Some existence results for advanced backward stochastic differential equations with a jump time

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Abstract

In this paper, we are interested by advanced backward stochastic differential equations (ABSDE), in a probability space equipped with a Brownian motion and a single jump process. The solution of the ABSDE is a triple (Y, Z, U) where Y is a semimartingale, Z is the diffusion coefficient and U the size of the jump. We allow the generator to depend on the future paths of the solution.

Keywords: Advanced Backward Stochastic Differential Equations, Single Jump, Immersion.

1 Introduction

In this paper, we are interested by backward stochastic differential equations of one of the following forms, called advanced backward stochastic differential equations (in short ABSDE)

$$\begin{cases} -dY_t = & f\big(t, Y_t, \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}], (\mathbb{E}^{\mathcal{G}_t}[Y_{t+s}])_{0\le s\le\delta}, Z_t, \mathbb{E}^{\mathcal{G}_t}[Z_{t+\delta}], (\mathbb{E}^{\mathcal{G}_t}[Z_{t+s}])_{0\le s\le\delta}, \\ & U_t, \mathbb{E}^{\mathcal{G}_t}[U_{t+\delta}], (\mathbb{E}^{\mathcal{G}_t}[U_{t+s}])_{0\le s\le\delta}\big)dt - Z_t dB_t - U_t dH_t\ , \quad 0 \le t \le T\ , \\ Y_{T+t} = & \xi_{T+t}\ , \quad 0 \le t \le \delta\ , \\ Z_{T+t} = & P_{T+t}\ , \quad U_{T+t} \ = \ Q_{T+t}\mathbb{1}_{\{T+t\le\tau\}}, \quad 0 < t \le \delta\ , \end{cases} \tag{1.1}$$

and

$$\begin{cases} -dY_t = & \mathbb{E}^{\mathcal{G}_t}\big[f(t, Y_t, Y_{t+\delta}, (Y_{t+s})_{0\le s\le\delta}, Z_t, Z_{t+\delta}, (Z_{t+s})_{0\le s\le\delta}, \\ & U_t, U_{t+\delta}, (U_{t+s})_{0\le s\le\delta})]dt - Z_t dB_t - U_t dH_t\ , \quad 0 \le t \le T\ , \\ Y_{T+t} = & \xi_{T+t}\ , \quad 0 \le t \le \delta\ , \\ Z_{T+t} = & P_{T+t}, \quad U_{T+t} \ = \ Q_{T+t}\mathbb{1}_{\{T+t\le\tau\}}, \quad 0 < t \le \delta\ , \end{cases} \tag{1.2}$$

where B is a Brownian motion and H is the process $H_t = \mathbb{1}_{\{\tau \le t\}}$ associated with a given random time τ. In this equation, for an integrable random variable X, we have used the notation $\mathbb{E}^{\mathcal{G}_t}[X] := \mathbb{E}[X|\mathcal{G}_t]$, where $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ is the filtration generated by B and H. The terminal conditions ξ, P and Q are given processes. We remark that the generators f of these ABSDEs depend on the values of the processes (Y, Z, U) for present time t as well as for future time $t+\delta$ and also of the trajectory of the processes on the interval $[t, t+\delta]$. The ABSDE (1.1) was introduced by Peng and Yang in [12] in a Brownian case setting (roughly speaking, for $\tau \equiv 0$). Øksendal *et al.* [11] have introduced ABSDEs of the form (1.2) when dealing with optimal control for delayed systems, taking into account a random Poisson measure, instead of a single jump process.

Using the methodology of BSDEs in an enlargement of filtration setting as in Kharroubi and Lim [8], we give conditions such that there exists a unique solution of (1.1) and of (1.2) under immersion hypothesis and in adequate spaces. This progressive enlargement is often considered as progressive adding of information given in form of a random time τ in a way which transforms τ to a stopping time with respect to the filtration $\mathbb{G}$. The topic of enlargement of filtration was initiated by Jacod, Jeulin and Yor (see [6, 7]). Naturally, the enlargement of filtration appears in credit risk and it has also been related recently to stochastic optimal control by Pham [13] and to mean-variance hedging by Kharroubi *et al.* [9] where the optimal strategy is described by non-standard BSDEs driven by a Brownian motion and a jump martingale in the enlarged filtration.

2 Framework

2.1 Classical results about progressive enlargement

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion B and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t\ge 0}$ the right-continuous and complete filtration generated by B. We consider on this space a random time τ and we introduce the right-continuous process $H := \mathbb{1}_{\{\tau \le .\}}$. Since τ is not supposed to be an $\mathbb{F}$-stopping time, we use the standard approach of filtration enlargement by considering the smallest right-continuous extension $\mathbb{G}$ of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time. More precisely, the filtration $\mathbb{G} := (\mathcal{G}_t)_{t\ge 0}$ is defined by

$$\mathcal{G}_t \quad := \quad \bigcap_{\varepsilon > 0} \tilde{\mathcal{G}}_{t+\varepsilon} \ ,$$

for any $t \ge 0$, where $\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(H_u \ , \ u \in [0, s])$, for any $s \ge 0$.
We denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable subsets of $\Omega \times \mathbb{R}_+$, i.e., the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes. We denote by $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{G})$) the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-optional subsets of $\Omega \times \mathbb{R}_+$, i.e., the σ-algebra generated by the right-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes.

We impose the following hypothesis introduced by Bremaud and Yor [2], which is classical in the filtration enlargement theory and is called $(\mathcal{H})$-hypothesis or immersion property .

Hypothesis 2.1 *The process B remains a $\mathbb{G}$-Brownian motion.*

We observe that, since the filtration $\mathbb{F}$ is generated by the Brownian motion B, Hypothesis 2.1 is equivalent to all $\mathbb{F}$-martingales are also $\mathbb{G}$-martingales. In particular, the stochastic integral $\int_0^t X_s dB_s$ is a well defined $\mathbb{G}$-local martingale for all $\mathcal{P}(\mathbb{G})$-measurable processes X such that $\int_0^t |X_s|^2 ds < \infty$, for all $t \geq 0$.

We also introduce another hypothesis, often called the Jacod equivalence hypothesis (see, e.g., [1, chapter 4], that the conditional law of τ is equivalent to the law of τ and that τ admits a density w.r.t. Lebesgue's measure, which will allow us to compute conditional expectations w.r.t. $\mathbb{G}$ in terms of conditional expectations w.r.t. $\mathbb{F}$.

Hypothesis 2.2 *We assume that there exists a strictly positive $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable function $(\omega, t, u) \to \alpha_t(\omega, u)$ continuous in t such that*

a) for any $\theta \geq 0$, the process $(\alpha_t(\theta))_{t\geq 0}$ is an $\mathbb{F}$-martingale,

b) for any $t \geq 0$, the measure $\alpha_t(\omega, \theta)d\theta$ is a version of $\mathbb{P}(\tau \in d\theta | \mathcal{F}_t)(\omega)$, that is for any Borel function f such that $f(\tau)$ is integrable, one has

$$\mathbb{E}[f(\tau)|\mathcal{F}_t] \quad = \quad \int_0^\infty f(\theta)\alpha_t(\theta)d\theta \ , \quad a.s.$$

In particular, the density of τ is α_0.

In all the paper, Hypotheses 2.1 and 2.2 are in force.

We now recall some standard results that will be important for our purpose and we refer to [3] for their proofs.

We introduce the $\mathbb{F}$-supermartingale G (called Azéma's supermartingale) defined as

$$G_t \quad := \quad \mathbb{P}(\tau > t \mid \mathcal{F}_t) = \int_t^\infty \alpha_t(\theta)d\theta \ , \quad t \geq 0 \ .$$

The supermartingale G is strictly positive, non-increasing and continuous. The process M defined by

$$M_t \quad := \quad H_t - \int_0^{t\wedge\tau} \frac{\alpha_s(s)}{G_s} ds \ , \quad t \geq 0 \ ,$$

is a $\mathbb{G}$-martingale, with a single jump at time τ. The $\mathbb{F}$-adapted process λ defined by

$$\lambda_t \quad := \quad \frac{\alpha_t(t)}{G_t} \ , \quad t \geq 0 \ , \tag{2.1}$$

is called the $\mathbb{F}$-intensity of τ. Under Hypotheses 2.1 and 2.2, we have, from [3, equality (11)],

$$\alpha_t(\theta) \quad = \quad \alpha_\theta(\theta) \ , \quad \forall \, t \geq \theta \ , \tag{2.2}$$

which implies

$$G_t \quad = \quad \exp\Big(- \int_0^t \lambda_s ds \Big) \ , \tag{2.3}$$

since, by definition of G and λ and the fact that (2.2) holds, we have the following equalities,

$$G_t = \int_t^\infty \alpha_t(\theta)d\theta = 1 - \int_0^t \alpha_t(\theta)d\theta = 1 - \int_0^t \alpha_\theta(\theta)d\theta = 1 - \int_0^t G_\theta \lambda_\theta d\theta$$

and $G_0 = 1$. Note that, from immersion

$$G_t \quad = \quad \mathbb{E}^{\mathcal{F}_s}(\mathbb{1}_{\{\tau > t\}}) \ , \quad \forall\, s > t \ . \tag{2.4}$$

Hypothesis 2.3 *We assume that the process λ is upper bounded by a constant k.*

Lemma 2.4 *For any $t \in [0, T]$, the random variable G_t is lower bounded by e^{-kt}, and for any $\theta \in [0, T]$ we have $0 < \alpha_t(\theta) \leq k$.*

Proof: The bound on G is obvious from (2.3). Let $\theta \in [0, T]$; then for any $t \geq \theta$, (2.1) and (2.2) lead to

$$\alpha_t(\theta) \quad = \quad \alpha_\theta(\theta) \ = \ \lambda_\theta G_\theta \ \leq \ k \ .$$

Moreover, since $\alpha(\theta)$ is a martingale, we get $\alpha_t(\theta) \leq \mathbb{E}^{\mathcal{F}_t}(\alpha_\theta(\theta)) \leq k$ for any $t \leq \theta$. □

Furthermore, if $Y \in \mathcal{G}_T$ is integrable, then we have

$$\mathbb{E}^{\mathcal{G}_t}[Y \mathbb{1}_{\{t<\tau\}}] \quad = \quad \frac{1}{G_t} \mathbb{1}_{\{t<\tau\}} \mathbb{E}^{\mathcal{F}_t}(Y \mathbb{1}_{\{t<\tau\}}) \ .$$

We recall a decomposition result for $\mathcal{P}(\mathbb{G})$-measurable processes, proved in [7, Lemma 4.4] for bounded processes. It can be easily extended to the case of unbounded processes.

Proposition 2.5 *Any $\mathcal{P}(\mathbb{G})$-measurable process $X = (X_t)_{t \geq 0}$ can be represented as*

$$X_t \quad = \quad X_t^b \mathbb{1}_{\{t \leq \tau\}} + X_t^a(\tau) \mathbb{1}_{\{t > \tau\}} \ ,$$

for all $t \geq 0$, where X^b is $\mathcal{P}(\mathbb{F})$-measurable and $X^a(\cdot)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

Here, the superscript b is for *before* τ and a for *after* τ. In particular, a $\mathbb{G}$-predictable process is equal to an $\mathbb{F}$-predictable process on the set $\{t \leq \tau\}$.

Song [14] has extended the previous result to the class of optional processes under some hypotheses, which are satisfied under equivalence Jacod's hypothesis.

Proposition 2.6 *Any $\mathcal{O}(\mathbb{G})$-measurable process $X = (X_t)_{t \geq 0}$ can be represented as*

$$X_t \quad = \quad X_t^b \mathbb{1}_{\{t < \tau\}} + X_t^a(\tau) \mathbb{1}_{\{t \geq \tau\}} \ ,$$

for all $t \geq 0$, where X^b is $\mathcal{O}(\mathbb{F})$-measurable and $X^a(\cdot)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

If the process X is bounded by a constant K, then the process X^b is bounded by K and one can also choose the process $X^a(\theta)$ bounded by K for any $\theta \geq 0$. We remark that the uniqueness of $X_t^a(\theta)$ is granted for $\theta \leq t$.

The process X^b is uniquely determined on $[0, T]$ by $X_t^b = \frac{1}{G_t} \mathbb{E}^{\mathcal{F}_t}(X_t \mathbb{1}_{\{t<\tau\}})$, this quantity will be called the pre-default part.

Lemma 2.7 *Let $Y_T(\tau)$ be a bounded $\mathcal{F}_T \otimes \sigma(\tau)$-measurable random variable. Then, for any $t \le T$, we have*

$$\mathbb{E}^{\mathcal{G}_t}[Y_T(\tau)] \quad = \quad Y_t^b \mathbb{1}_{\{t<\tau\}} + Y_t^a(\tau)\mathbb{1}_{\{\tau\le t\}} \quad a.s.$$

where

$$\begin{aligned} Y_t^b &= \frac{\mathbb{E}^{\mathcal{F}_t}\big[\int_t^\infty Y_T(u)\alpha_T(u)du\big]}{G_t} \quad a.s. \\ Y_t^a(\theta) &= \mathbb{E}^{\mathcal{F}_t}\big[Y_T(\theta)\big] \quad a.s. \ \ \textit{for any} \ \ \theta \le t \end{aligned} \tag{2.5}$$

which can be rewritten under the form

$$Y_t^a(\tau) \quad = \quad \mathbb{E}^{\mathcal{F}_t}\big[Y_T(\theta)\big]_{|\theta=\tau} \qquad a.s. \ \ on \ \ \tau \le t\ . \tag{2.6}$$

Proof: The proof of this Lemma is an application of Proposition 2.6 and that

$$Y_t^a(\theta) \quad = \quad \frac{\mathbb{E}^{\mathcal{F}_t}\big[Y_T(\theta)\alpha_T(\theta)\big]}{\alpha_t(\theta)} \quad a.s.$$

and $\alpha_t(\theta) = \alpha_\theta(\theta)$ for any $t \ge \theta$. □

Therefore, if $Y_T(\tau)$ is bounded by a constant K then the processes Y^b and $Y^a(\theta)$ are bounded by K for any $\theta \ge 0$.

We now give a decomposition result for Stochastic Differential Equations (SDEs) in $\mathbb{G}$ in terms of SDEs in $\mathbb{F}$.

Lemma 2.8 *If the process X satisfies the following stochastic differential equation*

$$dX_t \quad = \quad \mu(t, X_t, \eta_t)dt + \sigma(t, X_t, \eta_t)dB_t + \varphi(t, X_{t^-}, \eta_t)dH_t\ ,$$

where μ, σ are $\mathcal{O}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$-measurable maps, φ is a $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$ measurable map and η is a $\mathbb{G}$-predictable process, then X^a and X^b satisfy

$$\begin{cases} dX_t^a(\tau) = \ \mu^a(t,\tau,X_t^a(\tau),\eta_t^a(\tau))dt + \sigma^a(t,\tau,X_t^a(\tau),\eta_t^a(\tau))dB_t\ , \quad \tau \le t \le T\ , \\ dX_t^b = \ \mu^b(t,X_t^b,\eta_t^b)dt + \sigma^b(t,X_t^b,\eta_t^b)dB_t\ , \quad 0 \le t \le T\ , \\ X_t^a(t) - X_t^b = \ \varphi(t,X_t^b,\eta_t^b)\ , \quad 0 \le t \le \tau\ . \end{cases}$$

Proof: The proof of this Lemma is an application of Proposition 2.6 and for the last equality, we have used that if an $\mathbb{F}$-predictable process K satisfies $K_\tau = 0$, then $K_t = 0$ on $\{t \le \tau\}$ (see [10, Lemma 3, Chapter 1]). □

2.2 Notation

To define solutions to ABSDEs, we introduce the following spaces, where $s, t \in \mathbb{R}_+$ with $s \le t$, and $T < \infty$ is the terminal time and δ is a strictly positive constant.

- $\mathcal{S}^2_{\mathbb{G}}[s,t]$ (resp. $\mathcal{S}^2_{\mathbb{F}}[s,t]$) is the set of $\mathbb{R}$-valued $\mathcal{O}(\mathbb{G})$ (resp. $\mathcal{O}(\mathbb{F})$)-measurable processes $(Y_u)_{u\in[s,t]}$ such that

$$\|Y\|_{\mathcal{S}^2[s,t]} \quad := \quad \mathbb{E}[\sup_{u\in[s,t]} |Y_u|^2] \ < \ \infty\ .$$

- $L^2_{\mathbb{G}}[s,t]$ (resp. $L^2_{\mathbb{F}}[s,t]$) is the set of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$ (resp. $\mathcal{P}(\mathbb{F})$)-measurable processes $(Z_u)_{u\in[s,t]}$ such that

$$\|Z\|^2_{L^2[s,t]} \;:=\; \mathbb{E}\Big[\int_s^t |Z_u|^2 du\Big] \;<\; \infty\,.$$

- $L^2(\mathcal{F}_t)$ is the set of $\mathbb{R}$-valued square integrable $\mathcal{F}_t$-measurable random variables.
- L^2_τ is the set of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{F})$-measurable processes U such that $U_t = 0$ for $t > \tau$ and

$$||U||^2_{L^2_\tau} \;:=\; \mathbb{E}\Big[\int_0^T |U_s|^2 ds\Big] \;<\; \infty\,.$$

- $\mathbb{D}[0,\delta]$ is the set of càdlàg real valued maps defined on $[0,\delta]$. For $\mathcal{Y} \in \mathbb{D}[0,\delta]$, we denote $|\mathcal{Y}| := \frac{1}{\sqrt{\delta}}\int_0^\delta |\mathcal{Y}(s)|ds$.

2.3 Existence results for ABSDE in a Brownian filtration

We extend the results of Peng and Yang [12] to more general drivers. The proofs are based on standard methodologies, however they require careful majorizations. To simplify the writing we introduce some new notation for each Proposition, and the same notation $\vec{\mathbf{y}} \in \mathbf{A}$ is used in different meanings which are clear from the context.

Proposition 2.9 *Let $\mathbf{A} := \mathbb{R}^2\times\mathbb{D}[0,\delta]\times\mathbb{R}^2\times\mathbb{D}[0,\delta]$ and, for any $\vec{\mathbf{y}} = (y,\widehat{y},\mathcal{Y},z,\widehat{z},\mathcal{Z}) \in \mathbf{A}$ we define $|\vec{\mathbf{y}}|$ by*

$$|\vec{\mathbf{y}}| \;=\; |y| + |\widehat{y}| + |\mathcal{Y}| + |z| + |\widehat{z}| + |\mathcal{Z}|\,,$$

where $|\mathcal{Y}|$ is defined in Section 2.2. Let f be a map from $\Omega\times[0,T]\times\mathbf{A}$ valued in $\mathbb{R}$. Let p and q be given bounded $\mathbb{F}$-adapted processes.
The following ABSDE

$$\begin{cases} -dY_t = & f\big(t, Y_t, \mathbb{E}^{\mathcal{F}_t}[p_{t+\delta}Y_{t+\delta}], \{\mathbb{E}^{\mathcal{F}_t}(p_{t+s}Y_{t+s})\}_{0\le s\le\delta}, Z_t, \mathbb{E}^{\mathcal{F}_t}[q_{t+\delta}Z_{t+\delta}], \{\mathbb{E}^{\mathcal{F}_t}[q_{t+s}Z_{t+s}]\}_{0\le s\le\delta}\big)dt \\ & -\, Z_t dB_t\,, \quad 0\le t\le T\,, \\ Y_{T+t} = & \xi_{T+t}\,, \quad 0\le t\le\delta\,, \\ Z_{T+t} = & P_{T+t}\,, \quad 0< t\le\delta\,, \end{cases} \tag{2.7}$$

has a unique solution in $\mathcal{S}^2_{\mathbb{F}}[0,T+\delta]\times L^2_{\mathbb{F}}[0,T+\delta]$ if

a) the map $f(\cdot,\vec{\mathbf{y}})$ is optional for any $\vec{\mathbf{y}} \in \mathbf{A}$,

b) there exists $C > 0$ such that, for any $t\in[0,T]$, any $\vec{\mathbf{y}}\in\mathbf{A}$, we have

$$\big|f(t,\vec{\mathbf{y}}) - f(t,\vec{\mathbf{y}}')\big| \;\le\; C|\vec{\mathbf{y}} - \vec{\mathbf{y}}'|\,,$$

c) $\mathbb{E}[\int_0^T |f(s,\vec{\mathbf{0}})|^2 ds] < \infty$,

d) the terminal condition ξ belongs to $\mathcal{S}^2_{\mathbb{F}}[T,T+\delta]$ and P belongs to $L^2_{\mathbb{F}}[T,T+\delta]$.

Proof: In the driver, the map $\mathcal{Y} = (\mathcal{Y}_t(s) = \mathbb{E}^{\mathcal{F}_t}(p_{t+s}Y_{t+s}), 0 \leq s \leq \delta)$ is a family of $\mathcal{F}_t$-measurable random variables. Let us first introduce a norm in the Banach space $E := \mathcal{S}^2_{\mathbb{F}}[0, T+\delta] \times L^2_{\mathbb{F}}[0, T+\delta]$ for $\beta > 0$: for $(Y, Z) \in E$

$$||(Y,Z)||^2_\beta \ := \ \mathbb{E}\Big[\int_0^{T+\delta} e^{\beta t}(Y_t^2 + Z_t^2)dt\Big] ,$$

and define the mapping $\Phi : E \to E$ by $\Phi((y,z)) = (Y,Z)$ where (Y,Z) is defined by

$$\begin{cases} -dY_t = & f(t, \vec{\mathbf{y}}_t)dt - Z_t dB_t , \quad 0 \leq t \leq T , \\ Y_{T+t} = & \xi_{T+t} , \quad 0 \leq t \leq \delta , \\ Z_{T+t} = & P_{T+t} , \quad 0 < t \leq \delta , \end{cases}$$

where $\vec{\mathbf{y}}_t = (y_t, \mathbb{E}^{\mathcal{F}_t}[p_{t+\delta}y_{t+\delta}], \{\mathbb{E}^{\mathcal{F}_t}[p_{t+s}y_{t+s}]\}_{0\leq s\leq\delta}, z_t, \mathbb{E}^{\mathcal{F}_t}[q_{t+\delta}z_{t+\delta}], \{\mathbb{E}^{\mathcal{F}_t}[q_{t+s}z_{t+s}]\}_{0\leq s\leq\delta})$. We now prove Φ is a contraction in E under the norm $||.||_\beta$. For two arbitrary elements (y, z) and (y', z'), we denote their difference by

$$(\widetilde{y}, \widetilde{z}) \ = \ \big(y - y', z - z'\big) .$$

We can prove by using classical estimates we have

$$\mathbb{E}\Big[\int_0^T e^{\beta t}\Big(\frac{\beta}{2}\widetilde{Y}_t^2 + \widetilde{Z}_t^2\Big)dt\Big] \ \leq \ \frac{2}{\beta}\,\mathbb{E}\Big[\int_0^T e^{\beta t}\big|f(t,\vec{\mathbf{y}}_t) - f(t,\vec{\mathbf{y}}'_t)\big|^2 dt\Big] .$$

In the following inequalities, K is a constant which does not depend on β and may change from line to line. By Lipschitz property of the map f, standard majorization of the square of a sum (resp. integral) via the sum (resp. integral) of the square (up to a constant) and the boundness of p and q, it follows that

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T e^{\beta t}\Big(\frac{\beta}{2}\widetilde{Y}_t^2 + \widetilde{Z}_t^2\Big)dt\Big] \\ &\qquad \leq \frac{K}{\beta}\,\mathbb{E}\Big[\int_0^T e^{\beta t}\Big(\widetilde{y}_t^2 + \widetilde{z}_t^2 + \widetilde{y}_{t+\delta}^2 + \widetilde{z}_{t+\delta}^2 + \frac{1}{\delta}\int_0^\delta (\widetilde{y}_{t+s}^2 + \widetilde{z}_{t+s}^2)ds\Big)dt\Big] . \end{aligned} \tag{2.8}$$

By the change of variable $u = t + s$, we get

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T e^{\beta t}\Big(\frac{\beta}{2}\widetilde{Y}_t^2 + \widetilde{Z}_t^2\Big)dt\Big] \\ &\qquad \leq \frac{K}{\beta}\,\mathbb{E}\Big[\int_0^{T+\delta} e^{\beta t}(\widetilde{y}_t^2 + \widetilde{z}_t^2)dt + \frac{1}{\delta}\int_0^T e^{\beta t}\int_t^{t+\delta}(\widetilde{y}_u^2 + \widetilde{z}_u^2)du\,dt\Big] . \end{aligned} \tag{2.9}$$

Fubini's theorem leads to

$$\begin{aligned} \frac{1}{\delta}\int_0^T e^{\beta t}\int_t^{t+\delta}(\widetilde{y}_u^2 + \widetilde{z}_u^2)du\,dt \ &\leq \ \frac{1}{\delta}\int_0^{T+\delta}\Big(\int_{u-\delta}^u e^{\beta t}dt\Big)(\widetilde{y}_u^2 + \widetilde{z}_u^2)du \\ &\leq \ \frac{\big(1 - e^{-\beta\delta}\big)}{\beta\delta}\int_0^{T+\delta} e^{\beta u}(\widetilde{y}_u^2 + \widetilde{z}_u^2)du \\ &\leq \ \int_0^{T+\delta} e^{\beta u}(\widetilde{y}_u^2 + \widetilde{z}_u^2)du \end{aligned} \tag{2.10}$$

where we have used that $1 - e^{-\beta\delta} \le \beta\delta$.
Combining (2.10) with (2.9), we obtain for $\beta \ge 2$

$$\mathbb{E}\Big[\int_0^T e^{\beta t}\Big(\widetilde{Y}_t^2 + \widetilde{Z}_t^2\Big)dt\Big] \quad \le \quad \frac{K}{\beta}\,\mathbb{E}\Big[\int_0^{T+\delta} e^{\beta t}\big(\widetilde{y}_t^2 + \widetilde{z}_t^2\big)dt\Big]\ .$$

Consequently, since $\widetilde{Y} = \widetilde{Z} = 0$ for $t > T$, we get

$$||(\widetilde{Y}, \widetilde{Z})||_\beta^2 \quad \le \quad \frac{K}{\beta}\,||(\widetilde{y}, \widetilde{z})||_\beta^2\ ,$$

and Φ is a contraction on $\mathcal{S}_{\mathbb{F}}^2[0, T+\delta] \times L_{\mathbb{F}}^2[0, T+\delta]$ for β large enough to ensure that $K/\beta < 1$, and $\beta > 2$. $\square$

We now give an estimation of the solution of the ABSDE.

Proposition 2.10 *Suppose f satisfies the hypotheses of Proposition 2.9. Then there exists a strictly positive constant K that only depends on the Lipschitz constant C and on T such that for any $\xi \in \mathcal{S}_{\mathbb{F}}^2[T, T+\delta]$ and $P \in L_{\mathbb{F}}^2[T, T+\delta]$, the solution (Y, Z) of the ABSDE (2.7) satisfies*

$$\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{t\le s\le T} Y_s^2 + \int_t^T Z_s^2 ds\Big] \quad \le \quad K\ \mathbb{E}^{\mathcal{F}_t}\Big[\xi_T^2 + \int_T^{T+\delta}(\xi_s^2 + P_s^2)ds + \int_t^T f(s, \vec{\mathbf{0}})^2 ds\Big]\ ,$$

for any $t \in [0, T]$.

Proof: The proof is obtained with standard computations. For the sake of completeness, we give details in the Appendix. $\square$

Using the same methodology as in Proposition 2.9, one obtains the following result, fwhere $\mathbb{D}(t, [0, \delta])$ is the family of maps $\mathcal{Y}$ from $[0, \delta]$ to $\mathbb{R}$ such that $\mathcal{Y}(s)$ is $\mathcal{F}_{t+s}$-measurable, for any $s \in [0, \delta]$.

Proposition 2.11 *For any $t \in [0, T]$, let $\mathbf{A}_t = \mathbb{R} \times L^2(\mathcal{F}_{t+\delta}) \times \mathbb{D}(t, [0, \delta]) \times \mathbb{R} \times L^2(\mathcal{F}_{t+\delta}) \times \mathbb{D}(t, [0, \delta])$ and for any $\vec{\mathbf{y}} = (y, \zeta, \mathcal{Y}, z, \eta, \mathcal{Z}) \in \mathbf{A}_t$, $|\vec{\mathbf{y}}| = |y| + |z| + \mathbb{E}^{\mathcal{F}_t}(|\zeta| + |\eta| + |\mathcal{Y}| + |\mathcal{Z}|)$. For a map f such that $f(\omega, t) : \mathbf{A}_t \to \mathbb{R}$, the following ABSDE*

$$\begin{cases} -dY_t = & f(t, Y_t, Y_{t+\delta}, (Y_{t+s})_{0\le s\le\delta}, Z_t, Z_{t+\delta}, (Z_{t+s})_{0\le s\le\delta})dt - Z_t dB_t\ , \quad 0 \le t \le T\ , \\ Y_{T+t} = & \xi_{T+t}\ , \quad 0 \le t \le \delta\ , \\ Z_{T+t} = & P_{T+t}\ , \quad 0 < t \le \delta\ , \end{cases}$$

has a unique solution in $\mathcal{S}_{\mathbb{F}}^2[0, T+\delta] \times L_{\mathbb{F}}^2[0, T+\delta]$ if the map f satisfies:

a) for $\vec{\mathbf{y}}_t \in \mathbf{A}_t$, $f(t, \vec{\mathbf{y}}_t)$ is $\mathcal{F}_t$-measurable,

b) there exists C such that for any $t \in [0, T]$, any $\vec{\mathbf{y}}, \vec{\mathbf{y}}'$ in $\mathbf{A}_t$, one has

$$|f(t, \vec{\mathbf{y}}) - f(t, \vec{\mathbf{y}}')| \quad \le \quad C|\vec{\mathbf{y}} - \vec{\mathbf{y}}'|\ ,$$

c) $\mathbb{E}(\int_0^T |f(t, \vec{\mathbf{0}})|^2 dt) < \infty$, *d) the terminal condition ξ belongs to $\mathcal{S}^2_{\mathbb{F}}[T,T+\delta]$ and P belongs to $L^2_{\mathbb{F}}[T,T+\delta]$.*

Moreover, there exists a constant K such that we have

$$\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{t\leq s\leq T} Y_s^2+\int_t^T Z_s^2 ds\Big] \quad\leq\quad K\,\mathbb{E}^{\mathcal{F}_t}\Big[\xi_T^2+\int_T^{T+\delta}(\xi_s^2+P_s^2)ds+\int_t^T f(s,\vec{\mathbf{0}})^2 ds\Big]\ ,$$

for any $t\in[0,T]$.

Proof: We use similar arguments to the proofs of Proposition 2.9 and 2.10. $\square$

3 ABSDE with jump of type (1.1)

We assume that Hypotheses 2.1, 2.2 and 2.3 hold. We consider in this section an ABSDE of the following form: find a triple $(Y,Z,U)\in\mathcal{S}^2_{\mathbb{G}}[0,T+\delta]\times L^2_{\mathbb{G}}[0,T+\delta]\times L^2_\tau$ satisfying

$$\begin{cases} -dY_t = & f\big(t,Y_t,\mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}],\{\mathbb{E}^{\mathcal{G}_t}[Y_{t+s}]\}_{0\leq s\leq\delta},Z_t,\mathbb{E}^{\mathcal{G}_t}[Z_{t+\delta}],\{\mathbb{E}^{\mathcal{G}_t}[Z_{t+s}]\}_{0\leq s\leq\delta},\\ & U_t,\mathbb{E}^{\mathcal{G}_t}[U_{t+\delta}],\{\mathbb{E}^{\mathcal{G}_t}[U_{t+s}]\}_{0\leq s\leq\delta}\big)dt-Z_t dB_t-U_t dH_t\ ,\quad 0\leq t\leq T\ ,\\ Y_{T+t} = & \xi_{T+t}\ ,\quad 0\leq t\leq\delta\ ,\\ Z_{T+t} = & P_{T+t}\ ,\quad U_{T+t}\ =\ Q_{T+t}\mathbb{1}_{\{T+t\leq\tau\}}\ ,\quad 0<t\leq\delta\ . \end{cases} \tag{3.1}$$

From Propositions 2.5 and 2.6, all the involved processes can be decomposed in two parts, before and after τ. In particular, since ξ will be given as a $\mathbb{G}$-optional process and P as a $\mathbb{G}$-predictable process, we have for any $t\in[0,T]$

$$f(t,\vec{\mathbf{y}}) \quad=\quad f^b(t,\vec{\mathbf{y}})\mathbb{1}_{\{t<\tau\}}+f^a(t,\tau,\vec{\mathbf{y}})\mathbb{1}_{\{t\geq\tau\}}\quad\text{(optional decomposition)}\ ,$$

and we have for any $t\in[T,T+\delta]$

$$\begin{cases} \xi_t = & \xi_t^b\mathbb{1}_{\{t<\tau\}}+\xi_t^a(\tau)\mathbb{1}_{\{t\geq\tau\}}\quad\text{(optional decomposition)}\\ P_t = & P_t^b\mathbb{1}_{\{t\leq\tau\}}+P_t^a(\tau)\mathbb{1}_{\{t>\tau\}}\quad\text{(predictable decomposition)}\ . \end{cases}$$

We work under the following hypotheses:

Hypotheses 3.1 *Let $\mathbf{A}:=\mathbb{R}^2\times\mathbb{D}[0,\delta]\times\mathbb{R}^2\times\mathbb{D}[0,\delta]\times\mathbb{R}^2\times\mathbb{D}[0,\delta]$ and, for any $\vec{\mathbf{y}}\in\mathbf{A}$ we define $|\vec{\mathbf{y}}|$ by*

$$|\vec{\mathbf{y}}|\quad=\quad|y|+|\hat{y}|+|\mathcal{Y}|+|z|+|\hat{z}|+|\mathcal{Z}|+|u|+|\hat{u}|+|\mathcal{U}|\ .$$

a) The terminal conditions satisfy $\xi\in\mathcal{S}^2_{\mathbb{G}}[T,T+\delta]$, $P\in L^2_{\mathbb{G}}[T,T+\delta]$, $Q\in L^2_{\mathbb{F}}[T,T+\delta]$, there exists a constant K such that $\mathbb{E}[|\xi^a_u(\theta)|^2]\leq K$ and $\mathbb{E}[|P^a_u(\theta)|^2]\leq K$ for any $(\theta,u)\in[0,T]\times[T,T+\delta]$.

b) The generator $f:\Omega\times[0,T]\times\mathbf{A}\to\mathbb{R}$ of the ABSDE is Lipschitz, i.e., there exists a constant C such that, for any $t\in[0,T]$, any $\vec{\mathbf{y}}$ and $\vec{\mathbf{y}}'$ in $\mathbf{A}$, we have

$$|f(t,\vec{\mathbf{y}})-f(t,\vec{\mathbf{y}}')|\quad\leq\quad C|\vec{\mathbf{y}}-\vec{\mathbf{y}}'|\ .$$

c) For any $\vec{\mathbf{y}}\in\mathbf{A}$, the process $f(\cdot,\vec{\mathbf{y}})$ is $\mathbb{G}$-optional.

d) There exists a constant C' such that $|f(s,\vec{\mathbf{0}})| < C'$.

From Propositions 2.5 and 2.6, we can write

$$\begin{cases} Y_t = & Y_t^b \mathbb{1}_{\{t<\tau\}} + Y_t^a(\tau)\mathbb{1}_{\{t\geq\tau\}} \quad \text{(optional decomposition)} \\ Z_t = & Z_t^b \mathbb{1}_{\{t\leq\tau\}} + Z_t^a(\tau)\mathbb{1}_{\{t>\tau\}} \quad \text{(predictable decomposition)} \end{cases}$$

It follows, from Lemma 2.8 that

$$\begin{cases} -dY_t^a(\tau) = & f^a\big(t,\tau,Y_t^a(\tau),\mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}^a(\tau)],\{\mathbb{E}^{\mathcal{G}_t}[Y_{t+s}^a(\tau)]\}_{0\leq s\leq\delta}, Z_t^a(\tau),\mathbb{E}^{\mathcal{G}_t}[Z_{t+\delta}^a(\tau)], \\ & \{\mathbb{E}^{\mathcal{G}_t}[Z_{t+s}^a(\tau)]\}_{0\leq s\leq\delta},0,0,0\big)dt - Z_t^a(\tau)dB_t\ , \quad T\wedge\tau\leq t\leq T\ , \\ Y_{T+t}^a(\tau) = & \xi_{T+t}^a(\tau)\ , \quad 0\leq t\leq\delta\ , \\ Z_{T+t}^a(\tau) = & P_{T+t}^a(\tau)\ , \quad 0<t\leq\delta\ , \end{cases} \tag{3.2}$$

and

$$\begin{cases} -dY_t^b = & f^b\big(t,Y_t^b,\mathbb{E}^{\mathcal{F}_t}[Y_{t+\delta}],\{\mathbb{E}^{\mathcal{F}_t}[Y_{t+s}]\}_{0\leq s\leq\delta},Z_t^b,\mathbb{E}^{\mathcal{F}_t}[Z_{t+\delta}],\{\mathbb{E}^{\mathcal{F}_t}[Z_{t+s}]\}_{0\leq s\leq\delta}\big)dt \\ & -Z_t^b dB_t\ , \quad 0\leq t\leq T\ , \\ Y_{T+t}^b = & \xi_{T+t}^b\ , \quad 0\leq t\leq\delta\ , \\ Z_{T+t}^b = & P_{T+t}^b\ , \quad U_{T+t}^b \ = \ Q_{T+t}\ , \quad 0<t\leq\delta\ . \end{cases} \tag{3.3}$$

Furthermore, $U_t = \big[(Y_t^a(t)-Y_t^b)\mathbb{1}_{\{t\leq T\}} + Q_t\mathbb{1}_{\{T<t\leq T+\delta\}}\big]\mathbb{1}_{\{t\leq\tau\}}$.

3.1 Study of the Equation (3.2)

Our aim is to write (3.2) as a family of ABSDEs in the filtration $\mathbb{F}$. For that purpose, we note that, on the set $\{t\geq\tau\}$, we have from (2.6)

$$\mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}^a(\tau)] \quad = \quad \mathbb{E}^{\mathcal{F}_t}[Y_{t+\delta}^a(\theta)]_{|\theta=\tau}\ .$$

The same equality holds for the part involving $f(t,\vec{\mathbf{y}})$ and $Z_{t+\delta}^a(\tau)$. Therefore, we study the family of ABSDE

$$\begin{cases} -dY_t^a(\theta) = & f^a\big(t,\theta,Y_t^a(\theta),\mathbb{E}^{\mathcal{F}_t}[Y_{t+\delta}^a(\theta)],\{\mathbb{E}^{\mathcal{F}_t}[Y_{t+s}^a(\theta)]\}_{0\leq s\leq\delta}, Z_t^a(\theta),\mathbb{E}^{\mathcal{F}_t}[Z_{t+\delta}^a(\theta)], \\ & \{\mathbb{E}^{\mathcal{F}_t}[Z_{t+s}^a(\theta)]\}_{0\leq s\leq\delta},0,0,0\big)dt - Z_t^a(\theta)dB_t\ , \quad 0\leq t\leq T\ , \\ Y_{T+t}^a(\theta) = & \xi_{T+t}^a(\theta)\ , \quad 0\leq t\leq\delta\ , \\ Z_{T+t}^a(\theta) = & P_{T+t}^a(\theta)\ , \quad 0<t\leq\delta\ . \end{cases} \tag{3.4}$$

For any fixed $\theta\in[0,T]$, the map $F := f^a(\theta)$ defined as $F(t,\vec{\mathbf{y}}) = f^a(t,\theta,\vec{\mathbf{y}})$ inherits the Lipschitz conditions of Proposition 2.9 from the one of f. Due to the boundedness of $f(\cdot,\vec{\mathbf{0}})$, the map $F(\cdot,\vec{\mathbf{0}})$ is also bounded, and satisfies

$$\sup_{0\leq\theta\leq T}\mathbb{E}\Big[\int_0^T \big|f^a(t,\theta,\vec{\mathbf{0}})\big|^2 dt\Big] \quad < \quad \infty\ ,$$

and the existence of a solution follows from Proposition 2.9.

Using Proposition 2.10, there exists a constant K such that

$$\mathbb{E}^{\mathcal{F}_t}\Big(\sup_{t\leq s\leq T}(Y_s^a(\theta))^2 + \int_t^T (Z_s^a(\theta))^2 ds\Big) \quad \leq \quad K\,\mathbb{E}^{\mathcal{F}_t}\Big((\xi_T^a(\theta))^2 + \int_T^{T+\delta}\big((\xi_s^a(\theta))^2 + (P_s^a(\theta))^2\big)ds + \int_t^T (f^a(s,\theta,\vec{\mathbf{0}}))^2 ds\Big)\ . \tag{3.5}$$

3.2 Study of the Equation (3.3)

Our aim is to write (3.3) as an ABSDE in the filtration $\mathbb{F}$, that is to get rid of the quantities involving processes after time τ (as, e.g., $Y_{t+\delta}$ on $\{t+\delta>\tau\}$) and working only with conditional expectation w.r.t. $\mathbb{F}$. Obviously, for any $t\le u\le t+\delta$, we have

$$\mathbb{E}^{\mathcal{G}_t}[Y_u] \;=\; \mathbb{E}^{\mathcal{G}_t}[Y_u\mathbb{1}_{\{u<\tau\}}]+\mathbb{E}^{\mathcal{G}_t}[Y_u\mathbb{1}_{\{u\geq\tau\}}]\,.$$

Furthermore, from (2.5), we have

$$\mathbb{E}^{\mathcal{G}_t}[Y_u\mathbb{1}_{\{u<\tau\}}]\mathbb{1}_{\{t<\tau\}} \;=\; \mathbb{E}^{\mathcal{G}_t}[Y_u^b\mathbb{1}_{\{u<\tau\}}]\mathbb{1}_{\{t<\tau\}}=\frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}[Y_u^bG_u]\mathbb{1}_{\{t<\tau\}}\,, \tag{3.6}$$

and

$$\begin{aligned}\mathbb{E}^{\mathcal{G}_t}[Y_u\mathbb{1}_{\{u\geq\tau\}}]\mathbb{1}_{\{t<\tau\}} \;&=\; \mathbb{E}^{\mathcal{G}_t}[Y_u^a(\tau)\mathbb{1}_{\{u\geq\tau\}}]\mathbb{1}_{\{t<\tau\}}\\ &=\; \frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}\Big[\int_t^u Y_u^a(\theta)\alpha_u(\theta)d\theta\Big]\mathbb{1}_{\{t<\tau\}} \;=:\; J_t^{Y^a}(u)\,.\end{aligned} \tag{3.7}$$

The same equalities hold for the part involving Z^a. We are lead to consider, relying on the uniqueness of pre-default parts, the BSDE

$$\begin{cases} -dY_t^b = \; g(t,Y_t^b,Y_{t+\delta}^b,\{Y_{t+s}^b\}_{0\leq s\leq\delta},Z_t^b,Z_{t+\delta}^b,\{Z_{t+s}^b\}_{0\leq s\leq\delta})dt-Z_t^bdB_t\,, \quad 0\leq t\leq T\,,\\ Y_{T+t}^b = \; \xi_{T+t}^b\,, \quad 0\leq t\leq\delta\,,\\ Z_{T+t}^b = \; P_{T+t}^b\,, \quad 0<t\leq\delta\,.\end{cases} \tag{3.8}$$

Here, using the equalities (3.6) and (3.7), we obtain that g is the map $\Omega\times[0,T]\times\mathbb{R}\times L^2(\mathcal{F}_{.+\delta})\times\mathbb{D}(\cdot,[0,\delta])\times\mathbb{R}\times L^2(\mathcal{F}_{.+\delta})\times\mathbb{D}(\cdot,[0,\delta])\to\mathbb{R}$ defined, for y and z in $\mathbb{R}$, ζ and η in $L^2(\mathcal{F}_{.+\delta})$, and $\mathcal{Y}$ and $\mathcal{Z}$ in $\mathbb{D}(\cdot,[0,\delta])$, in terms of solution of the equation (3.4) by, for $\vec{\mathbf{y}}=(y,\zeta,\mathcal{Y},z,\eta,\mathcal{Z})$

$$g(t,\vec{\mathbf{y}}) \;=\; f^b\big(t,\vec{I}_t^1,\vec{I}_t^2,\vec{I}_t^3\big)$$

where, recalling the quantities J are defined in (3.7)

$$\begin{aligned}\vec{I}_t^1 \;&=\; \big(y,\frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}[\zeta G_{t+\delta}]+J_t^{Y^a}(t+\delta),\{\frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}(\mathcal{Y}_t(s)G_{t+s})+J_t^{Y^a}(t+s)\}_{0\leq s\leq\delta}\big)\,,\\ \vec{I}_t^2 \;&=\; \big(z,\frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}[\eta G_{t+\delta}]+J_t^{Z^a}(t+\delta),\,\{\frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}(\mathcal{Z}_t(s)G_{t+s})+J_t^{Z^a}(t+s)\}_{0\leq s\leq\delta}\big)\,,\\ \vec{I}_t^3 \;&=\; \big(Y_t^a(t)-y,\,\frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}[\mathbb{1}_{\{t+\delta\leq T\}}(Y_{t+\delta}^a(t+\delta)-\zeta)G_{t+\delta}+\mathbb{1}_{\{t+\delta>T\}}Q_{t+\delta}G_{t+\delta}]\,,\\ &\quad\frac{1}{G_t}\Big\{\mathbb{E}^{\mathcal{F}_t}[(Y_{t+s}^a(t+s)-\mathcal{Y}_t(s))G_{t+s}\mathbb{1}_{\{t+s\leq T\}}+Q_{t+s}G_{t+s}\mathbb{1}_{\{t+s>T\}}]\Big\}_{0\leq s\leq\delta}\big)\,.\end{aligned}$$

It is straightforward that g is $\mathbb{F}$-optional. We now show g satisfies Lipschitz conditions recalled in Proposition 2.9.
Since we have

$$f^b(t,\vec{\mathbf{y}}) \;=\; \frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}(f(t,\vec{\mathbf{y}})\mathbb{1}_{\{t<\tau\}})\,,$$

we obtain that, using the Lipschitz condition for f and that G is bounded, there exists a constant K such that

$$
\begin{aligned}
|g(t,\vec{\mathbf{y}}) - g(t,\vec{\mathbf{y}}')| \;\; &\le \;\; \frac{K}{G_t}\Big((|y-y'|+|z-z'|)\mathbb{E}^{\mathcal{F}_t}(\mathbb{1}_{\{t<\tau\}}) \\
&\quad + \mathbb{E}^{\mathcal{F}_t}[(|\zeta-\zeta'|+|\eta-\eta'|)G_{t+\delta}\mathbb{1}_{\{t<\tau\}}] \\
&\quad + \mathbb{E}^{\mathcal{F}_t}\left[(|\mathcal{Y}G-\mathcal{Y}'G|+|\mathcal{Z}G-\mathcal{Z}'G|)\mathbb{1}_{\{t<\tau\}}\right]\Big) .
\end{aligned}
$$

Since, for $X \in \mathcal{F}_s$ and $s > t$, one has from (2.4)

$$
\mathbb{E}^{\mathcal{F}_t}(X\mathbb{1}_{\{t<\tau\}}) = \mathbb{E}^{\mathcal{F}_t}(X\mathbb{E}^{\mathcal{F}_s}(\mathbb{1}_{\{t<\tau\}})) = \mathbb{E}^{\mathcal{F}_t}(XG_t) = G_t\mathbb{E}^{\mathcal{F}_t}(X) .
$$

We deduce

$$
\begin{aligned}
|g(t,\vec{\mathbf{y}}) - g(t,\vec{\mathbf{y}}')| \;\; &\le \;\; K\Big(|y-y'|+|z-z'|+\mathbb{E}^{\mathcal{F}_t}[(|\zeta-\zeta'|+|\eta-\eta'|)G_{t+\delta}] \\
&\quad + \mathbb{E}^{\mathcal{F}_t}[|\mathcal{Y}G-\mathcal{Y}'G|+|\mathcal{Z}G-\mathcal{Z}'G|]\Big) .
\end{aligned}
$$

Noting G is upper bounded by 1, the Lipschitz property of Proposition 2.9 for g holds.

We now check the integrability condition on $|g(t,\vec{\mathbf{0}})|^2$. We notice, using notation (3.7), we have

$$
\begin{aligned}
g(t,\vec{\mathbf{0}}) \;\; &= \;\; f^b\Big(t,\ 0,\ J_t^{Y^a}(t+\delta), \Big\{J_t^{Y^a}(t+s)\Big\}_{0\le s\le\delta}, 0, J_t^{Z^a}(t+\delta), \Big\{J_t^{Z^a}(t+s)\Big\}_{0\le s\le\delta}, Y_t^a(t), \\
&\quad \frac{1}{G_t}\ \mathbb{E}^{\mathcal{F}_t}[Y_{t+\delta}^a(t+\delta)G_{t+\delta}\mathbb{1}_{\{t+\delta\le T\}} + Q_{t+\delta}G_{t+\delta}\mathbb{1}_{\{t+\delta>T\}}], \\
&\quad \Big\{\frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}[Y_{t+s}^a(t+s)G_{t+s}\mathbb{1}_{\{t+s\le T\}} + Q_{t+s}G_{t+s}\mathbb{1}_{\{t+s>T\}}]\Big\}_{0\le s\le\delta}\Big) .
\end{aligned}
$$

From Lipschitz property of f, that $f(t,\vec{\mathbf{0}})$ is bounded and $G_t = \mathbb{E}^{\mathcal{F}_t}(\mathbb{1}_{\{t<\tau\}})$, we have

$$
f^b(t,\vec{\mathbf{y}}) \;\; \le \;\; \frac{1}{G_t}\Big(\mathbb{E}^{\mathcal{F}_t}[(f(t,\vec{\mathbf{0}}) + C|\,\vec{\mathbf{y}}|)\mathbb{1}_{\{t<\tau\}}]\ \Big) \;\; \le \;\; C_1 + C|\vec{\mathbf{y}}| .
$$

Using again that the square of a sum is bounded (up to a constant) by the sum of the squares, and using again the fact that G is lower bounded, the integrability condition of $|g(t,\vec{\mathbf{0}})|^2$ will follow from the boundedness of the quantities

$$
\mathbb{E}\left(\int_0^T \left(J_t^Y(t+\delta)\right)^2 dt\right), \quad \mathbb{E}\left(\int_0^T \int_0^\delta (J_t^Y(t+s))^2 ds\, dt\right) \tag{3.9}
$$

and similar expressions with J^Z, as well as

$$
\left\{
\begin{array}{l}
\mathbb{E}\left(\int_0^T \left(\mathbb{E}^{\mathcal{F}_t}(Y_{t+\delta}^a(t+\delta))\right)^2 dt\right) \\
\mathbb{E}\left(\int_0^T (Y_t^a(t))^2 dt\right) \\
\mathbb{E}\left(\int_T^{T+\delta} \left(\mathbb{E}^{\mathcal{F}_t}(Q_t)\right)^2 dt\right) \\
\mathbb{E}\left(\int_0^T \left[\int_0^\delta \left(Y_{t+s}^a(t+s)^2\mathbb{1}_{t+s\le T} + Q_{t+s}^2\mathbb{1}_{t+s>T}\right)ds\right]dt\right) .
\end{array}
\right.
$$

The quantities in (3.9) are bounded since α is bounded and

$$
\int_0^{T+\delta} d\theta \int_0^T \mathbb{1}_{\{t<\theta<t+\delta\}}\mathbb{E}((Y_{t+\delta}^a(\theta))^2)dt
$$

is bounded since

$$\sup_{0\le\theta\le T}\mathbb{E}((\sup_{0\le s\le T}Y_s^a(\theta))^2) \;\le\; K\sup_{0\le\theta\le T}\mathbb{E}\Big((\xi_T^a(\theta))^2+\int_T^{T+\delta}\big((\xi_s^a(\theta))^2+(P_s^a(\theta))^2\big)ds + \int_0^T(f^a(s,\theta,\vec{\mathbf{0}}))^2ds\Big)$$

and the assumed boundness of P and ξ. The other quantities are studied using the same methodology and that $Q\in L^2[T,T+\delta]$.

The existence of a unique solution (Y^b,Z^b) of the ABSDE (3.8) follows from Proposition 2.11. Moreover we have

$$\mathbb{E}^{\mathcal{F}_t}\Big(\sup_{t\le s\le T}(Y_s^b)^2+\int_t^T(Z_s^b)^2ds\Big) \;\le\; K\mathbb{E}^{\mathcal{F}_t}\Big(|\xi_T^b|^2+\int_T^{T+\delta}\big((\xi_s^b)^2+(P_s^b)^2\big)du + \int_t^T(f^b(s,\vec{\mathbf{0}}))^2ds\Big)\,. \tag{3.10}$$

3.3 Integrability of the solutions

In this part we consider the integrability of the solutions (Y,Z,U) where

$$\begin{aligned} Y_t &= Y_t^b\mathbb{1}_{\{t<\tau\}}+Y_t^a(\tau)\mathbb{1}_{\{t\ge\tau\}}\,,\\ Z_t &= Z_t^b\mathbb{1}_{\{t\le\tau\}}+Z_t^a(\tau)\mathbb{1}_{\{t>\tau\}}\,,\\ U_t &= (Y_t^a(t)-Y_t^b)\mathbb{1}_{\{t\le\tau\}}\,. \end{aligned}$$

From Subsections 3.1 and 3.2 we know (Y,Z,U) satisfy the ABSDE (3.1).

Proposition 3.2 *The process U belongs to L^2_τ.*

Proof: We have

$$\begin{aligned} \mathbb{E}\Big[\int_0^{(T+\delta)\wedge\tau}U_s^2ds\Big] &= \mathbb{E}\Big[\int_0^{T\wedge\tau}(Y_s^a(s)-Y_s^b)^2ds\Big]+\Big[\int_{T\wedge\tau}^{(T+\delta)\wedge\tau}Q_s^2ds\Big]\\ &\le 2\mathbb{E}\Big[\int_0^T(Y_s^a(s))^2ds\Big]+2\mathbb{E}\Big[\int_0^T(Y_s^b)^2ds\Big]+\Big[\int_T^{T+\delta}Q_s^2ds\Big]\\ &\le 2\int_0^T\mathbb{E}\Big[(Y_s^a(s))^2\Big]ds+2T\mathbb{E}\big[\sup_{0\le t\le T}(Y_t^b)^2\big]+\mathbb{E}\Big[\int_T^{T+\delta}Q_s^2ds\Big] \end{aligned}$$

and the quantities on the right-hand side are finite. □

Proposition 3.3 *There exists a strictly positive constant K such that the solution (Y,Z,U) of the ABSDE (3.1) satisfies*

$$\begin{aligned} &\mathbb{E}^{\mathcal{G}_t}\Big[\sup_{t\le s\le T}Y_s^2+\int_t^TZ_s^2ds\Big]\\ \le\;& K\mathbb{E}^{\mathcal{F}_t}\Big[(\xi_T^b)^2+\int_T^{T+\delta}\big((\xi_s^b)^2+(P_s^b)^2\big)ds+\int_t^T(f^b(s,\vec{0})^2)|ds\Big]\\ +\;&\frac{K}{\alpha_t(\tau)}\mathbb{E}^{\mathcal{F}_t}\Big[(\xi_T^a(\theta))^2+\int_T^{T+\delta}((\xi_s^a(\theta))^2+(P_s^a(\theta))^2)ds+\int_t^T(f^a(s,\theta,\vec{0}))^2ds\Big]_{\theta=\tau}\mathbb{1}_{\{\tau<t\}}\\ +\;&K\mathbb{1}_{\{t\le\tau\}}\mathbb{E}^{\mathcal{F}_t}\Big[\int_t^T\Big\{(\xi_T^a(\theta))^2+\int_T^{T+\delta}((\xi_s^a(\theta))^2+(P_s^a(\theta))^2)ds+\int_t^T(f^a(s,\theta,\vec{0}))^2ds\Big\}d\theta\Big] \end{aligned}$$

for any $t \in [0,T]$.

Proof: In the proof, the constant K can vary from line to line. We remark [1]

$$\begin{aligned}
& \mathbb{E}^{\mathcal{G}_t}\Big[\sup_{t\leq s\leq T} Y_s^2 + \int_t^T Z_s^2 ds\Big] \\
= \ & \mathbb{E}^{\mathcal{G}_t}\Big[\sup_{t\leq s\leq T} Y_s^2 + \int_t^{T\wedge\tau} (Z_s^b)^2 ds + \int_{T\wedge\tau}^T (Z_s^a(\tau))^2 ds\Big] \\
\leq \ & \mathbb{E}^{\mathcal{G}_t}\Big[\sup_{t\leq s\leq T} Y_s^2 + \int_t^{T} (Z_s^b)^2 ds + \int_{T\wedge\tau}^T (Z_s^a(\tau))^2 ds\Big] .
\end{aligned}$$

On the set $\{\tau < t\}$, we use that

$$\begin{aligned}
\mathbb{E}^{\mathcal{G}_t}\Big[\sup_{t\leq s\leq T} Y_s^2\Big] &= \mathbb{E}^{\mathcal{G}_t}\Big[\sup_{t\leq s\leq T} (Y_s^a(\tau))^2\Big] = \frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{t\leq s\leq T} (Y_s^a(\theta))^2\alpha_T(\theta)\Big] \\
&\leq ke^{kt}\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{t\leq s\leq T} (Y_s^a(\theta))^2\Big] \leq K\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{t\leq s\leq T} (Y_s^a(\theta))^2\Big] .
\end{aligned}$$

On the set $\{t \leq \tau\}$, we remark

$$\mathbb{E}^{\mathcal{G}_t}\Big[\sup_{t\leq s\leq T} Y_s^2\Big] \leq \mathbb{E}^{\mathcal{F}_t}\Big[\sup_{t\leq s\leq T} (Y_s^b)^2\Big] + \mathbb{E}^{\mathcal{G}_t}\Big[\sup_{T\wedge\tau\leq s\leq T} (Y_s^a(\tau))^2\Big] .$$

From $\mathbb{E}^{\mathcal{G}_t}\Big[\sup_{T\wedge\tau\leq s\leq T}(Y_s^a(\tau))^2\Big] = \frac{1}{\alpha_t(\tau)}\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{T\wedge\theta\leq s\leq T}(Y_s^a(\theta))^2\alpha_T(\theta)\Big]_{\theta=\tau}$ and that α is bounded, we have

$$\mathbb{E}^{\mathcal{G}_t}\Big[\sup_{T\wedge\tau\leq s\leq T} (Y_s^a(\tau))^2\Big] \leq \frac{K}{\alpha_t(\tau)}\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{T\wedge\theta\leq s\leq T} (Y_s^a(\theta))^2\Big]_{\theta=\tau} .$$

We proceed in the same way for the part $\int_{T\wedge\tau}^T (Z_s^a(\tau))^2 ds$.
Using (3.5)-(3.10) we can conclude. □

3.4 Uniqueness of the solution

In this part we consider the uniqueness of the solution of ABSDE (3.1). Suppose this ABSDE has two solutions (Y, Z, U) and $(\bar{Y}, \bar{Z}, \bar{U})$. Each process admits a unique decomposition under the form (Y^b, Z^b, U^b)-$(Y^a(\tau), Z^a(\tau))$ and $(\bar{Y}^b, \bar{Z}^b, \bar{U}^b)$-$(\bar{Y}^a(\tau), \bar{Z}^a(\tau))$. Moreover we know (Y^b, Z^b) and $(\bar{Y}^b, \bar{Z}^b)$ are solution of ABSDE (3.3), thus by uniqueness of the solution of ABSDE (3.3) from Proposition 2.11 we get that $Y^b = \bar{Y}^b$ and $Z^b = \bar{Z}^b$. We have with the same arguments $Y^a(\tau) = \bar{Y}^a(\tau)$ and $Z^a(\tau) = \bar{Z}^a(\tau)$. Moreover we have $U_t = (Y_t^a(t) - Y_t^b)\mathbb{1}_{\{t\leq\tau\}}$, thus $U = \bar{U}$. Finally we get the uniqueness of the solution of ABSDE (3.1).

[1] with the convention $\int_a^b .ds = 0$ if $b < a$

4 ABSDE with jump of type (1.2)

We assume that Hypotheses 2.1, 2.2 and 2.3 hold. We define, for any $t \in [0,T]$, $\mathbf{A}_t = \mathbb{R} \times L^2(\mathcal{F}_{t+\delta}) \times \mathbb{D}(t,[0,\delta]) \times \mathbb{R} \times L^2(\mathcal{F}_{t+\delta}) \times \mathbb{D}(t,[0,\delta]) \times \mathbb{R} \times L^2(\mathcal{F}_{t+\delta}) \times \mathbb{D}(t,[0,\delta])$. We consider in this section an ABSDE of the following form: find a triple $(Y,Z,U) \in \mathcal{S}^2_{\mathbb{G}}[0,T+\delta] \times L^2_{\mathbb{G}}[0,T+\delta] \times L^2_\tau$ satisfying

$$\begin{cases} -dY_t = & \mathbb{E}^{\mathcal{G}_t}\big[f(t,Y_t,Y_{t+\delta},\{Y_{t+s}\}_{0\le s\le\delta},Z_t,Z_{t+\delta},\{Z_{t+s}\}_{0\le s\le\delta},U_t,U_{t+\delta}, \\ & \{U_{t+s}\}_{0\le s\le\delta})]dt - Z_t dB_t - U_t dH_t\ , \quad 0\le t\le T\ , \\ Y_{T+s} = & \xi_{T+s}\ , \quad 0\le s\le\delta \\ Z_{T+s} = & P_{T+s}\ , \quad U_{T+s} \ = \ Q_{T+s}\mathbb{1}_{\{T+t\le\tau\}}\ , \quad 0\le s\le\delta\ , \end{cases} \tag{4.1}$$

with the following hypotheses

Hypotheses 4.1 *Suppose that*

a) The terminal conditions satisfy $\xi \in \mathcal{S}^2_{\mathbb{G}}[T,T+\delta]$, $P \in L^2_{\mathbb{G}}[T,T+\delta]$ and $Q \in L^2_{\mathbb{F}}[T,T+\delta]$, and $\sup_{0\le\theta\le T}\xi^a(\theta) \in \mathcal{S}^2_{\mathbb{F}}[T,T+\delta]$ and $\sup_{0\le\theta\le T}P^a(\theta) \in L^2_{\mathbb{F}}[T,T+\delta]$.

b) The generator $f : \Omega\times[0,T]\times\mathbf{A}_. \to \mathbb{R}$ is Lipschitz, that means there exists a constant C such that for any $t\in[0,T]$, for any $\vec{\mathbf{y}}$ and $\vec{\mathbf{y}}'$ in $\mathbf{A}_t$, one has

$$|f(t,\vec{\mathbf{y}}) - f(t,\vec{\mathbf{y}}')| \quad\le\quad C|\vec{\mathbf{y}}-\vec{\mathbf{y}}'|\,.$$

c) There exists a constant C' such that $|f(s,\vec{\mathbf{0}})| \le C'$.

Proceeding as before, we consider, on the set $\{\tau\le t\}$, the ABSDE

$$\begin{cases} -dY^a_t(\tau) = & \mathbb{E}^{\mathcal{G}_t}\big[f^a(t,\tau,Y^a_t(\tau),Y^a_{t+\delta}(\tau),\{Y^a_{t+s}(\tau)\}_{0\le s\le\delta},Z^a_t(\tau),Z^a_{t+\delta}(\tau), \\ & \{Z^a_{t+s}(\tau)\}_{0\le s\le\delta},0,0,0)]dt - Z^a_t(\tau)dB_t\ , \quad \tau\le t\le T\ , \\ Y^a_{T+s}(\tau) = & \xi^a_{T+s}(\tau)\ , \quad 0\le s\le\delta\ , \\ Z^a_{T+s}(\tau) = & P^a_{T+s}(\tau)\ , \quad 0< s\le\delta\ , \end{cases} \tag{4.2}$$

whereas, due to the uniqueness of pre-default parts we consider the ABSDE

$$\begin{cases} -dY^b_t = & \mathbb{E}^{\mathcal{G}_t}\big[f^b(t,Y^b_t,Y_{t+\delta},\{Y_{t+s}\}_{0\le s\le\delta},Z^b_t,Z_{t+\delta},\{Z_{t+s}\}_{0\le s\le\delta},U^b_t,U_{t+\delta},\{U_{t+s}\}_{0\le s\le\delta})]dt \\ & -\ Z^b_t dB_t\ , \quad 0\le t\le T\ , \\ Y^b_{T+s} = & \xi^b_{T+s}\ , \quad 0\le s\le\delta\ , \\ Z^b_{T+s} = & P_{T+s}\ , \quad U^b_{T+s} \ = \ Q_{T+s}\ , \quad 0< s\le\delta\ . \end{cases} \tag{4.3}$$

4.1 Study of the Equation (4.2)

Using the same arguments as in Subsection 3.1 we study the family of ABSDEs

$$\begin{cases} -dY^a_t(\theta) = & \mathbb{E}^{\mathcal{F}_t}\big[f^a(t,\theta,Y^a_t(\theta),Y^a_{t+\delta}(\theta),\{Y^a_{t+s}(\theta)\}_{0\le s\le\delta},Z^a_t(\theta),Z^a_{t+\delta}(\theta), \\ & \{Z^a_{t+s}(\theta)\}_{0\le s\le\delta},0,0,0)]dt - Z^a_t(\theta)dB_t\ , \quad \theta\le t\le T\ , \\ Y^a_{T+t}(\theta) = & \xi^a_{T+t}(\theta)\ , \quad 0\le t\le\delta\ , \\ Z^a_{T+t}(\theta) = & P^a_{T+t}(\theta)\ , \quad 0< t\le\delta\ . \end{cases}$$

This ABSDE can be written under the following form

$$\begin{cases} -dY_t^a(\theta) = & g(t,\theta,Y_t^a(\theta),Y_{t+\delta}^a(\theta),\{Y_{t+s}^a(\theta)\}_{0\le s\le\delta},Z_t^a(\theta),Z_{t+\delta}^a(\theta),\{Z_{t+s}^a(\theta)\}_{0\le s\le\delta})dt \\ & -\ Z_t^a(\theta)dB_t\ , \quad \theta\le t\le T\ , \\ Y_{T+t}^a(\theta) = & \xi_{T+t}^a(\theta)\ , \quad 0\le t\le\delta\ , \\ Z_{T+t}^a(\theta) = & P_{T+t}^a(\theta)\ , \quad 0< t\le\delta\ , \end{cases} \tag{4.4}$$

which is on the form of Proposition 2.11. The Lipschitz condition on g follows from the hypothesis on f. The square integrability of $g(t,\vec{\mathbf{0}}) = \mathbb{E}^{\mathcal{F}_t}\left[f^a(t,\theta,\vec{\mathbf{0}})\right]$ follows as in Subsection 3.1 from the boundedness hypothesis of $f(t,\vec{\mathbf{0}})$. Thus from Proposition 2.11 we get the existence of a unique solution to this ABSDE satisfying

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t}\Big(\sup_{t\le s\le T}(Y_s^a(\theta))^2+\int_t^T(Z_s^a(\theta))^2ds\Big) \ \le\ & C\mathbb{E}^{\mathcal{F}_t}\Big((\xi_T^a(\theta))^2+\int_T^{T+\delta}\big((\xi_s^a(\theta))^2+(P_s^a(\theta))^2\big)ds \\ & +\int_t^T(g(s,\theta,\vec{\mathbf{0}}))^2ds\Big)\ . \end{aligned} \tag{4.5}$$

4.2 Study of the Equation (4.3)

Using the same arguments as in Subsection 3.2, we are lead to consider

$$\begin{cases} -dY_t^b = & g(t,Y_t^b,Y_{t+\delta}^b,\{Y_{t+s}^b\}_{0\le s\le\delta},Z_t^b,Z_{t+\delta}^b,\{Z_{t+s}^b\}_{0\le s\le\delta})dt - Z_t^bdB_t\ , \quad 0\le t\le T \\ Y_{T+s}^b = & \xi_{T+s}^b\ , \quad 0\le s\le\delta\ , \\ Z_{T+s}^b = & P_{T+s}^b\ , \quad 0< s\le\delta\ , \end{cases} \tag{4.6}$$

where

$$\begin{aligned} g(t,y,\zeta,\mathcal{Y},z,\eta,\mathcal{Z}) \ =\ & \frac{1}{G_t}\int_t^{t+\delta}\mathbb{E}^{\mathcal{F}_t}\left[f^b\big(t,\vec{K}_t^1,\vec{K}_t^2,\vec{K}_t^3\big)\ \alpha_{t+\delta}(\theta)\right]d\theta \\ & +\frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}\left[f^b\big(t,y,\zeta,\mathcal{Y},z,\eta,\mathcal{Z},\vec{K}_t^4\big)\right]\ . \end{aligned}$$

with

$$\begin{aligned} \vec{K}_t^1 &= (y,Y_{t+\delta}^a(\theta),\{\mathcal{Y}_s\mathbb{1}_{t+s<\theta}+Y_{t+s}^a(\theta)\mathbb{1}_{t+s\ge\theta}\}_{0\le s\le\delta}) \\ \vec{K}_t^2 &= (z,Z_{t+\delta}^a(\theta),\{\mathcal{Z}_s\mathbb{1}_{t+s\le\theta}+Z_{t+s}^a(\theta)\mathbb{1}_{t+s>\theta}\}_{0\le s\le\delta}) \\ \vec{K}_t^3 &= (Y_t^a(t)-y,0,\Big\{((Y_{t+s}^a(t+s)-\mathcal{Y}_s)\mathbb{1}_{t+s<T}+Q_{t+s}\mathbb{1}_{t+s\ge T})\mathbb{1}_{t+s\le\theta}\Big\}_{0\le s\le\delta}) \\ \vec{K}_t^4 &= (Y_t^a(t)-y,\ (Y_{t+\delta}^a(t+\delta)-\zeta)\mathbb{1}_{\{t+\delta\le T\}}+Q_{t+\delta}\mathbb{1}_{\{t+\delta>T\}})G_{t+\delta}\ , \\ & \qquad \Big\{((Y_{t+s}^a(t+s)-\mathcal{Y}_s)\mathbb{1}_{\{t+s\le T\}}+Q_{t+s}\mathbb{1}_{\{t+s>T\}})G_{t+s}\Big\}_{0\le s\le\delta})\ . \end{aligned}$$

We show that the hypotheses of Proposition 2.11 are satisfied. First, we show that the driver is Lipschitz. Using that f is Lipschitz we get

$$|f^b(t,\vec{\mathbf{y}})-f^b(t,\vec{\mathbf{y}}')| \ \le\ \frac{1}{G_t}\Big|\mathbb{E}^{\mathcal{F}_t}[\ |f(t,\vec{\mathbf{y}})-f(t,\vec{\mathbf{y}}')|\mathbb{1}_{\{t<\tau\}}]\Big| \ \le\ C|\vec{\mathbf{y}}-\vec{\mathbf{y}}'|\ .$$

It follows that, setting $\vec{\mathbf{Y}} = (y, Y^a_{t+\delta}(\theta), \{\mathcal{Y}_s \mathbb{1}_{t+s<\theta} + Y^a_{t+s}(\theta)\mathbb{1}_{t+s\geq\theta}\}_{0\leq s\leq\delta}, z, Z^a_{t+\delta}(\theta), \{\mathcal{Z}_s \mathbb{1}_{t+s\leq\theta} + Z^a_{t+s}(\theta)\mathbb{1}_{t+s>\theta}\}_{0\leq s\leq\delta}, Y^a_t(t) - y, 0, \Big\{((Y^a_{t+s}(t+s) - \mathcal{Y}_s)\mathbb{1}_{t+s<T} + Q_{t+s}\mathbb{1}_{t+s\geq T})\mathbb{1}_{t+s\leq\theta}\Big\}_{0\leq s\leq\delta})$, there exists a constant C such that

$$\begin{aligned}
&\Big|\mathbb{E}^{\mathcal{F}_t}[f^b(t,\vec{\mathbf{Y}})\alpha_{t+\delta}(\theta)] - \mathbb{E}^{\mathcal{F}_t}[f^b(t,\vec{\mathbf{Y}}')\alpha_{t+\delta}(\theta)]\Big| \\
&\quad \leq C\Big((|y-y'| + |z-z'|)\mathbb{E}^{\mathcal{F}_t}(\alpha_{t+\delta}(\theta)) + |\mathcal{Y}\alpha(\theta) - \mathcal{Y}'\alpha(\theta)| + |\mathcal{Z}\alpha(\theta) - \mathcal{Z}'\alpha(\theta)|\Big) \\
&\quad \leq C(|y-y'| + |z-z'| + |\mathcal{Y}-\mathcal{Y}'| + |\mathcal{Z}-\mathcal{Z}'|)\alpha_t(\theta) \ ,
\end{aligned}$$

where we use that $\alpha(\theta)$ is a martingale. Hence, using that $\int_0^\infty \alpha_t(\theta)d\theta = 1$, we get

$$\int_t^{t+\delta} \mathbb{E}^{\mathcal{F}_t}[|f^b(t,\vec{\mathbf{Y}}) - f^b(t,\vec{\mathbf{Y}}')|\ \alpha_{t+\delta}(\theta)]d\theta \quad \leq \quad C(|y-y'| + |z-z'| + |\mathcal{Y}-\mathcal{Y}'| + |\mathcal{Z}-\mathcal{Z}'|) \ .$$

In the other hand, using the Lipschitz property of f^b, that G is upper bounded and denoting

$$\varphi(t,\vec{\mathbf{y}}) \quad := \quad \mathbb{E}^{\mathcal{F}_t}[f^b\,(t,\vec{\mathbf{y}})]$$

for $\vec{\mathbf{y}} = (y, \zeta, \mathcal{Y}, z, \eta, \mathcal{Z}, Y^a_t(t) - y, ((Y^a_{t+\delta}(t+\delta) - \zeta)\mathbb{1}_{\{t+\delta\leq T\}} + Q_{t+\delta}\mathbb{1}_{\{t+\delta>T\}})G_{t+\delta}, \Big\{((Y^a_{t+s}(t+s) - \mathcal{Y}_s)\mathbb{1}_{\{t+s\leq T\}} + Q_{t+s}\mathbb{1}_{\{t+s>T\}})G_{t+s}\Big\}_{0\leq s\leq\delta})$ there exists a constant K such that one has

$$|\varphi(t,\vec{\mathbf{y}}) - \varphi(t,\vec{\mathbf{y}}')| \quad \leq \quad K|\vec{\mathbf{y}} - \vec{\mathbf{y}}'| \ .$$

It follows (using one more time that G is lower bounded) that there exists a constant K such that

$$|g(t,\vec{\mathbf{y}}) - g(t,\vec{\mathbf{y}}')| \quad \leq \quad K|\vec{\mathbf{y}} - \vec{\mathbf{y}}'|$$

and the Lipschitz property holds.

4.3 Integrability

The integrability condition of

$$\begin{aligned}
g(t,\vec{\mathbf{0}}) \quad = \quad & \frac{1}{G_t}\int_t^{t+\delta} \mathbb{E}^{\mathcal{F}_t}[f^b(t, 0, Y^a_{t+\delta}(\theta), \{Y^a_{t+s}(\theta)\mathbb{1}_{t+s\geq\theta}\}_{0\leq s\leq\delta}, \\
& 0, Z^a_{t+\delta}(\theta), \{Z^a_{t+s}(\theta)\mathbb{1}_{t+s\geq\theta}\}_{0\leq s\leq\delta}, \\
& Y^a_t(t), 0, \{(Y^a_{t+s}(t+s)\mathbb{1}_{\{t+s\leq T\}} + Q_{t+s}\mathbb{1}_{\{t+s>T\}})\mathbb{1}_{t+s\leq\theta}\}_{0\leq s\leq\delta})\alpha_{t+\delta}(\theta)]d\theta \\
+ \quad & \frac{1}{G_t}\mathbb{E}^{\mathcal{F}_t}[f^b(t, 0, 0, 0, 0, Y^a_t(t),\ Y^a_{t+\delta}(t+\delta)\ G_{t+\delta}\mathbb{1}_{\{t+\delta<T\}} + Q_{t+\delta}\ G_{t+\delta}\mathbb{1}_{\{t+\delta\geq T\}})]
\end{aligned}$$

follows with the same arguments as in Section 3.2.

We also consider the integrability of the solutions (Y, Z, U) for ABSDE (4.1), where

$$\begin{aligned}
Y_t &= Y^b_t\mathbb{1}_{\{t<\tau\}} + Y^a_t(\tau)\mathbb{1}_{\{t\geq\tau\}} \ , \\
Z_t &= Z^b_t\mathbb{1}_{\{t\leq\tau\}} + Z^a_t(\tau)\mathbb{1}_{\{t>\tau\}} \ , \\
U_t &= (Y^a_t(t) - Y^b_t)\mathbb{1}_{\{t\leq\tau\}} \ .
\end{aligned}$$

One can apply the same methodology than the one in the previous section, since Proposition 2.10 is valid in the case of ABSDE (4.1) and obtain similar results.

5 Particular cases

5.1 Moving average term

We consider the case

$$
\begin{aligned}
-dY_t &= \mathbb{E}^{\mathcal{G}_t}\big[f(t, Y_t, \int_t^{t+\delta} a_s Y_s \eta(ds), Z_t, U_t)\big]dt - Z_t dB_t - U_t dH_t\ , \quad 0 \le t \le T\ , && (5.1)\\
Y_{T+t} &= \xi_{T+t}\ , \quad 0 \le t \le \delta\ , \\
Z_{T+t} &= P_{T+t}\ , && (5.2)\\
U_{T+t} &= Q_{T+t}\mathbb{1}_{\{T+t \le \tau\}} \quad 0 < t \le \delta\ ,
\end{aligned}
$$

where a is a bounded $\mathbb{F}$ adapted process and η is a measure of the form $\eta(ds) = \ell(s)ds + \sum_i k_i \epsilon_{s_i}(ds)$ where ℓ is a bounded Borel function, ϵ_a the Dirac measure at a, and $f(t, y, \widehat{y}, z, u)$ is Lipschitz. Since $\int_t^{t+\delta} |a_s||Y_s|\eta(ds) \le K\left(\int_t^{t+\delta} |Y_s|ds + \sum \mathbb{1}_{s_i \in [t,t+\delta]} Y_{s_i}\right)$, the driver satisfies the Lipschitz condition, and the above ABSDE has a solution.

5.2 Linear ABSDE

In this part we give a closed formula for the solution of linear ABSDEs. That means the driver f is linear w.r.t. Y, Z and U. We first give a result about the form of Y, part of the solution of a linear ABSDEs in the Brownian case.

Proposition 5.1 *Consider the following ABSDE*

$$
\begin{cases}
-dY_t = \Big[< \vec{\mathbf{a}}_t, \vec{\mathbf{Y}}_t > + l_t \Big] dt - Z_t dB_t\ , \quad t \in [0, T+\delta]\ , \\
Y_{T+t} = \xi_{T+t}\ , \quad 0 \le t \le \delta\ , \\
Z_{T+t} = P_{T+t}, \quad 0 < t \le \delta\ ,
\end{cases} \tag{5.3}
$$

where $< \cdot, \cdot >$ *is the scalar product,* $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{Y}}$ *are defined by*

$$
\begin{aligned}
\vec{\mathbf{a}}_t &= (\mu_t, \overline{\mu}_t, \underline{\mu}_t, \sigma_t, \overline{\sigma}_t, \underline{\sigma}_t)\ , \\
\vec{\mathbf{Y}}_t &= (Y_t, \mathbb{E}^{\mathcal{F}_t}[p_{t+\delta} Y_{t+\delta}], \mathbb{E}^{\mathcal{F}_t}\Big(\int_0^\delta p_{t+u} Y_{t+u} du\Big), Z_t, \mathbb{E}^{\mathcal{F}_t}[q_{t+\delta} Z_{t+\delta}], \mathbb{E}^{\mathcal{F}_t}\Big(\int_0^\delta q_{t+u} Z_{t+u} du]\Big)\ .
\end{aligned}
$$

and where the processes $\mu, \overline{\mu}, \underline{\mu}, \sigma, \overline{\sigma}, \underline{\sigma} \in L^2_{\mathbb{F}}[-\delta, T+\delta]$ *are assumed to be uniformly bounded and* $l \in L^2_{\mathbb{F}}[0, T]$.
Then, for any $t \in [0, T]$, *the solution* Y *is given by*

$$
\begin{aligned}
Y_t = \ & \mathbb{E}^{\mathcal{F}_t}\Big[X_T^t \xi_T + \int_t^T l_s X_s^t ds + \int_T^{T+\delta} \{\overline{\mu}_{s-\delta} p_s \xi_s + \overline{\sigma}_{s-\delta} q_s P_s\} X_{s-\delta}^t ds \\
& + \int_T^{T+\delta} \int_0^\delta \Big\{ \underline{\mu}_{s-u} p_s \xi_s + \underline{\sigma}_{s-u} q_s P_s \Big\} X_{s-u}^t du\, ds \Big]\ ,
\end{aligned}
$$

where

$$
\begin{cases}
dX_s^t = \Big[\mu_s X_s^t + p_s \overline{\mu}_{s-\delta} X_{s-\delta}^t + p_s \int_0^\delta \underline{\mu}_{s-u} X_{s-u}^t du \Big] ds \\
\qquad\qquad + \Big[\sigma_s X_s^t + q_s \overline{\sigma}_{s-\delta} X_{s-\delta}^t + q_s \int_0^\delta \underline{\sigma}_{s-u} X_{s-u}^t du \Big] dB_s, \quad t \le s \le T+\delta\ , \\
X_t^t = 1\ , \\
X_s^t = 0, \quad t-\delta \le s \le t\ .
\end{cases}
$$

Proof: The proof of this result is similar to the proof of Theorem 2.1 in [12]. We give some details in Appendix. □

We now extend the previous result to the case of an ABSDE with jump.

Proposition 5.2 *Consider the following ABSDE*

$$\begin{cases} -dY_t = & \big[< \vec{\mathbf{a}}_t, \vec{\mathbf{Y}}_t > + l_t \big] dt - Z_t dB_t - U_t dH_t \ , \quad 0 \le t \le T \ , \\ Y_{T+t} = & \xi_{T+t} \ , \quad 0 \le t \le \delta \ , \\ Z_{T+t} = & P_{T+t} \ , \quad U_{T+t} = Q_{T+t} \mathbb{1}_{\{T+t \le \tau\}} \ , \quad 0 < t \le \delta \ . \end{cases}$$

where $< \cdot, \cdot >$ is the scalar product, and $\vec{\mathbf{a}}$ and $\vec{\mathbf{Y}}$ are defined by

$$\begin{aligned} \vec{\mathbf{a}}_t = & \ (\mu_t, \overline{\mu}_t, \underline{\mu}_t, \sigma_t, \overline{\sigma_t}, \underline{\sigma}_t, \rho_t, \overline{\rho}_t, \underline{\rho}_t) \ , \\ \vec{\mathbf{Y}}_t = & \ (Y_t, \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}], \mathbb{E}^{\mathcal{G}_t}[\textstyle\int_0^\delta Y_{t+u} du], Z_t, \mathbb{E}^{\mathcal{G}_t}[Z_{t+\delta}], \mathbb{E}^{\mathcal{G}_t}[\int_0^\delta Z_{t+u} du], U_t, \mathbb{E}^{\mathcal{G}_t}[U_{t+\delta}], \mathbb{E}^{\mathcal{G}_t}[\int_0^\delta U_{t+u} du]) \ . \end{aligned}$$

We assume the functions $\mu, \overline{\mu}, \underline{\mu}, \sigma, \overline{\sigma}, \underline{\sigma}, \rho, \overline{\rho}, \underline{\rho} \in L^2_{\mathbb{G}}[-\delta, T+\delta]$ are assumed to be uniformly bounded and $l \in L^2_{\mathbb{G}}[0,T]$.
Then, for any $t \in [0,T]$, the solution Y is given by $Y_t = Y_t^b \mathbb{1}_{t<\tau} + Y_t^a(\tau) \mathbb{1}_{t \ge \tau}$ where Y^b and $Y^a(\theta)$, for any $\theta \in [0,T]$, are defined by

$$\begin{cases} Y_t^a(\theta) = & \mathbb{E}^{\mathcal{F}_t} \Big[X_T^{t,a}(\theta) \xi_T^a(\theta) + \displaystyle\int_t^T X_s^{t,a}(\theta) l_s^a(\theta) ds + \int_T^{T+\delta} (\xi_s^a(\theta) \overline{\mu}_{s-\delta}^a(\theta) \\ & \quad + P_s^a(\theta) \overline{\sigma}_{s-\delta}^a(\theta) X_{s-\delta}^{t,a}(\theta)) ds + \displaystyle\int_T^{T+\delta} \int_0^\delta \Big(\xi_s^a(\theta) \underline{\mu}_{s-u}^a(\theta) + P_s^a(\theta) \underline{\sigma}_{s-u}^a(\theta) \Big) X_{s-u}^{t,a}(\theta) du \, ds \Big] \ , \\ Y_t^b = & \mathbb{E}^{\mathcal{F}_t} \Big[X_T^{t,b} \xi_T^b + \displaystyle\int_t^T X_s^{t,b} L_s ds + \int_T^{T+\delta} \Big(G_s \xi_s^b \overline{\mu}_{s-\delta}^b + G_s P_s^b \overline{\sigma}_{s-\delta}^b \Big) X_{s-\delta}^{t,b} ds \\ & \quad + \displaystyle\int_T^{T+\delta} \int_0^\delta \Big(G_s \xi_s^b \overline{\mu}_{s-u}^b + G_s P_s^b \overline{\sigma}_{s-u}^b \Big) X_{s-u}^{t,b} du \, ds \Big] \end{cases}$$

with

$$\begin{aligned} L_t = & \ l_t^b + \overline{\mu}_t^b J_t^{Y^a}(t+\delta) + \underline{\mu}_t^b \int_0^\delta J_t^{Y^a}(t+u) du + \overline{\sigma}_t^b J_t^{Z^a}(t+\delta) + \underline{\sigma}_t^b \int_0^\delta J_t^{Z^a}(t+u) du \\ & + \rho_t^b Y_t^a(t) + \frac{\overline{\rho}_t^b}{G_t} \mathbb{E}^{\mathcal{F}_t} \left[G_{t+\delta} Y_{t+\delta}^a \mathbb{1}_{\{t+\delta \le T\}} + G_{t+\delta} Q_{t+\delta} \mathbb{1}_{\{t+\delta > T\}} \right] \\ & + \frac{\underline{\rho}_t^b}{G_t} \mathbb{E}^{\mathcal{F}_t} \Big[\int_0^\delta \left(Y_{t+u}^a(t+u) \mathbb{1}_{\{t+u \le T\}} + Q_{t+u} \mathbb{1}_{\{t+u > T\}} \right) G_{t+u} du \Big] \ , \end{aligned}$$

where $X^{t,a}(\theta)$ is the solution of the following linear stochastic delayed differential equation (SDDE)

$$\begin{cases} dX_s^{t,a}(\theta) = & \Big[\mu_s^a X_s^{t,a}(\theta) + \overline{\mu}_{s-\delta}(\theta) X_{s-\delta}^{t,a}(\theta) + \displaystyle\int_0^\delta \underline{\mu}_{s-u}^a(\theta) X_{s-u}^{t,a}(\theta) du \Big] ds \\ & \quad + \Big[\sigma_{s-\delta}^a(\theta) X_s^{t,a}(\theta) + \overline{\sigma}_{s-\delta}^a X_{s-\delta}^{t,a}(\theta) + \displaystyle\int_0^\delta \underline{\sigma}_{s-u}^a(\theta) X_{s-u}^{t,a}(\theta) du \Big] dB_s, \quad s \in [t, T+\delta] \ , \\ X_t^{t,a}(\theta) = & 1 \ , \\ X_s^{t,a}(\theta) = & 0, \quad s \in [t-\delta, t] \ , \end{cases}$$

and $X^{t,b}$ is the solution of the following linear SDDE

$$
\begin{cases}
dX_s^{t,b} = & \left[\left(\mu_s^b - \rho_s^b\right) X_s^{t,b} + \left(\dfrac{\overline{\mu}_{s-\delta}^b}{G_{s-\delta}} - \dfrac{\overline{\rho}_{s-\delta}^b}{G_{s-\delta}} \mathbb{1}_{\{s \leq T\}}\right) X_{s-\delta}^{t,b} \right. \\
& \left. + \displaystyle\int_0^\delta \left(\dfrac{\underline{\mu}_{s-u}^b}{G_{s-u}} - \dfrac{\underline{\rho}_{s-u}^b}{G_{s-u}} \mathbb{1}_{\{s-u \leq T\}}\right) X_{s-u}^{t,b} du \right] ds \\
& + \left[\sigma_s^b X_s^{t,b} + \dfrac{\overline{\sigma}_{s-\delta}^b}{G_{s-\delta}} X_{s-\delta}^{t,b} + \displaystyle\int_0^\delta \dfrac{\underline{\sigma}_{s-u}^b}{G_{s-u}} X_{s-u}^{t,b} du\right] dB_s, \quad s \in [t, T+\delta] \ , \\
X_t^{t,b} = & 1 \ , \\
X_s^{t,b} = & 0 \ , \quad s \in [t-\delta, t] \ .
\end{cases}
$$

Proof: The result is an application of the results of Section 3 and Proposition 5.1. □

6 Appendix

6.1 Proof of Proposition 2.10

Applying Itô's formula to $e^{\beta s} Y_s^2$, we obtain for any $s \in [0, T]$

$$
e^{\beta T} \xi_T^2 - e^{\beta t} Y_t^2 = \int_t^T e^{\beta s} \left(\beta Y_s^2 + Z_s^2\right) ds - 2 \int_t^T e^{\beta s} Y_s f(s, \vec{\mathbf{Y}}_s) ds + 2 \int_t^T e^{\beta s} Y_s Z_s dB_s \tag{6.1}
$$

where $\vec{\mathbf{Y}}_s = \big(Y_s, \mathbb{E}^{\mathcal{F}_s}[p_{s+\delta} Y_{s+\delta}], \{\mathbb{E}^{\mathcal{F}_s}[p_{s+u} Y_{s+u}]\}_{0 \leq u \leq \delta}, Z_s, \mathbb{E}^{\mathcal{F}_s}[q_{s+\delta} Z_{s+\delta}], \{\mathbb{E}^{\mathcal{F}_s}[q_{s+u} Z_{s+u}]\}_{0 \leq u \leq \delta}\big)$. By the Lipschitz assumption on f, using similar estimations as in the previous proofs and the boundness of p and q, we obtain there exists K such that

$$
\begin{aligned}
f(t, \vec{\mathbf{Y}}_t) &\leq \left| f(t, \vec{\mathbf{Y}}_t) - f(t, \vec{\mathbf{0}}) \right| + f(t, \vec{\mathbf{0}}) \\
&\leq K \left(|Y_t| + |Z_t| + \mathbb{E}^{\mathcal{F}_t}\big[\, |Y_{t+\delta}| + |Z_{t+\delta}| \,\big] + \mathbb{E}^{\mathcal{F}_t}\left[\int_0^\delta \left(|Y_{t+u}| + |Z_{t+u}|\right) du\right]\right) + f(t, \vec{\mathbf{0}}) \ .
\end{aligned} \tag{6.2}
$$

Combining (6.1) and (6.2), and using several times that $2|ab| \leq \frac{a^2}{c} + cb^2$ for different positive constants c, as for example

$$
2|Y_s| \mathbb{E}^{\mathcal{F}_s}(|Y_{s+\delta}|) \leq \frac{Y_s^2}{c_1} + c_1 (\mathbb{E}^{\mathcal{F}_s}(Y_{s+\delta}))^2 \ ,
$$

we get after some computations

$$
\begin{aligned}
& e^{\beta t} |Y_t|^2 + [\beta - (2 + c_1 + c_2 + c_3 + c_4 + c_5) K] \int_t^T e^{\beta s} Y_s^2 ds + [1 - \frac{K}{c_3}] \int_t^T e^{\beta s} Z_s^2 ds \\
\leq \ & e^{\beta T} \xi_T^2 - 2 \int_t^T e^{\beta s} Y_s Z_s dB_s + 2 \int_t^T e^{\beta s} Y_s f(s, \vec{\mathbf{0}}) ds + \frac{K}{c_1} \int_t^T e^{\beta s} \mathbb{E}^{\mathcal{F}_s}(Y_{s+\delta}^2) ds \\
& + \frac{K}{c_2} \int_t^T e^{\beta s} \int_0^\delta \mathbb{E}^{\mathcal{F}_s}(Y_{s+u}^2) du \, ds + \frac{K}{c_4} \int_t^T e^{\beta s} \mathbb{E}^{\mathcal{F}_s}(Z_{s+\delta}^2) ds + \frac{K}{c_5} \int_t^T e^{\beta s} \int_0^\delta \mathbb{E}^{\mathcal{F}_s}(Z_{s+u}^2) du \, ds \ .
\end{aligned}
$$

Taking conditional expectation w.r.t. $\mathcal{F}_t$, noting that from Fubini

$$\int_t^T e^{\beta s}\int_0^\delta Y_{s+u}^2 du\, ds \quad \leq \quad \delta \int_t^{T+\delta} e^{\beta s} Y_s^2 ds$$

and that

$$\int_t^T e^{\beta s} Y_{s+\delta}^2 ds \quad \leq \quad \int_t^{T+\delta} e^{\beta s} Y_s^2 ds$$

we obtain, after some simplifications, and using again $2|ab| \leq \frac{a^2}{c} + cb^2$ (with a new constant c_6!)

$$\begin{aligned}
& e^{\beta t}|Y_t|^2 + [\beta - (2 + c_1 + c_2 + c_3 + c_4 + c_5)K - \frac{K}{c_1} - \frac{\delta K}{c_2}K - c_6]\,\mathbb{E}^{\mathcal{F}_t}\big(\int_t^T e^{\beta s} Y_s^2 ds\big) \\
& \quad + [1 - \frac{K}{c_3} - \frac{K}{c_4} - \frac{\delta K}{c_5}]\,\mathbb{E}^{\mathcal{F}_t}\big(\int_t^T e^{\beta s} Z_s^2 ds\big) \\
\leq \quad & e^{\beta T}\mathbb{E}^{\mathcal{F}_t}(\xi_T^2) + \frac{1}{c_6}\mathbb{E}^{\mathcal{F}_t}\big(\int_t^T e^{\beta s}(f(s,\vec{\mathbf{0}}))^2 ds\big) + [\frac{\delta K}{c_2} + \frac{K}{c_1}]\mathbb{E}^{\mathcal{F}_t}\big(\int_T^{T+\delta} \xi_s^2 ds\big) \\
& \quad + [\frac{\delta K}{c_5} + \frac{K}{c_4}]\mathbb{E}^{\mathcal{F}_t}\big(\int_T^{T+\delta} P_s^2 ds\big)
\end{aligned}$$

It remains to choose the constants c_i such that $\beta = (2+c_1+c_2+c_3+c_4+c_5)K + \frac{K}{c_1} + \frac{\delta K}{c_2}K + c_6$ and $\frac{K}{c_3} + \frac{K}{c_4} + \frac{\delta K}{c_5} \leq 1$ to obtain that there exists a constant K such that

$$\mathbb{E}^{\mathcal{F}_t}\Big[\int_t^T Z_s^2 ds\Big] \quad \leq \quad K\,\mathbb{E}^{\mathcal{F}_t}\Big[\xi_T^2 + \int_T^{T+\delta}(\xi_s^2 + P_s^2)ds + \int_t^T (f(s,\vec{\mathbf{0}}))^2 ds\Big],$$

for any $t \in [0,T]$.

The proof for the term with the supremum on Y is obtained using the same methodology and BDG inequality.

6.2 Proof of Proposition 5.1

Applying Itô's formula to X^tY and taking the conditional expectation we get

$$\begin{aligned}
X_t^t Y_t = \ & \mathbb{E}^{\mathcal{F}_t}\Big[X_T^t Y_T + \int_t^T l_s X_s^t ds + \int_t^T \overline{\mu}_s X_s^t p_{s+\delta} Y_{s+\delta} ds - \int_t^T \overline{\mu}_{s-\delta} X_{s-\delta}^t p_s Y_s ds \\
& + \int_t^T \overline{\sigma}_s X_s^t q_{s+\delta} Z_{s+\delta} ds - \int_t^T \overline{\sigma}_{s-\delta} X_{s-\delta}^t q_s Z_s ds \\
& + \int_t^T \underline{\mu}_s X_s^t \int_0^\delta p_{s+u} Y_{s+u} du\, ds - \int_t^T p_s Y_s \int_0^\delta \underline{\mu}_{s-u} X_{s-u}^t du\, ds \\
& + \int_t^T \underline{\sigma}_s X_s^t \int_0^\delta q_{s+u} Z_{s+u} du\, ds - \int_t^T q_s Z_s \int_0^\delta \underline{\sigma}_{s-u} X_{s-u}^t du\, ds\Big].
\end{aligned}$$

This can be rewritten, by using $X_u^t = 0$ for $u \le t$, under the form

$$\begin{aligned}
X_t^t Y_t = \; & \mathbb{E}^{\mathcal{F}_t}\Big[X_T^t Y_T + \int_t^T l_s X_s^t ds + \int_t^{T+\delta} \overline{\mu}_{s-\delta} X_{s-\delta}^t p_s Y_s ds - \int_t^T \overline{\mu}_{s-\delta} X_{s-\delta}^t p_s Y_s ds \\
& + \int_t^{T+\delta} \overline{\sigma}_{s-\delta} X_{s-\delta}^t q_s Z_s ds - \int_t^T \overline{\sigma}_{s-\delta} X_{s-\delta}^t q_s Z_s ds + \int_t^{T+\delta} \int_0^\delta \underline{\mu}_{s-u} X_{s-u}^t p_s Y_s du\, ds \\
& - \int_t^T \int_0^\delta \underline{\mu}_{s-u} X_{s-u}^t p_s Y_s du\, ds + \int_t^{T+\delta} \int_0^\delta \underline{\sigma}_{s-u} X_{s-u}^t q_s Z_s du\, ds \\
& - \int_t^T \int_0^\delta \underline{\sigma}_{s-u} X_{s-u}^t q_s Z_s du\, ds\Big] .
\end{aligned}$$

Since $X_t^t = 1$, we conclude

$$\begin{aligned}
Y_t = \; & \mathbb{E}^{\mathcal{F}_t}\Big[X_T^t \xi_T + \int_t^T l_s X_s^t ds + \int_T^{T+\delta} \left\{\overline{\mu}_{s-\delta} p_s \xi_s + \overline{\sigma}_{s-\delta} q_s P_s\right\} X_{s-\delta}^t ds \\
& + \int_T^{T+\delta} \int_0^\delta \left\{\underline{\mu}_{s-u} p_s \xi_s + \underline{\sigma}_{s-u} q_s P_s\right\} X_{s-u}^t du\, ds\Big] .
\end{aligned}$$

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