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Higher-order expansion of misfit functional for defect identification in elastic solids

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Abstract. In this work, least-squares functionals commonly used for defect identification are expanded in powers of the small radius of a trial inclusion, in the context of time-harmonic elastodynamics, generalizing to higher orders the concept of topological derivative. Such expansion, whose derivation and evaluation are facilitated by using an adjoint state, provides a basis for the quantitative estimation of flaws whereby a region of interest may be exhaustively probed at reasonable computational cost.

Keywords: Topological derivative, identification, elastodynamics, asymptotic analysis

Problem statement. We consider a reference (i.e defect-free) 3D elastic solid Ω characterized by Hooke’s tensor $C$ and mass density $\rho$. The time-harmonic background displacement $u$ then solves

$$\langle u, w \rangle_\Omega - \omega^2 \langle u, w \rangle_\Omega = F(w) \quad \forall w \in W,$$  

(1)

where $\langle .. \rangle_D^\rho$ and $\langle .. \rangle_D^C$ denote the stiffness and mass bilinear forms associated to a given domain $D$ characterized by $(C, \rho)$, $W \subset H^1(\Omega)$ is the function space incorporating the relevant essential boundary conditions (if any), the linear form $F \in W'$ defines the applied time-harmonic loading and $\omega$ is the angular frequency.

Assuming the presence of a defect inside $\Omega$, and that we can measure the resulting displacement $u^{\text{ex}}$ on a surface $\Gamma$, we define the least-squares cost functional $J(w)$, with the elastodynamic displacement $w$ associated to the given excitation and a known trial defect, by:

$$J(w) = \frac{1}{2} \int_{\Gamma} |w(x) - u^{\text{ex}}(x)|^2 \, dS_x.$$  

(2)

We now consider a specific trial defect $B_a = z + aB$ (Fig 1), centered at $z \in \Omega$, of small size $a$ and a reference shape $B$. It is a perfectly bonded inclusion filled with a material characterized by its Hooke tensor $C^* = C + \Delta C$ and mass density $\rho^* = \rho + \Delta \rho$. We denote $u_a$ the displacement in the perturbed domain, and $v_a = u_a - u$ the displacement perturbation. $J(u_a)$ admits the exact expansion about $u$:

$$J(u_a) = J(u) + J'(u; v_a) + J''(u; v_a, v_a)$$

$$= J(u) + \Re \int_{\Gamma} (u - u^{\text{ex}}) \cdot v_a + \frac{1}{2} \int_{\Gamma} |v_a|^2$$  

(3)

The goal is now to expand $J(u_a)$ in powers of $a$. Similar expansions have been studied in e.g. [2] for rigid obstacles in 3D acoustic media and [5] for holes in 2D elastic bodies.

Define the adjoint field $p$ as the solution of

$$\langle p, w \rangle_\Omega - \omega^2 \langle p, w \rangle_\Omega = J'(u; w) \quad \forall w \in W.$$  

(4)

We can then compute $J'(u; v_a)$ as

$$J'(u; v_a) = -\langle p, u_a \rangle_{\Gamma B_a}^C + \omega^2 \langle p, u_a \rangle_{\Gamma B_a}^\rho.$$  

(5)

Expanding $v_a$ in powers of $a$ is now needed. As we will see, $J''(u; v_a, v_a)$ requires only the leading contribution of $v_a|_{\Gamma}$ whereas a higher-order expansion of $v_a|_{\Gamma B_a}$ is needed for evaluating (5).

Expansion of the solution perturbation. Following e.g. [3], $v_a$ solves the integro-differential Lippmann-Schwinger equation:

$$L_a[v_a](x) = -(u, G)^{\Delta C}_{B_a} + \omega^2 (u, G)^{\Delta \rho}_{B_a}$$  

(6)

with $L_a[v](x) := v(x) + (v, G)^{\Delta C}_{B_a} - \omega^2 (v, G)^{\Delta \rho}_{B_a}$ and $G = G(\cdot, x)$ is the elastodynamic Green’s tensor for a unit point force applied at $x$ and satisfying homogeneous boundary conditions consistent with problem (1) on $\partial \Omega$. Substituting the ansatz

$$v_a(x) = a V_1(\bar{x}) + a^2 V_2(\bar{x}) + \frac{1}{2} a^2 V_3(\bar{x})$$

$$+ \frac{1}{6} a^4 V_4(\bar{x}) + \delta_a(x), \quad x \in B_a$$  

(7)
(with $\vec{x} := (x-z)/a \in B$) into (6) and expanding the resulting equation in powers of $a$ (in particular using that $G(\xi, x) = a^{-1}G_{\infty}(\xi - \vec{x}) + O(1)$, with $G_{\infty}$ denoting the static full-space Kelvin fundamental solution) yields a sequence of integral equations for the $V_j$. These equations correspond to elastostatic problems for the normalized inclusion $B$ embedded in an unbounded reference medium, and are solved with the help of Eshelby’s equivalent inclusion method [4].

The operator $\delta a$ in (7) solves an integrodifferential equation of the form $L[\delta a] = \gamma_a$. The operator $L : H^1(B_a) \rightarrow H^1(B_a)$ is shown to be invertible with bounded inverse, while $\gamma_a$ can be estimated as $\|\gamma_a\|_{H^1(B_a)} = O(a^{11/2})$. Consequently, there exists a constant $C > 0$ independent of $a$ such that

$$\|\delta a\|_{H^1(B_a)} \leq Ca^{11/2}. \tag{8}$$

For $x \notin B_a$, plugging (7) in the form $v_a(x) \approx aV_{j}(\vec{x})$ into (6) yields the outer expansion

$$v_a(x) = -a^3 \left[ \nabla u(z) : A \cdot \nabla G(z, x) - \omega^2 \Delta \rho |B| u(z) \cdot G(z, x) \right] + o(a^3), \tag{9}$$

$A$ being the elastic moment tensor associated to $B$, $C$ and $\Delta C$ [1,3].

Cost functional expansion. Substituting (7) into (5) and (9) into (3), $J(u_a)$ is finally found to have an expansion of the form:

$$J(u_a) = J_0(a, z) + o(a^6) \tag{10}$$

with $J_0(a, z) = J(u) + a^3 T_3(z) + a^4 T_4(z) + a^5 T_5(z) + a^6 T_6(z)$,

the $o(a^6)$ estimate resulting from (8) and (9).

The $T_j(z)$ are found to be given in terms of (i) the background field $u$ and its derivatives at $z$, (ii) the adjoint field $p$ and its derivatives at $z$, (iii) $A$ and other elastic moment tensors that involve the material parameters, the shape $B$ and the angular frequency $\omega$, and (iv) the complementary part of $G$, i.e. $G = G_{\infty} = (G_{\infty, \omega} - G_{\infty}) + G_C$, where $G_{\infty, \omega}$ is the elastodynamic full-space fundamental solution and $G_C$ accounts for the boundedness of $\Omega$. In particular, $T_3(z)$ is the well-known topological derivative:

$$T_3(z) = -[\nabla u : A : \nabla p - \omega^2 \Delta \rho |B| u \cdot p](z).$$

Moreover, $T_1(z) = 0$ for any centrally-symmetric shape $B$. The complementary part $G_{\infty, \omega} - G_{\infty}$ (known analytically) is involved in $T_5(z)$ and

Figure 2: $T_3(z)$ and $J_0(a^{est}, z)$ plotted in the (XZ) plane around the obstacle (in white).

$T_6(z)$, while $G_C$ appears in $T_6(z)$ only. Since the exact computation of $G_C$ would require solving an elastodynamic problem on $\Omega$ for each trial location $z$, we plan to use an approximation method to save computational time.

Closed-form formulae for the $T_j$ can be obtained when $B$ is spherical (for which case we provide explicit expressions) or ellipsoidal.

Identification. Following [2], estimates of the location $z^{est}$ and size $a^{est}$ of the real defect can then be sought as minimizers of $J_0(a, z)$, with $z$ spanning a predefined sampling grid. This entails computing the $T_j(z)$ over the sampling grid and minimizing $a \rightarrow J_0(a, z)$ for each $z$, the latter step being very fast and straightforward.

A preliminary example is set in free space (so that $G_C = 0$) for a spherical scatterer of radius $0.1 \lambda_S$ illuminated by a plane P-wave travelling along the positive $x$-direction, with a discrete array of displacement sensors lying behind the scatterer. The above procedure yields the size estimate $a^{est} \approx 0.105 \lambda_S$; moreover, the estimated location $z^{est}$ is found to be very close to the true center of the scatterer. The contour plot of $J_0(a^{est}, z)$ (Fig 2) shows improved localisation (relative to the topological derivative $T_3(z)$) for this partial-aperture configuration.

References


