Monotonicity of bistable transition fronts in $\mathbb{R}^N$

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Dedicated to Professor David Kinderlehrer

Abstract

This paper is concerned with the monotonicity of transition fronts for bistable reaction-diffusion equations. Transition fronts generalize the standard notions of traveling fronts. Known examples of standard traveling fronts are the planar fronts and the fronts with conical-shaped or pyramidal level sets which are invariant in a moving frame. Other more general non-standard transition fronts with more complex level sets were constructed recently. In this paper, we prove the time monotonicity of all bistable transition fronts with non-zero global mean speed, whatever shape their level sets may have.

Keywords. Reaction-diffusion equations; transition fronts; monotonicity.

1 Introduction

This paper is concerned with the monotonicity of generalized fronts for the semilinear parabolic equation

$$u_t = \Delta u + f(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$ and Δ denotes the Laplace operator with respect to the space variables $x \in \mathbb{R}^N$. The function f is assumed to be of the bistable type, namely the states $u = 0$ and $u = 1$ are assumed to be both stable stationary states (more precise assumptions will be given later). A typical example is the cubic nonlinearity $f_\theta(s) = s(1 - s)(s - \theta)$ with $0 < \theta < 1$.

It is well known that in one dimension, under some assumptions on f, (1.1) admits standard traveling fronts, that is, solutions of the type

$$u(t, x) = \phi(x - c_f t)$$

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where the front speed $c_f \in \mathbb{R}$ and the front profile $\phi : \mathbb{R} \to [0,1]$ satisfy

$$\begin{cases} \phi'' + c_f \phi' + f(\phi) = 0, \\ \phi(-\infty) = 1, \ \ \phi(+\infty) = 0. \end{cases} \tag{1.2}$$

For precise conditions for the existence and non-existence, we refer to Fife and McLeod [6]. It has also been proved that if a front (c_f, ϕ) solving (1.2) exists, it is uniquely determined up to shifts for ϕ and there holds $\phi'(\xi) < 0$ for $\xi \in \mathbb{R}$. In particular, for such a traveling front $u(t,x) = \phi(x - c_f t)$, observe that the time derivative $u_t(t,x) = -c_f \phi'(x - c_f t)$ has a constant sign for all $(t,x) \in \mathbb{R} \times \mathbb{R}$.

For higher dimensions $N \geq 2$, an immediate extension of one-dimensional traveling fronts consists in planar traveling fronts

$$u(t,x) = \phi(x \cdot e - c_f t)$$

for any given unit vector e of $\mathbb{R}^N$, where c_f and ϕ are as above. The level sets of such traveling fronts are parallel hyperplanes which are orthogonal to the direction of propagation e. These fronts are invariant in the moving frame with speed c_f in the direction e. The existence and uniqueness of these fronts can be referred to the one-dimensional traveling fronts. Besides, in $\mathbb{R}^N$ with $N \geq 2$, more general traveling fronts exist, which have non-planar level sets. For instance, conical-shaped axisymmetric non-planar fronts are known to exist for some f, see [9, 18]. Fronts with non-axisymmetric shapes, such as pyramidal fronts, are also known to exist, see [26, 28]. For qualitative properties of these traveling fronts, we refer to [8, 9, 10, 18, 19, 22, 27, 28].

Even if the types of traveling fronts are various, they share some common properties. For all of them, the solutions u converge to the stable states 0 or 1 far away from their moving or stationary level sets, uniformly in time. This fact led to the introduction of a more general notion of traveling fronts, that is, transition fronts, see [2, 3, 7] and see [23] in the one-dimensional setting. In order to recall the notion of transition fronts, one needs to introduce a few notations. First, for any two subsets A and B of $\mathbb{R}^N$ and for $x \in \mathbb{R}^N$, we set

$$d(A,B) = \inf\{|y - z|;\ (y,z) \in A \times B\}$$

and $d(x,A) = d(\{x\},A)$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. Consider now two families $(\Omega_t^-)_{t\in\mathbb{R}}$ and $(\Omega_t^+)_{t\in\mathbb{R}}$ of open nonempty subsets of $\mathbb{R}^N$ such that

$$\forall t \in \mathbb{R}, \quad \begin{cases} \Omega_t^- \cap \Omega_t^+ = \emptyset, \\ \partial\Omega_t^- = \partial\Omega_t^+ =: \Gamma_t, \\ \Omega_t^- \cup \Gamma_t \cup \Omega_t^+ = \mathbb{R}^N, \\ \sup\{d(x,\Gamma_t);\ \ x \in \Omega_t^+\} = \sup\{d(x,\Gamma_t);\ \ x \in \Omega_t^-\} = +\infty \end{cases} \tag{1.3}$$

and

$$\begin{cases} \inf\Big\{\sup\{d(y,\Gamma_t);\ y \in \Omega_t^+,\ |y-x| \leq r\};\ \ t \in \mathbb{R},\ \ x \in \Gamma_t\Big\} \to +\infty \\ \inf\Big\{\sup\{d(y,\Gamma_t);\ y \in \Omega_t^-,\ |y-x| \leq r\};\ \ t \in \mathbb{R},\ \ x \in \Gamma_t\Big\} \to +\infty \end{cases} \text{as } \ r \to +\infty. \tag{1.4}$$

Notice that the condition (1.3) implies in particular that the interface Γ_t is not empty for every $t \in \mathbb{R}$. As far as (1.4) is concerned, it says that for any $M > 0$, there is $r_M > 0$ such that for any $t \in \mathbb{R}$ and $x \in \Gamma_t$, there are $y^\pm = y^\pm_{t,x} \in \mathbb{R}^N$ such that

$$y^\pm \in \Omega_t^\pm, \ \ |x - y^\pm| \le r_M \ \text{ and } \ d(y^\pm, \Gamma_t) \ge M. \tag{1.5}$$

that is, $y^\pm \in \overline{B(x, r_M)}$ and $B(y^\pm, M) \subset \Omega_t^\pm$, where $B(y,r)$ denotes the open Euclidean ball of center y and radius $r > 0$. In other words, not too far from any point $x \in \Gamma_t$, the sets $\Omega_t^\pm$ contain large balls. Moreover, the sets Γ_t are assumed to be made of a finite number of graphs: there is an integer $n \ge 1$ such that, for each $t \in \mathbb{R}$, there are n open subsets $\omega_{i,t} \subset \mathbb{R}^{N-1}$(for $1 \le i \le n$), n continuous maps $\psi_{i,t} : \omega_{i,t} \to \mathbb{R}$ and n rotations $R_{i,t}$ of $\mathbb{R}^N$, such that

$$\Gamma_t \subset \bigcup_{1\le i\le n} R_{i,t}\Big(\{x \in \mathbb{R}^N; \ \ x' \in \omega_{i,t}, \ \ x_N = \psi_{i,t}(x')\}\Big). \tag{1.6}$$

Definition 1.1 [2, 3] *For problem* (1.1)*, a transition front connecting* 0 *and* 1 *is a classical solution* $u : \mathbb{R}\times\mathbb{R}^N \to (0,1)$ *for which there exist some sets* $(\Omega_t^\pm)_{t\in\mathbb{R}}$ *and* $(\Gamma_t)_{t\in\mathbb{R}}$ *satisfying* (1.3), (1.4) *and* (1.6)*, and, for every* $\varepsilon > 0$*, there exists* $M_\varepsilon > 0$ *such that*

$$\begin{cases} \forall t \in \mathbb{R}, \ \ \forall x \in \Omega_t^+, \ \ (d(x,\Gamma_t) \ge M_\varepsilon) \Rightarrow (u(t,x) \ge 1 - \varepsilon), \\ \forall t \in \mathbb{R}, \ \ \forall x \in \Omega_t^-, \ \ (d(x,\Gamma_t) \ge M_\varepsilon) \Rightarrow (u(t,x) \le \varepsilon). \end{cases} \tag{1.7}$$

Furthermore, u is said to have a global mean speed γ (≥ 0) *if*

$$\frac{d(\Gamma_t, \Gamma_s)}{|t-s|} \to \gamma \ \ as \ \ |t-s| \to +\infty.$$

This definition has been shown in [2, 3, 7] to cover and unify all classical cases. Moreover, it was proved in [7] that, under some assumptions on f, any almost-planar transition front (in the sense that, for every $t \in \mathbb{R}$, Γ_t is a hyperplane) connecting 0 and 1 is truly planar, and that any transition front connecting 0 and 1 has a global mean speed γ, which is equal to $|c_f|$. Non-standard transition fronts which are not invariant in any moving frame were also constructed in [7]. For other properties of bistable transition fronts, we refer to [2, 3, 7]. There is now a large literature devoted to transition fronts in various homogeneous or heterogeneous settings or for other reaction terms, see e.g. [11, 12, 13, 14, 15, 16, 17, 20, 21, 24, 25, 29, 30, 31].

Referring to many works devoted to traveling fronts, we can notice that the monotonicity is actually an important factor for proving further properties of the traveling fronts, but the monotonicity also has its own interest. The aforementioned standard fronts, such as the planar fronts, the conical-shaped fronts and the pyramidal fronts, possess some monotonicity properties, especially they are all monotone in time and in their direction of propagation. Although in dimensions $N \ge 2$ the spatial monotonicity of a given transition front does not make sense in general, since the front may not have a privileged direction of propagation, it still makes sense to ask whether transition fronts of (1.1) are monotone in time. The main goal of this paper is actually to give a positive answer to this question for all transition fronts, whatever shape their level sets may have.

Let us now make more precise the assumptions on the function f. Throughout the paper, we assume the following conditions:

(F1) $f \in C^1([0,1])$ satisfies $f(0) = f(1) = 0$, $f'(0) < 0$ and $f'(1) < 0$.

(F2) There exist $c_f \neq 0$ and $\phi \in C^2(\mathbb{R}, [0,1])$ that satisfy (1.2).

Without loss of generality, we can then assume that

$$c_f > 0$$

even if it means replacing u by $1 - u$, $f(u)$ by $-f(1-u)$ and c_f by $-c_f$. For mathematical purposes, the function f is extended in $\mathbb{R}$ as a $C^1(\mathbb{R})$ function such that $f(s) = f'(0)s > 0$ for all $s \in (-\infty, 0)$ and $f(s) = f'(1)(s-1) < 0$ for all $s \in (0, +\infty)$. From (F1), there exists then a real number $\sigma \in (0, 1/2)$ such that

$$f \text{ is decreasing in } (-\infty, \sigma] \text{ and } [1-\sigma, +\infty),\ f < 0 \text{ in } (0, \sigma] \text{ and } f > 0 \text{ in } [1-\sigma, 1). \tag{1.8}$$

Notice that, in addition to (F1), condition (F2) is fulfilled in particular if there is $\theta \in (0,1)$ such that $f < 0$ in $(0, \theta)$, $f > 0$ on $(\theta, 1)$ and $\int_0^1 f(s)ds \neq 0$, see [1, 6]. However, conditions (F1)-(F2) may also cover other more general nonlinearities f with multiple zeroes in the interval $(0,1)$, see [6].

We always assume throughout the paper that

u is *any* transition front connecting 0 and 1

in the sense of Definition 1.1, with sets $(\Omega_t^{\pm})_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$ satisfying (1.3)-(1.6). We point out that u is *any* transition front, which may or may not be a standard traveling front with planar, conical-shaped or pyramidal level sets, or which may be of none of these types (examples of other fronts have been constructed in [7]).

The main result of the paper is to establish the time monotonicity of the transition front u, under these conditions (F1)-(F2).

Theorem 1.2 *Under assumptions* (F1)-(F2) *and* $c_f > 0$, *any transition front* u *connecting* 0 *and* 1 *is such that* $u_t(t,x) > 0$ *for all* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

Remark 1.3 When $c_f < 0$, it follows immediately from Theorem 1.2 that any transition front u connecting 0 and 1 satisfies $u_t < 0$ in $\mathbb{R} \times \mathbb{R}^N$. When $c_f = 0$ in (F2) with $f < 0$ in $(0, \theta)$ and $f > 0$ in $(\theta, 1)$ for some $\theta \in (0,1)$, then time-increasing fronts, time-decreasing fronts and stationary fronts are known to exist [1, 4, 6], as well as non-monotone fronts [5]. In other words, the condition (F2), i.e. $c_f \neq 0$, is optimal in order to get the time-monotonicity of all transition fronts connecting 0 and 1.

Outline of the paper. In the next section, we prove some auxiliary lemmas on estimates of some particular radially symmetric functions. Section 3 is devoted to the proof of Theorem 1.2.

2 Some preliminary lemmas

We first introduce auxiliary notations for some radially symmetric functions and we show some of their dynamical properties. We recall that f is assumed to satisfy (F1)-(F2). For any $R > 0$ and $\beta \in \mathbb{R}$, let $v_{R,\beta}$ denote the solution of the Cauchy problem

$$\begin{cases} (v_{R,\beta})_t = \Delta v_{R,\beta} + f(v_{R,\beta}), \quad t > 0, \ x \in \mathbb{R}^N, \\ v_{R,\beta}(0,x) = \begin{cases} \beta \ \text{ if } \ |x| < R, \\ 0 \ \text{ if } \ |x| \geq R. \end{cases} \end{cases} \tag{2.1}$$

Lemma 2.1 *For any $T > 0$, $\delta > 0$ and $\beta \in [1-\sigma, 1)$, where $\sigma > 0$ is given in* (1.8)*, there exists $R = R(T,\delta) > 0$ such that*

$$v_{2R,\beta}(t,x) \geq \beta - \delta \ \textit{ for all } 0 \leq t \leq T \textit{ and } |x| \leq R.$$

Proof. Let T, δ and β be fixed as in the statement. Let $\varrho_\beta : \mathbb{R} \to (0,1)$ denote the solution of

$$\begin{cases} \varrho_\beta'(t) = f(\varrho_\beta(t)) \\ \varrho_\beta(0) = \beta. \end{cases}$$

Since $\beta \in [1-\sigma, 1)$ and $f(s) > 0$ for $s \in [1-\sigma, 1)$, $\varrho_\beta(t)$ is increasing in t and $\varrho_\beta(t) \geq \beta$ for all $t \geq 0$. From the maximum principle and (F1), one infers that, for any $R > 0$, $1 \geq \varrho_\beta(t) \geq v_{2R,\beta}(t,x) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}^N$. Then, the following differential inequality holds

$$(\varrho_\beta - v_{2R,\beta})_t - \Delta(\varrho_\beta - v_{2R,\beta}) = f(\varrho_\beta) - f(v_{2R,\beta}) \leq L(\varrho_\beta - v_{2R,\beta}),$$

where $L = \max_{[0,1]} |f'|$. It follows from the maximum principle that

$$0 \leq \varrho_\beta(t) - v_{2R,\beta}(t,x) \leq \frac{e^{Lt}}{(4\pi t)^{N/2}} \int_{|y| \geq 2R} e^{-\frac{|x-y|^2}{4t}} dy \ \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^N.$$

For $0 < t \leq T$ and $|x| \leq R$, one has

$$0 \leq \varrho_\beta(t) - v_{2R,\beta}(t,x) \leq \frac{e^{Lt}}{(4\pi)^{N/2}} \int_{|z| \geq \frac{R}{\sqrt{t}}} e^{-\frac{|z|^2}{4}} dz \leq \frac{e^{LT}}{(4\pi)^{N/2}} \int_{|z| \geq \frac{R}{\sqrt{T}}} e^{-\frac{|z|^2}{4}} dz.$$

Thus, there exists $R = R(T,\delta) > 0$ large enough such that $0 \leq \varrho_\beta(t) - v_{2R,\beta}(t,x) \leq \delta$ for all $0 < t \leq T$ and $|x| \leq R$. Then, it follows that

$$v_{2R,\beta}(t,x) \geq \varrho_\beta(t) - \delta \geq \beta - \delta \ \text{ for all } 0 < t \leq T \text{ and } |x| \leq R.$$

Notice also that the inequality $v_{2R,\beta}(0,x) \geq \beta - \delta$ for $|x| \leq R$ is satisfied immediately at time $t = 0$. □

Remark 2.2 Notice from the proof that the radius $R(T,\delta)$ can be chosen independently of $\beta \in [1-\sigma, 1)$.

The proof of Lemma 2.1 only used the profile of the function f on the interval $[1-\sigma, 1]$. The conclusion was concerned with the behavior of the solution $v_{2R,\beta}$ locally in time. Let us now recall a brief version of [7, Lemma 4.1] (see also [1, Theorem 6.2]), which deals with the large-time behavior of the solutions $v_{R,\beta}$ and for which we recall that $c_f > 0$.

Lemma 2.3 [7] *Fix any $\beta \in [1-\sigma, 1)$, where $\sigma > 0$ is given in* (1.8). *There are some real numbers $R > 0$ and $T > 0$ such that*

$$v_{R,\beta}(t, x) \geq \beta \ \textit{for all}\ t \geq T \ \textit{and}\ |x| \leq R.$$

Let us roughly explain the above lemmas, since they are helpful to the understanding of the following proofs. On the one hand, Lemma 2.1 says that in a bounded time interval $[0, T]$, the function $v_{2R,\beta}$ can not decrease too much in a ball $B(0, R)$ by setting R large enough. On the other hand, Lemma 2.3 says that $v_{R,\beta}$ stays larger than β in a ball $B(0, R)$ at large time. For our transition front u, Lemma 2.1 says that the region where u is close to 1 can not reduce too much as time runs. Furthermore, we recall from [7] that

$$\frac{d(\Gamma_t, \Gamma_s)}{|t-s|} \to c_f > 0 \ \text{ as } |t-s| \to +\infty, \tag{2.2}$$

whence $d(\Gamma_t, \Gamma_s) \to +\infty$ as $|t-s| \to +\infty$. Finally, it can eventually only happen that the state 1 invades in some sense the state 0. These properties will be some essential steps in the proof of the monotonicity of u with respect to t. We show the explicit proofs in the following section.

3 Monotonicity: proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2 on the monotonicity in time of all transition fronts. We recall that f satisfies (F1)-(F2) with $c_f > 0$ and u is an arbitrary transition front connecting 0 and 1 in the sense of Definition 1.1, with sets $(\Omega_t^{\pm})_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$ satisfying (1.3)-(1.6). One can easily check that equation (1.1) and the function f satisfy all assumptions of [3, Theorem 1.11]. That means that, in order to get the time monotonicity and the conclusion of Theorem 1.2, it is sufficient to show that the transition front u is an invasion (of the state 0 by the state 1), in the sense that

$$\Omega_s^+ \subset \Omega_t^+ \text{ for all } s < t \ \text{ and } \ d(\Gamma_t, \Gamma_s) \to +\infty \text{ as } |t-s| \to +\infty. \tag{3.1}$$

Notice that, for the transition front u, the sets $(\Omega_t^{\pm})_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$ satisfying (1.3)-(1.7) are not uniquely determined, since bounded shifts of them still satisfy the same properties. It is therefore enough to show that, for our given transition front u, *some* families $(\Omega_t^{\pm})_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$ satisfy conditions (1.3)-(1.7), together with the invasion property (3.1), even if it means redefining the sets $(\Omega_t^{\pm})_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$.

In the following lemmas, we prove some properties of the sets $(\Omega_t^{\pm})_{t\in\mathbb{R}}$ of the transition front u. The first key property shows that the interfaces $(\Gamma_t)_{t\in\mathbb{R}}$ cannot move infinitely fast.

Lemma 3.1 *For any $T > 0$, there holds*

$$\sup\{d(x, \Gamma_{t-\tau});\ t \in \mathbb{R},\ 0 \le \tau \le T,\ x \in \Gamma_t\} < +\infty. \tag{3.2}$$

Proof. If the conclusion is not true, then, owing to (1.3), two cases may occur, that is, either

$$\sup\{d(x, \Gamma_{t-\tau});\ t \in \mathbb{R},\ 0 \le \tau \le T,\ x \in \Gamma_t \cap \Omega^+_{t-\tau}\} = +\infty,$$

or

$$\sup\{d(x, \Gamma_{t-\tau});\ t \in \mathbb{R},\ 0 \le \tau \le T,\ x \in \Gamma_t \cap \Omega^-_{t-\tau}\} = +\infty.$$

We only consider the first case, the second one can be handled similarly. Fix $\varepsilon \in (0, \sigma)$ (remember that $\sigma > 0$ is given in (1.8)), set $\beta = 1 - \varepsilon \in [1 - \sigma, 1)$ and let $R = R(T, \sigma - \varepsilon) > 0$ be sufficiently large such that the conclusion of Lemma 2.1 holds with $T > 0$ and $\delta = \sigma - \varepsilon > 0$. Then, as the first case above is here considered, there are $t_0 \in \mathbb{R}$, $\tau_0 \in [0, T]$ and a point

$$x_0 \in \Gamma_{t_0} \cap \Omega^+_{t_0-\tau_0} \tag{3.3}$$

such that

$$d(x_0, \Gamma_{t_0-\tau_0}) \ge r_{M_\varepsilon+2R} + M_\varepsilon + 2R, \tag{3.4}$$

where $M_\varepsilon > 0$ is given in (1.7) and $r_{M_\varepsilon+2R} > 0$ is given in the property (1.5) with $M = M_\varepsilon + 2R$. From (1.5), there exists $y_0 \in \mathbb{R}^N$ such that

$$y_0 \in \Omega^-_{t_0},\ \ |y_0 - x_0| \le r_{M_\varepsilon+2R} \ \text{ and } \ d(y_0, \Gamma_{t_0}) \ge M_\varepsilon + 2R,$$

which implies that

$$B(y_0, 2R) \subset \Omega^-_{t_0} \ \text{ and } \ d(B(y_0, 2R), \Gamma_{t_0}) \ge M_\varepsilon.$$

Thus,

$$u(t_0, y) \le \varepsilon < \sigma < 1 - \sigma \ \text{ for all } y \in B(y_0, 2R). \tag{3.5}$$

From (3.3), (3.4) and $|y_0 - x_0| \le r_{M_\varepsilon+2R}$, one also has

$$B(y_0, 2R) \subset \Omega^+_{t_0-\tau_0} \ \text{ and } \ d(B(y_0, 2R), \Gamma_{t_0-\tau_0}) \ge M_\varepsilon.$$

Thus,

$$u(t_0 - \tau_0, y) \ge 1 - \varepsilon \ \text{ for all } y \in B(y_0, 2R).$$

Let $v_{2R,1-\varepsilon}$ be as defined in (2.1) with $2R$ and $\beta = 1 - \varepsilon \in [1 - \sigma, 1)$. Since $u(t_0 - \tau_0, y) \ge v_{2R,1-\varepsilon}(0, y - y_0)$ for all $y \in \mathbb{R}^N$, it follows from the comparison principle that

$$u(t_0, y) \ge v_{2R,1-\varepsilon}(\tau_0, y - y_0) \ \text{ for all } y \in \mathbb{R}^N.$$

Furthermore, from Lemma 2.1 and the choice of R, we have

$$u(t_0, y) \ge v_{2R,1-\varepsilon}(\tau_0, y - y_0) \ge 1 - \varepsilon - (\sigma - \varepsilon) = 1 - \sigma \ \text{ for all } |y - y_0| \le R.$$

This contradicts (3.5). The proof of Lemma 3.1 is thereby complete. □

Remark 3.2 Similarly to Lemma 3.1, one can show that, for any $T > 0$, there holds

$$\sup\{d(x, \Gamma_{t+\tau});\ t \in \mathbb{R},\ 0 \le \tau \le T,\ x \in \Gamma_t\} < +\infty.$$

This property, which is a priori not equivalent to (3.2), will actually not be used in the sequel. But it is still stated since, together with Lemma 3.1, it implies that, for any $T > 0$, the Hausdorff distance between Γ_t and Γ_s is bounded uniformly with respect to $t \in \mathbb{R}$ and $s \in \mathbb{R}$ such that $|t - s| \le T$.

From [3, Theorem 1.2] and Lemma 3.1, we can get the following lemma immediately.

Lemma 3.3 *For any $C \ge 0$, the transition front u satisfies*

$$\begin{aligned} 0 &< \inf\{u(t, x);\ d(x, \Gamma_t) \le C,\ (t, x) \in \mathbb{R} \times \mathbb{R}^N\} \\ &\le \sup\{u(t, x);\ d(x, \Gamma_t) \le C,\ (t, x) \in \mathbb{R} \times \mathbb{R}^N\} < 1. \end{aligned}$$

The second key property of the sets $(\Omega_t^\pm)_{t\in\mathbb{R}}$ is their τ-monotonicity for large $\tau > 0$.

Lemma 3.4 *There exists $\tau_0 > 0$ such that, for any $t \in \mathbb{R}$ and $\tau \ge \tau_0$,*

$$\Omega_t^+ \subset \Omega_{t+\tau}^+.$$

Proof. First of all, property (1.7) and Lemma 3.3 yield the existence of $\varepsilon > 0$ such that

$$u(t, x) < 1 - \varepsilon \ \text{ for all } t \in \mathbb{R} \text{ and } x \in \Omega_t^- \cup \Gamma_t. \tag{3.6}$$

Without loss of generality, one can assume that $\varepsilon \le \sigma$, with $\sigma \in (0, 1/2)$ given in (1.8). Let then $R > 0$ and $T > 0$ be some real numbers such that Lemma 2.3 holds true with $\beta = 1-\varepsilon \in [1-\sigma, 1)$. Since $d(\Gamma_t, \Gamma_s) \to +\infty$ as $|t-s| \to +\infty$ by (2.2), there exists $\tau_0 > 0$ large enough such that $\tau_0 \ge T$ and

$$d(\Gamma_{t+\tau}, \Gamma_t) \ge r_{M_\varepsilon+R} + M_\varepsilon + R \ (> 0) \ \text{ for all } t \in \mathbb{R} \text{ and } \tau \ge \tau_0, \tag{3.7}$$

where $M_\varepsilon > 0$ and $r_{M_\varepsilon+R} > 0$ are given in (1.7) and (1.5) respectively.

In this paragraph, we fix any real number τ such that $\tau \ge \tau_0$. We claim that $\Gamma_t \subset \Omega_{t+\tau}^+$ for all $t \in \mathbb{R}$. Assume not. Then, remembering (1.3) and (3.7), there is $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^N$ such that

$$x_0 \in \Gamma_{t_0} \text{ and } x_0 \in \Omega_{t_0+\tau}^-. \tag{3.8}$$

Then there is $y_0 \in \mathbb{R}^N$ such that

$$y_0 \in \Omega_{t_0}^+,\ \ |y_0 - x_0| \le r_{M_\varepsilon+R} \ \text{ and } \ d(y_0, \Gamma_{t_0}) \ge M_\varepsilon + R, \tag{3.9}$$

which implies that $B(y_0, R) \subset \Omega_{t_0}^+$ and $d(B(y_0, R), \Gamma_{t_0}) \ge M_\varepsilon$. Therefore, $u(t_0, y) \ge 1 - \varepsilon$ for any $y \in B(y_0, R)$. Thus, $u(t_0, y) \ge v_{R,1-\varepsilon}(0, y - y_0)$ for any $y \in \mathbb{R}^N$ where $v_{R,1-\varepsilon}$ is defined in (2.1) with $\beta = 1 - \varepsilon$. From the comparison principle, one gets that

$$u(t_0 + t, y) \ge v_{R,1-\varepsilon}(t, y - y_0) \ \text{ for all } t > 0 \text{ and } y \in \mathbb{R}^N.$$

From Lemma 2.3 and the choice of $R > 0$ and $T > 0$, it follows that

$$u(t_0 + t, y_0) \geq v_{R,1-\varepsilon}(t, 0) \geq 1 - \varepsilon \text{ for all } t \geq T. \tag{3.10}$$

Meanwhile, from (3.7), (3.8) and (3.9), one has

$$y_0 \in \Omega^-_{t_0+\tau} \text{ and } d(y_0, \Gamma_{t_0+\tau}) \geq M_\varepsilon + R \geq M_\varepsilon.$$

This implies that $u(t_0 + \tau, y_0) \leq \varepsilon$. Since $\varepsilon \leq \sigma < 1/2 < 1 - \varepsilon$, this contradicts (3.10) with $t = \tau \geq \tau_0 \geq T$. So, we conclude that

$$\Gamma_t \subset \Omega^+_{t+\tau} \text{ for all } t \in \mathbb{R} \text{ and } \tau \geq \tau_0. \tag{3.11}$$

Finally, assume by contradiction that the conclusion of Lemma 3.4 does not hold with $\tau_0 > 0$ given above. Then, there are $t_1 \in \mathbb{R}$, $\tau_1 \geq \tau_0$ and $x_1 \in \Omega^+_{t_1}$ such that $x_1 \notin \Omega^+_{t_1+\tau_1}$. Since $x_1 \in \Gamma_{t_1+\tau_1} \cup \Omega^-_{t_1+\tau_1}$ and $\Gamma_{t_1} \subset \Omega^+_{t_1+\tau_1}$ by (3.11), one infers that $d(x_1, \Gamma_{t_1}) \geq d(\Gamma_{t_1+\tau_1}, \Gamma_{t_1})$. Hence, by (3.7),

$$d(x_1, \Gamma_{t_1}) \geq r_{M_\varepsilon+R} + M_\varepsilon + R \geq M_\varepsilon + R.$$

Since $x_1 \in \Omega^+_{t_1}$, this also implies that

$$B(x_1, R) \subset \Omega^+_{t_1} \text{ and } d(B(x_1, R), \Gamma_{t_1}) \geq M_\varepsilon.$$

Therefore, $u(t_1, x) \geq 1 - \varepsilon$ for all $x \in B(x_1, R)$ and $u(t_1, x) \geq v_{R,1-\varepsilon}(0, x - x_1)$ for all $x \in \mathbb{R}^N$. From the comparison principle, one gets that

$$u(t_1 + t, x) \geq v_{R,1-\varepsilon}(t, x - x_1) \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^N.$$

From Lemma 2.3 and the choice of R and T, it follows from $\tau_1 \geq \tau_0 \geq T > 0$ that

$$u(t_1 + \tau_1, x_1) \geq v_{R,1-\varepsilon}(\tau_1, 0) \geq 1 - \varepsilon.$$

Since $x_1 \in \Gamma_{t_1+\tau_1} \cup \Omega^-_{t_1+\tau_1}$, the above inequality contradicts (3.6). The proof of Lemma 3.4 is thereby complete. □

Now we are going to redefine the sets $(\Omega^\pm_t)_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$ so that the transition front u is an invasion in the sense of (3.1). To do so, let $\tau_0 > 0$ be as in Lemma 3.4 and set

$$\begin{cases} \widetilde{\Omega}^\pm_{k\tau_0+t} := \Omega^\pm_{k\tau_0} & \text{for any } k \in \mathbb{Z} \text{ and } 0 \leq t < \tau_0, \\ \widetilde{\Gamma}_t := \partial\widetilde{\Omega}^+_t = \partial\widetilde{\Omega}^-_t & \text{for any } t \in \mathbb{R}. \end{cases} \tag{3.12}$$

Proposition 3.5 *The solution u is a transition front with the families $(\widetilde{\Omega}^\pm_t)_{t\in\mathbb{R}}$ and $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$, and then it is an invasion in the sense of* (3.1) *with $(\widetilde{\Omega}^\pm_t)_{t\in\mathbb{R}}$ and $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$.*

Proof. Observe first that, owing to the definitions of the sets $\widetilde{\Omega}_t^{\pm}$ and $\widetilde{\Gamma}_t$, one has $d(\widetilde{\Gamma}_t, \widetilde{\Gamma}_s) \to +\infty$ as $|t-s| \to +\infty$, since $d(\Gamma_t, \Gamma_s) \to +\infty$ as $|t-s| \to +\infty$. Hence, from Lemma 3.4, we immediately get that u is an invasion with the families $(\widetilde{\Omega}_t^{\pm})_{t\in\mathbb{R}}$ and $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$, that is, these sets satisfy (3.1).

It is obvious that, owing to their definition, the sets $(\widetilde{\Omega}_t^{\pm})_{t\in\mathbb{R}}$ and $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$ satisfy the properties (1.3), (1.4) and (1.6). Therefore, we only need to show that u satisfies (1.7) with $(\widetilde{\Omega}_t^{\pm})_{t\in\mathbb{R}}$ and $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$, at least for all $\varepsilon > 0$ small enough.

First of all, we claim that there is $\varepsilon_0 \in (0, 1/2)$ such that

$$\forall\, \varepsilon \in (0, \varepsilon_0),\ \forall\, (t, x) \in \mathbb{R} \times \mathbb{R}^N,\ \forall\, s \in [0, \tau_0], \quad \begin{cases} (u(t, x) \geq 1 - \varepsilon) & \Longrightarrow \quad (u(t + s, x) > \varepsilon), \\ (u(t, x) \leq \varepsilon) & \Longrightarrow \quad (u(t + s, x) < 1 - \varepsilon). \end{cases} \tag{3.13}$$

We only show the first property (the second one can be proved similarly). If it does not hold, there exists a sequence $(t_n, x_n, s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^N\times[0, \tau_0]$ such that $u(t_n, x_n) \to 1$ and $u(t_n+s_n, x_n) \to 0$ as $n \to +\infty$. From standard parabolic estimates, the functions

$$u_n(t, x) := u(t + t_n, x + x_n)$$

converge in $C^{1,2}_{t,x;loc}(\mathbb{R} \times \mathbb{R}^N)$, up to extraction of a subsequence, to a solution $0 \leq u_\infty(t, x) \leq 1$ of (1.1) such that $u_\infty(0, 0) = 1$ and $u_\infty(s_\infty, 0) = 0$ for some $s_\infty \in [0, \tau_0]$. The strong maximum principle implies that $u_\infty = 1$ in $(-\infty, 0] \times \mathbb{R}^N$ and then in $\mathbb{R} \times \mathbb{R}^N$ by uniqueness of the solutions of the associated Cauchy problem. This is impossible, since $u_\infty(s_\infty, 0) = 0$. Thus, there is $\varepsilon_0 \in (0, 1/2)$ satisfying (3.13).

Now, set

$$D := \sup\{d(x, \Gamma_{k\tau_0});\ k \in \mathbb{Z},\ 0 \leq t < \tau_0,\ x \in \Gamma_{k\tau_0+t}\}, \tag{3.14}$$

which is a well-defined real number by Lemma 3.1. In the sequel, fix any real number ε such that $0 < \varepsilon < \varepsilon_0\ (< 1/2)$ and define

$$\widetilde{M}_\varepsilon := M_\varepsilon + D > 0, \tag{3.15}$$

where $M_\varepsilon > 0$ is given in (1.7) for the families $(\Omega_t^{\pm})_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$. Since any $t \in \mathbb{R}$ can be written as $t = k\tau_0 + s$ for some $k \in \mathbb{Z}$ and $0 \leq s < \tau_0$, it follows from (1.7) that we only need to show that

$$\forall\, k \in \mathbb{Z},\ \forall\, 0 \leq t < \tau_0,\ \forall x \in \widetilde{\Omega}^+_{k\tau_0+t},\ d(x, \widetilde{\Gamma}_{k\tau_0+t}) \geq \widetilde{M}_\varepsilon \Longrightarrow (x \in \Omega^+_{k\tau_0+t} \text{ and } d(x, \Gamma_{k\tau_0+t}) \geq M_\varepsilon) \tag{3.16}$$

and

$$\forall\, k \in \mathbb{Z},\ \forall\, 0 \leq t < \tau_0,\ \forall x \in \widetilde{\Omega}^-_{k\tau_0+t},\ d(x, \widetilde{\Gamma}_{k\tau_0+t}) \geq \widetilde{M}_\varepsilon \Longrightarrow (x \in \Omega^-_{k\tau_0+t} \text{ and } d(x, \Gamma_{k\tau_0+t}) \geq M_\varepsilon). \tag{3.17}$$

We only prove (3.16), the property (3.17) being proved similarly thanks to the second property in (3.13). To show (3.16), we first claim that, for any given $k \in \mathbb{Z}$ and $0 \leq t < \tau_0$, there holds

$$\Gamma_{k\tau_0+t} \subset \{x \in \mathbb{R}^N;\ d(x, \widetilde{\Gamma}_{k\tau_0+t}) \leq D\}. \tag{3.18}$$

Indeed, otherwise, there is a point $x_0 \in \Gamma_{k\tau_0+t}$ such that $d(x_0, \widetilde{\Gamma}_{k\tau_0+t}) > D$. Since $\widetilde{\Gamma}_{k\tau_0+t} = \Gamma_{k\tau_0}$ by definition, this yields $d(x_0, \Gamma_{k\tau_0}) > D$, which contradicts the definition of D in (3.14).

Then, we claim that, for any given $k \in \mathbb{Z}$ and $0 \le t < \tau_0$, there holds

$$\{x \in \widetilde{\Omega}^+_{k\tau_0+t};\ d(x, \widetilde{\Gamma}_{k\tau_0+t}) \ge \widetilde{M}_\varepsilon\} \subset \Omega^+_{k\tau_0+t}. \tag{3.19}$$

Indeed, let $x \in \widetilde{\Omega}^+_{k\tau_0+t}$ be such that $d(x, \widetilde{\Gamma}_{k\tau_0+t}) \ge \widetilde{M}_\varepsilon$. In other words, $x \in \Omega^+_{k\tau_0}$ and $d(x, \Gamma_{k\tau_0}) \ge \widetilde{M}_\varepsilon = M_\varepsilon + D$. Hence, $d(x, \Gamma_{k\tau_0}) \ge M_\varepsilon$ and

$$u(k\tau_0, x) \ge 1 - \varepsilon \tag{3.20}$$

by definition of M_ε. Furthermore, $d(x, \Gamma_{k\tau_0}) \ge \widetilde{M}_\varepsilon = M_\varepsilon + D$ and (3.14) imply that

$$d(x, \Gamma_{k\tau_0+t}) \ge M_\varepsilon.$$

Therefore, either $x \in \Omega^-_{k\tau_0+t}$ and $u(k\tau_0 + t, x) \le \varepsilon$, or $x \in \Omega^+_{k\tau_0+t}$ (and $u(k\tau_0 + t, x) \ge 1 - \varepsilon$). The former case is impossible due to (3.13) and (3.20). Thus, $x \in \Omega^+_{k\tau_0+t}$ and (3.19) is proved.

Finally, from (3.12), (3.15), (3.18), (3.19), we easily get (3.16). As already emphasized, the proof of Proposition 3.5 is thereby complete. □

Proof of Theorem 1.2. From [3, Theorem 1.11] and the fact that, by Proposition 3.5, the transition front u is an invasion in the sense of (3.1), with the sets $(\widetilde{\Omega}^\pm_t)_{t\in\mathbb{R}}$ and $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$, we immediately get the desired monotonicity property $u_t > 0$ in $\mathbb{R} \times \mathbb{R}^N$. □

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