightarrow{\lambda} & W \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{G}_{\mathrm{m}}^n & \xrightarrow{\lambda} & \mathbb{G}_{\mathrm{m}}^n \end{array} \tag{3.1}$$

the corresponding fibered product square.

Definition 3.1. The map π satisfies the *property PB (pullback)* if, for every isogeny λ of $\mathbb{G}_{\mathrm{m}}^n$, we have that $\lambda^* W$ is an irreducible variety.

The aforementioned result by Fuchs, Mantova and Zannier can be stated as follows.

Theorem 3.2. *Let W be an irreducible quasiprojective variety of dimension n and $\pi\colon W \to \mathbb{G}_{\mathrm{m}}^n$ a dominant map that is finite onto its image and that satisfies the property PB. There is a finite union $\mathcal{E}$ of proper subtori of $\mathbb{G}_{\mathrm{m}}^n$ such that, for every subtorus $T \subset \mathbb{G}_{\mathrm{m}}^n$ not contained in $\mathcal{E}$ and every point $p \in \mathbb{G}_{\mathrm{m}}^n(\mathbb{C}) = (\mathbb{C}^\times)^n$, we have that $\pi^{-1}(p \cdot T)$ is an irreducible subvariety of W.*

When the property PB is not verified, the conclusion of this theorem does not necessarily hold, as already pointed out in [Zan10].

Example 3.3. Let $F = z^2 - x_1x_2^2 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, z^{\pm 1}]$, set $W = V(F) \subset \mathbb{G}_{\mathrm{m}}^3$ and consider the map

$$\pi\colon W \longrightarrow \mathbb{G}_{\mathrm{m}}^2$$

defined by $\pi(x_1, x_2, z) = (x_1, x_2)$ for $(x_1, x_2, z) \in W$.

Since F is irreducible, the variety W is irreducible and, since F is monic in z, the map π is finite. However, it does not satisfy the property PB, since for the isogeny λ of $\mathbb{G}_{\mathrm{m}}^2$ defined by $\lambda(x_1, x_2) = (x_1^2, x_2)$,

$$\lambda^*W \simeq V(z - x_1x_2) \cup V(z + x_1x_2)$$

and so this pullback is reducible.

Indeed, this map does neither satisfy the conclusion of Theorem 3.2: given $(a_1, a_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ with a_1 even, let $T \subset \mathbb{G}_{\mathrm{m}}^2$ be the 1-dimensional subtorus given as the image of the map $t \mapsto (t^{a_1}, t^{a_2})$. Then

$$\pi^{-1}(T) = V(z - t^{a_1/2}t^{a_2}) \cup V(z + t^{a_1/2}t^{a_2}),$$

and so this fiber is not irreducible.

Theorem 1.2 extends this result to the situation where the property PB is not necessarily verified. Instead, the conclusion of Theorem 3.2 is replaced by an alternative that "explains" the possibility that a fiber is reducible by a factorization of this fiber through a reducible pullback of the variety W by an isogeny of $\mathbb{G}_{\mathrm{m}}^n$ in a certain finite set. We repeat the statement of this result, for the convenience of the reader.

Theorem 3.4 (Toric Bertini's theorem)**.** *Let W be an irreducible quasiprojective variety of dimension n and $\pi\colon W \to \mathbb{G}_{\mathrm{m}}^n$ a dominant map that is finite onto its image. There is a finite union $\mathcal{E}$ of proper subtori of $\mathbb{G}_{\mathrm{m}}^n$ and a finite set Λ of isogenies of $\mathbb{G}_{\mathrm{m}}^n$ such that, for a subtorus $T \subset \mathbb{G}_{\mathrm{m}}^n$ and a point $p \in \mathbb{G}_{\mathrm{m}}^n(\mathbb{C}) = (\mathbb{C}^\times)^n$, one of the next conditions holds:*

1. *$T \subseteq \mathcal{E}$;*
2. *there is $\lambda \in \Lambda$ with λ^*W reducible and a subtorus $T' \subset \mathbb{G}_{\mathrm{m}}^n$ with λ inducing an isomorphism $T' \to T$;*
3. *$\pi^{-1}(p \cdot T)$ is irreducible.*

Remark 3.5. When the condition (2) above is satisfied, there is a diagram

$$\begin{array}{ccccc}
\pi^{-1}(T') & \longrightarrow & \lambda^*W & \xrightarrow{\lambda} & W \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
T' & \xrightarrow{\iota} & \mathbb{G}_{\mathrm{m}}^n & \xrightarrow{\lambda} & \mathbb{G}_{\mathrm{m}}^n
\end{array}$$

with λ^*W reducible and $\lambda\colon T' \to T$ an isomorphism, and where ι denotes the inclusion of the subtorus T' into $\mathbb{G}_{\mathrm{m}}^n$.

Both inner squares in this diagram are fibered products, and so is the outer square. This implies that the fibers $\pi^{-1}(T)$ and $\pi^{-1}(T')$ are isomorphic. Thus $\pi^{-1}(T)$ can be identified with the fiber of a subtorus for the *reducible* cover $\lambda^*W \to \mathbb{G}_{\mathrm{m}}^n$, and so this fiber is expected to be reducible as well.

Example 3.6. We keep the notation from Example 3.3. In particular, $F = z^2 - x_1x_2^2 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, z^{\pm 1}]$, $W = V(F) \subset \mathbb{G}_{\mathrm{m}}^3$, and $\pi\colon W \to \mathbb{G}_{\mathrm{m}}^2$ the map defined by $\pi(x_1, x_2, z) = (x_1, x_2)$.

Let $(a_1, a_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ with a_1 even, and set $T \subset \mathbb{G}_{\mathrm{m}}^2$ for the 1-dimensional subtorus given as the image of the map $t \mapsto (t^{a_1}, t^{a_2})$. These vectors satisfy the condition (2) in Theorem 3.4 for the isogeny $\lambda\colon \mathbb{G}_{\mathrm{m}}^2 \to \mathbb{G}_{\mathrm{m}}^2$ defined by

$$\lambda(x_1, x_2) = (x_1^2, x_2).$$

Indeed, $\lambda^* W$ is reducible, and this isogeny induces an isomorphism $T' \to T$ with the subtorus $T' \subset \mathbb{G}_{\mathrm{m}}^2$ given as the image of the map $t \mapsto (t^{a_1/2}, t^{a_2})$.

We give the proof of this theorem after some auxiliary results. We first study the reducibility of pullbacks of varieties with respect to isogenies of tori.

Lemma 3.7. *Let $\pi\colon W \to X$ be a map of varieties and $\lambda\colon X \to X$ an étale map. Then $X \times_{\lambda,\pi} W$ is a variety.*

In particular, for a map $\pi\colon W \to \mathbb{G}_{\mathrm{m}}^n$ and an isogeny λ of $\mathbb{G}_{\mathrm{m}}^n$, we have that $\lambda^* W$ is a variety.

Proof. Since $\lambda\colon X \to X$ is étale, the map

$$\lambda\colon X \times_{\lambda,\pi} W \longrightarrow W \tag{3.2}$$

is also étale, because of the invariance of this property under base change [Har77, Chapter IV, Proposition 10.1(b)]. By [Har77, Chapter IV, Exercise 10.4], this implies that, for every closed point $q \in X \times_{\lambda,\pi} W$ and $p := \lambda(q) \in W$, the induced map of completed local rings

$$\widehat{\mathcal{O}}_p \longrightarrow \widehat{\mathcal{O}}_q \tag{3.3}$$

is an isomorphism. Since W is a variety, the local ring $\mathcal{O}_p$ is reduced and, by a theorem of Chevalley [ZS75, §8.13], the completion $\widehat{\mathcal{O}}_p$ is reduced too.

By the isomorphism in (3.3), the completed ring $\widehat{\mathcal{O}}_q$ is reduced. Since this is the completion of a ring with respect to a maximal ideal, the map $\mathcal{O}_q \to \widehat{\mathcal{O}}_q$ is injective, and so the local ring $\mathcal{O}_q$ is also reduced. Since the condition of being reduced is local, this implies that $X \times_{\lambda,\pi} W$ is a variety.

The last statement comes from the fact that the isogenies of algebraic groups over $\mathbb{C}$ are étale maps. □

Thanks to this result, $\lambda^* W$ can be identified with its underlying algebraic subset in the Cartesian product $\mathbb{G}_{\mathrm{m}}^n(\mathbb{C}) \times W(\mathbb{C})$, namely

$$\lambda^* W = \{(p, w) \in \mathbb{G}_{\mathrm{m}}^n(\mathbb{C}) \times W(\mathbb{C}) \mid \lambda(p) = \pi(w)\}. \tag{3.4}$$

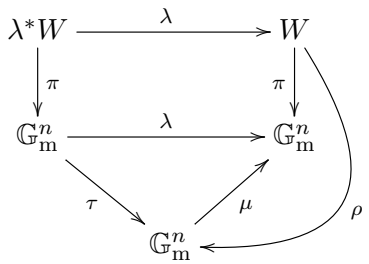
Hence, $\lambda^* W$ is irreducible if and only if this algebraic subset is irreducible. In particular, the map π satisfies the property PB if and only if for every isogeny λ of $\mathbb{G}_{\mathrm{m}}^n$, the pullback $\lambda^* W$ has a single irreducible component. By [Zan10, Proposition 2.1], it is enough to test this condition for $\lambda = [e]$, the multiplication map of $\mathbb{G}_{\mathrm{m}}^n$ by the integer $e = \deg(\pi)$.

The following proposition is implicit in the proof of [Zan10, Proposition 2.1].

Proposition 3.8. *Let $\pi\colon W \to \mathbb{G}_{\mathrm{m}}^n$ be a map from an irreducible variety W, and λ an isogeny of $\mathbb{G}_{\mathrm{m}}^n$. The following conditions are equivalent:*

1. *the pullback $\lambda^* W$ is reducible;*
2. *there is a factorization $\lambda = \mu \circ \tau$ with μ, τ isogenies of $\mathbb{G}_{\mathrm{m}}^n$ such that μ is not an isomorphism, and a map $\rho\colon W \to \mathbb{G}_{\mathrm{m}}^n$ such that $\pi = \mu \circ \rho$.*

In other terms, the condition (2) in the proposition above amounts to the existence of the commutative diagram extending (3.1) of the form



Proof. Suppose that the condition (2) holds. In this case, for $p \in \mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$ and $w \in W(\mathbb{C})$, the fact that $\lambda(p) = \pi(w)$ is equivalent to $\mu(\tau(p)) = \mu(\rho(w))$, and so this holds if and only if there is $\zeta \in \ker(\mu)$ with $\tau(p) = \zeta \cdot \rho(w)$. From (3.4), the pullback decomposes into disjoints subvarieties as

$$\lambda^*W = \bigcup_{\zeta \in \ker(\mu)} \mathbb{G}_{\mathrm{m}}^n \times_{\tau, \zeta\cdot\rho} W.$$

Since μ is not an isomorphism, this pullback is reducible, giving the condition (1).

Conversely, suppose that the condition (1) holds. Then λ^*W has a decomposition into irreducible components

$$\lambda^*W = \bigcup_{i=1}^{k} U_i$$

with $k \geq 2$. Similarly as in (3.2), the map $\lambda^*W \to W$ is étale, and so the U_i's are disjoint. Since λ is an isogeny, the map $\lambda^*W \to W$ is also finite.

The finite subgroup $\ker(\lambda)$ of $\mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$ acts on λ^*W *via* the maps $(p, w) \mapsto (\zeta \cdot p, w)$ for $\zeta \in \ker(\lambda)$, and this action respects the fibers of λ. The action is transitive on the fibers, and so it is also transitive on the U_i's.

Let $H \subset \ker(\lambda)$ be the stabilizer of the irreducible component U_1, and U_1/H the quotient variety. We have that H acts on U_1 transitively on the fibers and without fixed points. The induced map

$$U_1/H \longrightarrow W$$

is a finite étale map of degree 1, and so it is an isomorphism [Mum88, §III.10, Proposition 2].

Then we define the map $\rho\colon W \to \mathbb{G}_{\mathrm{m}}^n$ as the map obtained from the quotient map $U_1/H \to \mathbb{G}_{\mathrm{m}}^n/H$ and the identifications $U_1/H \simeq W$ and $\mathbb{G}_{\mathrm{m}}^n/H \simeq \mathbb{G}_{\mathrm{m}}^n$. In concrete terms and identifying $\mathbb{G}_{\mathrm{m}}^n/H \simeq \mathbb{G}_{\mathrm{m}}^n$, this map is defined, for $w \in W$, as $\rho(w) = \tau(p \cdot H)$ for any $p \in \mathbb{G}_{\mathrm{m}}^n$ such that $(p, w) \in U_1$.

Both $\mathbb{G}_{\mathrm{m}}^n/H$ and $\mathbb{G}_{\mathrm{m}}^n/\ker(\lambda)$ are isomorphic to $\mathbb{G}_{\mathrm{m}}^n$, and so there is a factorization

$$\lambda = \mu \circ \tau,$$

with τ and μ corresponding to the projections $\mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n/H$ and $\mathbb{G}_{\mathrm{m}}^n/H \to \mathbb{G}_{\mathrm{m}}^n/\ker(\lambda)$, respectively. For $w \in W$ and $(p, w) \in U_1$, we have that $\mu \circ \rho(w) = \mu \circ \tau(p) = \pi(w)$. Since the action of $\ker(\lambda)$ on the U_i's is transitive and $k \geq 2$, we have that $H \neq \ker(\lambda)$ and so μ is not an isomorphism, giving the condition (2). □

Remark 3.9. By this proof, if λ^*W is reducible, then the number of its irreducible components is equal to the maximum of the quantity $\deg(\mu) = [\ker(\lambda) : H]$ over all possible maps ρ as in the condition (2).

The next result allows to factorize the dominant map $\pi\colon W \to \mathbb{G}_{\mathrm{m}}^n$ as a map satisfying the property PB followed by an isogeny. It is a variant of [Zan10, Proposition 2.1], that states a similar property for dominant *rational* maps.

Corollary 3.10. *Let W be an irreducible variety of dimension n and $\pi\colon W \to \mathbb{G}_{\mathrm{m}}^n$ a dominant map. There are a map $\rho\colon W \to \mathbb{G}_{\mathrm{m}}^n$ satisfying the property PB and an isogeny λ of $\mathbb{G}_{\mathrm{m}}^n$ with $\pi = \lambda \circ \rho$.*

Proof. Choose ρ as a map $W \to \mathbb{G}_{\mathrm{m}}^n$ of minimal degree among those that give a factorization of the form $\pi = \lambda \circ \rho$ with λ an isogeny of $\mathbb{G}_{\mathrm{m}}^n$.

Suppose that there is a further isogeny ν such that $\nu^* W = \mathbb{G}_{\mathrm{m}}^n \times_{\nu,\rho} W$ is reducible. By Proposition 3.8, there would be an isogeny μ that is not an isomorphism and a map $\rho'\colon W \to \mathbb{G}_{\mathrm{m}}^n$ with $\rho = \mu \circ \rho'$. Hence

$$\pi = \lambda \circ \rho = (\lambda \circ \mu) \circ \rho' \quad \text{and} \quad \deg(\rho) = \#\ker(\mu) \cdot \deg(\rho') > \deg(\rho').$$

By construction, this is not possible. Hence $\nu^* W$ is irreducible for every isogeny ν of $\mathbb{G}_{\mathrm{m}}^n$, and so ρ satisfies the property PB. □

The next result gives a criterion to detect if the inclusion of a subtorus can be factored through a given isogeny as in Proposition 3.8(2).

Lemma 3.11. *Let $T \subset \mathbb{G}_{\mathrm{m}}^n$ be a subtorus and λ an isogeny of $\mathbb{G}_{\mathrm{m}}^n$. The following conditions are equivalent:*

1. *there is a subtorus $T' \subset \mathbb{G}_{\mathrm{m}}^n$ such that λ induces an isomorphism $T' \to T$;*
2. *$\lambda^{-1}(T)$ is the union of $\deg(\lambda)$ distinct torsion cosets.*

Proof. First suppose that $\lambda^{-1}(T)$ is the union of $\deg(\lambda) = \#\ker(\lambda)$ distinct torsion cosets, and denote by T' the one that contains the neutral element. Then T' is a subtorus and $T' \cap \ker(\lambda) = \{1\}$. It follows that $\lambda|_{T'}\colon T' \to T$ is an isogeny of degree 1 and hence an isomorphism, giving the first condition.

Conversely, let $T' \subset \mathbb{G}_{\mathrm{m}}^n$ be a subtorus such that $\lambda|_{T'}\colon T' \to T$ is an isomorphism. Then

$$\lambda^{-1}(T) = \ker(\lambda) \cdot T'.$$

Since $T' \cap \ker(\lambda) = \{1\}$, this fiber is the union of $\#\ker(\lambda) = \deg(\lambda)$ distinct torsion cosets, giving the second condition. □

Proof of Theorem 3.4. By Corollary 3.10, there are a map $\rho\colon W \to \mathbb{G}_{\mathrm{m}}^n$ satisfying the property PB and an isogeny λ of $\mathbb{G}_{\mathrm{m}}^n$ with $\pi = \lambda \circ \rho$.

For each subgroup H of $\ker(\lambda)$, both $\mathbb{G}_{\mathrm{m}}^n / H$ and $\mathbb{G}_{\mathrm{m}}^n / \ker(\lambda)$ are isomorphic to $\mathbb{G}_{\mathrm{m}}^n$, and we consider then a factorization

$$\lambda = \mu_H \circ \tau_H \tag{3.5}$$

with τ_H and μ_H corresponding to the projections $\mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n / H$ and $\mathbb{G}_{\mathrm{m}}^n / H \to \mathbb{G}_{\mathrm{m}}^n / \ker(\lambda)$, respectively. We set Λ as the finite set of isogenies of $\mathbb{G}_{\mathrm{m}}^n$ of the form μ_H as above, for a proper subgroup H of $\ker(\lambda)$.

Since $\rho\colon W \to \mathbb{G}_{\mathrm{m}}^n$ satisfies the property PB, by [FMZ17, Theorem 1.5] there is a finite union $\mathcal{E}'$ of proper subtori of $\mathbb{G}_{\mathrm{m}}^n$ such that, for every subtorus T of $\mathbb{G}_{\mathrm{m}}^n$ not contained in $\mathcal{E}'$ and every point $p \in \mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$, the fiber $\rho^{-1}(p \cdot T)$ is irreducible. Set $\mathcal{E} = \lambda(\mathcal{E}')$.

We next show that the pair $(\Lambda, \mathcal{E})$ satisfies the requirements of Theorem 3.4. Let T be a subtorus of $\mathbb{G}_{\mathrm{m}}^n$ that is not contained in $\mathcal{E}$ and write $\lambda^{-1}(T) = \bigcup_{i=1}^k T_i$ as a disjoint union of torsion cosets T_i of $\mathbb{G}_{\mathrm{m}}^n$.

When $k = 1$, we have that $\lambda^{-1}(T) = T_1$ is a subtorus of $\mathbb{G}_{\mathrm{m}}^n$ that is not contained in $\mathcal{E}$. Hence, $\pi^{-1}(T) = \rho^{-1}(T_1)$ is irreducible.

Otherwise, $k \geq 2$. Let $H \subset \ker(\lambda)$ be the stabilizer of the (unique) subtori in this decomposition, say T_1. This is a proper subgroup, because $\ker(\lambda)$ acts transitively on this collection of torsion cosets and $k \geq 2$.

Consider the factorization $\lambda = \mu_H \circ \tau_H$ as in (3.5). Then $\mu_H \in \Lambda$ and $\mu_H^{-1}(T)$ splits as an union of $k = [\ker(\lambda) : H] = \deg(\mu_H)$ distinct torsion cosets. By Lemma 3.11, μ_H induces an isomorphism between a subtorus T' of $\mathbb{G}_{\mathrm{m}}^n$ and T. Moreover, Proposition 3.8(2) applied to the map $\tau_H \circ \rho$ and the isogeny μ_H shows that the pullback $\mu_H^* W$ is reducible, completing the proof. □

We next prove a polynomial variant of this result. To state it, we first introduce some further notation and auxiliary results.

Let $\boldsymbol{t} = (t_1, \ldots, t_k)$ be a group of k variables. A matrix $A = (a_{i,j})_{i,j} \in \mathbb{Z}^{n \times k}$ defines the family of n monomials in the variables $\boldsymbol{t}$ given by

$$\boldsymbol{t}^A = \Big(\prod_{j=1}^k t_j^{a_{1,j}}, \ldots, \prod_{j=1}^k t_j^{a_{n,j}} \Big).$$

The rule $\boldsymbol{t} \mapsto \boldsymbol{t}^A$ defines a k-parameter monomial map $\mathbb{G}_{\mathrm{m}}^k \to \mathbb{G}_{\mathrm{m}}^n$. This is a group morphism and indeed, every group morphism from $\mathbb{G}_{\mathrm{m}}^k$ to $\mathbb{G}_{\mathrm{m}}^n$ is of this form. The isogenies of $\mathbb{G}_{\mathrm{m}}^n$ correspond to the nonsingular matrices of $\mathbb{Z}^{n \times n}$.

Given $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we can consider it as a row vector, that is, as a matrix in $\mathbb{Z}^{1 \times n}$. In this case,

$$\boldsymbol{x}^{\boldsymbol{a}} = \prod_{j=1}^n x_j^{a_j}$$

is an n-variate monomial. Row vectors give characters of $\mathbb{G}_{\mathrm{m}}^n$, that is, group morphisms $\mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}$. When $\boldsymbol{a}$ is primitive, the kernel of its associated character is a subtorus of $\mathbb{G}_{\mathrm{m}}^n$ of codimension 1, and every such subtorus arises in this way.

Else, we can consider $\boldsymbol{a}$ as a column vector, that is, as a matrix in $\mathbb{Z}^{n \times 1}$. Then

$$t^{\boldsymbol{a}} = (t^{a_1}, \ldots, t^{a_n})$$

is a collection of n univariate monomials in a variable t. Column vectors give group morphisms $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}^n$. When $\boldsymbol{a} \neq 0$, the image of such a morphims is a subtorus of $\mathbb{G}_{\mathrm{m}}^n$ of dimension 1, that we denote by $T_{\boldsymbol{a}}$. When $\boldsymbol{a}$ is primitive, the associated group morphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}^n$ gives an isomorphism between $\mathbb{G}_{\mathrm{m}}$ and $T_{\boldsymbol{a}}$.

For subvarieties of tori, fibered products like those in (3.1) can be expressed in more concrete terms. The next lemma gives such an expression for the case of hypersurfaces.

Lemma 3.12. *Let $F \in \mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}]$, $G \in \mathbb{C}[\boldsymbol{x}^{\pm 1}] \setminus \{0\}$, and $A \in \mathbb{Z}^{n \times k}$. Let W be the hypersurface of $\mathbb{G}_{\mathrm{m}}^{n+1} \setminus V(G)$ defined by F, $\pi \colon W \to \mathbb{G}_{\mathrm{m}}^n$ the map defined by $\pi(\boldsymbol{x}, z) = \boldsymbol{x}$, and $\lambda \colon \mathbb{G}_{\mathrm{m}}^k \to \mathbb{G}_{\mathrm{m}}^n$ the group morphism defined by $\lambda(\boldsymbol{t}) = \boldsymbol{t}^A$. Then $\mathbb{G}_{\mathrm{m}}^k \times_{\lambda,\pi} W$ is isomorphic to the subscheme of $\mathbb{G}_{\mathrm{m}}^{k+1} \setminus V(G(\boldsymbol{t}^A))$ defined by $F(\boldsymbol{t}^A, z)$.*

Proof. The maps π and λ correspond to the morphisms of $\mathbb{C}$-algebras

$$\mathbb{C}[\boldsymbol{x}^{\pm 1}] \longrightarrow \mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}]_G / F \quad \text{and} \quad \mathbb{C}[\boldsymbol{x}^{\pm 1}] \longrightarrow \mathbb{C}[\boldsymbol{t}^{\pm 1}] \simeq \mathbb{C}[\boldsymbol{x}^{\pm 1}, \boldsymbol{t}^{\pm 1}]/(\boldsymbol{x} - \boldsymbol{t}^A),$$

and the fibered product $\mathbb{G}_{\mathrm{m}}^k \times_{\lambda,\pi} W$ is the scheme associated to the tensor product

$$\mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}]_G / F \otimes_{\mathbb{C}[\boldsymbol{x}^{\pm 1}]} \mathbb{C}[\boldsymbol{x}^{\pm 1}, \boldsymbol{t}^{\pm 1}]/(\boldsymbol{x} - \boldsymbol{t}^A).$$

This tensor product is isomorphic to the $\mathbb{C}$-algebra

$$\mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}, \boldsymbol{t}^{\pm 1}]_G/(F, \boldsymbol{x} - \boldsymbol{t}^A) \simeq \mathbb{C}[z^{\pm 1}, \boldsymbol{t}^{\pm 1}]_{G(\boldsymbol{t}^A)}/(F(\boldsymbol{t}^A, z)),$$

which gives the statement. □

Lemma 3.13. *Let $f \in \mathbb{C}(t)[z]$ be an irreducible polynomial of degree $d \geq 1$, and such that $f(t^m, z)$ is reducible for some $m \in \mathbb{N}$. There is $e \in \mathbb{N}$ dividing $\gcd(m, d)$ such that $f(t^e, z)$ is also reducible.*

Proof. The proof relies on the action of torsion points on irreducible factors as in [Zan10, Proposition 2.1].

By Lemma 3.12, the subscheme of $\mathbb{G}_{\mathrm{m}}^2$ defined by $f(t^m, z)$ is isomorphic to the pullback $[m]^*V(f)$, with $[m]$ the m-th multiplication map of $\mathbb{G}_{\mathrm{m}}$. By Lemma 3.7, this pullback is reduced, and so $f(t^m, z)$ is separable. Consider its decomposition into distinct irreducible factors

$$f(t^m, z) = \prod_{i=1}^{k} p_i, \tag{3.6}$$

with $k \geq 2$.

The group μ_m of m-th roots of the unity acts on the set of these irreducible factors by $p_i(t, z) \mapsto p_i(\zeta \cdot t, z)$, $i = 1, \ldots, k$, for $\zeta \in \mu_m$. Let $\mathcal{P} \subset \{p_1, \ldots, p_k\}$ be a nonempty orbit of this action. The polynomial

$$\prod_{p \in \mathcal{P}} p$$

is invariant under the action of μ_m, and so it is of the form $g(t^m, z)$ with $g \in \mathbb{C}(t)[z]$. This product is a nontrivial factor of $f(t^m, z)$, and so g coincides with f up to a scalar. It follows that $\mathcal{P} = \{p_1, \ldots, p_k\}$ and so the action is transitive. In particular, all the p_i's have the same degree in the variable z, and so this degree is positive and $k|d$.

The stabilizer of an irreducible factor p_i is a subgroup of μ_m, hence it is of the form μ_l with $l|m$. Since the action is transitive and μ_m is abelian, this subgroup does not depend on the choice of p_i. Moreover, m/l is equal to k, the number of irreducible factors of $f(t^m, z)$, also because of the transitivity of the action.

By the invariance of each p_i under the action of μ_l, there is $q_i \in \mathbb{C}(t)[z] \setminus \mathbb{C}(t)$ with $p_i = q_i(t^l, z)$. It follows from (3.6) that

$$f(t^e, z) = \prod_{i=1}^{k} q_i(t, z),$$

with $e = m/l$. Clearly $e|m$ and as explained, $e = k$, and so this quantity also divides d, completing the proof. □

Lemma 3.14. *Let $F \in \mathbb{C}[\boldsymbol{x}, z]$ be an irreducible polynomial of degree $d \geq 1$ in the variable z, and $G \in \mathbb{C}[\boldsymbol{x}] \setminus \{0\}$ its leading coefficient.*

1. *Let $W = V(F) \setminus V(G) \subset \mathbb{G}_{\mathrm{m}}^{n+1}$ and $\pi \colon W \to \mathbb{G}_{\mathrm{m}}^n$ the map defined by $\pi(\boldsymbol{x}, z) = \boldsymbol{x}$. The image of π is the open subset $\mathbb{G}_{\mathrm{m}}^n \setminus V(G)$ of $\mathbb{G}_{\mathrm{m}}^n$, and this map is finite onto this open subset.*
2. *There is a finite subset Δ_F of $\mathbb{Z}^n$ such that for $\boldsymbol{a} \in \mathbb{Z}^n$ with $\langle \boldsymbol{c}, \boldsymbol{a} \rangle \neq 0$ for all $\boldsymbol{c} \in \Delta_F$, the polynomial $F(t^{\boldsymbol{a}}, z)$ has degree d in the variable z.*
3. *If $A \in \mathbb{Z}^{n \times n}$ is nonsingular, then $F(\boldsymbol{x}^A, z)$ has no nontrivial factors in $\mathbb{C}[\boldsymbol{x}^{\pm 1}]_G$.*

Proof. For the first statement, the image of the map π is contained in the open set $U = \mathbb{G}_{\mathrm{m}}^n \setminus V(G)$. The induced map $W \to U$ corresponds to the morphism of $\mathbb{C}$-algebras

$$\mathbb{C}[\boldsymbol{x}^{\pm 1}]_G \hookrightarrow \mathbb{C}[\boldsymbol{x}^{\pm 1}, z]_G/(F).$$

This morphism is an integral extension because the leading term G is invertible in $\mathbb{C}[\boldsymbol{x}^{\pm 1}]_G$, and so the map $W \to U$ is finite and, *a fortiori*, surjective.

For the second statement, write $G = \sum_{j=1}^r G_j \boldsymbol{x}^{\boldsymbol{c}_j}$ with $G_j \in \mathbb{C}^\times$ and $\boldsymbol{c}_j \in \mathbb{N}^n$, $j = 1, \ldots, r$, and consider the finite subset of $\mathbb{Z}^n$ given by

$$\Delta_F = \{\boldsymbol{c}_j - \boldsymbol{c}_1 \mid j = 2, \ldots, r\}.$$

For $\boldsymbol{a} \in \mathbb{Z}^n$ with $\langle \boldsymbol{c}, \boldsymbol{a} \rangle \neq 0$ for all $\boldsymbol{c} \in \Delta_F$, we have that $G(t^{\boldsymbol{a}}) \neq 0$ and so $\deg_z(F(t^{\boldsymbol{a}}, z)) = d$.

As for Lemma 3.13, the proof of the last assertion relies on the action of torsion points on irreducible factors, and so we only sketch it. Using Lemmas 3.12 and 3.7, we show that $F(\boldsymbol{x}^A, z)$ is separable. Let

$$F(\boldsymbol{x}^A, z) = \prod_{i=1}^k P_i$$

the decomposition of this Laurent polynomial into distinct irreducible factors. The action of the finite group $\{\boldsymbol{x} \in \mathbb{G}_{\mathrm{m}}^n \mid \boldsymbol{x}^A = 1\}$ on the the sets of these irreducible factors is transitive, and so the P_i's have the same degree with respect to the variable z. Hence for $i = 1, \ldots, k$, we have that $k \deg_z(P_i) = d \geq 1$. In particular, $\deg_z(P_i) \geq 1$, proving the statement. □

Theorem 3.15 (Polynomial toric Bertini's theorem)**.** *Let $F \in \mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}] \setminus \mathbb{C}[\boldsymbol{x}^{\pm 1}]$ be an irreducible Laurent polynomial, and $G \in \mathbb{C}[\boldsymbol{x}^{\pm 1}]$ the coefficient of the term of highest degree in the variable z. There are finite subsets $\Phi \subset \mathbb{Z}^{n \times n}$ of nonsingular matrices and $\Sigma \subset \mathbb{Z}^n$ of nonzero vectors such that, for $\boldsymbol{a} \in \mathbb{Z}^n$, one of the next alternatives hold:*

1. *there is $\boldsymbol{c} \in \Sigma$ such that $\langle \boldsymbol{c}, \boldsymbol{a} \rangle = 0$;*
2. *there is $M \in \Phi$ such that $\boldsymbol{a} \in \mathrm{im}(M)$ and $F(\boldsymbol{x}^M, z)$ is reducible;*
3. *the Laurent polynomial $F(t^{\boldsymbol{a}}, z) \in \mathbb{C}[t^{\pm 1}, z^{\pm 1}]$ is irreducible in $\mathbb{C}[t^{\pm 1}, z^{\pm 1}]_{G(t^{\boldsymbol{a}})}$.*

Proof. This statement, restricted to *primitive* vectors $\boldsymbol{a} \in \mathbb{Z}^n$, is a specialization of Theorem 3.4. To see this, first reduce, multiplying by a suitable monomial, to the case when F is an irreducible polynomial in $\mathbb{C}[\boldsymbol{x}, z]$ of degree $d \geq 1$ in the variable z. Set $W = V(F) \setminus V(G)$ and consider the map

$$\pi \colon W \longrightarrow \mathbb{G}_{\mathrm{m}}^n$$

defined by $\pi(\boldsymbol{x}, z) = \boldsymbol{x}$ for $(\boldsymbol{x}, z) \in W$. The quasi-projective variety W is irreducible and, by Lemma 3.14(1), this map is dominant and finite onto its image, the open subset $U = \mathbb{G}_{\mathrm{m}}^n \setminus V(G)$ of $\mathbb{G}_{\mathrm{m}}^n$.

Let Λ be a finite subset of isogenies of $\mathbb{G}_{\mathrm{m}}^n$ and $\mathcal{E}$ a finite union of proper subtori of $\mathbb{G}_{\mathrm{m}}^n$ satisfying the conclusion of Theorem 3.4 applied to this map. Set then Φ_1 for the finite subset of nonsingular matrices in $\mathbb{Z}^{n \times n}$ corresponding to the isogenies in Λ, and Σ_1 for a finite subset of nonzero vectors of $\mathbb{Z}^n$ such that

$$\mathcal{E} \subset \bigcup_{\boldsymbol{c} \in \Sigma_1} V(\boldsymbol{x}^{\boldsymbol{c}} - 1). \tag{3.7}$$

For a primitive vector $\boldsymbol{a} \in \mathbb{Z}^n$, set $T_{\boldsymbol{a}}$ for the 1-dimensional subtorus defined as the image of the group morphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}^n$. This map gives an isomorphism between $\mathbb{G}_{\mathrm{m}}$ and $T_{\boldsymbol{a}}$. By Lemma 3.12, the fiber $\pi^{-1}(T_{\boldsymbol{a}})$ is isomorphic to the subscheme of

$\mathbb{G}_{\mathrm{m}}^2 \setminus V(G(t^{\boldsymbol{a}}))$ defined by $F(t^{\boldsymbol{a}}, z)$. For the isogeny λ associated to a nonsingular matrix $M \in \Phi_1$, the same result shows that $\lambda^* W$ is isomorphic to the subscheme of $\mathbb{G}_{\mathrm{m}}^{n+1} \setminus V(G)$ defined by $F(\boldsymbol{x}^M, z)$.

The three alternatives from Theorem 3.4 applied to the map π, the subtorus $T_{\boldsymbol{a}}$ and the point $p = (1, \dots, 1) \in \mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$, then boil down to those in the theorem under examination, as explained below.

(1) Suppose that $T_{\boldsymbol{a}} \subset \mathcal{E}$. By (3.7), there is $\boldsymbol{c} \in \Sigma_1$ with $\langle \boldsymbol{c}, \boldsymbol{a} \rangle = 0$.
(2) Else suppose that there is an isogeny $\lambda \in \Lambda$ with $\lambda^* W$ reducible and a subtorus T' of $\mathbb{G}_{\mathrm{m}}^n$ with λ inducing an isomorphism between T' and $T_{\boldsymbol{a}}$. For $M \in \Phi_1$ the nonsingular matrix associated to λ, we have that $\boldsymbol{a} \in \mathrm{im}(B)$ and, by Lemma 3.12, $F(\boldsymbol{x}^M, z)$ is reducible.
(3) Else suppose that $\pi^{-1}(T_{\boldsymbol{a}})$ is irreducible in $\mathbb{G}_{\mathrm{m}}^2 \setminus V(G)$. By Lemma 3.12, this implies that $F(t^{\boldsymbol{a}}, z)$ is irreducible in $\mathbb{C}(t)[z^{\pm 1}]$.

We next enlarge these finite sets to cover the rest of the cases. Let $d \geq 1$ be the degree of F in the variable z, and let e be a divisor of d. If $F(\boldsymbol{x}^e, z)$ is irreducible, we respectively denote by Φ_e and Σ_e the finite subsets of nonsingular matrices in $\mathbb{Z}^{n \times n}$ and of nonzero vectors of $\mathbb{Z}^n$ given by the application of Theorem 3.4 to this polynomial. Otherwise, we set $\Phi_e = \{I_n\}$ with I_n the identity matrix of $\mathbb{Z}^{n \times n}$, and $\Sigma_e = \emptyset$. Set also Δ for the finite subset of nonzero vectors in $\mathbb{Z}^n$ associated to F by Lemma 3.14(2). Set then

$$\Phi = \bigcup_{e|d} e\,\Phi_e \quad \text{and} \quad \Sigma = \Delta \cup \bigcup_{e|d} \Sigma_e.$$

By Theorem 3.4 and the previous analysis, the statement holds for all vectors of the form $e\,\boldsymbol{b}$ with $\boldsymbol{b} \in \mathbb{Z}^n$ primitive and $e|d$.

Given an arbitrary vector $\boldsymbol{a} \in \mathbb{Z}^n$, write $\boldsymbol{a} = m\boldsymbol{b}$ with $m \in \mathbb{N}$ and $\boldsymbol{b} \in \mathbb{Z}^n$ primitive, and set

$$f = F(t^{\boldsymbol{b}}, z) \in \mathbb{C}[t^{\pm 1}, z].$$

Suppose that neither (1) nor (2) hold for $\boldsymbol{a}$. Let $e \in \mathbb{N}$ be a common divisor of d and m. *A fortiori*, these conditions do neither hold for $e\,\boldsymbol{b}$ and, as explained before,

$$f(t^e, z) = F(t^{e\boldsymbol{b}}, z)$$

is irreducible in $\mathbb{C}(t)[z]$. By Lemma 3.13, we have that $F_{\boldsymbol{a}} = f(t^m, z)$ is irreducible in $\mathbb{C}(t)[z]$, giving the condition (3) for $\boldsymbol{a}$ and concluding the proof. □

Remark 3.16. Using the toric Bertini's theorem 3.4 for cosets of arbitrary dimension, the present polynomial version in Theorem 3.15 might be extended to k-parameter monomial maps for any k, and also to arbitrary translates of these monomial maps.

We have kept the present more restricted statement for the sake of simplicity, and also because it is sufficient for our application.

4. Factorization of sparse polynomials

Here we prove the results on the factorization of Laurent polynomials announced in the introduction and in Section 2. Theorem 2.4 is easily seen to be implied by the following statement.

Theorem 4.1. *Let $F \in \mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}]$ without nontrivial factors in $\mathbb{C}[\boldsymbol{x}^{\pm 1}]$. There are finite sets $\Omega^0 \subset \mathbb{Z}^{n \times n}$ of nonsingular matrices and $\Gamma \subset \mathbb{Z}^n$ of nonzero vectors satisfying the property that, for $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$, one of the next alternatives holds:*

(1) *there is $\boldsymbol{c} \in \Gamma$ with $\langle \boldsymbol{c}, \boldsymbol{a} \rangle = 0$;*

(2) *there are* $M \in \Omega^0$ *and* $\boldsymbol{b} \in \mathbb{Z}^n$ *with* $\boldsymbol{a} = M\boldsymbol{b}$ *such that if*

$$F(\boldsymbol{x}^M, z) = \prod_P P^{e_P}$$

is the irreducible factorization of $F(\boldsymbol{x}^M, z)$ *in* $\mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}]$*, then*

$$F(t^{\boldsymbol{a}}, z) = \prod_P P(t^{\boldsymbol{b}}, z)^{e_P}$$

is the irreducible factorization of $F(t^{\boldsymbol{a}}, z)$ *in* $\mathbb{C}(t)[z^{\pm 1}]$.

Proof of Theorem 4.1. We proceed by induction on $\deg_z(F)$. When $\deg_z(F) = 0$, the statement is trivial, and so we assume that $\deg_z(F) \geq 1$.

If F is irreducible, we respectively denote by Φ and Σ the finite sets of nonsingular matrices in $\mathbb{Z}^{n\times n}$ and of nonzero vectors in $\mathbb{Z}^n$ from Theorem 4.1 applied to this Laurent polynomial. If F is reducible, we set $\Phi = \{I_n\}$ and $\Sigma = \emptyset$.

Let $\boldsymbol{a} \in \mathbb{Z}^n$. When F is irreducible, if the condition (1) in Theorem 3.15 holds, then the condition (1) in Theorem 4.1 also holds by taking Γ as any finite set containing Σ. Still in the irreducible case, if the condition (3) in Theorem 3.15 holds, the Laurent polynomial $F(t^{\boldsymbol{a}}, z)$ is irreducible, and the condition (1) in Theorem 4.1 holds provided that Ω^0 contains I_n.

Else, suppose that the condition (2) in Theorem 3.15 holds, that is, there are $M \in \Phi$ and $\boldsymbol{b} \in \mathbb{Z}^n$ with $\boldsymbol{a} = M\boldsymbol{b}$ and $F(\boldsymbol{x}^M, z)$ is reducible. Let

$$F(\boldsymbol{x}^M, z) = F_1\, F_2$$

be a nontrivial factorization. By Lemma 3.14(3), $F(\boldsymbol{x}^M, z)$ has no factors in $\mathbb{C}[\boldsymbol{x}^{\pm 1}]$. Hence $\deg_z(F_i) < \deg_z(F)$, $i = 1, 2$, and by induction, Theorem 4.1 holds for these Laurent polynomials. Let Ω_i^0 and Γ_i respectively denote the finite sets of nonsingular matrices in $\mathbb{Z}^{n\times n}$ and of nonzero vectors in $\mathbb{Z}^n$ whose existence is assured by this theorem.

By construction, either there is a vector $\boldsymbol{c} \in \Gamma_1 \cup \Gamma_2$ with $\langle \boldsymbol{c}, \boldsymbol{b}\rangle = 0$, or we can find $M_i \in \Omega_i$ and $\boldsymbol{b}_i \in \mathbb{Z}^n$ with $\boldsymbol{b} = M_i\boldsymbol{b}_i$ and a decomposition

$$F_i(\boldsymbol{x}^{M_i}, z) = \prod_{j=1}^{k_i} F_{i,j}$$

with $F_{i,j}(t^{\boldsymbol{b}_i}, z)$ irreducible in $\mathbb{C}(t)[s^{\pm 1}]$ for all i, j.

Set

$$\Gamma = \{\mathrm{adj}(M)\boldsymbol{c} \mid M \in \Phi, \boldsymbol{c} \in \Gamma_1 \cup \Gamma_2\} \tag{4.1}$$

with $\mathrm{adj}(M)$ the adjoint matrix of M. If $\langle \boldsymbol{c}', \boldsymbol{a}\rangle \neq 0$ for all $\boldsymbol{c}' \in \Gamma$, then $\langle \boldsymbol{c}, \boldsymbol{b}\rangle \neq 0$ for all $\boldsymbol{c} \in \Gamma_1 \cup \Gamma_2$ and so $F_{i,j}(t^{\boldsymbol{b}_i}, z)$ is irreducible in $\mathbb{C}(t)[s^{\pm 1}]$ for all i, j.

Consider the lattices $K_i = \mathrm{im}(M_i)$, $i = 1, 2$, and set $K = K_1 \cap K_2$. Since K is also a lattice, there is a nonsingular matrix $M' \in \mathbb{Z}^{n\times n}$ with $K = \mathrm{im}(M')$ and, since $K \subseteq K_i$, there are nonsingular matrices N_i, $i = 1, 2$, with $M' = M_iN_i$. Furthermore, $\boldsymbol{b} \in K$ implies that there is $\boldsymbol{b}' \in \mathbb{Z}^n$ with $\boldsymbol{b} = M'\boldsymbol{b}' = M_iN_i\boldsymbol{b}'$. Hence $\boldsymbol{b}_i = M_i^{-1}\boldsymbol{b} = N_i\boldsymbol{b}'$ and

$$F(\boldsymbol{x}^{MM'}, z) = F_1(\boldsymbol{x}^{M_1N_1}, z)F_2(\boldsymbol{x}^{M_2N_2}, z) = \prod_{i=1}^{2}\prod_{j=1}^{r_i} F_{i,j}(\boldsymbol{x}^{N_i}, z).$$

Set $M'' = MM'$, $G_{i,j} = F_{i,j}(\boldsymbol{x}^{B_i}, z)$ and consider the decomposition

$$F(\boldsymbol{x}^{M''}, z) = \prod_{i=1}^{2} \prod_{j=1}^{r_i} G_{i,j}.$$

We have $\boldsymbol{a} = M\boldsymbol{b} = M''\boldsymbol{b}'$ and

$$G_{i,j}(t^{\boldsymbol{b}'}, z) = F_{i,j}(t^{B_i\boldsymbol{b}'}, z) = F_{i,j}(t^{\boldsymbol{b}_i}, z)$$

is irreducible in $\mathbb{C}(t)[s^{\pm 1}]$ for all i, j. The statement follows by taking Ω^0 as any finite set containing all the matrices of the form MM' for $M \in \Phi$, and Γ as in (4.1). □

We conclude by giving the proof of our main result.

Proof of Theorem 1.1. We proceed by induction on n. When $n = 0$ the statement is trivial, and so we assume that $n \geq 1$. Let $F \in \mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}]$ and write

$$F = CF'$$

with $C \in \mathbb{C}[\boldsymbol{x}^{\pm 1}]$ and $F' \in \mathbb{C}[\boldsymbol{x}^{\pm 1}, z^{\pm 1}]$ without nontrivial factors in $\mathbb{C}[\boldsymbol{x}^{\pm 1}]$. By Lemma 3.14(2), there is a finite subset $\Delta \subset \mathbb{Z}^n$ such that $C(t^{\boldsymbol{b}}) \neq 0$ for all $\boldsymbol{b} \in \mathbb{Z}^n$ with $\langle \boldsymbol{c}, \boldsymbol{b} \rangle \neq 0$ for all $\boldsymbol{c} \in \Delta$. Let also $\Omega^0 \subset \mathbb{Z}^{n \times n}$ and $\Gamma \in \mathbb{Z}^n$ be the finite subsets given by Theorem 4.1 applied to F'.

Let $\boldsymbol{a} \in \mathbb{Z}^n$. When $\langle \boldsymbol{c}, \boldsymbol{a} \rangle \neq 0$ for all $\boldsymbol{c} \in \Gamma \cup \Delta$, Theorem 4.1(2) implies the statement, provided that we choose any finite subset $\Omega \subset \mathbb{Z}^{n \times n}$ containing Ω^0.

Otherwise, suppose that there is $\boldsymbol{c} \in \Gamma \cup \Delta$ with $\langle \boldsymbol{c}, \boldsymbol{a} \rangle = 0$. If $C(t^{\boldsymbol{a}}, z) = 0$, we add to the finite set Ω the matrix $M \in \mathbb{Z}^{n \times n}$ made by adding to $n-1$ zero columns to the vector $\boldsymbol{a}$. Otherwise, choose a matrix $L \in \mathbb{Z}^{n \times (n-1)}$ defining a linear map $\mathbb{Z}^{n-1} \to \mathbb{Z}^n$ whose image is the submodule $c^{\perp} \cap \mathbb{Z}^n$, and a vector $\boldsymbol{d} \in \mathbb{Z}^{n-1}$ with $\boldsymbol{a} = L\boldsymbol{d}$. Let $\boldsymbol{u} = (u_1, \ldots, u_{n-1})$ be a group of $n-1$ variables and set

$$G = F'(\boldsymbol{u}^L) \in \mathbb{C}[\boldsymbol{u}^{\pm 1}, z^{\pm 1}].$$

By the inductive hypothesis, there is a finite subset $\Omega_{\boldsymbol{c}} \subset \mathbb{Z}^{(n-1)\times(n-1)}$ satisfying the statement of Theorem 1.1 applied to this Laurent polynomial. In particular, there are $N \in \Omega_{\boldsymbol{c}}$ and $\boldsymbol{e} \in \mathbb{Z}^{n-1}$ with $\boldsymbol{d} = N\boldsymbol{e}$ such that, for an irreducible factor Q of $G(\boldsymbol{u}^N, z)$, we have that $Q(t^{\boldsymbol{e}}, z)$ is, as a Laurent polynomial in $\mathbb{C}(t)[z^{\pm 1}]$, either a unit or an irreducible factor of $G(t^{\boldsymbol{d}}, z)$.

We have that $G(\boldsymbol{u}^N, z) = F'(\boldsymbol{u}^{LN}, z)$, and so Q is an irreducible factor of this latter Laurent polynomial. Moreover, $\boldsymbol{a} = LN\boldsymbol{e}$. Enlarging the matrix $LN \in \mathbb{Z}^{n \times (n-1)}$ to a matrix $M \in \mathbb{Z}^{n \times n}$ by adding to it a zero column at the end, and similarly enlarging the vector $\boldsymbol{e}$ to a vector $\boldsymbol{b} \in \mathbb{Z}^n$ by adding to it a zero entry at the end, the previous equalities are preserved with M and $\boldsymbol{b}$ in the place of LN and $\boldsymbol{e}$. Hence, $\boldsymbol{a} = M\boldsymbol{b}$ and, if $Q(x_1, \ldots, x_{n-1})$ is an irreducible factor of $F'(\boldsymbol{x}^M, z)$, then $Q(t^{\boldsymbol{e}}, z) = Q(t^{\boldsymbol{b}}, z)$ is, as a Laurent polynomial in $\mathbb{C}(t)[z^{\pm 1}]$, either a unit or an irreducible factor of $G(t^{\boldsymbol{d}}, z) = F(t^{\boldsymbol{a}}, z)$.

The statement then follows by also also adding to Ω all the matrices $M \in \mathbb{Z}^{n \times n}$ constructed in this way. □

Remark 4.2. In the setting of Theorem 1.1, the bivariate Laurent polynomials $F_{\boldsymbol{a}}$ can be defined as the pullback of the multivariate Laurent polynomial F by the 2-parameter

monomial map $(t,z) \mapsto (t,z)^A$ given by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ a_1 & 0 \\ \vdots & \vdots \\ a_n & 0 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times 2}.$$

In Conjecture 1.3 applied to F and $k = 2$, one can consider *all* matrices in $\mathbb{Z}^{(n+1)\times 2}$, and so its setting is more general than that of Theorem 1.1.

On the other hand, the conclusion of Conjecture 1.3 in this situation is slightly weaker than that of Theorem 1.1, since it does not give the irreducible factorization of the $F_{\boldsymbol{a}}$ in $\mathbb{C}(t)[z^{\pm 1}]$, but rather its irreducible factorization modulo the Laurent polynomials of the form $f(t^{d_1} z^{d_2})$ for a univariate f and $d_1, d_2 \in \mathbb{Z}$.

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