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# Fluid-structure interaction system with Coulomb's law

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## Abstract

We propose a new model in a fluid-structure system composed by a rigid body and a viscous incompressible fluid using a boundary condition based on Coulomb's law. This boundary condition allows the fluid to slip on the boundary if the tangential component of the stress is too large. In the opposite case, we recover the standard Dirichlet boundary condition. The governing equations are the Navier-Stokes system for the fluid and the Newton laws for the body. The corresponding coupled system can be written as a variational inequality. We prove that there exists a weak solution of this system.

**Key words:** Fluid-structure system, Navier-Stokes system, Cauchy theory, Coulomb's law

**2010 Mathematics Subject Classification:** 35Q30, 76D03, 76D27

## 1 Introduction

We consider the system composed by a rigid body immersed into a viscous incompressible fluid. The domain of the rigid body at time  $t$  is denoted by  $\mathcal{S}(t)$ . It can be obtained from the initial domain  $\mathcal{S}^0$  by using a translation and a rotation. The fluid domain at time  $t$  is denoted by  $\mathcal{F}(t)$  and is the relative complement of  $\mathcal{S}(t)$  in  $\Omega$ . We thus have the following formulas:

$$\mathcal{S}(t) := h(t) + R(t)\mathcal{S}^0, \quad \mathcal{F}(t) := \Omega \setminus \overline{\mathcal{S}(t)}. \quad (1.1)$$

More generally, we write for  $h \in \mathbb{R}^3$  and for  $R \in SO(3)$ ,

$$\mathcal{S}(h, R) := h + R\mathcal{S}^0, \quad \mathcal{F}(h, R) := \Omega \setminus \overline{\mathcal{S}(h, R)}, \quad (1.2)$$

where  $SO(3)$  denotes the classical rotation group in  $\mathbb{R}^3$ .

We model the fluid motion by the classical Navier-Stokes system and the motion of the rigid body by the Newton laws:

$$\frac{\partial u_F}{\partial t} + (u_F \cdot \nabla)u_F - \operatorname{div} \sigma(u_F, p_F) = 0 \quad t > 0, \quad x \in \mathcal{F}(t), \quad (1.3)$$

$$\operatorname{div} u_F = 0 \quad t > 0, \quad x \in \mathcal{F}(t), \quad (1.4)$$

$$mh''(t) = - \int_{\partial\mathcal{S}(t)} \sigma(u_F, p_F)n \, d\gamma \quad t > 0, \quad (1.5)$$

$$(J\omega)'(t) = - \int_{\partial\mathcal{S}(t)} (x - h) \times \sigma(u_F, p_F)n \, d\gamma \quad t > 0, \quad (1.6)$$

$$R'(t) = \mathbb{A}(\omega(t))R(t) \quad t > 0. \quad (1.7)$$

We have denoted by  $\sigma$  the Cauchy stress tensor:

$$\sigma(u_F, p_F) := 2\mu D(u_F) - p_F I_3, \quad (1.8)$$

with

$$D(u_F) := \frac{1}{2} ((\nabla u_F) + (\nabla u_F)^*).$$

We recall the formula

$$\operatorname{div} \sigma(u_F, p_F) = \mu \Delta u_F - \nabla p_F.$$

We have also denoted by  $n := n(t, x)$  the exterior unit normal of  $\partial\mathcal{F}(t)$ .

We write for any  $\omega \in \mathbb{R}^3$ ,

$$\mathbb{A}(\omega) := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \quad (1.9)$$

The mass  $m$  and the inertia matrix  $J$  are defined through the density  $\rho_S$  of the rigid body. To simplify, we assume in this paper that this density is a positive constant. In that case,

$$m = \rho_S |\mathcal{S}^0|$$

and

$$J(t) := \rho_S \int_{\mathcal{S}(t)} (|x - h(t)|^2 - (x - h(t)) \otimes (x - h(t))) \, dx. \quad (1.10)$$

We need to complete the above system with boundary conditions for the Navier-Stokes system. It is usual to impose the no-slip boundary conditions. A lot of works have been devoted to the corresponding system and we give here a non-exhaustive lists of papers: [2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 26, 27, 28, 29, 30, 31, 33, 34, 35], etc.

In [26], the authors show that if the solid touches the exterior boundary, then this contact is done with null relative velocity and null relative acceleration. Such a result, which is related to the Dirichlet boundary condition, suggests that if at initial time two bodies are not touching, they will never be in contact at finite time. This result has been rigorously proved in [22] in the case of symmetric rigid body falling over a plane (see also [23] for the 3D case). Such a property indicates that this model should be modified in order to recover collisions between rigid bodies.

In the case of a perfect incompressible fluid, it is shown in [24] that collisions occur in finite time. If the fluid is viscous, Gérard-Varet and Hillairet have proposed two solutions to this problem. In [17], they analyse the case when the structure domain is not regular. In that case, they prove that collisions can occur and conclude that roughness is a possible explanation for the lack of collisions: the boundary of the rigid body should always have some irregularities and this should allow collisions. Another possibility that these two authors consider in [18] is to change the boundary conditions: instead of a no-slip boundary condition, they take the Navier condition. In that case, in [19] they prove one can again recover collisions.

In this paper, we propose a different model, with an hybrid boundary condition that follows the Coulomb laws (see [7, p.134]). With the Coulomb boundary condition, if the tangential component of  $\sigma(u_F, p_F)n$  does not exceed a threshold, then the boundary condition remains the standard Dirichlet boundary condition, whereas if it equals to this threshold, then the boundary condition corresponds to a generalized Navier boundary condition. With such a model, we impose a physical upper bound on the tangential component of  $\sigma(u_F, p_F)n$ , under of which the classical Dirichlet boundary condition used for the Navier-Stokes system is kept. In our recent paper [1], we have studied the well-posedness of the Navier-Stokes system with this hybrid boundary condition in absence of solids.

Let us precisely write the Coulomb law. To this end, we introduce the following notation: for any vector  $a \in \mathbb{R}^3$ , we consider respectively its normal and tangential components

$$a_n := (a \cdot n)n, \quad a_\tau := n \times (a \times n). \quad (1.11)$$

In particular, we have the following orthogonal decomposition

$$a = a_n + a_\tau. \quad (1.12)$$

Let us describe the Coulomb boundary conditions on  $\partial\Omega$ :

- we assume that the normal component of the fluid velocity  $u_F$  is equal to 0;
- for the tangential component, we assume that if  $|(\sigma(u_F, p_F)n)_\tau| < g$ , where  $g$  is some positive constant, then  $(u_F)_\tau = 0$  and if  $|(\sigma(u_F, p_F)n)_\tau| = g$  then  $(u_F)_\tau$  is collinear to  $(\sigma(u_F, p_F)n)_\tau$  with the opposite direction.

This boundary condition can be written as follows (see Appendix A below):

$$(u_F)_n = 0 \quad t > 0, x \in \partial\Omega, \quad (1.13)$$

$$-(u_F)_\tau \in \partial I_{\overline{B}(0,g)}((\sigma(u_F, p_F)n)_\tau) \quad t > 0, x \in \partial\Omega, \quad (1.14)$$

where  $I_{\overline{B}(0,g)}$  is the characteristic function of the closed ball  $\overline{B}(0,g)$  given by

$$\begin{aligned} I_{\overline{B}(0,g)} : \mathbb{R}^3 &\rightarrow \mathbb{R} \cup \{+\infty\} \\ x &\mapsto I_{\overline{B}(0,g)}(x) = \begin{cases} 0 & \text{if } |x| \leq g, \\ +\infty & \text{if } |x| > g. \end{cases} \end{aligned} \quad (1.15)$$

We recall that its subdifferential is given by

$$\partial I_{\overline{B}(0,g)}(x_0) = \begin{cases} \{0\} & \text{if } |x_0| < g, \\ \{\alpha x_0 ; \alpha \geq 0\} & \text{if } |x_0| = g, \\ \emptyset & \text{if } |x_0| > g. \end{cases} \quad (1.16)$$

On the moving interface  $\partial\mathcal{S}(t)$ , we impose a similar condition, but for the relative velocity  $u_F - u_R$ , where

$$u_R(t, x) := h'(t) + \omega(t) \times (x - h(t)). \quad (1.17)$$

The corresponding boundary conditions write

$$(u_F)_n = (u_R)_n \quad t > 0, x \in \partial\mathcal{S}(t), \quad (1.18)$$

$$-\left((u_F)_\tau - (u_R)_\tau\right) \in \partial I_{\overline{B}(0,g)}((\sigma(u_F, p_F)n)_\tau) \quad t > 0, x \in \partial\mathcal{S}(t). \quad (1.19)$$

Finally, we have to add the initial boundary conditions:

$$h(0) = 0, \quad R(0) = I_3, \quad (1.20)$$

$$h'(0) = \ell^0, \quad \omega(0) = \omega^0, \quad (1.21)$$

$$u_F(0, x) = u_F^0(x) \quad x \in \mathcal{F}^0. \quad (1.22)$$

The aim of this paper is to prove the existence of weak solutions for the system composed by the equations (1.1), (1.3)–(1.7), (1.13)–(1.14), (1.17)–(1.22). For shortness, we call **(SYS)** this system in what follows.

Let us describe the structure of the paper. We compute a weak formulation in Section 2 and state the main result given in Theorem 2.2 below which asserts the existence of a weak solution up to a contact. Due to the Coulomb boundary condition, the weak formulation corresponds to a variational inequality. One of the main difficulties in the proof of Theorem 2.2, with respect to the case of the Dirichlet boundary condition, comes from the fact that the “global” velocity field could be discontinuous at the interface between the fluid and the structure. It is a piecewise function defined as the fluid velocity field in  $\mathcal{F}(t)$  and the solid velocity in  $\mathcal{S}(t)$ . Such a difficulty will be overcome by using some approximations in order to obtain a global velocity in  $H^1$ . This is done in [18] and in Appendix B we state two crucial lemmas in that direction. In Section 3, we introduce approximations of the weak formulation given in Section 2: we use a Galerkin method and add a penalization term that allows to avoid a free-boundary problem due to the motion of the structure. Finally, we regularize the underlying convex function associated to the Coulomb law. In that case, the variational inequality is equivalent to a variational equality. Section 4 is devoted to pass to the limit in the Galerkin approximation. Finally, in Section 5 we prove the main result by passing to the limit for the two other approximations.

## 2 Weak formulation and main result

In this section, we give some notation used in this paper. We then introduce a notion of weak solution and we state the main result concerning the existence of a weak solution.

### 2.1 Notation

Let us denote by  $L^\alpha$ ,  $H^k$  the classical Lebesgue and Sobolev spaces. We also denote by  $\mathcal{C}^k$  the space of  $k$ -times continuous differentiable functions. We write  $\mathcal{C}_c^k$  the set of all functions in  $\mathcal{C}^k$  with compact support.

For  $\alpha \in \mathbb{R}^+$  and for a smooth domain  $A$ , let us introduce the following Hilbert space

$$V_n^\alpha(A) = \{v \in H^\alpha(A) : \operatorname{div}(v) = 0 \text{ in } A, v_n = 0 \text{ on } \partial A\}.$$

We have in particular

$$V_n^0(A) = \{v \in L^2(A) : \operatorname{div}(v) = 0 \text{ in } A, v_n = 0 \text{ on } \partial A\}$$

and

$$V_n^1(A) = \{v \in H^1(A) : \operatorname{div}(v) = 0 \text{ in } A, v_n = 0 \text{ on } \partial A\}. \quad (2.1)$$

If moreover,  $\partial A$  is bounded, we also consider the space

$$V^1(A) = \left\{ v \in H^1(A) : \operatorname{div}(v) = 0 \text{ in } A, \int_{\partial A} v_n \, d\gamma = 0 \right\}. \quad (2.2)$$

We denote by  $\mathcal{R}$  the space of rigid velocities:

$$\mathcal{R} := \{\ell + \omega \times x : \ell, \omega \in \mathbb{R}^3\}$$

and for any domain  $A \subset \Omega$ , we define the orthogonal projection  $P_A : L^2(\Omega) \rightarrow \mathcal{R}$ , i.e. for all  $u \in L^2(\Omega)$ ,

$$\int_A (u - P_A(u)) \cdot v \, dx = 0 \quad \forall v \in \mathcal{R}. \quad (2.3)$$

For any domain  $\mathcal{S} \Subset \Omega$ , we define the space

$$H_R(\mathcal{S}) = \{v \in V_n^0(\Omega) : \exists v_R \in \mathcal{R}, \exists v_F \in H^1(\Omega), v = v_F \text{ in } \Omega \setminus \overline{\mathcal{S}}, v = v_R \text{ in } \mathcal{S}\}.$$

In particular, for any  $v \in H_R(\mathcal{S})$ , there exist two vectors  $\ell_v, \omega_v$  such that

$$v_R(x) = \ell_v + \omega_v \times x \quad \forall x \in \mathbb{R}^3, \quad (2.4)$$

$$(v_F - v_R)_n(x) = 0 \quad \forall x \in \partial \mathcal{S}. \quad (2.5)$$

For  $h \in \mathbb{R}^3$  and for  $R \in SO(3)$ , we consider the operators  $\Xi_{h,R}$  and  $\widehat{\Xi}_{h,R}$  defined by

$$\Xi_{h,R}(v)(y) := R^*v(h + Ry), \quad y \in \mathbb{R}^3, \quad (2.6)$$

$$\widehat{\Xi}_{h,R}(v)(x) := Rv(R^*(x - h)), \quad x \in \mathbb{R}^3, \quad (2.7)$$

where  $R^*$  is the transposed matrix of  $R$ .

Let consider the following space

$$\mathcal{T}_T := \left\{ v \in \mathcal{C}_c^0([0, T]; V_n^0(\Omega)) : \exists v_R \in \mathcal{C}^1([0, T]; \mathcal{R}), \exists v_F \in \mathcal{C}^1([0, T]; H^1(\Omega)) \right. \\ \left. \text{and } v = v_F \text{ in } \mathcal{F}(t), v = v_R \text{ in } \mathcal{S}(t) \right\}. \quad (2.8)$$

For any set  $A \subset \mathbb{R}^3$ , let us denote

$$A_\delta := \{x \in A : \operatorname{dist}(x, \partial A) > \delta\} \quad (2.9)$$

and

$$A^\delta := \{x \in \Omega : \operatorname{dist}(x, A) < \delta\}. \quad (2.10)$$

## 2.2 Weak formulation

In order to obtain a weak formulation of **(SYS)**, we define  $u$  by

$$u(t, \cdot) = u_F(t, \cdot) \text{ in } \mathcal{F}(t), \quad u(t, \cdot) = u_R(t, \cdot) \text{ in } \mathcal{S}(t).$$

In particular, we can consider that the solution satisfies  $u(t, \cdot) \in H_R(\mathcal{S}(t))$ .

We formally multiply equation (1.3) by  $v(t, \cdot) \in H_R(\mathcal{S}(t))$  and we integrate in  $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$ , obtaining

$$\int_{\mathcal{F}(t)} \left[ \left( \frac{\partial}{\partial t} + (u_F \cdot \nabla) \right) u_F \right] \cdot v_F \, dx + 2\mu \int_{\mathcal{F}(t)} D(u_F) : D(v_F) \, dx \\ = \int_{\partial \Omega} \sigma(u_F, p_F) n \cdot v_F \, d\gamma + \int_{\partial \mathcal{S}(t)} \sigma(u_F, p_F) n \cdot v_F \, d\gamma. \quad (2.11)$$

On  $\partial\Omega$ , we have  $(v_F)_n = 0$  and  $(u_F)_n = 0$ , thus we can write

$$\int_{\partial\Omega} \sigma(u_F, p_F) n \cdot v_F d\gamma = \int_{\partial\Omega} (\sigma(u_F, p_F) n)_\tau \cdot (v_F)_\tau d\gamma \geq \int_{\partial\Omega} \left[ g|(u_F)_\tau| - g|(v_F)_\tau + (u_F)_\tau \right] d\gamma, \quad (2.12)$$

where we have used Lemma A.1 below and (1.14).

On the other hand, on  $\partial\mathcal{S}(t)$ , we have

$$\int_{\partial\mathcal{S}(t)} \sigma(u_F, p_F) n \cdot v_F d\gamma = \int_{\partial\mathcal{S}(t)} \sigma(u_F, p_F) n \cdot v_R d\gamma + \int_{\partial\mathcal{S}(t)} \sigma(u_F, p_F) n \cdot (v_F - v_R) d\gamma.$$

Thus, using identity (2.4), we deduce that

$$\begin{aligned} \int_{\partial\mathcal{S}(t)} \sigma(u_F, p_F) n \cdot v_F d\gamma &= \int_{\partial\mathcal{S}(t)} \sigma(u_F, p_F) n \cdot (\ell_v + \omega_v \times (x - h)) d\gamma + \int_{\partial\mathcal{S}(t)} \sigma(u_F, p_F) n \cdot (v_F - v_R) d\gamma \\ &= -mh''(t) \cdot \ell_v - (J\omega)'(t) \cdot \omega_v + \int_{\partial\mathcal{S}(t)} [\sigma(u_F, p_F) n]_\tau \cdot (v_F - v_R)_\tau d\gamma. \end{aligned} \quad (2.13)$$

Using Lemma A.1 below and (1.19) in (2.13), we obtain

$$\begin{aligned} \int_{\partial\mathcal{S}(t)} \sigma(u_F, p_F) n \cdot v_F d\gamma &\geq -mh''(t) \cdot \ell_v - (J\omega)'(t) \cdot \omega_v \\ &\quad + \int_{\partial\mathcal{S}(t)} g|(u_F - u_R)_\tau| d\gamma - \int_{\partial\mathcal{S}(t)} g|(v_F - v_R + u_F - u_R)_\tau| d\gamma. \end{aligned} \quad (2.14)$$

Let us assume that  $v \in \mathcal{T}_T$ , where  $\mathcal{T}_T$  is defined in (2.8). In this case, for any  $t \in [0, T]$ , we get  $v(t, \cdot) \in H_R(\mathcal{S}(t))$ . Then, using (1.4), (1.13) and (1.18), we deduce that

$$\int_{\mathcal{F}(t)} \left( \frac{\partial}{\partial t} + (u_F \cdot \nabla) \right) u_F \cdot v_F dx = \frac{d}{dt} \int_{\mathcal{F}(t)} u_F \cdot v_F dx - \int_{\mathcal{F}(t)} u_F \cdot \left[ \frac{\partial v_F}{\partial t} + [(u_F \cdot \nabla) v_F] \right] dx. \quad (2.15)$$

Gathering (2.11)–(2.14) and (2.15), we deduce

$$\begin{aligned} &\frac{d}{dt} \int_{\mathcal{F}(t)} u_F \cdot v_F dx - \int_{\mathcal{F}(t)} u_F \cdot \left[ \frac{\partial v_F}{\partial t} + [(u_F \cdot \nabla) v_F] \right] dx \\ &\quad + 2\mu \int_{\mathcal{F}(t)} D(u_F) : D(v_F) dx + \frac{d}{dt} (mh'(t) \cdot \ell_v) - mh'(t) \cdot \ell'_v + \frac{d}{dt} [(J\omega)(t) \cdot \omega_v] - (J\omega)(t) \cdot \omega'_v \\ &\quad + \int_{\partial\mathcal{S}(t)} g|(u_F - u_R + v_F - v_R)_\tau| d\gamma - \int_{\partial\mathcal{S}(t)} g|(u_F - u_R)_\tau| d\gamma + \int_{\partial\Omega} [g|(u_F + v_F)_\tau| - g|(u_F)_\tau|] d\gamma \geq 0. \end{aligned} \quad (2.16)$$

We can then integrate (2.16) on  $[0, T]$  to obtain

$$\begin{aligned} &-\int_{\mathcal{F}^0} u_F^0 \cdot v_F(0, \cdot) dx - m\ell^0 \cdot \ell_v(0) - J^0 \omega^0 \cdot \omega_v(0) \\ &\quad - \int_0^T \int_{\mathcal{F}(t)} u_F \cdot \left[ \frac{\partial v_F}{\partial t} + [(u_F \cdot \nabla) v_F] \right] dx dt - \int_0^T mh'(t) \cdot \ell'_v(t) dt - \int_0^T J(t)\omega(t) \cdot \omega'_v(t) dt \\ &\quad + 2\mu \int_0^T \int_{\mathcal{F}(t)} D(u_F) : D(v_F) dx dt + \int_0^T \int_{\partial\Omega} [g|(u_F + v_F)_\tau| - g|(u_F)_\tau|] d\gamma dt \\ &\quad + \int_0^T \int_{\partial\mathcal{S}(t)} g|(u_F - u_R + v_F - v_R)_\tau| d\gamma dt - \int_0^T \int_{\partial\mathcal{S}(t)} g|(u_F - u_R)_\tau| d\gamma dt \geq 0. \end{aligned} \quad (2.17)$$

Setting

$$u_R^0(x) := \ell^0 + \omega^0 \times x,$$

we can rewrite the above system as follows:

$$\begin{aligned}
& - \int_{\mathcal{F}^0} u_F^0 \cdot v_F(0, \cdot) \, dx - \int_{S^0} \rho_S u_R^0 \cdot v_R(0, \cdot) \, dx \\
& - \int_0^T \int_{\mathcal{F}(t)} u_F \cdot \left[ \frac{\partial v_F}{\partial t} + [(u_F \cdot \nabla) v_F] \right] \, dx \, dt - \int_0^T \int_{S(t)} \rho_S u_R \cdot \left[ \frac{\partial v_R}{\partial t} + [(u_R \cdot \nabla) v_R] \right] \, dx \, dt \\
& + 2\mu \int_0^T \int_{\mathcal{F}(t)} D(u_F) : D(v_F) \, dx \, dt + \int_0^T \int_{\partial\Omega} \left[ g|(u_F + v_F)_\tau| - g|(u_F)_\tau| \right] \, d\gamma \, dt \\
& + \int_0^T \int_{\partial S(t)} g|(u_F - u_R + v_F - v_R)_\tau| \, d\gamma \, dt - \int_0^T \int_{\partial S(t)} g|(u_F - u_R)_\tau| \, d\gamma \, dt \geq 0. \quad (2.18)
\end{aligned}$$

This weak formulation allows us to introduce a notion of weak solution for **(SYS)**:

**Definition 2.1.** A weak solution of **(SYS)** is a triplet  $(h, R, u)$  with the following properties:

$$\begin{aligned}
& (h, R) \in W^{1,\infty}(0, T; \mathbb{R}^3 \times SO(3)), \quad \mathcal{S}(t) \Subset \Omega \quad (t \in (0, T)), \\
& u \in L^\infty(0, T; V_n^0(\Omega)), \quad u(t, \cdot) \in H_R(\mathcal{S}(t)) \quad \text{a.e. in } (0, T), \\
& u_F \in L^2(0, T; H^1(\Omega)), \quad u_R(t, x) = h'(t) + \omega(t) \times (x - h(t)),
\end{aligned}$$

(1.7) and (1.20) hold true, (2.18) holds true for any  $v \in \mathcal{T}_T$  and

$$\begin{aligned}
& \frac{1}{2} \int_{\mathcal{F}} |u_F(t, \cdot)|^2 \, dx + \frac{\rho_S}{2} \int_S |u_R(t, \cdot)|^2 \, dx + 2\mu \int_0^t \int_{\mathcal{F}(t)} |D(u_F)|^2 \, dx \, dt + \int_0^t \int_{\partial\Omega} g|u_F| \, d\gamma \, dt \\
& + \int_0^t \int_{\partial S(t)} g|u_F - u_R| \, d\gamma \, dt \leq \frac{1}{2} \int_{\mathcal{F}^0} |u_F^0|^2 \, dx + \frac{\rho_S}{2} \int_{S^0} |u_R^0|^2 \, dx, \quad (2.19)
\end{aligned}$$

for almost every  $t \in (0, T)$ .

We are now able to state the main result of this paper:

**Theorem 2.2.** Assume  $\mathcal{S}^0 \Subset \Omega$ ,  $\partial\mathcal{S}^0$  and  $\partial\Omega$  are of class  $\mathcal{C}^{1,1}$ ,  $u_F^0 \in V_n^0(\Omega)$ ,  $u_R^0 \in \mathcal{R}$  with  $(u_F^0)_n = (u_R^0)_n$  on  $\partial\mathcal{S}^0$ . Then, there exist  $T \in (0, \infty]$  and a weak solution of **(SYS)** in  $(0, T)$ . Moreover, we can choose  $T$  such that one of the following alternatives holds true:

1.  $T = \infty$ ;
2.  $\lim_{t \rightarrow T} \text{dist}(\mathcal{S}(t), \partial\Omega) = 0$ .

Let us give some remarks on this result.

1. We could present a 2D version of our model and we could prove the same result as Theorem 2.2 for this case.
2. The system **(SYS)** is a free-boundary fluid-structure interaction system: the position of the rigid body is unknown and thus the fluid domain is variable in time and is one of the unknowns. Among the classical methods developed in the literature, one can quote the introduction of penalization term in the approximations of the governing equations: one can approach the fluid-structure system by a fluid system in the whole domain  $\Omega$  with an additional term on the structure part in order to recover at the limit the rigid motion. This penalization can be seen as a viscosity term: the rigid body is approximated by a fluid with a large viscosity. This is the method introduced in [26]. The penalization term could be also obtained as an approximation of the orthogonal projection on the space of rigid velocities defined in (2.3). This is the method introduced in [2] and the one we consider here.
3. Another difficulty here, due to the boundary condition, is the fact that the global velocity  $u$ , which is equal to  $u_R$  in  $\mathcal{S}(t)$  and to  $u_F$  in  $\mathcal{F}(t)$ , may not be in  $H^1$  since only the normal traces coincide.
4. The test functions (which belong to  $\mathcal{T}_T$ ) could not be in  $H^1$  space. In order to overcome this difficulty, we will need some tools to approximate the test functions by more regular functions. This technical part was developed in [18] and we apply several of their results.
5. Finally, the last difficulty comes from the nonlinearity due the boundary conditions. This corresponds to the boundary integrals in (2.18) and the fact that this relation is an inequality. In order to deal with it, we regularize the convex function  $x \mapsto |x|$  by a smooth convex function. In that case, the inequality becomes an equality.

### 3 Approximations of weak formulation of (SYS)

We consider an approximation of weak formulation of (SYS) parametrized by three parameters  $\varepsilon > 0$ ,  $M, k \in \mathbb{N}^*$ . The parameter  $\varepsilon > 0$  corresponds to the regularization of the function  $j : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $x \mapsto |x|$  in (2.18), the parameter  $M$  corresponds to the Galerkin method (cardinal of the basis) and  $k$  corresponds to the penalization term that allows to avoid a free-boundary problem in the approximation.

For any  $\varepsilon > 0$ , we introduce convex regularized functions  $j_\varepsilon \in \mathcal{C}^1(\mathbb{R}^3)$  that approximate  $j$  and are defined by

$$j_\varepsilon(x) = \begin{cases} |x| & \text{if } |x| > \varepsilon, \\ \frac{1}{2\varepsilon}|x|^2 + \frac{\varepsilon}{2} & \text{if } |x| \leq \varepsilon. \end{cases} \quad (3.1)$$

In particular,

$$\nabla j_\varepsilon(x) \cdot x \geq 0 \quad \forall x \in \mathbb{R}^3, \quad (3.2)$$

$$|\nabla j_\varepsilon(x)| \leq 1 \quad \forall x \in \mathbb{R}^3, \quad (3.3)$$

$$|x| \leq j_\varepsilon(x) \leq |x| + \frac{\varepsilon}{2} \quad \forall x \in \mathbb{R}^3. \quad (3.4)$$

We use a Galerkin method. To this end, we introduce an orthonormal basis  $(v_j)_{j \geq 1}$  of  $V_n^0(\Omega)$  such that  $v_j \in V_n^1(\Omega)$  and we write

$$V_M = \text{span}\{v_1, \dots, v_M\}.$$

Then we consider the approximate problem: find  $h^{\varepsilon, k, M} \in \mathcal{C}^1([0, T]; \mathbb{R}^3)$ ,  $R^{\varepsilon, k, M} \in \mathcal{C}^1([0, T]; SO_3(\mathbb{R}))$ ,  $a_j \in \mathcal{C}^1(0, T)$  such that

$$\mathcal{S}^{\varepsilon, k, M}(t) := h^{\varepsilon, k, M}(t) + R^{\varepsilon, k, M}(t)\mathcal{S}^0, \quad (3.5)$$

$$u^{\varepsilon, k, M}(t, x) \in V_M \quad (3.6)$$

satisfying the equations

$$\frac{dh^{\varepsilon, k, M}}{dt}(t) = \frac{1}{m} \int_{\mathcal{S}^{\varepsilon, k, M}} \rho_{\mathcal{S}} u^{\varepsilon, k, M}(t, x) \, dx, \quad (3.7)$$

$$\frac{dR^{\varepsilon, k, M}}{dt}(t) = \mathbb{A} \left( J_{\mathcal{S}^{\varepsilon, k, M}}^{-1} \int_{\mathcal{S}^{\varepsilon, k, M}} \rho_{\mathcal{S}} (x - h^{\varepsilon, k, M}) \times u^{\varepsilon, k, M}(t, x) \, dx \right) R^{\varepsilon, k, M}, \quad (3.8)$$

$$h^{\varepsilon, k, M}(0) = 0, \quad R^{\varepsilon, k, M}(0) = I_3 \quad (3.9)$$

and

$$\begin{aligned} \int_{\Omega} \rho^{\varepsilon, k, M} \frac{\partial u^{\varepsilon, k, M}}{\partial t} \cdot v \, dx + \int_{\Omega} \rho^{\varepsilon, k, M} [(Q_{\mathcal{S}^{\varepsilon, k, M}(t)} u^{\varepsilon, k, M} \cdot \nabla) u^{\varepsilon, k, M}] \cdot v \, dx \\ + 2\mu \int_{\mathcal{F}^{\varepsilon, k, M}(t)} D(u^{\varepsilon, k, M}) : D(v) \, dx + \int_{\partial\Omega} g \nabla j_\varepsilon(u^{\varepsilon, k, M}) \cdot v \, d\gamma \\ + \int_{\partial\mathcal{S}^{\varepsilon, k, M}(t)} \nabla j_\varepsilon(u^{\varepsilon, k, M} - P_{\mathcal{S}^{\varepsilon, k, M}(t)} u^{\varepsilon, k, M}) \cdot (v - P_{\mathcal{S}^{\varepsilon, k, M}(t)} v) \, d\gamma \\ + k \int_{\mathcal{S}^{\varepsilon, k, M}(t)} (u^{\varepsilon, k, M} - P_{\mathcal{S}^{\varepsilon, k, M}(t)} u^{\varepsilon, k, M}) \cdot (v - P_{\mathcal{S}^{\varepsilon, k, M}(t)} v) \, dx = 0, \end{aligned} \quad (3.10)$$

for any  $v \in V_M$ , with the initial condition  $u^{\varepsilon, k, M}(0, x)$  being the orthogonal projection of  $u^0$  on  $V_M$ . Here, we have written

$$\rho^{k, \varepsilon, M} := (1 - \mathbb{1}_{\mathcal{S}^{k, \varepsilon, M}}) + \rho_{\mathcal{S}} \mathbb{1}_{\mathcal{S}^{k, \varepsilon, M}}, \quad (3.11)$$

$$\mathcal{F}^{\varepsilon, k, M}(t) := \Omega \setminus \overline{\mathcal{S}^{\varepsilon, k, M}(t)}$$

and

$$J_{\mathcal{S}^{\varepsilon, k, M}} := \rho_{\mathcal{S}} \int_{\mathcal{S}^{\varepsilon, k, M}} (|x - h^{\varepsilon, k, M}(t)|^2 - (x - h^{\varepsilon, k, M}(t)) \otimes (x - h^{\varepsilon, k, M}(t))) \, dx.$$

We have also used the operator

$$Q_{\mathcal{S}^{k, \varepsilon, M}} : L^2(0, T; V_0^1(\Omega)) \rightarrow L^2(0, T; V_0^1(\mathbb{R}^3))$$

defined as follows: for any  $v \in V_0^1(\Omega)$  we extend it to a function in  $V_0^1(\mathbb{R}^3)$ . Then, we consider the change of variables:

$$V(t, \cdot) := \Xi_{h^{\varepsilon,k,M}(t), R^{\varepsilon,k,M}(t)}(v(t, \cdot)),$$

where  $\Xi$  is defined by (2.6). Thus, we can define

$$V^{\delta,k} := \Lambda^{\delta,\delta/k}[V, P_{S^0}V],$$

where  $\Lambda^{\delta_1,\delta_2}$  is defined in Proposition B.1 from Appendix and  $P_{S^0}$  is defined by (2.3). Finally,

$$(Q_{S^{\varepsilon,k,M}v})(t, \cdot) := \widehat{\Xi}_{h^{\varepsilon,k,M}(t), R^{\varepsilon,k,M}(t)}(V^{\delta,k}(t, \cdot)),$$

where  $\widehat{\Xi}$  is defined by (2.7).

Note that

$$Q_{S^{\varepsilon,k,M}v} = v \quad \text{in } \Omega \setminus (\mathcal{S}^{k,\varepsilon,M})^\delta$$

and

$$Q_{S^{\varepsilon,k,M}v} = P_{S^{\varepsilon,k,M}v} \quad \text{in } \mathcal{S}^{k,\varepsilon,M}.$$

From (3.6), we can write

$$u^{\varepsilon,k,M}(t, x) = \sum_{j=1}^M a_j(t) v_j(x),$$

where coefficients  $a_j$  depend on  $\varepsilon, k, M$ . The system (3.5)–(3.10) can be written as follows:

$$\frac{d}{dt}(h^{k,\varepsilon,M}, R^{k,\varepsilon,M}, a) = F(h^{k,\varepsilon,M}, R^{k,\varepsilon,M}, a),$$

with  $F$  a Lipschitz continuous function. Thus, using the Cauchy-Lipschitz Theorem, we deduce the existence and uniqueness of a local solution  $(h^{k,\varepsilon,M}, R^{k,\varepsilon,M}, a)$  and  $u_{k,\varepsilon,M}$  of the system (3.5)–(3.10).

Moreover, we can deduce the following energy equality:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho^{\varepsilon,k,M} |u^{\varepsilon,k,M}(t)|^2 dx + 2\mu \int_0^t \int_{\mathcal{F}^{\varepsilon,k,M}(t)} |D(u^{\varepsilon,k,M})|^2 dx + \int_0^t \int_{\partial\Omega} g \nabla j_{\varepsilon}(u^{\varepsilon,k,M}) \cdot u^{\varepsilon,k,M} d\gamma \\ & + \int_0^t \int_{\partial S^{\varepsilon,k,M}(t)} g \nabla j_{\varepsilon}(u^{\varepsilon,k,M} - P_{S^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}) \cdot (u^{\varepsilon,k,M} - P_{S^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}) d\gamma \\ & + k \int_0^t \int_{S^{\varepsilon,k,M}(t)} |u^{\varepsilon,k,M} - P_{S^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}|^2 dx = \frac{1}{2} \int_{\Omega} \rho^{\varepsilon,k,M}(0) |u^{\varepsilon,k,M}(0)|^2 dx. \end{aligned} \quad (3.12)$$

This identity and (3.2) yield the global existence in time of the solution  $(h^{k,\varepsilon,M}, R^{k,\varepsilon,M}, u_{k,\varepsilon,M})$ .

We can write

$$P_{S^{\varepsilon,k,M}} u^{\varepsilon,k,M} = \ell^{\varepsilon,k,M} + \omega^{\varepsilon,k,M} \times (x - h^{\varepsilon,k,M})$$

and in that case, equations (3.7) and (3.8) write as

$$(h^{\varepsilon,k,M})' = \ell^{\varepsilon,k,M} \quad \text{and} \quad (R^{\varepsilon,k,M})' = \mathbb{A}(\omega^{\varepsilon,k,M}) R^{\varepsilon,k,M}.$$

By definition, we also have

$$Q_{S^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M} = \ell^{\varepsilon,k,M} + \omega^{\varepsilon,k,M} \times (x - h^{\varepsilon,k,M}) \quad \text{in } S^{\varepsilon,k,M}(t),$$

then, from (3.5) and (3.11), we deduce that

$$\frac{\partial \rho^{\varepsilon,k,M}}{\partial t} + Q_{S^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M} \cdot \nabla \rho^{\varepsilon,k,M} = 0. \quad (3.13)$$

Therefore,

$$\begin{aligned} & \int_{\Omega} \rho^{\varepsilon,k,M} \frac{\partial u^{\varepsilon,k,M}}{\partial t} \cdot v dx + \int_{\Omega} \rho^{\varepsilon,k,M} [(Q_{S^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M} \cdot \nabla) u^{\varepsilon,k,M}] \cdot v dx \\ & = \int_{\Omega} \left[ \frac{\partial}{\partial t} + (Q_{S^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M} \cdot \nabla) \right] (\rho^{\varepsilon,k,M} u^{\varepsilon,k,M}) \cdot v dx. \end{aligned}$$

Assume now that  $v \in C_c^1([0, T]; V_M)$ . Using the above relation, we get

$$\begin{aligned} & \int_{\Omega} \rho^{\varepsilon, k, M} \frac{\partial u^{\varepsilon, k, M}}{\partial t} \cdot v \, dx + \int_{\Omega} \rho^{\varepsilon, k, M} [(Q_{\mathcal{S}^{\varepsilon, k, M}(t)} u^{\varepsilon, k, M} \cdot \nabla) u^{\varepsilon, k, M}] \cdot v \, dx \\ &= \frac{d}{dt} \int_{\Omega} \rho^{\varepsilon, k, M} u^{\varepsilon, k, M} \cdot v \, dx - \int_{\Omega} \left[ \frac{\partial}{\partial t} + (Q_{\mathcal{S}^{\varepsilon, k, M}(t)} u^{\varepsilon, k, M} \cdot \nabla) \right] v \cdot (\rho^{\varepsilon, k, M} u^{\varepsilon, k, M}) \, dx. \end{aligned} \quad (3.14)$$

Inserting (3.14) in identity (3.10) and integrating in  $[0, T]$ , we deduce

$$\begin{aligned} & - \int_{\Omega} \rho^0 u^{\varepsilon, k, M}(0) \cdot v(0) \, dx - \int_0^T \int_{\Omega} \left[ \frac{\partial}{\partial t} + (Q_{\mathcal{S}^{\varepsilon, k, M}(t)} u^{\varepsilon, k, M} \cdot \nabla) \right] v \cdot (\rho^{\varepsilon, k, M} u^{\varepsilon, k, M}) \, dx \, dt \\ &+ 2\mu \int_0^T \int_{\mathcal{F}^{\varepsilon, k, M}(t)} D(u^{\varepsilon, k, M}) : D(v) \, dx \, dt + \int_0^T \int_{\partial\Omega} g \nabla j_{\varepsilon}(u^{\varepsilon, k, M}) \cdot v \, d\gamma \, dt \\ &+ \int_0^T \int_{\partial\mathcal{S}^{\varepsilon, k, M}(t)} g \nabla j_{\varepsilon}(u^{\varepsilon, k, M} - P_{\mathcal{S}^{\varepsilon, k, M}(t)} u^{\varepsilon, k, M}) \cdot (v - P_{\mathcal{S}^{\varepsilon, k, M}(t)} v) \, d\gamma \, dt \\ &+ k \int_0^T \int_{\mathcal{S}^{\varepsilon, k, M}(t)} (u^{\varepsilon, k, M} - P_{\mathcal{S}^{\varepsilon, k, M}(t)} u^{\varepsilon, k, M}) \cdot (v - P_{\mathcal{S}^{\varepsilon, k, M}(t)} v) \, dx \, dt = 0. \end{aligned} \quad (3.15)$$

## 4 Passing to the limit in Galerkin method

Our aim is to pass to the limit in (3.15), as  $M \rightarrow \infty$ . This section is similar to Section 4.3 in [18], and for this reason, we only point out the main steps and the differences.

Using (3.12), there exist

$$u^{\varepsilon, k} \in L^\infty(0, T, V_n^0(\Omega)) \cap L^2(0, T, V_n^1(\Omega))$$

and a sequence  $M = (M_j) > 0$  such that

$$u^{\varepsilon, k, M} \rightharpoonup u^{\varepsilon, k} \quad \text{weakly } * \text{ in } L^\infty(0, T, V_n^0(\Omega)), \quad (4.1)$$

$$u^{\varepsilon, k, M} \rightharpoonup u^{\varepsilon, k} \quad \text{weakly in } L^2(0, T, V_n^1(\Omega)). \quad (4.2)$$

In order to simplify, we write

$$u_R^{\varepsilon, k, M} := P_{\mathcal{S}^{\varepsilon, k, M}} u^{\varepsilon, k, M}, \quad (4.3)$$

so that

$$u_R^{\varepsilon, k, M}(t, x) = \ell^{\varepsilon, k, M}(t) + \omega^{\varepsilon, k, M}(t) \times (x - h^{\varepsilon, k, M}(t)).$$

Using again (3.12), we deduce the existence of

$$u_R^{\varepsilon, k} \in L^2(0, T; \mathcal{R})$$

such that

$$u_R^{\varepsilon, k, M} \rightharpoonup u_R^{\varepsilon, k} \quad \text{weakly in } L^2(0, T; \mathcal{R}). \quad (4.4)$$

Since

$$\partial_t \mathbb{1}_{\mathcal{S}^{\varepsilon, k, M}} + \operatorname{div} \left( u_R^{\varepsilon, k, M} \mathbb{1}_{\mathcal{S}^{\varepsilon, k, M}} \right) = 0, \quad \mathbb{1}_{\mathcal{S}^{\varepsilon, k, M}}(0, \cdot) = \mathbb{1}_{\mathcal{S}^0}, \quad (4.5)$$

we deduce from (4.4) and from the compactness result due to DiPerna-Lions (see [6], [25] and also [26]) that

$$\mathbb{1}_{\mathcal{S}^{\varepsilon, k, M}} \rightharpoonup \mathbb{1}_{\mathcal{S}^{\varepsilon, k}} \quad \text{weakly } * \text{ in } L^\infty(0, T, L^\infty(\Omega)), \quad (4.6)$$

$$\mathbb{1}_{\mathcal{S}^{\varepsilon, k, M}} \rightarrow \mathbb{1}_{\mathcal{S}^{\varepsilon, k}} \quad \text{strongly in } C^0([0, T]; L^p(\Omega)), \quad 1 \leq p < \infty, \quad (4.7)$$

where the characteristic function  $\mathbb{1}_{\mathcal{S}^{\varepsilon, k}}$  satisfies the equation

$$\partial_t \mathbb{1}_{\mathcal{S}^{\varepsilon, k}} + \operatorname{div} \left( u_R^{\varepsilon, k} \mathbb{1}_{\mathcal{S}^{\varepsilon, k}} \right) = 0, \quad \mathbb{1}_{\mathcal{S}^{\varepsilon, k}}(0, \cdot) = \mathbb{1}_{\mathcal{S}^0}. \quad (4.8)$$

Moreover, there exist  $h^{\varepsilon, k}$  and  $R^{\varepsilon, k}$  such that

$$\mathcal{S}^{\varepsilon, k}(t) = \mathcal{S}(h^{\varepsilon, k}(t), R^{\varepsilon, k}(t)) \quad \forall t \in [0, T]. \quad (4.9)$$

Combining (4.2), (4.4) and (4.7), we deduce

$$P_{\mathcal{S}^{\varepsilon,k,M}} u^{\varepsilon,k,M} \rightharpoonup P_{\mathcal{S}^{\varepsilon,k}} u^{\varepsilon,k} \quad \text{weakly in } L^2(0, T, \mathcal{R}), \quad (4.10)$$

$$P_{\mathcal{S}^{\varepsilon,k,M}} u^{\varepsilon,k,M} \rightharpoonup P_{\mathcal{S}^{\varepsilon,k}} u^{\varepsilon,k} \quad \text{weakly * in } L^\infty(0, T, \mathcal{R}), \quad (4.11)$$

$$h^{\varepsilon,k,M} \rightharpoonup h^{\varepsilon,k,M} \quad \text{weakly * in } W^{1,\infty}(0, T; \mathbb{R}^3), \quad (4.12)$$

$$R^{\varepsilon,k,M} \rightharpoonup R^{\varepsilon,k,M} \quad \text{weakly * in } W^{1,\infty}(0, T; SO(3)) \quad (4.13)$$

and the following relations

$$\frac{dh^{\varepsilon,k}}{dt}(t) = \frac{1}{m} \int_{\mathcal{S}^{\varepsilon,k}} \rho_{\mathcal{S}} u^{\varepsilon,k}(t, x) dx, \quad (4.14)$$

$$\frac{dR^{\varepsilon,k}}{dt}(t) = \mathbb{A} \left( J_{\mathcal{S}^{\varepsilon,k}}^{-1} \int_{\mathcal{S}^{\varepsilon,k}} \rho_{\mathcal{S}}(x - h^{\varepsilon,k}) \times u^{\varepsilon,k}(t, x) dx \right) R^{\varepsilon,k}, \quad (4.15)$$

$$h^{\varepsilon,k}(0) = 0, \quad R^{\varepsilon,k}(0) = I_3. \quad (4.16)$$

Let us prove now that

$$Q_{\mathcal{S}^{\varepsilon,k,M}} u^{\varepsilon,k,M} \rightharpoonup Q_{\mathcal{S}^{\varepsilon,k}} u^{\varepsilon,k} \quad \text{weakly in } L^2(0, T, V_n^1(\Omega)). \quad (4.17)$$

We follow the construction of the operator  $Q_{\mathcal{S}^{\varepsilon,k,M}}$ : we consider

$$U^{\varepsilon,k,M} = \Xi_{h^{\varepsilon,k,M}, R^{\varepsilon,k,M}}(u^{\varepsilon,k,M}), \quad U^{\varepsilon,k} = \Xi_{h^{\varepsilon,k}, R^{\varepsilon,k}}(u^{\varepsilon,k}).$$

Using (4.2), (4.12), (4.13) and Lemma A.2 in [18], we deduce that

$$U^{\varepsilon,k,M} \rightharpoonup U^{\varepsilon,k} \quad \text{weakly in } L^2(0, T, V_n^1(\mathbb{R}^3)).$$

Thus,

$$\Lambda^{\delta,\delta/k}[U^{\varepsilon,k,M}, P_{\mathcal{S}^0} U^{\varepsilon,k,M}] \rightharpoonup \Lambda^{\delta,\delta/k}[U^{\varepsilon,k}, P_{\mathcal{S}^0} U^{\varepsilon,k}] \quad \text{weakly in } L^2(0, T, V_n^1(\mathbb{R}^3))$$

and finally we obtain (4.17) by using again Lemma A.2 from [18].

Due to convergences (4.6)–(4.7) and identity (3.11), we deduce

$$\rho^{\varepsilon,k,M} \rightharpoonup \rho^{\varepsilon,k} \quad \text{weakly * in } L^\infty(0, T, L^\infty(\Omega)), \quad (4.18)$$

$$\rho^{\varepsilon,k,M} \rightarrow \rho^{\varepsilon,k} \quad \text{strongly in } C^0([0, T]; L^p(\Omega)), \quad 1 \leq p < \infty. \quad (4.19)$$

In particular, we deduce that

$$\rho^{\varepsilon,k,M} u^{\varepsilon,k,M} \rightharpoonup \rho^{\varepsilon,k} u^{\varepsilon,k} \quad \text{weakly * in } L^\infty(0, T, L^2(\Omega)). \quad (4.20)$$

Let us fix  $i \geq 1$ . We write  $\widehat{P} : L^2(\Omega) \rightarrow L_\sigma^2(\Omega)$  and  $\widehat{P}_i : L^2(\Omega) \rightarrow V_i$  the orthogonal projections. From (3.10), we deduce that, for  $M \geq i$ ,

$$\frac{\partial}{\partial t} \widehat{P}_i(\rho^{\varepsilon,k,M} u^{\varepsilon,k,M}) + \widehat{P}_i F^{\varepsilon,k,M} = 0 \quad \text{in } \mathcal{D}'(0, T; [H_\sigma^1(\Omega)]^*), \quad (4.21)$$

where for  $v \in \mathcal{D}(0, T; H_\sigma^1(\Omega))$ , we use the notation

$$\begin{aligned} \langle F^{\varepsilon,k,M}, v \rangle &:= - \int_0^T \int_\Omega \rho^{\varepsilon,k,M} [(Q_{\mathcal{S}^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M} \cdot \nabla)v] \cdot u^{\varepsilon,k,M} dx dt \\ &+ 2\mu \int_0^T \int_{\mathcal{F}^{\varepsilon,k,M}(t)} D(u^{\varepsilon,k,M}) : D(v) dx dt + \int_0^T \int_{\partial\Omega} g \nabla j_\varepsilon(u^{\varepsilon,k,M}) \cdot v d\gamma dt \\ &+ \int_0^T \int_{\partial\mathcal{S}^{\varepsilon,k,M}(t)} g \nabla j_\varepsilon(u^{\varepsilon,k,M} - P_{\mathcal{S}^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}) \cdot (v - P_{\mathcal{S}^{\varepsilon,k,M}(t)} v) d\gamma dt \\ &+ k \int_0^T \int_{\mathcal{S}^{\varepsilon,k,M}(t)} (u^{\varepsilon,k,M} - P_{\mathcal{S}^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}) \cdot (v - P_{\mathcal{S}^{\varepsilon,k,M}(t)} v) dx dt. \end{aligned} \quad (4.22)$$

One can show that  $(\widehat{P}_i F^{\varepsilon,k,M})_M$  is bounded in  $L^2(0, T; [H_\sigma^1(\Omega)]^*)$  (with a bound that may depend on  $\varepsilon$  and on  $k$ ). In this step, we use the property (3.3) and the trace theorem for the boundaries terms. The other terms can be estimated in a standard way. Using the Aubin-Lions compactness result (and (4.20)), we deduce

$$\widehat{P}_i(\rho^{\varepsilon,k,M} u^{\varepsilon,k,M}) \rightarrow \widehat{P}_i(\rho^{\varepsilon,k} u^{\varepsilon,k}) \quad \text{strongly in } L^2(0, T; [H_\sigma^1(\Omega)]^*),$$

as  $M \rightarrow \infty$ . Using the uniform bound (in  $M$ ) of  $(\rho^{\varepsilon,k,M} u^{\varepsilon,k,M})$ , we deduce that

$$\widehat{P}(\rho^{\varepsilon,k,M} u^{\varepsilon,k,M}) \rightarrow \widehat{P}(\rho^{\varepsilon,k} u^{\varepsilon,k}) \quad \text{strongly in } L^2(0, T; [H_\sigma^1(\Omega)]^*).$$

Then, using the same method as in [25, p.47], we deduce that

$$\sqrt{\rho^{\varepsilon,k,M}} u^{\varepsilon,k,M} \rightarrow \sqrt{\rho^{\varepsilon,k}} u^{\varepsilon,k} \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \quad (4.23)$$

From (3.11), we have

$$\frac{1}{\sqrt{\rho^{\varepsilon,k,M}}} = (1 - \mathbb{1}_{\mathcal{S}^{k,\varepsilon,M}}) + \frac{1}{\sqrt{\rho_S}} \mathbb{1}_{\mathcal{S}^{k,\varepsilon,M}},$$

then due to convergence (4.7), we obtain

$$\frac{1}{\sqrt{\rho^{\varepsilon,k,M}}} \rightarrow \frac{1}{\sqrt{\rho^{\varepsilon,k}}} \quad \text{strongly in } \mathcal{C}^0([0, T]; L^3(\Omega)). \quad (4.24)$$

Combining the above convergence and (4.23), we deduce

$$u^{\varepsilon,k,M} \rightarrow u^{\varepsilon,k} \quad \text{strongly in } L^2(0, T; L^{6/5}(\Omega)).$$

The above convergence and the fact that  $(u^{\varepsilon,k,M})_M$  is bounded in  $L^2(0, T; L^6(\Omega))$ , yield that

$$u^{\varepsilon,k,M} \rightarrow u^{\varepsilon,k} \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

The above convergence and the boundedness of  $(u^{\varepsilon,k,M})_M$  in  $L^2(0, T; H^1(\Omega))$  imply that

$$u^{\varepsilon,k,M} \rightarrow u^{\varepsilon,k} \quad \text{strongly in } L^2(0, T; V_n^{3/4}(\Omega)). \quad (4.25)$$

We are now in position to pass to the limit in the boundary terms in (3.15). In order to deal with the term

$$\int_{\partial \mathcal{S}^{\varepsilon,k,M}(t)} \nabla j_\varepsilon(u^{\varepsilon,k,M} - P_{\mathcal{S}^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}) \cdot (v - P_{\mathcal{S}^{\varepsilon,k,M}(t)} v) \, d\gamma,$$

we make a change of variables for writing the integral on a fixed domain: we extend  $u^{\varepsilon,k,M}$  and  $u^{\varepsilon,k}$  by 0 at the exterior of  $\Omega$  and we write

$$\begin{aligned} U^{\varepsilon,k,M} &:= \Xi_{h^{\varepsilon,k,M}, R^{\varepsilon,k,M}}(u^{\varepsilon,k,M}), & U^{\varepsilon,k} &:= \Xi_{h^{\varepsilon,k}, R^{\varepsilon,k}}(u^{\varepsilon,k}), \\ U_R^{\varepsilon,k,M} &:= \Xi_{h^{\varepsilon,k,M}, R^{\varepsilon,k,M}}(P_{\mathcal{S}^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}), & U_R^{\varepsilon,k} &:= \Xi_{h^{\varepsilon,k}, R^{\varepsilon,k}}(P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k}), \\ V^{\varepsilon,k,M} &:= \Xi_{h^{\varepsilon,k,M}, R^{\varepsilon,k,M}}(v), & V^{\varepsilon,k} &:= \Xi_{h^{\varepsilon,k}, R^{\varepsilon,k}}(v), \\ V_R^{\varepsilon,k,M} &:= \Xi_{h^{\varepsilon,k,M}, R^{\varepsilon,k,M}}(P_{\mathcal{S}^{\varepsilon,k,M}(t)} v), & V_R^{\varepsilon,k} &:= \Xi_{h^{\varepsilon,k}, R^{\varepsilon,k}}(P_{\mathcal{S}^{\varepsilon,k}(t)} v). \end{aligned}$$

Then, we get

$$\begin{aligned} &\int_{\partial \mathcal{S}^{\varepsilon,k,M}(t)} \nabla j_\varepsilon(u^{\varepsilon,k,M} - P_{\mathcal{S}^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}) \cdot (v - P_{\mathcal{S}^{\varepsilon,k,M}(t)} v) \, d\gamma \\ &= \int_{\partial \mathcal{S}^0} \nabla j_\varepsilon(R^{\varepsilon,k,M} U^{\varepsilon,k,M} - R^{\varepsilon,k,M} U_R^{\varepsilon,k,M}) \cdot [R^{\varepsilon,k,M} V^{\varepsilon,k,M} - R^{\varepsilon,k,M} V_R^{\varepsilon,k,M}] \, d\gamma. \end{aligned}$$

Applying Lemma A.2 from [18], we obtain

$$U^{\varepsilon,k,M} \rightarrow U^{\varepsilon,k} \quad \text{strongly in } L^2(0, T; H_{loc}^{3/4}(\mathbb{R}^3)), \quad (4.26)$$

$$U_R^{\varepsilon,k,M} \rightarrow U_R^{\varepsilon,k} \quad \text{strongly in } L^2(0, T; H_{loc}^1(\mathbb{R}^3)), \quad (4.27)$$

$$V^{\varepsilon,k,M} \rightarrow V^{\varepsilon,k} \quad \text{strongly in } L^2(0, T; H_{loc}^1(\mathbb{R}^3)), \quad (4.28)$$

$$V_R^{\varepsilon,k,M} \rightarrow V_R^{\varepsilon,k} \quad \text{strongly in } L^2(0, T; H_{loc}^1(\mathbb{R}^3)). \quad (4.29)$$

Using (4.13) and that  $\nabla j_\varepsilon$  is a bounded continuous function, we deduce

$$\begin{aligned} \int_0^T \int_{\partial S^{\varepsilon,k,M}(t)} \nabla j_\varepsilon(u^{\varepsilon,k,M} - P_{S^{\varepsilon,k,M}(t)} u^{\varepsilon,k,M}) \cdot (v - P_{S^{\varepsilon,k,M}(t)} v) \, d\gamma \, dt \\ \rightarrow \int_0^T \int_{\partial S^{\varepsilon,k}(t)} \nabla j_\varepsilon(u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v - P_{S^{\varepsilon,k}(t)} v) \, d\gamma \, dt. \end{aligned}$$

Using similar arguments and standard techniques for the Navier-Stokes system with Dirichlet boundary conditions, one can pass to the limit as  $M \rightarrow \infty$  in all the other terms in (3.15). We thus obtain that  $(h^{\varepsilon,k}, R^{\varepsilon,k}, u^{\varepsilon,k})$  satisfies the following identity, for any  $v \in \mathcal{C}_c^1([0, T]; V_n^1(\Omega))$ :

$$\begin{aligned} - \int_\Omega \rho^0 u^0 \cdot v(0) \, dx - \int_0^T \int_\Omega \left[ \frac{\partial}{\partial t} + (Q_{S^{\varepsilon,k}(t)} u^{\varepsilon,k} \cdot \nabla) \right] v \cdot (\rho^{\varepsilon,k} u^{\varepsilon,k}) \, dx \, dt \\ + 2\mu \int_0^T \int_{\mathcal{F}^{\varepsilon,k}(t)} D(u^{\varepsilon,k}) : D(v) \, dx \, dt + \int_0^T \int_{\partial\Omega} g \nabla j_\varepsilon(u^{\varepsilon,k}) \cdot v \, d\gamma \, dt \\ + \int_0^T \int_{\partial S^{\varepsilon,k}(t)} g \nabla j_\varepsilon(u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v - P_{S^{\varepsilon,k}(t)} v) \, d\gamma \, dt \\ + k \int_0^T \int_{S^{\varepsilon,k}(t)} (u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v - P_{S^{\varepsilon,k}(t)} v) \, dx \, dt = 0. \quad (4.30) \end{aligned}$$

By a density argument, such a relation holds true for any  $v \in W^{1,\infty}(0, T; V_n^0(\Omega)) \cap \mathcal{C}_c^0([0, T]; V_n^1(\Omega))$ .

Since

$$\mathbb{1}_{\mathcal{F}^{\varepsilon,k,M}} D(u^{\varepsilon,k,M}) \rightharpoonup \mathbb{1}_{\mathcal{F}^{\varepsilon,k}} D(u^{\varepsilon,k}) \quad \text{weakly in } L^2(0, T; L^2(\Omega)),$$

we deduce

$$2\mu \int_0^T \int_{\mathcal{F}^{\varepsilon,k}(t)} |D(u^{\varepsilon,k})|^2 \, dx \, dt \leq \liminf 2\mu \int_0^T \int_{\mathcal{F}^{\varepsilon,k,M}(t)} |D(u^{\varepsilon,k,M})|^2 \, dx \, dt. \quad (4.31)$$

Using standard techniques (see, for instance, [32, pp.290-291] and similar arguments as the one above to deal with the boundary terms, we deduce from (3.12) the following energy estimate:

$$\begin{aligned} \frac{1}{2} \int_\Omega \rho^{\varepsilon,k}(t, \cdot) |u^{\varepsilon,k}(t, \cdot)|^2 \, dx + 2\mu \int_0^t \int_{\mathcal{F}^{\varepsilon,k}(t)} |D(u^{\varepsilon,k})|^2 \, dx \, dt + \int_0^t \int_{\partial\Omega} g \nabla j_\varepsilon(u^{\varepsilon,k}) \cdot u^{\varepsilon,k} \, d\gamma \, dt \\ + \int_0^t \int_{\partial S^{\varepsilon,k}(t)} g \nabla j_\varepsilon(u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \, d\gamma \, dt \\ + k \int_0^t \int_{S^{\varepsilon,k}(t)} |u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}|^2 \, dx \, dt \leq \frac{1}{2} \int_\Omega \rho^0 |u^0|^2 \, dx, \quad (4.32) \end{aligned}$$

for almost every  $t \in (0, T)$ .

## 5 Proof of Theorem 2.2

In this section, we pass to the limit in  $k$  and  $\varepsilon$  in the relations of the previous sections.

### 5.1 First convergences

In order to deal only with the variable  $k$ , let us take

$$\varepsilon := \frac{1}{k}.$$

This precise relation is not used below, we only need that  $\varepsilon$  goes to 0, as  $k$  goes to  $\infty$ . In all what follows, our convergences correspond to subsequences  $(k_j)_j$ , with  $\lim_j k_j = \infty$ , whereas  $\varepsilon_{k_j}$  is obtained from  $k_j$  through the above relation.

For any  $(h, R)$  such that  $\text{dist}(\mathcal{S}(h, R), \partial\Omega) \geq 2\delta$ , we can construct an extension operator

$$E_{\mathcal{F}(h,R)} : V^1(\mathcal{F}(h, R)) \rightarrow V_n^1(\Omega) \quad (5.1)$$

such that

$$\|E_{\mathcal{F}(h,R)}u\|_{H^1(\Omega)}^2 \leq C_\delta \left[ \int_{\mathcal{F}(h,R)} |D(u)|^2 dx + \int_{\mathcal{F}(h,R)} |u|^2 dx \right], \quad (5.2)$$

with  $C_\delta$  independent of  $(h, R)$ .

Taking  $\delta$  such that

$$0 < \delta < \frac{1}{4} \text{dist}(\mathcal{S}^0, \partial\Omega), \quad (5.3)$$

we deduce from (4.14)–(4.16) and from (4.32) that there exists a time  $T > 0$ , uniform in  $\varepsilon$  and  $k$ , such that

$$\text{dist}(\mathcal{S}^{\varepsilon,k}(t), \partial\Omega) \geq 2\delta \quad (t \in [0, T]). \quad (5.4)$$

We write

$$u_F^{\varepsilon,k} := E_{\mathcal{F}(h^{\varepsilon,k}, R^{\varepsilon,k})} u^{\varepsilon,k}.$$

We deduce from (5.2) and (4.32) that, up to a subsequence,

$$u^{\varepsilon,k} \rightharpoonup u \quad \text{weakly } * \text{ in } L^\infty(0, T, V_n^0(\Omega)), \quad (5.5)$$

$$u_F^{\varepsilon,k} \rightharpoonup u_F \quad \text{weakly in } L^2(0, T, V_n^1(\Omega)). \quad (5.6)$$

We also deduce that  $u_R^{\varepsilon,k} = P_{\mathcal{S}^{\varepsilon,k}} u^{\varepsilon,k}$  is bounded in  $L^2(0, T; \mathcal{R})$ , so that

$$u_R^{\varepsilon,k} \rightharpoonup u_R \quad \text{weakly in } L^2(0, T, H^1(\Omega)). \quad (5.7)$$

Thus, we deduce

$$\mathbb{1}_{\mathcal{S}^{\varepsilon,k}} \rightharpoonup \mathbb{1}_{\mathcal{S}} \quad \text{weakly } * \text{ in } L^\infty(0, T, L^\infty(\Omega)), \quad (5.8)$$

$$\mathbb{1}_{\mathcal{S}^{\varepsilon,k}} \rightarrow \mathbb{1}_{\mathcal{S}} \quad \text{strongly in } \mathcal{C}^0([0, T]; L^p(\Omega)), \quad 1 \leq p < \infty, \quad (5.9)$$

where the characteristic function  $\mathbb{1}_{\mathcal{S}}$  satisfies the equation

$$\partial_t \mathbb{1}_{\mathcal{S}} + \text{div}(u_R \mathbb{1}_{\mathcal{S}}) = 0, \quad \mathbb{1}_{\mathcal{S}}(0, \cdot) = \mathbb{1}_{\mathcal{S}^0}. \quad (5.10)$$

Moreover, there exist  $h$  and  $R$  such that

$$\mathcal{S}(t) = \mathcal{S}(h(t), R(t)) \quad \forall t \in [0, T], \quad (5.11)$$

$$h^{\varepsilon,k} \rightharpoonup h \quad \text{weakly } * \text{ in } W^{1,\infty}(0, T), \quad (5.12)$$

$$R^{\varepsilon,k} \rightharpoonup R \quad \text{weakly } * \text{ in } W^{1,\infty}(0, T) \quad (5.13)$$

and

$$\frac{dh}{dt}(t) = \frac{1}{m} \int_{\mathcal{S}} \rho_{\mathcal{S}} u(t, x) dx, \quad (5.14)$$

$$\frac{dR}{dt}(t) = \mathbb{A} \left( J_{\mathcal{S}}^{-1} \int_{\mathcal{S}} \rho_{\mathcal{S}}(x - h) \times u(t, x) dx \right) R, \quad (5.15)$$

$$h(0) = 0, \quad R(0) = I_3. \quad (5.16)$$

We can also check that

$$u_R = P_{\mathcal{S}} u,$$

then (5.7) writes

$$P_{\mathcal{S}^{\varepsilon,k}} u^{\varepsilon,k} \rightharpoonup P_{\mathcal{S}} u \quad \text{weakly in } L^2(0, T, H^1(\Omega)). \quad (5.17)$$

From (4.32), we also obtain

$$\|\mathbb{1}_{\mathcal{S}^{\varepsilon,k}}(u^{\varepsilon,k} - P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k})\|_{L^2((0,T) \times \Omega)} \leq \frac{C}{k^{1/2}} \quad (5.18)$$

and thus, using the decomposition

$$u^{\varepsilon,k} = (1 - \mathbb{1}_{\mathcal{S}^{\varepsilon,k}})u^{\varepsilon,k} + \mathbb{1}_{\mathcal{S}^{\varepsilon,k}}(u^{\varepsilon,k} - P_{\mathcal{S}^{\varepsilon,k}}u^{\varepsilon,k}) + \mathbb{1}_{\mathcal{S}^{\varepsilon,k}}(P_{\mathcal{S}^{\varepsilon,k}}u^{\varepsilon,k}), \quad (5.19)$$

together with (5.6), (5.7), (5.9) and (5.18), we get

$$u = (1 - \mathbb{1}_{\mathcal{S}})u_F + \mathbb{1}_{\mathcal{S}}u_R. \quad (5.20)$$

We now define

$$V_R^s(\Omega, \mathcal{S}) := \{v \in V_n^s(\Omega) ; D(v) = 0 \text{ in } \mathcal{S}\} \quad (5.21)$$

and we consider the orthogonal projection (in  $V_n^s(\Omega)$ )

$$P^s[\mathcal{S}] : V_n^s(\Omega) \rightarrow V_R^s(\Omega, \mathcal{S}).$$

Following the arguments of Proposition 7.1 from [26] (see also Section 5.5 of [18]), we can obtain that for  $s \in (0, 1/3)$ , there exists  $\beta_0 > 0$  such that for any  $\beta \in (0, \beta_0)$ ,

$$\lim_{k \rightarrow \infty} \int_0^T \int_{\Omega} \rho^{\varepsilon,k} u^{\varepsilon,k} \cdot P^s[(\mathcal{S}^{\varepsilon,k}(t))_{\beta}] u^{\varepsilon,k} \, dx \, dt = \int_0^T \int_{\Omega} \rho u \cdot P^s[(\mathcal{S}(t))_{\beta}] u \, dx \, dt. \quad (5.22)$$

The idea of this result is that for a small time interval,  $\mathcal{S}^{\varepsilon,k}(t) \subset (\mathcal{S}(t))_{\beta}$ , so that for any test function in  $V_R^s(\Omega, \mathcal{S}(t))$ , the penalization term in (4.30) disappears and since  $j_{\varepsilon}$  satisfies (3.3), we can apply the Aubin-Lions compactness result.

Then, using Lemma 5.3 of [18], we can deduce that

$$\lim_{k \rightarrow \infty} \int_0^T \int_{\Omega} \rho^{\varepsilon,k} |u^{\varepsilon,k}|^2 \, dx \, dt = \int_0^T \int_{\Omega} \rho |u|^2 \, dx \, dt, \quad (5.23)$$

and thus, we finally obtain

$$u^{\varepsilon,k} \rightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (5.24)$$

We deduce from (5.24) that

$$u^{\varepsilon,k}|_{(\partial\Omega)^{\delta}} \rightarrow u|_{(\partial\Omega)^{\delta}} \text{ strongly in } L^2(0, T; L^2((\partial\Omega)^{\delta})), \quad (5.25)$$

where  $(\partial\Omega)^{\delta}$  has been defined in (2.10). Moreover, since  $\text{dist}(\mathcal{S}^{\varepsilon,k}, \partial\Omega) > \delta$ , the energy inequality (4.32) yields that the sequence  $(u^{\varepsilon,k}|_{(\partial\Omega)^{\delta}})$  is bounded in  $L^2(0, T; H^1((\partial\Omega)^{\delta}))$ . Thus,

$$u^{\varepsilon,k}|_{(\partial\Omega)^{\delta}} \rightarrow (u_F)|_{(\partial\Omega)^{\delta}} \text{ strongly in } L^2(0, T; H^s((\partial\Omega)^{\delta})), \quad s < 1 \quad (5.26)$$

and, in particular,

$$u^{\varepsilon,k} \rightarrow u_F \text{ strongly in } L^2(0, T; L^2(\partial\Omega)). \quad (5.27)$$

Let us write

$$U := \Xi_{h,R}(u), \quad U^{\varepsilon,k} := \Xi_{h^{\varepsilon,k}, R^{\varepsilon,k}}(u^{\varepsilon,k}). \quad (5.28)$$

We deduce from (5.24) and from the continuity of translations in  $L^2$  that

$$U^{\varepsilon,k}|_{(\mathcal{S}^0)^{\delta} \setminus \overline{\mathcal{S}^0}} \rightarrow U|_{(\mathcal{S}^0)^{\delta} \setminus \overline{\mathcal{S}^0}} \text{ strongly in } L^2(0, T; L^2((\mathcal{S}^0)^{\delta} \setminus \overline{\mathcal{S}^0})). \quad (5.29)$$

Moreover, since  $\text{dist}(\mathcal{S}^{\varepsilon,k}, \partial\Omega) > \delta$ , the energy inequality (4.32) and a standard calculation yield that the sequence  $(U^{\varepsilon,k}|_{(\mathcal{S}^0)^{\delta} \setminus \overline{\mathcal{S}^0}})$  is bounded in  $L^2(0, T; H^1((\mathcal{S}^0)^{\delta} \setminus \overline{\mathcal{S}^0}))$ . Thus,

$$R^{\varepsilon,k} U^{\varepsilon,k}|_{(\mathcal{S}^0)^{\delta} \setminus \overline{\mathcal{S}^0}} \rightarrow RU|_{(\mathcal{S}^0)^{\delta} \setminus \overline{\mathcal{S}^0}} \text{ strongly in } L^2(0, T; H^s((\mathcal{S}^0)^{\delta} \setminus \overline{\mathcal{S}^0})), \quad s < 1 \quad (5.30)$$

and, in particular,

$$R^{\varepsilon,k} U^{\varepsilon,k} \rightarrow RU \text{ strongly in } L^2(0, T; L^2(\partial\mathcal{S}^0)). \quad (5.31)$$

We recall that

$$(P_{\mathcal{S}^{\varepsilon,k,M}} u^{\varepsilon,k,M})(t, x) = \ell^{\varepsilon,k,M}(t) + \omega^{\varepsilon,k,M}(t) \times (x - h^{\varepsilon,k,M}(t)) \quad (5.32)$$

and

$$(P_{\mathcal{S}} u)(t, x) = \ell + \omega \times (x - h), \quad (5.33)$$

with

$$\varrho^{\varepsilon,k,M}(t) = \frac{1}{m} \int_{\mathcal{S}^{\varepsilon,k,M}} \rho_{\mathcal{S}} u^{\varepsilon,k,M}(t,x) dx, \quad (5.34)$$

$$\omega^{\varepsilon,k,M}(t) = J_{\mathcal{S}^{\varepsilon,k,M}}(t)^{-1} \int_{\mathcal{S}^{\varepsilon,k,M}} \rho_{\mathcal{S}}(x - h^{\varepsilon,k,M}(t)) \times u^{\varepsilon,k,M}(t,x) dx, \quad (5.35)$$

$$\ell(t) = \frac{1}{m} \int_{\mathcal{S}} \rho_{\mathcal{S}} u dx, \quad (5.36)$$

$$\omega(t) = J(t)^{-1} \int_{\mathcal{S}} \rho_{\mathcal{S}}(x - h(t)) \times u(t,x) dx. \quad (5.37)$$

From formula (1.10) and convergences (5.12)–(5.13), we deduce that

$$J_{\mathcal{S}^{\varepsilon,k,M}}^{-1} \rightarrow J^{-1} \quad \text{strongly in } L^\infty(0,T).$$

Using the above convergence, (5.34)–(5.37), (5.9) and (5.24), we deduce

$$P_{\mathcal{S}^{\varepsilon,k,M}} u^{\varepsilon,k,M} \rightarrow P_{\mathcal{S}} u \quad \text{strongly in } L^2(0,T; \mathcal{R}). \quad (5.38)$$

## 5.2 Approximations of the test functions

We prove here a result concerning approximations of the functions in  $\mathcal{T}_T$ . A similar result was considered in [18].

**Proposition 5.1.** *Let us fix  $\alpha > 3/2$  and  $\eta > 0$ . Assume  $v \in \mathcal{T}_T$  (see (2.8)) and that  $v_F$  has its support in  $\Omega_\eta$ . Let us assume that, for all  $\varepsilon$  and  $k$  large enough,*

$$\|h - h^{\varepsilon,k}\|_{C^0([0,T]; \mathbb{R}^3)} + \|R - R^{\varepsilon,k}\|_{C^0([0,T]; SO(3))} \leq \frac{\eta}{2}. \quad (5.39)$$

Let us write

$$v_F^{\varepsilon,k}(t,x) := \widehat{\Xi}_{h^{\varepsilon,k}, R^{\varepsilon,k}} \circ \Xi_{h,R}(v_F), \quad v_R^{\varepsilon,k}(t,x) := \widehat{\Xi}_{h^{\varepsilon,k}, R^{\varepsilon,k}} \circ \Xi_{h,R}(v_R).$$

Then, there exists a sequence

$$v^{\varepsilon,k} \in W^{1,\infty}(0,T; V_n^0(\Omega)) \cap C_c^0([0,T]; V_n^1(\Omega))$$

with the following properties

$$v^{\varepsilon,k} \rightarrow v \quad \text{strongly in } C^0([0,T]; L^4(\Omega)), \quad (5.40)$$

$$\mathbb{1}_{\mathcal{F}^{\varepsilon,k}} D(v^{\varepsilon,k}) \rightarrow \mathbb{1}_{\mathcal{F}} D(v_F) \quad \text{strongly in } C^0([0,T]; L^2(\Omega)), \quad (5.41)$$

$$\|v^{\varepsilon,k} - v_R^{\varepsilon,k}\|_{C^0([0,T]; L^2(\mathcal{S}^{\varepsilon,k}))} \leq C_\delta k^{-\alpha/3} \left( \|v_F\|_{C^0([0,T]; H^1(\Omega))} + \|v_R\|_{C^0([0,T]; \mathcal{R})} \right), \quad (5.42)$$

$$\|v^{\varepsilon,k}\|_{L^\infty(0,T; V(\Omega))} = O(k^{2\alpha/3}), \quad (5.43)$$

$$v^{\varepsilon,k} \rightarrow v \quad \text{strongly in } L^2(0,T; L^2(\partial\Omega)), \quad (5.44)$$

$$v^{\varepsilon,k} = v_F^{\varepsilon,k} \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S}^{\varepsilon,k}(t), \quad (5.45)$$

$$\mathbb{1}_{\mathcal{S}^{\varepsilon,k}} \left[ \frac{\partial}{\partial t} + (P_{\mathcal{S}^{\varepsilon,k}}(t) u^{\varepsilon,k} \cdot \nabla) \right] v^{\varepsilon,k} \rightarrow \mathbb{1}_{\mathcal{S}} \left[ \frac{\partial}{\partial t} + (P_{\mathcal{S}}(t) u \cdot \nabla) \right] v_R \quad \text{strongly in } L^2(0,T; L^2(\Omega)), \quad (5.46)$$

$$\mathbb{1}_{\mathcal{F}^{\varepsilon,k}} \frac{\partial v^{\varepsilon,k}}{\partial t} \rightarrow \mathbb{1}_{\mathcal{F}} \frac{\partial v_F}{\partial t} \quad \text{strongly in } L^2(0,T; L^2(\Omega)). \quad (5.47)$$

*Proof.* We extend  $v$  as a function of  $C^0([0,T]; V_n^0(\mathbb{R}^3))$  and we use the change of variables

$$V := \Xi_{h,R}(v), \quad V_F := \Xi_{h,R}(v_F), \quad V_R := \Xi_{h,R}(v_R). \quad (5.48)$$

Then, we can define

$$V^k := \widetilde{\Lambda}^{\delta, \delta/k^\alpha} [V_F, V_R],$$

where  $\widetilde{\Lambda}^{\delta_1, \delta_2}$  is defined in Proposition B.2 from Appendix. We then use the change of variables

$$v^{\varepsilon,k} := \widehat{\Xi}_{h^{\varepsilon,k}, R^{\varepsilon,k}}(V^k), \quad v_F^{\varepsilon,k}(t,x) := \widehat{\Xi}_{h^{\varepsilon,k}, R^{\varepsilon,k}}(V_F), \quad v_R^{\varepsilon,k}(t,x) := \widehat{\Xi}_{h^{\varepsilon,k}, R^{\varepsilon,k}}(V_R). \quad (5.49)$$

From the properties of  $\tilde{\Lambda}^{\delta, \delta/k^\alpha}$ , we have

$$\begin{aligned} v^{\varepsilon, k} &= v_R^{\varepsilon, k} && \text{in } (\mathcal{S}^{\varepsilon, k}(t))_\delta, \\ v^{\varepsilon, k} &= v_F^{\varepsilon, k} && \text{in } \mathbb{R}^3 \setminus \mathcal{S}^{\varepsilon, k}(t) \end{aligned}$$

and

$$\begin{aligned} \left\| v^{\varepsilon, k} - v_R^{\varepsilon, k} \right\|_{\mathcal{C}^0([0, T]; L^p(\mathcal{S}^{\varepsilon, k}))} &\leq C_{\delta, p} k^{-\alpha(1/p-1/6)} \left( \|v_F\|_{\mathcal{C}^0([0, T]; H^1(\Omega))} + \|v_R\|_{\mathcal{C}^0([0, T]; \mathcal{R})} \right) \quad \forall p \leq 4, \\ \left\| v^{\varepsilon, k} - v_R^{\varepsilon, k} \right\|_{\mathcal{C}^0([0, T]; H^1(\mathcal{S}^{\varepsilon, k}))} &\leq C_{\delta, p} k^{2\alpha/3} \left( \|v_F\|_{\mathcal{C}^0([0, T]; H^1(\Omega))} + \|v_R\|_{\mathcal{C}^0([0, T]; \mathcal{R})} \right). \end{aligned}$$

Here we have used that  $(v_R^{\varepsilon, k})_n = (v_F^{\varepsilon, k})_n$ .

Using the continuity of translations in  $L^4$  and convergences (5.12)–(5.13), we deduce that

$$v_F^{\varepsilon, k} \rightarrow v_F \quad \text{strongly in } \mathcal{C}^0([0, T]; L^4(\Omega)), \quad v_R^{\varepsilon, k} \rightarrow v_R \quad \text{strongly in } \mathcal{C}^0([0, T]; L^4(\Omega)). \quad (5.50)$$

Using this property and the estimates on  $v^{\varepsilon, k} - v_R^{\varepsilon, k}$ , we deduce (5.40).

Note that from (5.48), (5.49) and (5.39), the support of  $v_F^{\varepsilon, k}$  is included in  $\Omega_{\eta/2}$ , so that  $v^{\varepsilon, k}(t, \cdot) \in V_n^1(\Omega)$ .

Using again the continuity of translations in  $L^p$ , we deduce (5.41). In particular, in a neighborhood  $(\partial\Omega)^\delta$  of  $\partial\Omega$ ,

$$v^{\varepsilon, k} \rightarrow v \quad \text{strongly in } L^2(0, T; H^1((\partial\Omega)^\delta))$$

and thus we obtain (5.44).

From (5.49), we deduce

$$\widehat{\Xi}_{h^{\varepsilon, k}, R^{\varepsilon, k}} \frac{\partial V^k}{\partial t} = \frac{\partial v^{\varepsilon, k}}{\partial t} - \omega^{\varepsilon, k} \times v^{\varepsilon, k} + (P_{\mathcal{S}^{\varepsilon, k}} u^{\varepsilon, k} \cdot \nabla) v^{\varepsilon, k}, \quad (5.51)$$

$$\widehat{\Xi}_{h^{\varepsilon, k}, R^{\varepsilon, k}} \frac{\partial V_F}{\partial t} = \frac{\partial v_F^{\varepsilon, k}}{\partial t} - \omega^{\varepsilon, k} \times v_F^{\varepsilon, k} + (P_{\mathcal{S}^{\varepsilon, k}} u^{\varepsilon, k} \cdot \nabla) v_F^{\varepsilon, k}, \quad (5.52)$$

$$\widehat{\Xi}_{h^{\varepsilon, k}, R^{\varepsilon, k}} \frac{\partial V_R}{\partial t} = \frac{\partial v_R^{\varepsilon, k}}{\partial t} - \omega^{\varepsilon, k} \times v_R^{\varepsilon, k} + (P_{\mathcal{S}^{\varepsilon, k}} u^{\varepsilon, k} \cdot \nabla) v_R^{\varepsilon, k}. \quad (5.53)$$

Combining the identity

$$\frac{\partial V^k}{\partial t} = \tilde{\Lambda}^{\delta, k^\alpha} \left[ \frac{\partial V_F}{\partial t}, \frac{\partial V_R}{\partial t} \right]$$

with (B.14), we deduce that

$$\mathbb{1}_{\mathcal{S}^{\varepsilon, k}} \left\{ \left( \frac{\partial}{\partial t} + P_{\mathcal{S}^{\varepsilon, k}} u^{\varepsilon, k} \cdot \nabla \right) (v^{\varepsilon, k} - v_R^{\varepsilon, k}) - \omega^{\varepsilon, k} \times (v^{\varepsilon, k} - v_R^{\varepsilon, k}) \right\} \rightarrow 0 \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

Since  $(\omega^{\varepsilon, k})$  is bounded, (5.50) and the above relation yield to the following convergence:

$$\mathbb{1}_{\mathcal{S}^{\varepsilon, k}} \left\{ \left( \frac{\partial}{\partial t} + P_{\mathcal{S}^{\varepsilon, k}} u^{\varepsilon, k} \cdot \nabla \right) (v^{\varepsilon, k} - v_R^{\varepsilon, k}) \right\} \rightarrow 0 \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \quad (5.54)$$

From (5.48), we also have

$$\widehat{\Xi}_{h, R} \frac{\partial V_R}{\partial t} = \frac{\partial v_R}{\partial t} - \omega \times v_R + (P_S u \cdot \nabla) v_R,$$

so that, using Lemma A.1 from [18],

$$\frac{\partial v_R^{\varepsilon, k}}{\partial t} - \omega^{\varepsilon, k} \times v_R^{\varepsilon, k} + (P_{\mathcal{S}^{\varepsilon, k}} u^{\varepsilon, k} \cdot \nabla) v_R^{\varepsilon, k} \rightarrow \frac{\partial v_R}{\partial t} - \omega \times v_R + (P_S u \cdot \nabla) v_R \quad \text{strongly in } \mathcal{C}^0([0, T]; H^1(\Omega)).$$

Combining the above limit with (5.50), (5.38) and (5.9), we deduce

$$\mathbb{1}_{\mathcal{S}^{\varepsilon, k}} \left\{ \frac{\partial v_R^{\varepsilon, k}}{\partial t} + (P_{\mathcal{S}^{\varepsilon, k}} u^{\varepsilon, k} \cdot \nabla) v_R^{\varepsilon, k} \right\} \rightarrow \mathbb{1}_S \left\{ \frac{\partial v_R}{\partial t} + (P_S u \cdot \nabla) v_R \right\} \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

The above relation and (5.54) yield to convergence (5.46).

Similarly, we have

$$\frac{\partial v_F^{\varepsilon, k}}{\partial t} - \omega^{\varepsilon, k} \times v_F^{\varepsilon, k} + (P_{\mathcal{S}^{\varepsilon, k}} u^{\varepsilon, k} \cdot \nabla) v_F^{\varepsilon, k} \rightarrow \frac{\partial v_F}{\partial t} - \omega \times v_F + (P_S u \cdot \nabla) v_F \quad \text{strongly in } \mathcal{C}^0([0, T]; H^1(\Omega))$$

and using (5.41), (5.38), (5.50) and (5.9), we deduce the convergence (5.47).  $\square$

### 5.3 Passing to the limit

We fix  $v \in \mathcal{T}_T$  (see (2.8)) and a sequence  $(v^{\varepsilon,k})$  as in Proposition 5.1. Taking  $v^{\varepsilon,k}$  as a test function in (4.30), we obtain

$$\begin{aligned}
& - \int_{\Omega} \rho^0 u^0 \cdot v^{\varepsilon,k}(0) \, dx - \int_0^T \int_{\Omega} \left[ \frac{\partial}{\partial t} + (Q_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} \cdot \nabla) \right] v^{\varepsilon,k} \cdot (\rho^{\varepsilon,k} u^{\varepsilon,k}) \, dx \, dt \\
& \quad + 2\mu \int_0^T \int_{\mathcal{F}^{\varepsilon,k}(t)} D(u^{\varepsilon,k}) : D(v^{\varepsilon,k}) \, dx \, dt + \int_0^T \int_{\partial\Omega} g \nabla j_{\varepsilon}(u^{\varepsilon,k}) \cdot v^{\varepsilon,k} \, d\gamma \, dt \\
& \quad + \int_0^T \int_{\partial\mathcal{S}^{\varepsilon,k}(t)} g \nabla j_{\varepsilon}(u^{\varepsilon,k} - P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v^{\varepsilon,k} - P_{\mathcal{S}^{\varepsilon,k}(t)} v^{\varepsilon,k}) \, d\gamma \, dt \\
& \quad + k \int_0^T \int_{\mathcal{S}^{\varepsilon,k}(t)} (u^{\varepsilon,k} - P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v^{\varepsilon,k} - P_{\mathcal{S}^{\varepsilon,k}(t)} v^{\varepsilon,k}) \, dx \, dt = 0. \quad (5.55)
\end{aligned}$$

**Step 1:** From (5.40), we deduce

$$\int_{\Omega} \rho^0 u^0 \cdot v^{\varepsilon,k}(0) \, dx \rightarrow \int_{\Omega} \rho^0 u^0 \cdot v(0) \, dx.$$

**Step 2:** We write

$$2\mu \int_0^T \int_{\mathcal{F}^{\varepsilon,k}(t)} D(u^{\varepsilon,k}) : D(v^{\varepsilon,k}) \, dx \, dt = 2\mu \int_0^T \int_{\Omega} \mathbb{1}_{\mathcal{F}^{\varepsilon,k}} D(u_F^{\varepsilon,k}) : D(v^{\varepsilon,k}) \, dx \, dt.$$

The above relation combined with (5.6) and (5.41) yield to

$$2\mu \int_0^T \int_{\mathcal{F}^{\varepsilon,k}(t)} D(u^{\varepsilon,k}) : D(v^{\varepsilon,k}) \, dx \, dt \rightarrow 2\mu \int_0^T \int_{\mathcal{F}(t)} D(u_F) : D(v_F) \, dx \, dt. \quad (5.56)$$

**Step 3:** Since  $j_{\varepsilon}$  is a convex function, we can write

$$\nabla j_{\varepsilon}(u^{\varepsilon,k}) \cdot v^{\varepsilon,k} + j_{\varepsilon}(u^{\varepsilon,k}) \leq j_{\varepsilon}(v^{\varepsilon,k} + u^{\varepsilon,k}). \quad (5.57)$$

On the other hand, from (3.4), we deduce that

$$\left| \int_0^T \int_{\partial\Omega} (j_{\varepsilon}(u^{\varepsilon,k}) - |u_F|) \, dx \, dt \right| \leq \frac{\varepsilon}{2} T |\partial\Omega| + \int_0^T \int_{\partial\Omega} |u^{\varepsilon,k} - u_F| \, dx \, dt. \quad (5.58)$$

The above inequality combined with (5.27) yield to the following convergence:

$$\int_0^T \int_{\partial\Omega} j_{\varepsilon}(u^{\varepsilon,k}) \, dx \, dt \rightarrow \int_0^T \int_{\partial\Omega} |u_F| \, dx \, dt. \quad (5.59)$$

Similarly, using (5.44) and (5.27), we obtain

$$\int_0^T \int_{\partial\Omega} j_{\varepsilon}(u^{\varepsilon,k} + v^{\varepsilon,k}) \, dx \, dt \rightarrow \int_0^T \int_{\partial\Omega} |u_F + v_F| \, dx \, dt. \quad (5.60)$$

Consequently, we get

$$\int_0^T \int_{\partial\Omega} g |u_F + v_F| \, dx \, dt - \int_0^T \int_{\partial\Omega} g |u_F| \, dx \, dt \geq \limsup \int_0^T \int_{\partial\Omega} g \nabla j_{\varepsilon}(u^{\varepsilon,k}) \cdot v^{\varepsilon,k} \, d\gamma \, dt. \quad (5.61)$$

**Step 4:** In order to deal with the term on  $\partial\mathcal{S}^{\varepsilon,k}(t)$ , we use the change of variables (5.28) and (5.49). We write

$$W^{\varepsilon,k} := \Xi_{h^{\varepsilon,k}, R^{\varepsilon,k}}(P_{\mathcal{S}^{\varepsilon,k}} u^{\varepsilon,k}), \quad W := \Xi_{h,R}(P_S u).$$

From (5.12)–(5.13) and (5.38), we obtain that

$$W^{\varepsilon,k} \rightarrow W \quad \text{strongly in } L^2(0, T; L^2(\partial\mathcal{S}^0)). \quad (5.62)$$

We also write

$$Z^{\varepsilon,k} := \Xi_{h^{\varepsilon,k}, R^{\varepsilon,k}}(P_{S^{\varepsilon,k}} v^{\varepsilon,k}), \quad Z := \Xi_{h,R}(P_S v)$$

and from (5.42), (5.12)–(5.13) and the fact that  $v_R^{\varepsilon,k}(t, \cdot) \in \mathcal{R}$ , we conclude

$$Z^{\varepsilon,k} \rightarrow Z \quad \text{strongly in } C^0([0, T]; L^2(\partial S^0)). \quad (5.63)$$

Using the change of variables and the convexity of  $j_\varepsilon$ , we obtain

$$\begin{aligned} & \int_0^T \int_{\partial S^{\varepsilon,k}(t)} g \nabla j_\varepsilon(u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} v^{\varepsilon,k}) \, d\gamma \, dt \\ &= \int_0^T \int_{\partial S^0} g \nabla j_\varepsilon(R^{\varepsilon,k} U^{\varepsilon,k} - R^{\varepsilon,k} W^{\varepsilon,k}) \cdot (R^{\varepsilon,k} V^{\varepsilon,k} - R^{\varepsilon,k} Z^{\varepsilon,k}) \, d\gamma \, dt \\ &\leq \int_0^T \int_{\partial S^0} g j_\varepsilon(R^{\varepsilon,k} U^{\varepsilon,k} - R^{\varepsilon,k} W^{\varepsilon,k} + R^{\varepsilon,k} V^{\varepsilon,k} - R^{\varepsilon,k} Z^{\varepsilon,k}) \, d\gamma \, dt \\ &\quad - \int_0^T \int_{\partial S^0} g j_\varepsilon(R^{\varepsilon,k} U^{\varepsilon,k} - R^{\varepsilon,k} W^{\varepsilon,k}) \, d\gamma \, dt \end{aligned} \quad (5.64)$$

and we deduce from (5.31), (5.62)–(5.63) and (5.45) that

$$\begin{aligned} & \int_0^T \int_{\partial S^0} g j_\varepsilon(R^{\varepsilon,k} U^{\varepsilon,k} - R^{\varepsilon,k} W^{\varepsilon,k} + R^{\varepsilon,k} V^{\varepsilon,k} - R^{\varepsilon,k} Z^{\varepsilon,k}) \, d\gamma \, dt \\ & \rightarrow \int_0^T \int_{\partial S^0} g |RU - RW + RV_F - RZ| \, d\gamma \, dt = \int_0^T \int_{\partial S(t)} g |u_F - P_{S(t)} u + v_F - P_{S(t)} v| \, d\gamma \, dt \end{aligned} \quad (5.65)$$

and

$$\begin{aligned} & \int_0^T \int_{\partial S^0} g j_\varepsilon(R^{\varepsilon,k} U^{\varepsilon,k} - R^{\varepsilon,k} W^{\varepsilon,k}) \, d\gamma \, dt \\ & \rightarrow \int_0^T \int_{\partial S^0} g |RU - RW| \, d\gamma \, dt = \int_0^T \int_{\partial S(t)} g |u_F - P_{S(t)} u| \, d\gamma \, dt. \end{aligned} \quad (5.66)$$

Consequently, we get

$$\begin{aligned} & \int_0^T \int_{\partial S(t)} g |u_F - P_{S(t)} u + v_F - P_{S(t)} v| \, d\gamma \, dt - \int_0^T \int_{\partial S(t)} g |u_F - P_{S(t)} u| \, d\gamma \, dt \\ & \geq \limsup \int_0^T \int_{\partial S^{\varepsilon,k}(t)} g \nabla j_\varepsilon(u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} v^{\varepsilon,k}) \, d\gamma \, dt. \end{aligned} \quad (5.67)$$

**Step 5:** Gathering (5.18) and (5.42), we deduce

$$\begin{aligned} & \left| k \int_0^T \int_{S^{\varepsilon,k}(t)} (u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} v^{\varepsilon,k}) \, dx \, dt \right| \\ & \leq k \|u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}\|_{L^2(0,T;L^2(S^{\varepsilon,k}(t)))} \|v^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} v^{\varepsilon,k}\|_{L^2(0,T;L^2(S^{\varepsilon,k}(t)))} \leq C k^{1/2-\alpha/3} \end{aligned} \quad (5.68)$$

and, since  $\alpha > 3/2$ , it follows that

$$k \int_0^T \int_{S^{\varepsilon,k}(t)} (u^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} u^{\varepsilon,k}) \cdot (v^{\varepsilon,k} - P_{S^{\varepsilon,k}(t)} v^{\varepsilon,k}) \, dx \, dt \rightarrow 0. \quad (5.69)$$

**Step 6:** We first write

$$\begin{aligned} & - \int_0^T \int_\Omega \left[ \frac{\partial}{\partial t} + (Q_{S^{\varepsilon,k}(t)} u^{\varepsilon,k} \cdot \nabla) \right] v^{\varepsilon,k} \cdot (\rho^{\varepsilon,k} u^{\varepsilon,k}) \, dx \, dt \\ &= - \int_0^T \int_\Omega \mathbb{1}_{\mathcal{F}^{\varepsilon,k}} \left[ \frac{\partial}{\partial t} + (Q_{S^{\varepsilon,k}(t)} u^{\varepsilon,k} \cdot \nabla) \right] v^{\varepsilon,k} \cdot u^{\varepsilon,k} \, dx \, dt \\ & \quad - \int_0^T \int_\Omega \rho_S \mathbb{1}_{S^{\varepsilon,k}} \left[ \frac{\partial}{\partial t} + (Q_{S^{\varepsilon,k}(t)} u^{\varepsilon,k} \cdot \nabla) \right] v^{\varepsilon,k} \cdot u^{\varepsilon,k} \, dx \, dt. \end{aligned}$$

In  $\mathcal{S}^{\varepsilon,k}$ , we have the relation

$$Q_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} = P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k}.$$

From (5.46), the above remark and (5.24), we deduce that

$$-\int_0^T \int_{\Omega} \rho_S \mathbb{1}_{\mathcal{S}^{\varepsilon,k}} \left[ \frac{\partial}{\partial t} + (Q_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} \cdot \nabla) \right] v^{\varepsilon,k} \cdot u^{\varepsilon,k} dx dt \rightarrow -\int_0^T \int_{\Omega} \rho_S \mathbb{1}_{\mathcal{S}} \left[ \frac{\partial}{\partial t} + (P_S u \cdot \nabla) \right] v_R \cdot u dx dt.$$

Using (5.47) and (5.24), we obtain that

$$-\int_0^T \int_{\Omega} \mathbb{1}_{\mathcal{F}^{\varepsilon,k}} \frac{\partial v^{\varepsilon,k}}{\partial t} \cdot u^{\varepsilon,k} dx dt \rightarrow -\int_0^T \int_{\Omega} \mathbb{1}_{\mathcal{F}} \frac{\partial v_F}{\partial t} \cdot u dx dt.$$

For the last term, we first use (5.41):

$$\mathbb{1}_{\mathcal{F}^{\varepsilon,k}} \nabla v^{\varepsilon,k} \rightarrow \mathbb{1}_{\mathcal{F}} \nabla v_F \quad \text{strongly in } \mathcal{C}^0([0, T]; L^2(\Omega)). \quad (5.70)$$

From (4.32) and (5.4), we obtain that  $(u^{\varepsilon,k}_{|\mathcal{F}^{\varepsilon,k}})$  is bounded in  $L^2(0, T; L^6(\mathcal{F}^{\varepsilon,k}))$ . Using that  $\mathcal{R}$  is finite-dimensional, we also deduce from (4.32) that  $(u^{\varepsilon,k}_{|\mathcal{S}^{\varepsilon,k}})$  is bounded in  $L^2(0, T; L^6(\mathcal{S}^{\varepsilon,k}))$ .

Combining this remark with the convergence (5.24), we deduce that

$$u^{\varepsilon,k} \rightarrow u \quad \text{strongly in } L^2(0, T; L^5(\Omega)). \quad (5.71)$$

On the other hand, using (B.1), we have

$$\begin{aligned} \int_0^T \|\mathbb{1}_{\mathcal{F}^{\varepsilon,k}} [Q_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} - u^{\varepsilon,k}]\|_{L^p(\Omega)}^2 dt &\leq C_{\delta,p}^2 \int_0^T \|(P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k})_n - (u^{\varepsilon,k})_n\|_{L^p(\partial\mathcal{S}^{\varepsilon,k})}^2 dt \\ &\quad + C \left(\frac{\delta}{k}\right)^{2(1/p-1/6)} \int_0^T \left( \|u^{\varepsilon,k}\|_{H^1(\mathcal{F}^{\varepsilon,k})}^2 + \|P_{\mathcal{S}^{\varepsilon,k}} u^{\varepsilon,k}\|_{\mathcal{R}}^2 \right) dt. \end{aligned} \quad (5.72)$$

Using (4.32) and traces theorems, we have

$$\int_0^T \|(P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k})_n - (u^{\varepsilon,k})_n\|_{H^{-1/2}(\partial\mathcal{S}^{\varepsilon,k})}^2 dt \leq C \int_0^T \|P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} - u^{\varepsilon,k}\|_{L^2(\mathcal{S}^{\varepsilon,k})}^2 dt \leq \frac{C}{k}$$

and

$$\int_0^T \|(P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k})_n - (u^{\varepsilon,k})_n\|_{H^{1/2}(\partial\mathcal{S}^{\varepsilon,k})}^2 dt \leq C \int_0^T \left( \|P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k}\|_{\mathcal{R}}^2 + \|u^{\varepsilon,k}\|_{H^1(\mathcal{F}^{\varepsilon,k})}^2 \right) dt \leq C.$$

Combining the two above estimates with Sobolev embeddings theorems and with an interpolation inequality, we deduce that

$$\int_0^T \|(P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k})_n - (u^{\varepsilon,k})_n\|_{L^p(\partial\mathcal{S}^{\varepsilon,k})}^2 dt \leq \frac{C}{k^{2/p-1/2}}.$$

The above estimate and (5.72) yield

$$\mathbb{1}_{\mathcal{F}^{\varepsilon,k}} [Q_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} - u^{\varepsilon,k}] \rightarrow 0 \quad \text{strongly in } L^2(0, T; L^{10/3}(\Omega)). \quad (5.73)$$

The limits (5.71) and (5.9) yield to

$$\mathbb{1}_{\mathcal{F}^{\varepsilon,k}} u^{\varepsilon,k} \rightarrow \mathbb{1}_{\mathcal{F}} u_F \quad \text{strongly in } L^2(0, T; L^{10/3}(\Omega)).$$

Gathering the above relation and (5.73), we deduce

$$\mathbb{1}_{\mathcal{F}^{\varepsilon,k}} Q_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} \rightarrow \mathbb{1}_{\mathcal{F}} u_F \quad \text{strongly in } L^2(0, T; L^{10/3}(\Omega)). \quad (5.74)$$

Due to (5.70), (5.71) and (5.74), we find

$$-\int_0^T \int_{\Omega} \mathbb{1}_{\mathcal{F}^{\varepsilon,k}} [(Q_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} \cdot \nabla)] v^{\varepsilon,k} \cdot u^{\varepsilon,k} dx dt \rightarrow -\int_0^T \int_{\Omega} \mathbb{1}_{\mathcal{F}} [(u_F \cdot \nabla)] v_F \cdot u_F dx dt.$$

Finally, using all previous convergences we obtain that

$$\begin{aligned} & - \int_0^T \int_{\Omega} \left[ \frac{\partial}{\partial t} + (Q_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k} \cdot \nabla) \right] v^{\varepsilon,k} \cdot (\rho^{\varepsilon,k} u^{\varepsilon,k}) \, dx \, dt \\ & \quad \rightarrow - \int_0^T \int_{\mathcal{F}(t)} u_F \cdot \left[ \frac{\partial v_F}{\partial t} + [(u_F \cdot \nabla) v_F] \right] \, dx \, dt - \int_0^T \int_{\mathcal{S}(t)} \rho_S u_R \cdot \left[ \frac{\partial v_R}{\partial t} + [(u_R \cdot \nabla) v_R] \right] \, dx \, dt. \end{aligned}$$

Gathering Step 1 to Step 6, we conclude that (2.18) holds true for any  $v \in \mathcal{T}_T$  such that  $v_F$  has its support in  $\Omega_\eta$ . Since  $\eta > 0$  is arbitrary, (2.18) holds true for any  $v \in \mathcal{T}_T$ .

From (4.32), the convexity of  $j_\varepsilon$  and relation (3.4), we deduce

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho^{\varepsilon,k}(t, \cdot) |u^{\varepsilon,k}(t, \cdot)|^2 \, dx + 2\mu \int_0^t \int_{\mathcal{F}^{\varepsilon,k}(t)} |D(u^{\varepsilon,k})|^2 \, dx \, dt + \int_0^t \int_{\partial\Omega} g j_\varepsilon(u^{\varepsilon,k}) \, d\gamma \, dt \\ & \quad + \int_0^t \int_{\partial\mathcal{S}^{\varepsilon,k}(t)} g j_\varepsilon(u^{\varepsilon,k} - P_{\mathcal{S}^{\varepsilon,k}(t)} u^{\varepsilon,k}) \, d\gamma \, dt \leq g\varepsilon + \frac{1}{2} \int_{\Omega} \rho^0 |u^0|^2 \, dx, \end{aligned} \quad (5.75)$$

for almost every  $t \in (0, T)$ .

From (5.9), we have

$$\rho^{\varepsilon,k} \rightarrow \rho \quad \text{strongly in } \mathcal{C}^0([0, T]; L^p(\Omega)), \quad 1 \leq p < \infty,$$

where

$$\rho := \mathbb{1}_{\mathcal{F}} + \rho_S \mathbb{1}_{\mathcal{S}}.$$

The above convergence, (5.5) and (5.75) yield that (up to a subsequence)

$$\sqrt{\rho^{\varepsilon,k}} u^{\varepsilon,k} \rightharpoonup \sqrt{\rho} u \quad \text{weakly } * \text{ in } L^\infty(0, T, V_n^0(\Omega)). \quad (5.76)$$

Similarly, from (5.6), (5.9) and (5.75), we obtain

$$\mathbb{1}_{\mathcal{F}^{\varepsilon,k}} D(u^{\varepsilon,k}) \rightharpoonup \mathbb{1}_{\mathcal{F}^{\varepsilon,k}} D(u_F) \quad \text{weakly in } L^2(0, T, V_n^1(\Omega)). \quad (5.77)$$

Using (5.59), (5.66), (5.76) and (5.77), we can pass to the inferior limit in (5.75) and we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho(t, \cdot) |u(t, \cdot)|^2 \, dx + 2\mu \int_0^t \int_{\mathcal{F}(t)} |D(u_F)|^2 \, dx \, dt + \int_0^t \int_{\partial\Omega} g |u_F| \, d\gamma \, dt \\ & \quad + \int_0^t \int_{\partial\mathcal{S}(t)} g |u_F - u_R| \, d\gamma \, dt \leq \frac{1}{2} \int_{\Omega} \rho^0 |u^0|^2 \, dx, \end{aligned} \quad (5.78)$$

for almost every  $t \in (0, T)$ . This last inequality yields to the energy estimates (2.19).

Finally, one can conclude the proof of Theorem 2.2 by showing that one of the alternatives holds. This can be done in a standard way (see for instance [2]): we assume that both alternatives are false and from the above energy estimate, we obtain a sequence  $t_n \rightarrow T < \infty$  with

$$\frac{1}{2} \int_{\Omega} \rho(t_i, \cdot) |u(t_i, \cdot)|^2 \, dx \leq \frac{1}{2} \int_{\Omega} \rho^0 |u^0|^2 \, dx$$

and

$$\text{dist}(\mathcal{S}(t_i), \partial\Omega) \geq 2\delta > 0.$$

This allows us to extend the weak solution on  $[(t_i, t_i + \tilde{T})$ , with  $\tilde{T} > 0$  independent of  $i$ , and this leads to a contradiction.

## A Subdifferential

Let  $x_0 \in \overline{B}(0, g)$ . We have

$$b \in \partial I_{\overline{B}(0, g)}(x_0) \iff I_{\overline{B}(0, g)}(x) \geq I_{\overline{B}(0, g)}(x_0) + b \cdot (x - x_0) \quad \forall x \in \mathbb{R}^3 \quad (\text{A.1})$$

$$\iff 0 \geq b \cdot (x - x_0) \quad \forall x \in \overline{B}(0, g). \quad (\text{A.2})$$

By distinguishing the case  $|x_0| < g$  and  $|x_0| = g$ , we obtain

$$\partial I_{\overline{B}(0,g)}(x_0) = \begin{cases} \{0\} & \text{if } |x_0| < g, \\ \{\mu x_0; \mu \geq 0\} & \text{if } |x_0| = g, \\ \emptyset & \text{if } |x_0| > g. \end{cases} \quad (\text{A.3})$$

Moreover,

$$I_{\overline{B}(0,g)}^*(y) := \sup_x y \cdot x - I_{\overline{B}(0,g)}(x) = g|y| \quad (\text{A.4})$$

and thus

$$d \in \partial I_{\overline{B}(0,g)}^*(u) \iff g|y| \geq g|u| + d \cdot (y - u) \quad \forall y \in \mathbb{R}^3. \quad (\text{A.5})$$

**Lemma A.1.** 1. Relation (1.14) is equivalent to

$$(\sigma(u_F, p_F)n)_\tau \cdot y \geq g|(u_F)_\tau| - g|(u_F)_\tau + y| \quad \forall y \in \mathbb{R}^3 \quad \text{on } \partial\Omega. \quad (\text{A.6})$$

2. Relation (1.19) is equivalent to

$$(\sigma(u_F, p_F)n)_\tau \cdot y \geq g|(u_F)_\tau - (u_S)_\tau| - g|(u_F)_\tau - (u_S)_\tau + y| \quad \forall y \in \mathbb{R}^3 \quad \text{on } \partial S. \quad (\text{A.7})$$

*Proof.* 1. By duality, relation (1.14) is equivalent to

$$(\sigma(u_F, p_F)n)_\tau \in \partial I_{\overline{B}(0,g)}^*(-(u_F)_\tau) \quad \text{on } \partial\Omega. \quad (\text{A.8})$$

Combining this relation with (A.5), we deduce that (1.14) is equivalent to

$$g|y| \geq g|(u_F)_\tau| + (\sigma(u_F, p_F)n)_\tau \cdot (-y + (u_F)_\tau) \quad \forall y \in \mathbb{R}^3,$$

which is equivalent to (A.6).

2. Similarly, one can prove the second relation. □

## B Junction of solenoidal vector fields

Here we state two technical results whose proofs are direct consequences of Corollary 4.3, Section 5.2 and Proposition 5.1 from [18]. We recall that  $V^1(A)$  is defined by (2.2).

**Proposition B.1.** *Assume  $\delta_1 > \delta_2 > 0$  small enough. There exists a family of bounded operators*

$$\Lambda^{\delta_1, \delta_2} : V^1(\mathbb{R}^3 \setminus \overline{\mathcal{S}^0}) \times V^1(\mathcal{S}^0) \rightarrow V^1(\mathbb{R}^3)$$

such that for any  $(U^{(1)}, U^{(2)}) \in V^1(\mathbb{R}^3 \setminus \overline{\mathcal{S}^0}) \times V^1(\mathcal{S}^0)$ ,

$$\begin{aligned} \Lambda^{\delta_1, \delta_2}[U^{(1)}, U^{(2)}] &= U^{(2)} && \text{in } \mathcal{S}^0, \\ \Lambda^{\delta_1, \delta_2}[U^{(1)}, U^{(2)}] &= U^{(1)} && \text{in } \mathbb{R}^3 \setminus (\mathcal{S}^0)^{\delta_1} \end{aligned}$$

and

$$\begin{aligned} \left\| \Lambda^{\delta_1, \delta_2}[U^{(1)}, U^{(2)}] - U^{(1)} \right\|_{L^p((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} &\leq C_{\delta_1, p} \left( \|(U^{(1)} - U^{(2)}) \cdot n\|_{L^p(\partial \mathcal{S}^0)} \right. \\ &\quad \left. + \delta_2^{1/p-1/6} \left( \|U^{(1)}\|_{H^1((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} + \|U^{(2)}\|_{H^1(\mathcal{S}^0)} \right) \right) \quad \forall p \in [2, 4]. \quad (\text{B.1}) \end{aligned}$$

*Proof.* Following the method introduced in [18, Section 5.2], we begin by extending the function  $U_2$  to  $(\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0$  and we construct the function  $\Lambda^{\delta_1, \delta_2}$  as follows:

$$\Lambda^{\delta_1, \delta_2} = V_1 + V_2 + V_3,$$

where  $V_1$  and  $V_2$  are defined by

$$V_1 = U_1 + \left\{ [U_2 - U_1] - [(U_2 - U_1) \cdot e_z] e_z \right\} \varphi(z), \quad (\text{B.2})$$

$$V_2 = \left\{ (U_2 - U_1) \cdot e_z \right\}_{|z=0} e_z(z) \varphi(z). \quad (\text{B.3})$$

Here,  $z$  is the third component of an orthogonal curvilinear coordinates system  $(s_1, s_2, z)$  defined around  $\partial\mathcal{S}_0$  such that  $\partial\mathcal{S}_0 = \{\vec{r}(s_1, s_2, z) : z = 0\}$ . We have also denoted by  $(e_1, e_2, e_z)$  a direct orthonormal basis associated with the coordinates system  $(s_1, s_2, z)$ . We can extend these vector fields in a tubular domain  $\{\vec{r}(s_1, s_2, z) : z \in [0, \delta_1]\}$  for  $\delta_1$  small enough. Moreover for  $\delta_1$  small enough,  $\{\vec{r}(s_1, s_2, z) : z = \delta_1\} = \partial(\mathcal{S}_0)^{\delta_1}$ .

In the definition of  $V_1$  and  $V_2$ , we have taken  $\varphi$  a  $\mathcal{C}^\infty(\mathbb{R}; [0, 1])$  function such that  $\varphi(z) = 1 \ \forall z \leq 0$  and  $\varphi(z) = 0 \ \forall z \geq \delta_2$ . Finally,  $V_3$  is a function satisfying the following properties:

$$\operatorname{div} V_3 = -\operatorname{div}(V_1 + V_2) \quad \text{in } (\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0, \quad (\text{B.4})$$

$$V_3 = 0 \quad \text{on } \partial\mathcal{S}^0 \cup \partial(\mathcal{S}^0)^{\delta_1}. \quad (\text{B.5})$$

Using these definitions and the properties of  $\varphi$ , it follows that

$$V_1 + V_2 + V_3 = U_2 \quad \text{on } \partial\mathcal{S}^0, \quad (\text{B.6})$$

$$V_1 + V_2 + V_3 = U_1 \quad \text{on } \partial(\mathcal{S}^0)^{\delta_1}. \quad (\text{B.7})$$

Since  $\left\{ [U_2 - U_1] - [(U_2 - U_1) \cdot e_z] e_z \right\} \perp e_z$ , we have that  $V_1$  satisfies the following properties:

$$\operatorname{div} V_1 = -\varphi(z) \operatorname{div} \left\{ [(U_2 - U_1) \cdot e_z] e_z \right\} \quad \text{in } (\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0, \quad (\text{B.8})$$

$$|V_1 - U_1| \leq |U_2 - U_1| \quad \text{in } (\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0. \quad (\text{B.9})$$

Since  $\partial\mathcal{S}_0$  is of class  $\mathcal{C}^2$ , we have

$$|(\nabla e_z)_{ij}| = \left| \frac{1}{h_j} \frac{\partial e_z}{\partial s_j} \cdot e_i \right| \leq C,$$

where  $h_j$  is the scale factor associated with the orthonormal curvilinear coordinates system.

Due to (B.8) and the above estimate, we obtain

$$\|\operatorname{div} V_1\|_{L^2((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} \leq C_{\mathcal{S}_0} \|U_2 - U_1\|_{H^1((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)}. \quad (\text{B.10})$$

Using (B.9), we get

$$\|V_1 - U_1\|_{L^p((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} \leq \|U_2 - U_1\|_{L^p((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} \quad \forall p \in [1, 6].$$

Moreover, for  $V_2$  we have the following properties:

$$\operatorname{div} V_2 = \left\{ (U_2 - U_1) \cdot e_z \right\}_{|z=0} \left( \varphi(z) \operatorname{div} e_z + e_z \cdot \nabla \varphi(z) \right) \quad \text{in } (\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0, \quad (\text{B.11})$$

$$|V_2| \leq |U_2 - U_1|_{|z=0} \quad \text{in } (\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0. \quad (\text{B.12})$$

Due to (B.11) and the fact that  $\partial\mathcal{S}_0$  is of class  $\mathcal{C}^2$ , we obtain

$$\|\operatorname{div} V_2\|_{L^2((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} \leq C_{\mathcal{S}_0} (1 + \delta_2^{-1/2}) \|(U_2 - U_1) \cdot n\|_{L^2(\partial\mathcal{S}^0)}. \quad (\text{B.13})$$

Using (B.12), we get

$$\|V_2\|_{L^p((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} \leq \delta_2^{1/p} \|(U_2 - U_1) \cdot n\|_{L^p(\partial\mathcal{S}^0)} \quad \forall p \in [1, 4].$$

From Proposition 4.1 in [18], we have

$$\|V_3\|_{H^1((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} \leq C_{\mathcal{S}_0, \delta_1} \left( \|\operatorname{div} V_1\|_{L^2((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} + \|\operatorname{div} V_2\|_{L^2((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} \right).$$

Then, using (B.10) and (B.13), we deduce that

$$\|V_3\|_{H^1((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} \leq C_{\mathcal{S}_0, \delta_1} \left( \|U_2 - U_1\|_{H^1((\mathcal{S}^0)^{\delta_1} \setminus \mathcal{S}^0)} + (1 + \delta_2^{-1/2}) \|(U_2 - U_1) \cdot n\|_{L^2(\partial\mathcal{S}^0)} \right).$$

Therefore, applying Sobolev embedding injection, we conclude the estimate (B.1).  $\square$

Similarly, one can prove the following result:

**Proposition B.2.** *Assume  $\delta_1 > \delta_2 > 0$  small enough. There exists a family of bounded operators*

$$\tilde{\Lambda}^{\delta_1, \delta_2} : V^1(\mathbb{R}^3 \setminus \overline{\mathcal{S}^0}) \times V^1(\mathcal{S}^0) \rightarrow V^1(\mathbb{R}^3)$$

such that for any  $(U^{(1)}, U^{(2)}) \in V^1(\mathbb{R}^3 \setminus \overline{\mathcal{S}^0}) \times V^1(\mathcal{S}^0)$ ,

$$\begin{aligned} \tilde{\Lambda}^{\delta_1, \delta_2}[U^{(1)}, U^{(2)}] &= U^{(2)} && \text{in } (\mathcal{S}^0)_{\delta_1}, \\ \tilde{\Lambda}^{\delta_1, \delta_2}[U^{(1)}, U^{(2)}] &= U^{(1)} && \text{in } \mathbb{R}^3 \setminus \mathcal{S}^0 \end{aligned}$$

and

$$\begin{aligned} \left\| \tilde{\Lambda}^{\delta_1, \delta_2}[U^{(1)}, U^{(2)}] - U^{(2)} \right\|_{L^p(\mathcal{S}^0)} &\leq C_{\delta_1, p} \left( \|(U^{(1)} - U^{(2)}) \cdot n\|_{L^p(\partial\mathcal{S}^0)} \right. \\ &\quad \left. + \delta_2^{1/p-1/6} \left( \|U^{(1)}\|_{H^1(\mathbb{R}^3 \setminus \mathcal{S}^0)} + \|U^{(2)}\|_{H^1(\mathcal{S}^0)} \right) \right) \quad \forall p \in [2, 4]. \end{aligned} \quad (\text{B.14})$$

$$\left\| \tilde{\Lambda}^{\delta_1, \delta_2}[U^{(1)}, U^{(2)}] \right\|_{H^1(\mathcal{S}^0)} \leq C_{\delta_1} \left( \|(U^{(1)} - U^{(2)}) \cdot n\|_{L^2(\partial\mathcal{S}^0)} + \delta_2^{-2/3} \left( \|U^{(1)}\|_{H^1(\mathcal{S}^0)} + \|U^{(2)}\|_{H^1(\mathcal{S}^0)} \right) \right). \quad (\text{B.15})$$

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