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SMALL BERGMAN-ORLICZ AND HARDY-ORLICZ SPACES, AND THEIR COMPOSITION OPERATORS

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ABSTRACT. We show that the weighted Bergman-Orlicz space A^{ψ}_{α} coincides with some weighted Banach space of holomorphic functions if and only if the Orlicz function ψ satisfies the so-called Δ^2 -condition. In addition we prove that this condition characterizes those A^{ψ}_{α} on which every composition operator is bounded or order bounded into the Orlicz space L^{ψ}_{α} . This provides us with estimates of the norm and the essential norm of composition operators on such spaces. We also prove that when ψ satisfies the Δ^2 -condition, a composition operator is compact on A^{ψ}_{α} if and only if it is order bounded into the so-called Morse-Transue space M^{ψ}_{α} . Our results stand in the unit ball of \mathbb{C}^N .

1. Introduction and first definitions

Hardy-Orlicz and Bergman-Orlicz spaces are natural generalizations of the classical Hardy H^p and Bergman spaces A^p . They have been rather well studied when the defining Orlicz function grows slowly, i.e. when they are larger than H^1 and, roughly speaking, similar to the Nevanlinna class \mathcal{N} , which is one instance of such a space. Nevertheless, there are not so many papers dealing with small Hardy-Orlicz and Bergman-Orlicz spaces. One of the interests in looking at these spaces is that, due to the wide range of Orlicz functions, they provide an as well wide and refined scale of natural Banach spaces of holomorphic functions between A^1 and H^{∞} . And by many aspects H^{∞} radically differs from Hardy and Bergman spaces. This can be seen for instance from the point of view of operator theory and more specifically from that of composition operators. We recall here that a composition operator C_{ϕ} , with symbol ϕ , is defined as $C_{\phi}(f) = f \circ \phi$, where f lies in some Banach space of holomorphic functions and ϕ is an appropriate holomorphic map. If $\mathbb{B}_N = \left\{z = (z_1, \ldots, z_N) \in \mathbb{C}^N, |z| := \left(|z_1|^2 + \ldots + |z_N|^2\right)^{1/2} < 1\right\}$ is the unit ball of \mathbb{C}^N , every composition operator is trivially bounded on the space $H^{\infty}(\mathbb{B}_N)$ of bounded holomorphic functions and \mathbb{B}_N and where N is the Little scale \mathbb{C}^N in \mathbb{C}^N . phic functions on \mathbb{B}_N and, when N=1, the Littlewood Subordination Principle says that this is also true on every Hardy and Bergman space [39]. Yet things get more complicated whenever $N \geq 2$ since even the simple symbol $\varphi(z) = (N^{N/2}z_1z_2...z_N,0')$, 0' being the null N-1-tuple, does not induce a bounded composition operator on $H^2(\mathbb{B}_N)$ (see [15] for other examples). Besides, whatever the dimension, while the compactness of C_{ϕ} on $H^{p}(\mathbb{B}_{N})$, $1 \leq p < \infty$, does not depend on p and can occur when $\phi(\mathbb{B}_N)$ touches the boundary of \mathbb{B}_N [15, 40], it is compact on $H^{\infty}(\mathbb{B}_N)$ if and only if $|\phi(z)| \leq r < 1$ for every $z \in \mathbb{B}_N$ [38]. These observations recently motivated some authors to study composition operators on the whole scale of Hardy-Orlicz and Bergman-Orlicz spaces. This study turned out to be rich, [25, 26, 27, 28] on the unit disc \mathbb{D} and [10, 11, 13, 14] on the unit ball \mathbb{B}_N , $N \geq 1$. In the present paper we will mainly focus on *small* Hardy-Orlicz and Bergman-Orlicz spaces, namely those which are close to $H^{\infty}(\mathbb{B}_N)$. A surprising characterization of the latter ones will be obtained and used to refine some previously known results. We hope that it will highlight how properties of composition operators are related to the *structure* of these spaces.

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Let us give the definitions of Hardy-Orlicz and weighted Bergman-Orlicz spaces. Given an Orlicz function ψ , i.e. a strictly convex function $\psi:[0,+\infty)\to[0,+\infty)$ which vanishes at 0, is continuous at 0 and satisfies $\frac{\psi(x)}{x}\xrightarrow[x\to+\infty]{}+\infty$, and given a probability space (Ω,\mathbb{P}) , the Orlicz space $L^{\psi}(\Omega,\mathbb{P})$ associated with ψ on (Ω,\mathbb{P}) is defined as the set of all (equivalence classes of) measurable functions f on Ω for which there exists some C>0, such that $\int_{\Omega}\psi\left(\frac{|f|}{C}\right)d\mathbb{P}$ is finite. It is also usual to define the Morse-Transue space $M^{\psi}(\Omega,\mathbb{P})$ as the subspace of $L^{\psi}(\Omega,\mathbb{P})$ for which $\int_{\Omega}\psi\left(\frac{|f|}{C}\right)d\mathbb{P}$ is finite for any constant C>0. $L^{\psi}(\Omega,\mathbb{P})$ and $M^{\psi}(\Omega,\mathbb{P})$, endowed with the $Luxemburg\ norm\ \|\cdot\|_{\psi}$ given by

$$||f||_{\psi} = \inf \left\{ C > 0, \int_{\Omega} \psi \left(\frac{|f|}{C} \right) d\mathbb{P} \le 1 \right\},$$

are Banach spaces. Then for $\alpha > -1$, we introduce the weighted Bergman-Orlicz space $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ and the little weighted Bergman-Orlicz space $AM_{\alpha}^{\psi}(\mathbb{B}_{N})$ as the spaces of those functions in $H(\mathbb{B}_{N})$, the algebra of all holomorphic functions on \mathbb{B}_{N} , which also belongs to $L^{\psi}(\mathbb{B}_{N}, v_{\alpha})$ and $M^{\psi}(\mathbb{B}_{N}, v_{\alpha})$ respectively, where $dv_{\alpha} = c_{\alpha}(1 - |z|^{2})^{\alpha}dv$ is the normalized weighted Lebesgue measure on the ball. For any Orlicz function ψ , $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ and $AM_{\alpha}^{\psi}(\mathbb{B}_{N})$, $\alpha > -1$, endowed with the Luxemburg norm $\|\cdot\|_{A_{\alpha}^{\psi}}$ inherited from $L^{\psi}(\mathbb{B}_{N}, v_{\alpha})$, are Banach spaces.

Similarly generalizing the usual Hardy spaces we define the Hardy-Orlicz space $H^{\psi}(\mathbb{B}_N)$ as follows:

$$H^{\psi}(\mathbb{B}_{N}) := \left\{ f \in H(\mathbb{B}_{N}), \|f\|_{H^{\psi}} := \sup_{0 < r < 1} \|f_{r}\|_{\psi} < \infty \right\},$$

where $\|\cdot\|_{\psi}$ is the Luxembourg norm on the Orlicz space $L^{\psi}(\mathbb{S}_N, \sigma_N)$ and f_r is given by $f_r(z) = f(rz)$. Here σ_N stands for the normalized rotation-invariant positive Borel measure on the unit sphere $\mathbb{S}_N = \partial \mathbb{B}_N$. For any Orlicz function ψ , $H^{\psi}(\mathbb{B}_N)$ can be identified with a closed subspace of $L^{\psi}(\mathbb{S}_N, \sigma_N)$. We then define the *little Hardy-Orlicz space* $HM^{\psi}(\mathbb{B}_N)$ as $H^{\psi}(\mathbb{B}_N) \cap M^{\psi}(\mathbb{S}_N, \sigma_N)$. Endowed with $\|\cdot\|_{H^{\psi}}$, $H^{\psi}(\mathbb{B}_N)$ and $HM^{\psi}(\mathbb{B}_N)$ are Banach spaces [11].

When $\psi(x) = x^p$, $1 \leq p < \infty$, $A^{\psi}_{\alpha}(\mathbb{B}_N)$ and $H^{\psi}(\mathbb{B}_N)$ are the classical weighted Bergman space $A^p_{\alpha}(\mathbb{B}_N)$ and Hardy space $H^p(\mathbb{B}_N)$, respectively. Moreover we have the following [36]

$$H^{\infty} = \bigcap_{\psi \text{ Orlicz}} A_{\alpha}^{\psi}(\mathbb{B}_{N}) = \bigcap_{\psi \text{ Orlicz}} H^{\psi}(\mathbb{B}_{N}),$$

while trivially $A^1_{\alpha}(\mathbb{B}_N) = \bigcup_{\psi \text{ Orlicz}} A^{\psi}_{\alpha}(\mathbb{B}_N)$ and $H^1(\mathbb{B}_N) = \bigcup_{\psi \text{ Orlicz}} H^{\psi}(\mathbb{B}_N)$. We refer to [14, 28] for an introduction to weighted Bergman-Orlicz spaces and to [11, 25] for a discussion about Hardy-Orlicz spaces (on the disc and on the ball); to learn more about the general theory of Orlicz spaces, we refer to [22, 36].

As $A_{\alpha}^{\psi}(\mathbb{B}_N)$ and $H^{\psi}(\mathbb{B}_N)$ are continuously embedded into $H(\mathbb{B}_N)$, some sharp inclusions

$$A_{\alpha}^{\psi}\left(\mathbb{B}_{N}\right)\subseteq H_{v}^{\infty}\left(\mathbb{B}_{N}\right) \text{ and } H^{\psi}\left(\mathbb{B}_{N}\right)\subseteq H_{w}^{\infty}\left(\mathbb{B}_{N}\right)$$

hold for some typical radial weights v and w, where $H_v^{\infty}(\mathbb{B}_N)$ is the weighted Banach space - or growth space- associated with v:

$$(1.1) H_v^{\infty} := \left\{ f \in H\left(\mathbb{B}_N\right), \|f\|_v := \sup_{z \in \mathbb{B}_N} |f(z)| \, v(z) < \infty \right\}.$$

We recall that a weight v on \mathbb{B}_N is a continuous (strictly) positive function on \mathbb{B}_N , which is radial if v(z) = v(|z|) for all $z \in \mathbb{B}_N$, and typical if $v(z) \to 0$ as $|z| \to 1$.

Our first result will say that when the Orlicz function ψ grows fast, then $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ coincides with $H_{v}^{\infty}(\mathbb{B}_{N})$ for $v(z) = \left[\psi^{-1}\left(\frac{1}{1-|z|}\right)\right]^{-1}$. For instance, if $\psi(x) = e^{x} - 1$, then $A_{\alpha}^{\psi}(\mathbb{B}_{N}) = 0$

 $H_v^{\infty}(\mathbb{B}_N)$ with $v(z) = \left[\log(\frac{e}{1-|z|})\right]^{-1}$, which is the typical growth for the functions in the Bloch space. This result echoes Kaptanoğlu's observation that some weighted Bloch spaces (not the true ones) coincide with some other weighted Banach spaces [21]. An immediate consequence is that weighted and unweighted *small* Bergman-Orlicz are the same. We should mention that weighted Banach spaces and their composition operators have been far more studied than Bergman-Orlicz spaces, [2, 5, 6, 7, 9, 18, 29, 30, 31, 34] and the references therein. We will mention from time to time in the forthcoming sections that similar results were obtained independently for $H_v^{\infty}(\mathbb{D})$ and $A_{\alpha}^{\psi}(\mathbb{D})$ ($\mathbb{D} := \mathbb{B}_1$), and turn out to be identical for ψ and v as above.

We will also derive easily from the previous that every composition operator is bounded on $A^{\psi}_{\alpha}(\mathbb{B}_N)$ whenever ψ grows fast, a result which was already stated in [14] and where the machinery of Carleson measures was heavily used. Actually, the proof of our first result will point out that, more than bounded, every composition operator is order bounded into $L^{\psi}(\mathbb{B}_N, v_{\alpha})$ (ψ growing fast). We recall that an operator T from a Banach space X to a Banach lattice Y is order bounded into some subspace Z of Y if there exists $y \in Z$ positive such that $\sup_{\|x\|\leq 1} |Tx|\leq y$. The notion of order boundedness is often related to that of p-summing, nuclear or Dunford-Pettis operators [16], as illustrated by the result of A. Shields, L. Wallen and J. Williams, asserting that an operator from a normed space to some L^p space, which is order bounded into L^p , is p-summing. From the point of view of composition operators, the notion was investigated in several papers [17, 19, 20, 42]. On $H^{\infty}(\mathbb{B}_N)$ every bounded operator is trivially order bounded into $L^{\infty}(\mathbb{B}_N)$. For the classical Hardy and Bergman spaces $H^p(\mathbb{D})$, it is shown in [40] that C_{ϕ} is order bounded into $L^p(\partial \mathbb{D})$ if and only if it p-summing, when $2 \le p < \infty$ (i.e. Hilbert-Schmidt when p = 2). The same holds in \mathbb{B}_N and for the Bergman spaces. The situation becomes radically different when one studies the order boundedness of composition operators on Hardy-Orlicz and Bergman-Orlicz spaces which are neither reflexive nor separable (i.e. when $H^{\psi}(\mathbb{B}_N)$ and $A^{\psi}_{\alpha}(\mathbb{B}_N)$ differs from $HM^{\psi}(\mathbb{B}_N)$ and $AM^{\psi}_{\alpha}(\mathbb{B}_N)$ respectively). Indeed it appears, in such cases, that the order boundedness of C_{ϕ} acting on $H^{\psi}(\mathbb{B}_N)$ (resp. $A_{\alpha}^{\psi}(\mathbb{B}_N)$ into $L^{\psi}(\mathbb{S}_N, \sigma_N)$ (resp. $L^{\psi}(\mathbb{B}_N, v_{\alpha})$) is no longer stronger than its compactness (by the foregoing, this is already clear when ψ grows fast, since every composition operator is then order bounded into $L^{\psi}(\mathbb{B}_N, v_{\alpha})$, while the identity fails to be compact). Though C_{ϕ} is still compact for any ψ whenever it is order bounded into $M^{\psi}(\mathbb{S}_N, \sigma_N)$ (resp. $M^{\psi}(\mathbb{B}_N, v_{\alpha})$), the latter does not imply any more that it is p-summing, for any $p < \infty$ [25]. Note that on H^{∞} , C_{ϕ} is compact if and only if it is p-summing [23].

In these directions, our two main results state as follows. The first one (Theorem 3.20) completes what has been said above.

Theorem. Let $M \ge 1$, N > 1, $\alpha > -1$, and ψ be an Orlicz function. The following assertions are equivalent.

- (1) ψ satisfies the Δ^2 -condition (a fast growth condition, see Section 2.1)
- (2) $A_{\alpha}^{\psi}(\mathbb{B}_{M}) = H_{v}^{\infty}(\mathbb{B}_{M}), \text{ where } v(z) = \left[\psi^{-1}\left(\frac{1}{1-|z|}\right)\right]^{-1};$
- (3) For every $\phi : \mathbb{B}_M \to \mathbb{B}_M$ holomorphic, C_{ϕ} acting on $A^{\psi}_{\alpha}(\mathbb{B}_M)$ is order-bounded into $L^{\psi}(\mathbb{B}_M, v_{\alpha})$;
- (4) For every $\phi : \mathbb{B}_N \to \mathbb{B}_N$ holomorphic, C_{ϕ} is bounded on $A_{\alpha}^{\psi}(\mathbb{B}_N)$;
- (5) C_{φ} is bounded on $A_{\alpha}^{\psi}(\mathbb{B}_N)$, where $\varphi(z) = (N^{N/2}z_1z_2...z_N, 0')$.

The second one (Theorem 3.23) is a generalization of [25, Theorem 3.24] to $N \geq 2$ and to the Bergman-Orlicz setting.

Theorem. Let $\alpha > -1$, let ψ be an Orlicz function satisfying the Δ^2 -condition and let $\phi: \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. The following assertions are equivalent.

- (1) C_{ϕ} is compact on $H^{\psi}(\mathbb{B}_{N})$ (resp. $A_{\alpha}^{\psi}(\mathbb{B}_{N})$); (2) C_{ϕ} acting on $H^{\psi}(\mathbb{B}_{N})$ (resp. $A_{\alpha}^{\psi}(\mathbb{B}_{N})$) is order bounded into $M^{\psi}(\mathbb{S}_{N}, \sigma_{N})$ (resp. $M^{\psi}(\mathbb{B}_N, v_{\alpha}));$
- (3) C_{ϕ} is weakly compact on $H^{\psi}(\mathbb{B}_N)$ (resp. $A_{\alpha}^{\psi}(\mathbb{B}_N)$).

By many aspects the behavior of a composition operator C_{ϕ} is related to how frequently and sharply $\phi(\mathbb{B}_N)$ touches the boundary of \mathbb{B}_N . In particular, it is known that if $\phi(\mathbb{B}_N)$ is contained in some Korányi approach regions (i.e. Stolz domains for N=1), then C_{ϕ} is bounded or compact on $H^p(\mathbb{B}_N)$ and $A^p_{\alpha}(\mathbb{B}_N)$, depending on the opening of the Korányi region [15, 39]. In fact the condition on the opening of the Korányi region K which ensures that $\phi(\mathbb{B}_N) \subset K$ implies C_{ϕ} compact also ensures that $\phi(\mathbb{B}_N) \subset K$ implies C_{ϕ} order bounded into $L^{\psi}(\mathbb{S}_N, \sigma_N)$. In [13] it was proven that $\phi(\mathbb{B}_N)$ included in any such region does no longer automatically imply the compactness of C_{ϕ} on $H^{\psi}(\mathbb{B}_N)$ or $A_{\alpha}^{\psi}(\mathbb{B}_N)$. In the last section, we will geometrically illustrate the idea developed in the previous paragraph by showing that if $\phi(\mathbb{B}_N)$ is included in some Korányi region, then C_{ϕ} is order bounded into $L^{\psi}(\mathbb{B}_N, v_{\alpha})$ (resp. $L^{\psi}(\mathbb{S}_N, \sigma_N)$, whatever ψ . The faster the Orlicz function will grow, the less restrictive will be the condition on the opening of the region.

We shall mention that we will make use of Carleson measure techniques only for the equivalence between (1) and (3) in our second main theorem (through Corollary 3.3).

The paper is organized as follows: in the next section we will provide the remaining definitions and show that *small* weighted Bergman-Orlicz spaces coincide with some weighted Banach spaces. In Section 3, we will focus on the boundedness, the compactness, and the order boundedness of composition operators acting on small weighted Bergman-Orlicz spaces and Hardy-Orlicz spaces. The last section will be devoted to some geometric illustration of the general idea of the paper.

Notation 1.1. Given two points $z, w \in \mathbb{C}^N$, the euclidean inner product of z and w will be denoted by $\langle z, w \rangle$, that is $\langle z, w \rangle = \sum_{i=1}^N z_i \overline{w_i}$; the notation $|\cdot|$ will stand for the associated norm, as well as for the modulus of a complex number.

The notation \mathbb{T} will be used to denote the unit sphere $\mathbb{S}_1 = \partial \mathbb{D}$.

We recall that $dv_{\alpha} = c_{\alpha}(1-|z|^2)^{\alpha}dv$ is the normalized weighted Lebesgue measure on the ball and so that $c_{\alpha} = \frac{\Gamma(N+\alpha+1)}{N!\Gamma(\alpha+1)}$, where Γ is the Gamma function.

 $d\sigma_N$ will stand for the normalized rotation-invariant positive Borel measure on \mathbb{S}_N .

We will use the notations \lesssim and \gtrsim for one-sided estimates up to an absolute constant, and the notation \approx for two-sided estimates up to absolute constants.

2. A menagerie of spaces

2.1. Weighted Bergman-Orlicz spaces and Hardy-Orlicz spaces.

Some important classes of Orlicz functions. Orlicz-type spaces can be distinguished by the regularity and the growth at infinity of the defining Orlicz function. We recall that two equivalent Orlicz functions define the same Orlicz space, where two Orlicz functions Ψ_1 and Ψ_2 are said to be equivalent if there exists some constant c such that

$$c\Psi_1(cx) \le \Psi_2(x) \le c^{-1}\Psi_1(c^{-1}x),$$

for any x large enough. In the sequel, we will consider Orlicz functions essentially of four types:

- ψ satisfies the Δ_2 -condition (or belongs to the Δ_2 -class) if there exists C > 0 such that $\psi(2x) \leq C\psi(x)$ for any x large enough.
- ψ satisfies the Δ^1 -condition (or belongs to the Δ^1 -class) if there exists C > 0 such that $x\psi(x) \leq \psi(Cx)$ for any x large enough.
- ψ satisfies the Δ^2 -condition (or belongs to the Δ^2 -class) if there exists C > 0, such that $\psi(x)^2 \leq \psi(Cx)$ for any x large enough.
- ψ satisfies the ∇_2 -condition (or belongs to the ∇_2 -class) if its complementary function Φ belongs to the Δ_2 -class, where Φ is defined by

$$\Phi(y) = \sup_{x \ge 0} \{xy - \psi(x)\}, y \ge 0.$$

The ∇_2 -condition is a regularity condition satisfied by most of the Orlicz functions, and Δ_2 , Δ^1 and Δ^2 -conditions are growth conditions. It is easily seen that the Δ^2 -condition implies the Δ^1 -condition which in its turn implies the ∇_2 . The Δ_2 -class is disjoint from both Δ^1 and Δ^2 -classes.

Using that Orlicz functions are increasing convex functions and an easy induction, we get the following characterizations of Δ_2 , Δ^1 and Δ^2 -classes [10, 22].

Proposition 2.1. Let ψ be an Orlicz function.

- (1) ψ belongs to the Δ_2 -class if and only if for one b > 1 (or equivalently all b > 1) there exists C > 0 such that $\psi(bx) \leq C\psi(x)$ for any x large enough.
- (2) ψ belongs to the Δ^1 -class if and only if for one b > 1 (or equivalently all b > 1) there exists C > 0 such that $x^b \psi(x) \leq \psi(Cx)$ for any x large enough.
- (3) ψ belongs to the Δ^2 -class if and only if for one b > 1 (or equivalently all b > 1) there exists C > 0 such that $\psi(x)^b \leq \psi(Cx)$ for any x large enough.

If $\psi \in \Delta_2$, then $x^q \lesssim \psi(x) \lesssim x^p$ for some $1 \leq q \leq p < +\infty$ and x large enough; typical examples are $x \mapsto ax^p \left((\log(1+x))^b \right)$, 1 , <math>a > 0 and b > 0. If $\psi \in \Delta^1$, then $\psi(x) \geq \exp\left(a(\log x)^2\right)$ for some a > 0 and x large enough. The Δ^2 -class consists of functions ψ satisfying $\psi(x) \geq e^{x^a}$ for some a > 0 and for any x large enough. Orlicz functions of the form $x \mapsto \exp\left(a(\log(x+1))^b\right) - 1$ with a > 0 and $b \geq 2$ are examples of Δ^1 -functions, which are not in the Δ^2 -class. Moreover $x \mapsto e^{ax^b} - 1$ belongs to the Δ^2 -class, whenever a > 0 and $b \geq 1$. Note that $x \mapsto x^p$, $1 , satisfies the <math>\nabla_2$ -condition, and not the Δ^1 -one.

We do not go into more details and we refer to [13, 25] for information sufficient for our purpose. Actually the classification of Orlicz functions is much more refined; to learn more about the general theory of Orlicz spaces, see [22] or [36].

As in the classical case, it can be convenient for the presentation to sometimes look at Hardy-Orlicz spaces as a limit case of weighted Bergman-Orlicz spaces $A^{\psi}_{\alpha}(\mathbb{B}_N)$ when α tends to -1. The following standard lemma gives some sense to that.

Lemma 2.2. Let $f : \mathbb{B}_{\mathbb{N}} \mapsto [0, +\infty[$ be continuous and such that the function $r \mapsto \int_{\mathbb{S}_N} f(rz) d\sigma_N(z)$, $r \in [0, 1)$, is increasing. Then

$$\lim_{r \to 1} \int_{\mathbb{B}_N} f(r\zeta) d\sigma_N(\zeta) = \lim_{\alpha \to -1} \int_{\mathbb{B}_N} f(z) dv_\alpha(z),$$

where the limits may be ∞ .

Proof. Let f be as in the lemma and let us write, for $-1 < \alpha < 0$ and $0 < \eta < 1$,

$$(2.1) \int_{\mathbb{B}_N} f(z)dv_{\alpha}(z) = 2Nc_{\alpha} \int_0^{1-\eta} \int_{\mathbb{S}_N} f(r\zeta)d\sigma_N(\zeta)r^{2N-1} \left(1-r^2\right)^{\alpha} dr + 2Nc_{\alpha} \int_{1-\eta}^1 \int_{\mathbb{S}_N} f(r\zeta)d\sigma_N(\zeta)r^{2N-1} \left(1-r^2\right)^{\alpha} dr.$$

Observe that because $c_{\alpha} \to 0$ as $\alpha \to -1$, the first term of the right-hand side tends to 0. Since $F: r \mapsto \int_{\mathbb{S}_N} f(rz) d\sigma_N(z)$ is increasing, either F is convergent, or it increases to ∞ as $r \to 1$. Assume first that $F(r) \to \infty$ as $r \to 1$. Then for any A > 0 there exists $\eta(A) \in (0, 1)$, close enough to 1, such that

$$\int_{S_N} f(r\zeta)d\sigma_N(\zeta) \ge A, \ r \in (1 - \eta, 1).$$

Moreover $2Nc_{\alpha} \int_{0}^{1} (1-r^{2})^{\alpha} r^{2N-1} dr = 1$ and $2Nc_{\alpha} \int_{0}^{1-\eta} (1-r^{2})^{\alpha} r^{2N-1} dr \to 0$ as $\alpha \to -1$, hence

(2.2)
$$2Nc_{\alpha} \int_{1-\eta}^{1} (1-r^2)^{\alpha} r^{2N-1} dr \to 1 \text{ as } \alpha \to -1,$$

for any $\eta \in (0,1)$. Thus

$$2Nc_{\alpha} \int_{1-\eta(A)}^{1} \int_{\mathbb{S}_{N}} f(r\zeta) d\sigma_{N}(\zeta) r^{2N-1} \left(1-r^{2}\right)^{\alpha} dr \geq 2ANc_{\alpha} \int_{1-\eta(A)}^{1} \left(1-r^{2}\right)^{\alpha} r^{2N-1} dr,$$

which tends to A as $\alpha \to -1$. Since the first term of the right-hand side of (2.1) is non-negative, we finally get

$$\lim_{\alpha \to -1} \int_{\mathbb{R}_N} f(z) dv_{\alpha}(z) \ge A$$

for any A positive, that is

$$\lim_{\alpha \to -1} \int_{\mathbb{R}_N} f(z) dv_{\alpha}(z) = \lim_{r \to 1} \int_{\mathbb{S}_N} f(rz) d\sigma_N(z) = \infty.$$

Let now assume that $\lim_{r\to 1} F(r)$ exists in $(0,\infty)$. For every $\varepsilon > 0$ one can find $\eta(\varepsilon) \in (0,1)$ close enough to 1 such that

$$\int_{S_N} f(r\zeta)d\sigma_N(\zeta) - \lim_{r \to 1} F(r) \le \varepsilon, \ r \in (1 - \eta, 1).$$

Using (2.2) we get, for any $\varepsilon > 0$,

$$\lim_{\alpha \to -1} \int_{\mathbb{B}_N} f(z) dv_{\alpha}(z) - \lim_{r \to 1} F(r)$$

$$= \lim_{\alpha \to -1} 2Nc_{\alpha} \int_{1-\eta}^{1} \left(\int_{\mathbb{S}_N} f(r\zeta) - \lim_{r \to 1} F(r) \right) \left(1 - r^2 \right)^{\alpha} r^{2N-1} dr \le \varepsilon,$$

as expected.

In particular, given $g \in H(\mathbb{B}_N)$, ψ an Orlicz function and a > 0, and applying Lemma 2.2 to the subharmonic function $f = \psi\left(\frac{|g|}{a}\right)$ we get $\|g\|_{H^{\psi}} = \lim_{\alpha \to -1} \|g\|_{A^{\psi}_{\alpha}}$.

For now on we will use the unique notation $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ to denote the weighted Bergman-Orlicz space when $\alpha > -1$, and the Hardy-Orlicz space $H^{\psi}(\mathbb{B}_{N})$ when $\alpha = -1$. More abusively, L_{α}^{ψ} (resp. M_{α}^{ψ}), $\alpha \geq -1$, will often stand for $L^{\psi}(\mathbb{B}_{N}, v_{\alpha})$ (resp. $M^{\psi}(\mathbb{B}_{N}, v_{\alpha})$) when $\alpha > -1$ and for $L^{\psi}(\mathbb{S}_{N}, \sigma_{N})$ (resp. $M^{\psi}(\mathbb{S}_{N}, \sigma_{N})$) when $\alpha = -1$, v_{α} being σ_{N} when $\alpha = -1$.

The following result specifies the topological and dual properties of Hardy-Orlicz and Bergman-Orlicz spaces, pointing out that if we intend to generalize the classical Hardy or Bergman spaces and provide with a refined scale of spaces up to H^{∞} , then we must consider Banach spaces with less nice properties.

Theorem 2.3 ([11, 14]). Let $\alpha \geq -1$ and let ψ be an Orlicz function.

- (1) $AM_{\alpha}^{\psi}(\mathbb{B}_{N})$ is the closure of H^{∞} in L_{α}^{ψ} ; in particular, the set of all polynomials is dense in $AM^{\psi}(\mathbb{B}_{N})$.
- (2) On the unit ball of $AM_{\alpha}^{\psi}(\mathbb{B}_N)$ (or equivalently on every ball), the induced weak-star topology coincides with that of uniform convergence on compacta of \mathbb{B}_N .
- (3) $A^{\psi}_{\alpha}(\mathbb{B}_N)$ is weak-star closed in L^{ψ}_{α} .
- (4) Assume that ψ satisfies the ∇_2 -condition. Then $A^{\psi}_{\alpha}(\mathbb{B}_N)$ can be isometrically identified with $(AM^{\psi}_{\alpha}(\mathbb{B}_N))^{**}$. Moreover, C_{ϕ} is equal to the biadjoint of $C_{\phi}|_{AM^{\psi}_{\alpha}(\mathbb{B}_N)}$.

Note that when $\psi(x) = x^p$, $1 \le p < +\infty$, the previous theorem reminds that the classical Hardy and Bergman spaces are separable and reflexive, a situation which no longer occurs when the Orlicz function does not satisfy the Δ_2 -condition. Nevertheless it also points out some similarities with Bloch-type spaces or weighted Banach spaces, and more generally with M-ideals [35].

We finish this paragraph by recalling sharp estimates for the point evaluation functionals on $A^{\psi}_{\alpha}(\mathbb{B}_N)$, $\alpha \geq -1$.

Proposition 2.4 ([11, 14]). Let $\alpha \geq -1$ and let ψ be an Orlicz function. For any $a \in \mathbb{B}_N$, the point evaluation functional δ_a at a is bounded on $A^{\psi}_{\alpha}(\mathbb{B}_N)$ and we have

$$\frac{1}{4^{N+\alpha+1}}\psi^{-1}\left(\left(\frac{1+|a|}{1-|a|}\right)^{N+\alpha+1}\right) \le \|\delta_a\|_{\left(A_{\alpha}^{\psi}\right)^*} = \|\delta_a\|_{\left(AM_{\alpha}^{\psi}\right)^*} \le \psi^{-1}\left(\left(\frac{1+|a|}{1-|a|}\right)^{N+\alpha+1}\right).$$

2.2. A first characterization of small weighted Bergman-Orlicz spaces. We start with the following observation.

Lemma 2.5. Let $\alpha > -1$ and ψ be an Orlicz function satisfying the Δ^2 -condition. Then the function $\psi^{-1}\left(\frac{1}{(1-|z|)^a}\right)$ belongs to $L^{\psi}\left(\mathbb{B}_N, v_{\alpha}\right)$ for every a > 0.

Proof. We recall that $(1-|z|)^{\beta}$ belongs to $L^{1}(\mathbb{B}_{N},v)$ if and only if $\beta > -1$. Then it is enough to show that for any a > 0, there exists some constant C > 0 and some $\beta > -1$ such that

$$\psi\left(C\psi^{-1}\left(\frac{1}{(1-|z|)^a}\right)\right)(1-|z|)^{\alpha} \le (1-|z|)^{\beta}$$

for every |z| close enough to 1. Since ψ satisfies the Δ^2 -condition, Proposition 2.1 (3) ensures that for any b>1 there exists C>0 such that $\psi(x)^b\leq \psi(Cx)$ for any x large enough. Using that ψ is increasing, the last inequality becomes $\psi\left(C^{-1}\psi^{-1}\left(\psi(x)^b\right)\right)\leq \psi(x)$, and setting $y=(\psi(x))^b$ gives $\psi\left(C^{-1}\psi^{-1}(y)\right)\leq y^{1/b}$ for any y large enough. Thus, replacing y with $1/\left(1-|z|\right)^a$, we obtain

$$\psi\left(C^{-1}\psi^{-1}\left(\frac{1}{(1-|z|)^a}\right)\right)(1-|z|)^{\alpha} \le (1-|z|)^{\alpha-a/b}$$

for every |z| close enough to 1. The lemma follows by choosing b large enough in order that $\alpha - a/b > -1$ (possible for $\alpha > -1$).

Let us give another easy lemma.

Lemma 2.6. Let ψ be an Orlicz function satisfying the Δ^2 -condition. For every $a \ge 1$, there exists a constant C > 0 such that $\psi^{-1}(y) \le \psi^{-1}(y^a) \le C\psi^{-1}(y)$ for any y large enough.

Proof. The first inequality is valid for any $y \ge 1$ since ψ^{-1} is increasing and $a \ge 1$. For the second inequality, for b > 1 let C > 0 be given by Proposition 2.1 (3). We have $\psi(x)^b \le \psi(Cx)$ for any x large enough, hence setting $y = (\psi(x))^{b/a}$ as in the previous proof and composing the last inequality by the increasing function ψ^{-1} , we get $\psi^{-1}(y^a) \le C\psi^{-1}(y^{a/b})$ for any y large enough. To finish we choose b so that a/b is smaller than 1 and we use again that ψ^{-1} is increasing.

For any Orlicz function ψ and and $\gamma \geq 1$, let us define the following typical radial weight:

(2.3)
$$w^{\psi,\gamma}(z) = \left[\psi^{-1} \left(\frac{1}{(1-|z|)^{\gamma}} \right) \right]^{-1}, z \in \mathbb{B}_N.$$

When $\gamma=1$, we will denote $w^{\psi}=w^{\psi,\gamma}$. Proposition 2.4 tells that $A_{\alpha}^{\psi}(\mathbb{B}_{N})\subset H_{w^{\psi,N+\alpha+1}}^{\infty}(\mathbb{B}_{N})$ for every ψ and every $\alpha\geq -1$. Moreover, Lemma 2.5 ensures that if $\psi\in\Delta^{2}$ and $\alpha>-1$, then any f in $H_{w^{\psi,N+\alpha+1}}^{\infty}(\mathbb{B}_{N})$ lies also in $A_{\alpha}^{\psi}(\mathbb{B}_{N},v_{\alpha})$. We deduce that, whenever $\psi\in\Delta^{2}$ and $\alpha>-1$, $A_{\alpha}^{\psi}(\mathbb{B}_{N},v_{\alpha})=H_{w^{\psi,N+\alpha+1}}^{\infty}(\mathbb{B}_{N})$ and finally by Lemma 2.6 that $A_{\alpha}^{\psi}(\mathbb{B}_{N},v_{\alpha})=H_{w^{\psi}}^{\infty}(\mathbb{B}_{N})$ (see (1.1) in the introduction for the definition of the weighted Banach space $H_{v}^{\infty}(\mathbb{B}_{N})$). Actually the following theorem tells us a bit more.

Let us recall first that the vanishing weighted Banach space -or vanishing growth spaceassociated with v is defined by

$$H_v^0 := \left\{ f \in H(\mathbb{B}_N), \lim_{|z| \to 1} |f(z)| v(z) = 0 \right\},$$

and, endowed with the norm $\|\cdot\|_v$, is a closed subspace of H_v^{∞} .

Theorem 2.7. Let $\alpha > -1$ and let ψ be an Orlicz function satisfying the Δ^2 -condition. Then the weighted Bergman-Orlicz space $A^{\psi}_{\alpha}(\mathbb{B}_N)$ coincides with the growth space $H^{\infty}_{w^{\psi}}(\mathbb{B}_N)$, with an equivalent norm. Moreover $AM^{\psi}_{\alpha}(\mathbb{B}_N) = H^0_{w^{\psi}}(\mathbb{B}_N)$.

Proof. It remains to prove that the norms of $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ and $H_{w^{\psi}}^{\infty}(\mathbb{B}_{N})$ are equivalent and that $AM_{\alpha}^{\psi}(\mathbb{B}_{N}) = H_{w^{\psi}}^{0}(\mathbb{B}_{N})$. Let us assume for a while that the first assertion holds. Then the topologies of $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ and $H_{w^{\psi}}^{\infty}(\mathbb{B}_{N})$ are the same and it is known that $AM_{\alpha}^{\psi}(\mathbb{B}_{N})$ and $H_{w^{\psi}}^{0}(\mathbb{B}_{N})$ are the closure of the set of polynomials for this topology (Theorem 2.3 and [41]), hence the second assertion.

Let us now prove that the norms of $A^{\psi}_{\alpha}(\mathbb{B}_N)$ and $H^{\infty}_{w^{\psi}}(\mathbb{B}_N)$ are indeed equivalent. According to Proposition 2.4 and Lemma 2.6 we have

$$|f(z)| \left[\psi^{-1} \left(\frac{1}{1 - |z|} \right) \right]^{-1} \le 2^{N + \alpha + 1} C \|f\|_{A_{\alpha}^{\psi}}$$

for any $f \in A_{\alpha}^{\psi}(\mathbb{B}_N)$ and any $z \in \mathbb{B}_N$ (note that we have used the concavity of ψ^{-1}), where C > 0 is the constant given by Lemma 2.6 and which only depends on ψ and α . This gives $\|\cdot\|_{w^{\psi}} \lesssim \|\cdot\|_{A_{\alpha}^{\psi}}$. For the other inequality, we simply observe that for $f \in A_{\alpha}^{\psi}(\mathbb{B}_N) = H_{w^{\psi}}^{\infty}(\mathbb{B}_N)$,

$$\frac{|f(z)|}{\|f\|_{w^{\psi}}} = \frac{|f(z)|}{\sup_{z \in \mathbb{B}_N} |f(z)w^{\psi}(z)|} \le \psi^{-1} \left(\frac{1}{1 - |z|}\right), \ z \in \mathbb{B}_N.$$

By Lemma 2.5 $\psi^{-1}\left(\frac{1}{1-|z|}\right)$ is in $L^{\psi}(\mathbb{B}_N, v_{\alpha})$ so there exists a constant C' > 0 (depending only on ψ and α) such that

$$\int_{\mathbb{B}_N} \psi\left(\frac{|f(z)|}{C' \|f\|_{w^{\psi}}}\right) dv_{\alpha}(z) \le \int_{\mathbb{B}_N} \psi\left(\frac{1}{C'} \psi^{-1} \left(\frac{1}{1-|z|}\right)\right) dv_{\alpha}(z) \le 1.$$

The expected inequality $||f||_{A_{\alpha}^{\psi}} \leq C' ||f||_{w^{\psi}}$ follows.

Remark. 1) We will show in the next section that the Δ^2 -condition actually characterizes those Orlicz functions ψ for which the associated weighted Bergman-Orlicz space $A^{\psi}_{\alpha}(\mathbb{B}_N)$ coincides with the weighted Banach space $H^{\infty}_{w^{\psi}}(\mathbb{B}_N)$. We will also see that this result does not hold for Hardy-Orlicz spaces.

- 2) Note that, by Theorem 2.7, $A^{\psi}_{\alpha}(\mathbb{B}_N) = A^{\psi}_{\beta}(\mathbb{B}_N)$ for any $\alpha, \beta > -1$ whenever the Orlicz function ψ satisfies the Δ^2 -condition; in other words, the corresponding weighted Bergman-Orlicz spaces coincide with the non-weighted ones.
- 3) We shall recall Kaptanoğlu's result [21] telling that $\mathcal{B}^{\gamma}(\mathbb{B}_N) = H_{v^{\gamma}}^{\infty}(\mathbb{B}_N)$ and $\mathcal{B}_0^{\gamma}(\mathbb{B}_N) = H_{v^{\gamma}}^{0}(\mathbb{B}_N)$ whenever $\gamma > 0$, where v(z) = 1 |z| and $\mathcal{B}^{\gamma}(\mathbb{B}_N)$ stands for the weighted Bloch space. Theorem 2.7 completes his description for the weighted Banach spaces with slow growth.

The following corollary immediately proceeds from Proposition 2.4 and Lemma 2.5.

Corollary 2.8. Let $\alpha > -1$ and ψ be an Orlicz function satisfying the Δ^2 -condition. Then the function $z \mapsto \|\delta_z\|_{(A_{\alpha}^{\psi})^*}$ belongs to $L^{\psi}(\mathbb{B}_N, v_{\alpha})$.

The previous provides another characterization of weighted Bergman-Orlicz spaces associated with Δ^2 Orlicz functions.

Corollary 2.9. Let $\alpha > -1$ and let ψ be an Orlicz function satisfying the Δ^2 -condition. $A^{\psi}_{\alpha}(\mathbb{B}_N)$ coincides with the set of all functions f holomorphic on \mathbb{B}_N such that there exists A > 0 such that

(2.4)
$$c_{\alpha} \int_{0}^{1} \sup_{\zeta \in S_{N}} \psi\left(\frac{|f(r\zeta)|}{A}\right) r^{2N-1} \left(1 - r^{2}\right)^{\alpha} dr < +\infty.$$

Moreover, if we define $\|\cdot\|_{A^{\psi}_{\alpha,\infty}}$ as the infimum of those constants A>0 such that the left hand-side of (2.4) is less than or equal to 1, then $\|\cdot\|_{A^{\psi}_{\alpha,\infty}}$ is a norm on $A^{\psi}_{\alpha}(\mathbb{B}_N)$ and we have

$$||f||_{A_{\alpha}^{\psi}} \le ||f||_{A_{\alpha,\infty}^{\psi}} \le K ||f||_{A_{\alpha}^{\psi}},$$

for any $f \in A_{\alpha}^{\psi}(\mathbb{B}_N)$, and some constant K > 0.

Proof. According to Corollary 2.8, the maximum modulus principle and an integration in polar coordinates, there exists a function $g \in L^{\psi}([0,1], c_{\alpha}r^{2N-1}(1-r^2)^{\alpha}dr)$ such that for any $f \in A_{\alpha}^{\psi}(\mathbb{B}_N)$,

(2.6)
$$\sup_{|z| \le r} |f(z)| \le g(r) \|f\|_{A_{\alpha}^{\psi}},$$

hence the first assertion. For the last assertions, observe first that the inequality $||f||_{A_{\alpha}^{\psi}} \le ||f||_{A_{\alpha,\infty}^{\psi}}$ is trivial for any f holomorphic on \mathbb{B}_N . Let now A > 0 and $f \in A_{\alpha}^{\psi}(\mathbb{B}_N)$. By (2.6) we get

$$\int_0^1 \sup_{\zeta \in \mathbb{S}_N} \psi\left(\frac{|f(r\zeta)|}{A}\right) r^{2N-1} \left(1-r^2\right)^\alpha dr \leq \int_0^1 \psi\left(\frac{g(r)\left\|f\right\|_{A_\alpha^\psi}}{A}\right) r^{2N-1} \left(1-r^2\right)^\alpha dr.$$

Let K be the norm of g in $L^{\psi}([0,1], c_{\alpha}r^{2N-1}(1-r^2)^{\alpha}dr)$. If $A > ||f||_{A_{\alpha}^{\psi}}K$, then the left hand-side of the previous inequality is less than or equal to 1, so $A \ge ||f||_{A_{\alpha}^{\psi}}$ hence (2.5).

The fact that $\|\cdot\|_{A^{\psi}_{\alpha,\infty}}$ is a norm is easily checked.

Remark 2.10. Notice that the constant K above can be taken equal to the norm in L_{α}^{ψ} of $z \mapsto \|\delta_z\|_{(A_{\alpha}^{\psi})^*}$, up to some constant depending only on N and α .

In the next section we will apply those observations to composition operators, and furthermore we will be able to refine Theorem 2.7. Of course, in view of Theorem 2.7, the forthcoming results involving $A^{\psi}_{\alpha}(\mathbb{B}_N)$ with $\psi \in \Delta^2$, can be re-read with $H^{\infty}_{w^{\psi}}(\mathbb{B}_N)$ instead of $A^{\psi}_{\alpha}(\mathbb{B}_N)$.

3. Application to composition operators

In the sequel we will use the notations $w^{\psi,\gamma}$ and w^{ψ} , see (2.3).

3.1. Preliminary results.

3.1.1. Composition operators on weighted Bergman-Orlicz and Hardy-Orlicz spaces of \mathbb{B}_N . A useful tool to study the boundedness and the compactness of composition operators acting on Hardy-Orlicz and Bergman-Orlicz spaces are Carleson embedding theorems. In this paper we will use these techniques only to deal with the compactness. They involve geometric notions that we will briefly introduce here. For $\zeta \in \mathbb{B}_N$ and 0 < h < 1, let us denote by $S(\zeta, h)$ and $S(\zeta, h)$ the non-isotropic "balls", respectively in \mathbb{B}_N and $\overline{\mathbb{B}}_N$, defined by

$$S(\zeta, h) = \{z \in \mathbb{B}_N, |1 - \langle z, \zeta \rangle| < h\} \text{ and } S(\zeta, h) = \{z \in \overline{\mathbb{B}_N}, |1 - \langle z, \zeta \rangle| < h\}.$$

For $\phi : \mathbb{B}_N \to \mathbb{B}_N$, we denote by $\mu_{\phi,\alpha}$ the pull-back measure of v_{α} by ϕ and by μ_{ϕ} that of σ_N by the boundary limit ϕ^* of ϕ . To be precise, given $E \subset \mathbb{B}_N$ (resp. $E \subset \overline{\mathbb{B}_N}$),

$$\mu_{\phi,\alpha}\left(E\right) = v_{\alpha}\left(\phi^{-1}\left(E\right)\right) \text{ (resp. } \mu_{\phi}\left(E\right) = \sigma_{N}\left(\left(\phi^{*}\right)^{-1}\left(E\right) \cap \mathbb{S}_{N}\right)\right).$$

In view of the end of Paragraph 2.1, we will unify the notations denoting also by $\mu_{\phi,\alpha}$, with $\alpha = -1$, the measure μ_{ϕ} (the latter acting on $\overline{\mathbb{B}_N}$). Then we define the functions $\varrho_{\phi,\alpha}$ on the interval (0,1) by

$$\varrho_{\phi,\alpha}(h) = \begin{cases} \sup_{\zeta \in \mathbb{S}_N} \mu_{\phi,\alpha}(S(\zeta,h)) & \text{if } \alpha > -1\\ \sup_{\zeta \in \mathbb{S}_N} \mu_{\phi,\alpha}(S(\zeta,h)) & \text{if } \alpha = -1 \end{cases}$$

First of all, by testing the boundedness of C_{ϕ} on a standard function, one easily gets the following necessary condition.

Fact 3.1. If C_{ϕ} is bounded on $A_{\alpha}^{\psi}(\mathbb{B}_N)$ then there exist A > 0 and $0 < \eta < 1$, such that

$$\varrho_{\phi,\alpha}(h) \le \frac{1}{\psi\left(A\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)}$$

for every $h \in (0, \eta)$.

The next theorem, stated for ψ in the Δ^2 -class, is a particular case of the one obtained in [11, 14] for a larger class of Orlicz functions (see also [25, 28] for the unit disc).

Theorem 3.2. Let $\alpha \geq -1$, let ψ be an Orlicz function satisfying the Δ^2 -condition, and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. The composition operator C_{ϕ} is compact from A_{α}^{ψ} into itself if and only if for every A > 0, there exists $h_A \in (0,1)$ such that

$$\varrho_{\phi,\alpha}(h) \le \frac{1}{\psi\left(A\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)}$$

for every $h \in (0, h_A)$.

For $\alpha \geq -1$ and $a \in \mathbb{B}_N$, we define the function $f_a \in H^{\infty}$ by

$$f_a(z) = (1 - |a|)^{N+\alpha+1} \left(\frac{1 - |a|^2}{(1 - \langle z, a \rangle)^2} \right)^{N+\alpha+1}.$$

Then $|f_a(z)| = (1 - |a|)^{N+\alpha+1} H_a(z)$ where H_a is the Berezin kernel. The following corollary will be useful for the proof of $(2) \Leftrightarrow (3)$ in Theorem 3.23.

Corollary 3.3. Let $\alpha \geq -1$, let ψ be an Orlicz function satisfying the Δ^2 -condition and let $\phi: \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. C_{ϕ} is compact from $A_{\alpha}^{\psi}(\mathbb{B}_N)$ into itself if and only if

(3.1)
$$||f_a \circ \phi||_{A_{\alpha}^{\psi}(\mathbb{B}_N)} = o_{|a| \to 1} \left(\frac{1}{\psi^{-1} \left(1/\left(1 - |a|\right)^{N + \alpha + 1} \right)} \right).$$

Remark. This result was stated in [25] for an arbitrary Orlicz function, when $\alpha = -1$ and N=1. Theorem 3.2, Corollary 3.3 and its proof below, work for a class of Orlicz functions much larger than the Δ^2 -class, namely for Orlicz functions satisfying the so-called ∇_0 -condition; we refer to [11, 14].

Proof of the Corollary. For the only if part, we introduce the function

$$g_a(z) = \frac{\psi^{-1} \left(1/\left(1 - |a|\right)^{N+\alpha+1} \right)}{2^{N+\alpha+1}} f_a(z), z \in \mathbb{B}_N.$$

 g_a lies in the unit ball of $A_{\alpha}^{\psi}(\mathbb{B}_N)$ (see [25, Lemma 3.9] and the proofs of [11, 14, Propositions 1.9 and 1.6 respectively) and tends to 0 uniformly on every compact set of \mathbb{B}_N , as |a| tends to 1. In particular $g_a \circ \phi$ converges pointwise to 0 as $|a| \to 1$. C_{ϕ} being compact on $A_{\alpha}^{\psi}(\mathbb{B}_N)$, up to take a subsequence, we may assume that $g_a \circ \phi$ tends in $A_\alpha^{\psi}(\mathbb{B}_N)$ to some function g as $|a| \to 1$, hence g = 0 since the convergence in $A^{\psi}_{\alpha}(\mathbb{B}_N)$ implies the pointwise one. This ends the proof of the only if part.

For the converse, we only deal with $\alpha > -1$, the case $\alpha = -1$ being the same up to change the non-isotropic balls. It follows from the proof of [14, Theorem 2.5 (1)] that, given $\zeta \in \mathbb{S}_N$ and $a = |a|\zeta \in \mathbb{B}_N$, the following estimate holds:

$$\mu_{\phi,\alpha}(S(\zeta,1-|a|)) \le \frac{1}{\psi\left(K/\|f_a \circ \phi\|_{A_{\alpha}^{\psi}}\right)}$$

for some constant K>0 independent of ζ . So, if (3.1) holds, then for every $\varepsilon>0$, there exists $h_{\varepsilon} \in (0,1)$ such that for every $\zeta \in \mathbb{S}_N$ and every $a = |a|\zeta \in \mathbb{B}_N$ with $|a| \in (1 - h_{\varepsilon}, 1)$ we have

(3.2)
$$\mu_{\phi,\alpha}(S(\zeta, 1 - |a|)) \le \frac{1}{\psi\left(\frac{K}{\varepsilon}\psi^{-1}\left(1/(1 - |a|)^{N+\alpha+1}\right)\right)},$$

hence C_{ϕ} is compact on $A_{\alpha}^{\psi}(\mathbb{B}_N)$ by Theorem 3.2.

We deduce from the previous corollary some sufficient conditions for the compactness of C_{ϕ} on $A_{\alpha}^{\psi}(\mathbb{B}_N)$.

Proposition 3.4. Let $\alpha \geq -1$, let ψ be an Orlicz function satisfying the Δ^2 -condition and let $\phi: \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. Then C_{ϕ} is compact on $A_{\alpha}^{\psi}(\mathbb{B}_N)$ whenever one of the two following conditions is satisfied:

- (i) $1/(1-|\phi|) \in L_{\alpha}^{\psi}$; (ii) $\sum_{n=0}^{\infty} ||\phi|^n||_{L_{\alpha}^{\psi}} < \infty$.

Proof. Observe that

$$\limsup_{|a|\to 1} \psi^{-1} \left(\frac{1}{(1-|a|)^{N+\alpha+1}} \right) \|f_a \circ \phi\|_{A_{\alpha}^{\psi}} \\
\lesssim \limsup_{|a|\to 1} \psi^{-1} \left(\frac{1}{(1-|a|)^{N+\alpha+1}} \right) (1-|a|)^{N+\alpha+1} \left\| \frac{1}{(1-|\phi|)^{N+\alpha+1}} \right\|_{L_{\alpha}^{\psi}}.$$

Since ψ is an Orlicz function,

$$\lim_{|a|\to 1} \sup \psi^{-1} \left(\frac{1}{(1-|a|)^{N+\alpha+1}} \right) (1-|a|)^{N+\alpha+1} = 0,$$

and, by Corollary 3.3, C_{ϕ} is compact on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$, $\alpha \geq -1$, whenever $1/\left(1-|\phi|\right)^{N+\alpha+1}$ is in L_{α}^{ψ} . Now since ψ satisfies the Δ^{2} -condition, Lemma 2.6 makes it clear that (i) is indeed sufficient for the compactness of C_{ϕ} . To deal with (ii), we may just notice that

$$\left\| \frac{1}{1 - |\phi(z)|} \right\|_{L_{\alpha}^{\psi}} \le \sum_{n=0}^{\infty} \||\phi|^{n}\|_{L_{\alpha}^{\psi}}.$$

Remark. (1) Like Theorem 3.2 and Corollary 3.3, the previous proposition actually holds for Orlicz functions satisfying the so-called ∇_0 -condition.

(2) In Section 3.4, we shall see some conditions similar to (i) or (ii) above, but necessary and sufficient for the compactness of C_{ϕ} on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ when ψ satisfies the Δ^{2} -condition.

3.1.2. Composition operators on weighted Banach spaces of \mathbb{B}_N . In [8], the authors give a mild condition on a typical weight v on the unit disc \mathbb{D} to ensure that every composition operators acting the weighted Banach space $H_v^{\infty}(\mathbb{D})$ is bounded. We provide below a straightforward proof of this fact for the specific weights $w^{\psi,\gamma}$ on the unit ball \mathbb{B}_N . We need two lemmas and some definition.

For a typical weight v on \mathbb{B}_N , let $B_v = \left\{ f \in H(\mathbb{B}_N); |f| \leq \frac{1}{v} \right\}$ denote the unit ball of

 $H_v^{\infty}(\mathbb{B}_N)$. We define $\widetilde{v}(z) = \left(\sup_{\|f\|_v < 1} \{|f(z)|\}\right)^{-1}$, $z \in \mathbb{B}_N$, and say that v is essential if there exist two constants c and C such that

$$c\widetilde{v}(z) \le v(z) \le C\widetilde{v}(z)$$

for any $z \in \mathbb{B}_N$.

Lemma 3.5. For $N \geq 1$, $\gamma \geq N$ and ψ an Orlicz function, the weight $w^{\psi,\gamma}$ is essential.

Proof. The inequality $w^{\psi,\gamma}(z) \leq w^{\psi,\gamma}(z)$ follows from the definition of $w^{\psi,\gamma}(z)$. For the other one, let us choose $\alpha \geq -1$ such that $\gamma = N + \alpha + 1$ (this is possible since $\gamma \geq N$). We now use Proposition 2.4 to observe first that $\|\cdot\|_{w^{\psi,\gamma}} \leq 2^{N+\alpha+1} \|\cdot\|_{A^{\psi}_{\alpha}}$ and second that for every $z \in \mathbb{B}_N$, there exists f in the unit ball of $A^{\psi}_{\alpha}(\mathbb{B}_N)$ such that $|f(z)| \geq \frac{1}{4^{N+\alpha+1}w^{\psi,\gamma}(z)}$; hence the function $g := f/2^{N+\alpha+1}$ belongs to the unit ball $B_{w^{\psi,\gamma}}$ of $H^{\infty}_{w^{\psi,\gamma}}$ and satisfies $|g(z)| \geq \frac{1}{8^{N+\alpha+1}w^{\psi,\gamma}(z)}$. Then for every $z \in \mathbb{B}_N$, $\frac{1}{w^{\psi,\gamma}(z)} \geq \frac{1}{8^{N+\alpha+1}w^{\psi,\gamma}(z)}$, as expected. \square

Note that the previous proof underlines the fact that a weight v is essential if and only if there exists a Banach space of holomorphic functions X on \mathbb{B}_N , continuously embedded into $H_v^{\infty}(\mathbb{B}_N)$, such that for any $z \in \mathbb{B}_N$, 1/v(z) is the norm of the evaluation at z on X, up to some constants independent of z.

More generally, some conditions for a weight on \mathbb{D} to be essential were exhibited in [6] and, more recently, some general characterizations of such weights have been given in [1].

Lemma 3.6. Let ψ be an Orlicz function, $\gamma \geq 1$ and ϕ a holomorphic self-map of \mathbb{B}_N . Then

$$\sup_{z \in \mathbb{B}_N} \frac{\psi^{-1}\left(\frac{1}{(1-|\phi(z)|)^{\gamma}}\right)}{\psi^{-1}\left(\frac{1}{(1-|z|)^{\gamma}}\right)} < +\infty.$$

Proof. Let $a = \phi(0)$ and let φ_a be an automorphism of \mathbb{B}_N which vanishes at a. Since φ_a is an involution, $\phi = \varphi_a \circ \varphi_a \circ \phi$, and $\varphi_a \circ \phi(0) = 0$. Recall that for any automorphism φ_a of \mathbb{B}_N , we have

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle a, z \rangle|^2}$$

for every $z \in \mathbb{B}_N$ [37, Theorem 2.2.2]. Then, applying the Schwarz lemma to $\varphi_a \circ \phi$ and using the convexity of ψ , it follows that

$$\psi^{-1}\left(\frac{1}{(1-|\phi(z)|)^{\gamma}}\right) \leq 2^{\gamma}\psi^{-1}\left(\frac{1}{\left(1-|\varphi_{a}\circ\varphi_{a}\circ\phi(z)|^{2}\right)^{\gamma}}\right)$$

$$= 2^{\gamma}\psi^{-1}\left(\left(\frac{|1-\langle a,\varphi_{a}\circ\phi(z)\rangle|^{2}}{\left(1-|a|^{2}\right)\left(1-|\varphi_{a}\circ\phi(z)|^{2}\right)}\right)^{\gamma}\right)$$

$$\leq \left(\frac{2}{1-|a|}\right)^{\gamma}\psi^{-1}\left(\frac{1}{(1-|z|)^{\gamma}}\right).$$

Let $f \in H^{\infty}_{w^{\psi,\gamma}}(\mathbb{B}_N)$. By Lemma 3.6 we have $w^{\psi,\gamma}(z) \leq Cw^{\psi,\gamma}(\phi(z))$ for some constant C > 0. Thus

$$(3.3) |f \circ \phi(z)| \, w^{\psi,\gamma}(z) = |f \circ \phi(z)| \, w^{\psi,\gamma}(\phi(z)) \frac{w^{\psi,\gamma}(z)}{w^{\psi,\gamma}(\phi(z))} \le C \, ||f||_{H^{\infty}_{w^{\psi,\gamma}}(\phi(z))}$$

and we deduce:

Proposition 3.7. Let ψ be an Orlicz function and $\gamma \geq 1$. For every holomorphic self-map ϕ of \mathbb{B}_N , the composition operator C_{ϕ} is bounded on $H^{\infty}_{w^{\psi,\gamma}}(\mathbb{B}_N)$.

Since $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ coincides with $H_{w^{\psi}}^{\infty}(\mathbb{B}_{N})$ whenever ψ satisfies the Δ^{2} -condition (Theorem 2.7), the first assertion of the following corollary immediately proceeds from the previous proposition.

Corollary 3.8. Let $\alpha > -1$ and ψ be an Orlicz function satisfying the Δ^2 -condition. For every holomorphic self-map ϕ of \mathbb{B}_N , C_{ϕ} is bounded on $A_{\alpha}^{\psi}(\mathbb{B}_N)$. Moreover

$$(3.4) ||C_{\phi}||_{A_{\alpha}^{\psi} \to A_{\alpha}^{\psi}} \simeq \sup_{z \in \mathbb{B}_{N}} \frac{\psi^{-1} \left(\frac{1}{(1 - |\phi(z)|)^{N + \alpha + 1}}\right)}{\psi^{-1} \left(\frac{1}{(1 - |z|)^{N + \alpha + 1}}\right)} \simeq \sup_{z \in \mathbb{B}_{N}} \frac{\psi^{-1} \left(\frac{1}{1 - |\phi(z)|}\right)}{\psi^{-1} \left(\frac{1}{1 - |z|}\right)}.$$

Proof. Only (3.4) needs to be checked. Now, starting with (3.3), the first estimate is a consequence of Theorem 2.7 and Lemma 3.5 (applied to $\gamma = N + \alpha + 1$) and the second then follows from Lemma 2.6.

Remark. (1) Using Carleson measure theorems, it was already stated in [11, 14] that every composition operator is bounded on $A^{\psi}_{\alpha}(\mathbb{B}_N)$ and $H^{\psi}(\mathbb{B}_N)$ whenever ψ satisfies the Δ^2 -condition. Yet, whatever the Orlicz function, no explicit estimate of the norm of a composition operator where known for N > 1.

- (2) We will see in the next paragraph that the estimates in (3.4) does not hold whenever ψ does not satisfy the Δ^2 -condition; indeed, while the right-hand side of the estimate is always bounded (Lemma 3.6), we will see that, if $\psi \notin \Delta^2$, then there exist unbounded composition operators.
- (3) More generally, it is easily seen that the norm of C_{ϕ} on H_{v}^{∞} , v a typical radial weight, is equal to $\sup_{z \in \mathbb{B}_{N}} \frac{v(z)}{\widetilde{v}(\phi(z))}$, where \widetilde{v} is defined similarly as in Lemma 3.5 (see [7, 8] for N = 1) and can be replaced with v, up to some constant, whenever v is essential.

In the next paragraph, we will show that the Δ^2 -condition in fact characterizes these extreme behaviors.

- 3.2. Order boundedness of composition operators. Let us briefly recall that an operator T from a Banach space X to a Banach lattice Y is order bounded into some subspace Z of Y if there exists $y \in Z$ positive such that $|Tx| \leq y$ for every x in the unit ball of X. The following lemma is immediate and its proof is left to the reader.
- **Lemma 3.9.** Let Y be a Banach lattice and X a closed sublattice of Y. If the canonical embedding I from X into Y is order-bounded into Y, then a linear map $T: X \to Y$ with $T(X) \subset X$ is order-bounded into Y if and only if it is bounded (from X into X).

In the sequel X will stand for $A_{\alpha}^{\psi}(\mathbb{B}_{N})$, Y for L_{α}^{ψ} and Z for either Y or M_{α}^{ψ} , with $\alpha \geq -1$. Corollary 2.8 can be reformulated by saying that, if ψ satisfies the Δ^{2} -condition, then the canonical embedding I from $A_{\alpha}^{\psi}(\mathbb{B}_{N}, v_{\alpha})$ into $L^{\psi}(\mathbb{B}_{N}, v_{\alpha})$ is order bounded into $L^{\psi}(\mathbb{B}_{N}, v_{\alpha})$. Thus, Lemma 3.9 together with Corollary 3.8 already yields:

Theorem 3.10. Let $\alpha > -1$ and let ψ be an Orlicz function satisfying the Δ^2 -condition. Every composition operator acting on $A^{\psi}_{\alpha}(\mathbb{B}_N)$ is order bounded into $L^{\psi}(\mathbb{B}_N, v_{\alpha})$.

Remark 3.11. Since $H^{\psi}(\mathbb{B}_N)$ is canonically embedded into $A^{\psi}_{\alpha}(\mathbb{B}_N)$, it immediately follows from the previous result that the canonical injection $H^{\psi}(\mathbb{B}_N) \hookrightarrow A^{\psi}_{\alpha}(\mathbb{B}_N)$ is order bounded into L^{ψ}_{α} for any $\alpha > -1$. Yet, as we will see (Remark 3.16), the canonical embedding from $H^{\psi}(\mathbb{B}_N)$ into L^{ψ}_{α} with $\alpha = -1$ is not order-bounded into L^{ψ}_{α} , and Theorem 3.10 does not hold for Hardy-Orlicz spaces.

Moreover, proceeding as in [25, page 18, the Remark], the next proposition can be easily checked.

Proposition 3.12. Let $\alpha \geq -1$ and let ψ be an Orlicz function. Every composition operator acting on $A^{\psi}_{\alpha}(\mathbb{B}_N)$ which is order bounded into M^{ψ}_{α} is compact.

By definition, if δ_z stands for the functional of evaluation at the point z and ϕ denotes a holomorphic self-map of \mathbb{B}_N , then the composition operator C_{ϕ} acting on $A_{\alpha}^{\psi}(\mathbb{B}_N)$, $\alpha \geq -1$, is order bounded into L_{α}^{ψ} (resp. M_{α}^{ψ}) if and only if the function $z \mapsto \|\delta_{\phi(z)}\|_{(A_{\alpha}^{\psi})^*}$ belongs to L_{α}^{ψ} (resp. M_{α}^{ψ}). According to Proposition 2.4, we thus get the following.

Proposition 3.13. Let $\alpha \geq -1$, let ψ be an Orlicz function and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic.

(1) C_{ϕ} acting on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ is order-bounded into L_{α}^{ψ} if and only if

(3.5)
$$\psi^{-1}\left(\frac{1}{(1-|\phi|)^{N+\alpha+1}}\right) \in L_{\alpha}^{\psi};$$

(2) C_{ϕ} acting on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ is order-bounded into $M_{\alpha}^{\psi}(\mathbb{B}_{N})$ if and only if

(3.6)
$$\psi^{-1}\left(\frac{1}{(1-|\phi|)^{N+\alpha+1}}\right) \in M_{\alpha}^{\psi}.$$

If $\psi(x) = x^p$, Assertions (1) and (2) are the same and we recall that, in this case, Condition (3.5) (or (3.6)) is equivalent to C_{ϕ} being Hilbert-Schmidt on $A^2_{\alpha}(\mathbb{B}_N)$ (see the introduction for more precisions).

Remark 3.14. Let ψ be any Orlicz function and $\alpha \geq -1$. We immediately deduce from Proposition 3.13 the following observations:

- (i) If $C_{\phi}: A^{p}_{\alpha}(\mathbb{B}_{N}) \to A^{p}_{\alpha}(\mathbb{B}_{N})$ is order bounded into L^{p}_{α} for some $1 \leq p < +\infty$ (equivalently Hilbert-Schmidt if p = 2) then $C_{\phi}: A^{\psi}_{\alpha}(\mathbb{B}_{N}) \to A^{\psi}_{\alpha}(\mathbb{B}_{N})$ is order bounded into L^{ψ}_{α} ;
- (ii) If $C_{\phi}: A_{\alpha}^{\psi}(\mathbb{B}_{N}) \to A_{\alpha}^{\psi}(\mathbb{B}_{N})$ is order bounded into M_{α}^{ψ} then $C_{\phi}: A_{\alpha}^{p}(\mathbb{B}_{N}) \to A_{\alpha}^{p}(\mathbb{B}_{N})$ is order bounded into L_{α}^{p} (or equivalently into M_{α}^{p}).

In order to prove our main results, we need to complete Proposition 3.13 with a more geometric understanding of the order boundedness of composition operators. For $\alpha \geq -1$ and $h \in (0,1)$, let us denote by $C_{\alpha}(h)$ the corona defined by

(3.7)
$$C_{\alpha}(h) = \begin{cases} \{z \in \mathbb{B}_{N}, 1 - |z| < h\} & \text{if } \alpha > -1 \\ \{z \in \overline{\mathbb{B}}_{N}, 1 - |z| < h\} & \text{if } \alpha = -1 \end{cases}$$

We extend [25, Theorem 3.15] to dimension N > 1 and all $\alpha \ge -1$, providing a useful alternative to Proposition 3.13.

Theorem 3.15. Let $\alpha \geq -1$, let ψ be an Orlicz function and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic.

(1) (a) If C_{ϕ} acting on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ is order bounded, then there exist A > 0 and $\eta \in (0,1)$ such that

(3.8)
$$\mu_{\phi}(C_{\alpha}(h)) \leq \frac{1}{\psi(A\psi^{-1}(1/h^{N+\alpha+1}))}$$

for any $h \in (0, \eta)$.

- (b) If ψ satisfies the Δ^1 -condition, then (3.8) is sufficient for the order boundedness of C_{ϕ} .
- (2) (a) If C_{ϕ} acting on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ is order bounded into $AM_{\alpha}^{\psi}(\mathbb{B}_{N})$, then for every A > 0, there exist $C_{A} > 0$ and $h_{A} \in (0,1)$ such that

(3.9)
$$\mu_{\phi}\left(C_{\alpha}\left(h\right)\right) \leq \frac{C_{A}}{\psi\left(A\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)}$$

for any $h \in (0, h_A)$.

(b) If ψ satisfies the Δ^1 -condition, then (3.9) is sufficient for the order boundedness of C_{ϕ} into $AM_{\alpha}^{\psi}(\mathbb{B}_N)$.

Remark 3.16. By Theorem 3.15, Theorem 3.10 is trivially false for $\alpha = -1$, i.e. for $H^{\psi}(\mathbb{B}_N)$: if $\phi(z) = z$ for every $z \in \mathbb{B}_N$, $\mu_{\phi}(C_{-1}(h)) = 1$ hence C_{ϕ} is not order bounded. Thus we also deduce that, whatever the Orlicz function ψ , $H^{\psi}(\mathbb{B}_N)$ does not coincide with any weighted Banach space.

The proof of Theorem 3.15 is an adaptation of that of [25, Theorem 3.15]. It relies on the introduction of the *weak*-Orlicz space:

Definition 3.17. Given an Orlicz function ψ and a probability space (Ω, \mathbb{P}) , the weak-Orlicz space $L^{\psi,\infty}(\Omega, \mathbb{P})$ is the space of all measurable functions $f: \Omega \to \mathbb{C}$ such that, for some constant c > 0, one has

$$\mathbb{P}(|f| > t) \le \frac{1}{\psi(ct)},$$

for every t > 0.

We then have the following [25, Proposition 3.18] which is the key of the proof of [25, Theorem 3.15].

Proposition 3.18. If ψ is an Orlicz function satisfying the Δ^1 -condition, then $L^{\psi}(\Omega, \mathbb{P}) = L^{\psi,\infty}(\Omega, \mathbb{P})$.

Proof of Theorem 3.15. For $\alpha > -1$, the both only if parts proceed from Proposition 3.13 and the Markov's inequality

$$\mu_{\phi}(C_{\alpha}(h)) \leq v_{\alpha} \left(\psi \left(A \psi^{-1} \left(\frac{1}{(1 - |\phi|)^{N + \alpha + 1}} \right) \right) > \psi \left(A \psi^{-1} \left(\frac{1}{h^{N + \alpha + 1}} \right) \right) \right)$$

$$\leq \frac{1}{\psi \left(A \psi^{-1} \left(1/h^{N + \alpha + 1} \right) \right)} \int_{\mathbb{B}_{N}} \psi \left(A \psi^{-1} \left(\frac{1}{(1 - |\phi|)^{N + \alpha + 1}} \right) \right) dv_{\alpha}.$$

For $\alpha = -1$, it is enough to replace \mathbb{B}_N by \mathbb{S}_N in the previous inequalities (and to remind the notations).

We now assume that (3.8) (resp. (3.9)) is satisfied and that $L^{\psi}(\Omega, \mathbb{P}) = L^{\psi,\infty}(\Omega, \mathbb{P})$ (Proposition 3.18). Without loss of generality, we may assume that $\psi \in \mathcal{C}^1(\mathbb{R})$. According to [25, Lemma 3.19], there exists B > 1 such that

$$\int_{1}^{\infty} \frac{\psi'(u)}{\psi(Bu)} du = \int_{\psi(1)}^{\infty} \frac{1}{\psi(B\psi^{-1}(x))} dx < +\infty.$$

For C > 0, we denote $\chi_{C,\alpha}(x) = \psi(C\psi^{-1}(x^{N+\alpha+1})) \in \mathcal{C}^1(\mathbb{R})$. Using (3.8) (resp. (3.9), there exists A > 0 such that (resp. for every A > 0)

$$\int_{\mathbb{B}_{N}} \chi_{C,\alpha} \left(\frac{1}{1 - |\phi|} \right) dv_{\alpha} = \chi_{C,\alpha}(1) + \int_{1}^{\infty} v_{\alpha} \left(1 - |\phi| < 1/t \right) \chi'_{C,\alpha}(t) dt$$

$$\leq \chi_{C,\alpha}(1) + K \int_{1}^{\infty} \frac{\chi'_{C,\alpha}(t)}{\chi_{A,\alpha}(t)} dt,$$

for some $K < \infty$. Setting C = A/B, the change of variable $u = \chi_{C,\alpha}(t)$ gives $\chi_{A,\alpha}(t) = \chi_{B,\alpha}(u)$ and

$$\int_{\mathbb{B}_N} \chi_{C,\alpha} \left(\frac{1}{1 - |\phi|} \right) dv_{\alpha} \le \chi_{C,\alpha}(1) + K \int_{\chi_{C,\alpha}(1)}^{\infty} \frac{du}{\chi_{B,\alpha}(u)} < +\infty.$$

Thus (3.5) (resp. 3.6) is satisfied and we are done.

We are now ready to characterize *small* Bergman-Orlicz spaces.

3.3. A more complete characterization of *small* Bergman-Orlicz spaces. We recall that the function w^{ψ} is defined in (2.3). In order to prove the main result of this paragraph, we need the following lemma.

Lemma 3.19. Let $M \ge 1$, $\alpha \ge -1$, and ψ be an Orlicz function. Set $\gamma = N + \alpha + 1$. The following assertions are equivalent.

- (1) $A^{\psi}_{\alpha}(\mathbb{B}_{M}) = H^{\infty}_{w^{\psi,\gamma}}(\mathbb{B}_{M})$ with equivalent norms;
- (2) There exist finitely many functions f_1, \ldots, f_d in $A^{\psi}_{\alpha}(\mathbb{B}_M)$ such that, for any f in the unit ball of $A^{\psi}_{\alpha}(\mathbb{B}_M)$,

$$|f(z)| \le |f_1(z)| + \ldots + |f_d(z)|,$$

for any $z \in \mathbb{B}_M$.

Proof. Let us assume that (1) holds. We recall first that if ψ is an Orlicz function, then $w^{\psi,\gamma}$ is an essential weight on \mathbb{B}_N by Lemma 3.5, so it is equivalent to a log-convex weight by [6] (see also [18, Lemma 2.1]). By [1, Theorem 1.3] (see also [2, Corollary 12]), (2) holds for $H^{\infty}_{w^{\psi,\gamma}}(\mathbb{B}_M)$ instead of $A^{\psi}_{\alpha}(\mathbb{B}_M)$, hence the first implication. Conversely, if (2) holds it is clear from Proposition 2.4 that any function in $H^{\infty}_{m^{\psi,\gamma}}(\mathbb{B}_N)$ also belongs to $A^{\psi}_{\alpha}(\mathbb{B}_N)$.

We introduce the symbol

where 0' is the null (N-1)-tuple and $\pi(z) = N^{N/2} z_1 z_2 \dots z_N$ for $z = (z_1, z_2, \dots, z_N) \in \mathbb{B}_N$. Note that π maps \mathbb{B}_N onto \mathbb{D} and $\overline{\mathbb{B}_N}$ onto $\overline{\mathbb{D}}$ by the Arithmetic Mean Inequality. Our first main result states as follows.

Theorem 3.20. Let $M \geq 1$, N > 1, $\alpha > -1$, and let ψ be an Orlicz function. Set $\gamma = N + \alpha + 1$. The following assertions are equivalent.

- (1) ψ belongs to the Δ^2 -class;
- (2) $A_{\alpha}^{\psi}(\mathbb{B}_{M}) = H_{w^{\psi,\gamma}}^{\infty}(\mathbb{B}_{M})$ with equivalent norms; (3) $A_{\alpha}^{\psi}(\mathbb{B}_{M}) = H_{w^{\psi}}^{\infty}(\mathbb{B}_{M})$ with equivalent norms;
- (4) There exist finitely many functions f_1, \ldots, f_d in $A_{\alpha}^{\psi}(\mathbb{B}_M)$ such that, for any f in the unit ball of $A^{\psi}_{\alpha}(\mathbb{B}_M)$,

$$|f(z)| \le |f_1(z)| + \ldots + |f_d(z)|, z \in \mathbb{B}_M;$$

- (5) For every $\phi: \mathbb{B}_M \to \mathbb{B}_M$ holomorphic, C_{ϕ} acting on $A_{\alpha}^{\psi}(\mathbb{B}_M)$ is order-bounded into
- (6) For every $\phi : \mathbb{B}_N \to \mathbb{B}_N$ holomorphic, C_{ϕ} is bounded on $A_{\alpha}^{\psi}(\mathbb{B}_N)$;
- (7) For every $\phi: \mathbb{B}_N \to \mathbb{B}_N$ holomorphic, C_{ϕ} is bounded on $AM_{\alpha}^{\psi}(\mathbb{B}_N)$;
- (8) C_{φ} with symbol φ defined in (3.10) is bounded on $A_{\alpha}^{\psi}(\mathbb{B}_N)$.

Proof. Let M be fixed as in the statement of the theorem. We will prove that $(1) \Leftrightarrow (2)$ \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) and that (1) \Leftrightarrow (6) \Leftrightarrow (7). Since (1) does not depend on M nor N, this will be enough. $(1) \Rightarrow (2)$ was proven in Paragraph 2.2. $(2) \Leftrightarrow (4)$ is Lemma 3.19. If (4) holds, since it is equivalent to (2), then Proposition 3.7 ensures that every composition operator is bounded from $A^{\psi}_{\alpha}(\mathbb{B}_{M})$ into itself (note by passing that it gives (6) for the value M). Moreover, (4) trivially implies that the canonical embedding from $A^{\psi}_{\alpha}(\mathbb{B}_{M})$ into L^{ψ}_{α} is order-bounded into L_{α}^{ψ} . Thus, Lemma 3.9 gives (5). We divide the proof of (5) \Rightarrow (1) into two parts. First, we show that whatever N > 1, (6) always imply (1). Since (5) \Rightarrow (6) whenever M = N, it will give $(5) \Rightarrow (1)$ in the case M > 1. Because (6) trivially implies (8), we only have to prove (8) \Rightarrow (1). We recall that φ is defined in (3.10) by $\varphi(z) = (\pi(z), 0')$, where $\pi(z) = N^{N/2} z_1 z_2 \dots z_N$. Now, as computed in [33, Pages 904-905] (by the means of a result due to Ahern [3, Theorem 1]),

(3.11)
$$\mu_{\varphi,\alpha}\left(S\left(e_{1},h\right)\right) \sim h^{\frac{2\alpha+N+3}{2}}$$

Hence, according to Theorem 3.2, $C_{\phi_{\beta}}$ is unbounded on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ whenever for any A > 0, there exists $(h_n)_n$ decreasing to 0 such that

$$\frac{1}{\psi\left(A\psi^{-1}\left(1/h_n^{N+\alpha+1}\right)\right)} = o\left(h_n^{\frac{2\alpha+N+3}{2}}\right) \text{ as } n \to \infty.$$

Using that ψ^{-1} is increasing and setting $y_n = A\psi^{-1} \left(1/h_n^{N+\alpha+1}\right)$, this condition is realized as soon as for every A > 0, every C > 0, and every $y_0 > 0$ there exists $y \ge y_0$ such that

$$\psi\left(y\right)^{\frac{2N+2\alpha+2}{2\alpha+N+3}} \ge C\psi\left(\frac{y}{A}\right).$$

Now, since N > 1, we have $\frac{2N+2\alpha+2}{2\alpha+N+3} > 1$. By Proposition 2.1, the last condition is satisfied if ψ does not belong to the Δ^2 -class, which proves $(8) \Rightarrow (1)$ for any value of N > 1.

We now turn to the proof of $(5) \Rightarrow (1)$ in the case M = 1. It is similar to the previous one but even simpler: it is enough to take $\phi(z) = z$, $z \in \mathbb{D}$. Indeed, in this case, $\mu_{\phi}(C_{\alpha}(h)) = 1 - v_{\alpha}(D(0, 1 - h)) \sim h^{1+\alpha}$ as h tends to 0. Now, since $\frac{2+\alpha}{1+\alpha} > 1$, we can prove as above that, if ψ does not satisfy the Δ^2 -condition, then for any A > 0, there exists $(h_n)_n$ decreasing to 0 such that

$$\frac{1}{\psi\left(A\psi^{-1}\left(1/h_n^{2+\alpha}\right)\right)} = o\left(h_n^{1+\alpha}\right) \text{ as } n \to \infty.$$

This means that C_{ϕ} is not order bounded on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ by (1)–(a) of Theorem 3.15. Thus (5) \Rightarrow (1) also holds in the case M=1.

At this point, we have proven $(1) \Leftrightarrow (2) \Leftrightarrow (4) \Leftrightarrow (5)$. To get that (3) is also equivalent to the previous assertions, it is enough to recall that $(1) \Rightarrow (3)$ was also shown in Paragraph 2.2, and to observe that $(3) \Rightarrow (4)$ can be obtained as $(2) \Rightarrow (4)$ in Lemma 3.19.

To finish, we remark that $(6) \Leftrightarrow (7)$ is contained in Theorem 2.3 (4), that $(1) \Rightarrow (6)$ is given by Corollary 3.8, that $(6) \Rightarrow (8)$ is trivial, and that $(8) \Rightarrow (1)$ has been proven just above. This concludes the proof.

Remark. 1) Note that the symbol φ appearing in the eighth assertion of Theorem 3.20 is the simplest and most classical example of symbol inducing an unbounded composition operator on $H^2(\mathbb{B}_N)$ and $A^2_{\alpha}(\mathbb{B}_N)$. One of the striking aspect of Theorem 3.20 is that the boundedness of this composition operator on $A^{\psi}_{\alpha}(\mathbb{B}_M)$ entirely determines the *nature* of this space, and the boundedness of *all* composition operators on it.

- 2) In the previous theorem, the eight assertions are also equivalent to the following nineth one:
 - (9) There exists an essential (radial) weight v such that $A_{\alpha}^{\psi}(\mathbb{B}_{M}) = H_{v}^{\infty}(\mathbb{B}_{M})$ with equivalent norms.

Indeed, the proof of $(9) \Rightarrow (4)$ works the same as that of $(2) \Rightarrow (4)$ (or equivalently $(1) \Rightarrow (2)$ in Lemma 3.19).

3) As already said in Remark 3.16, Assertion (2) in Theorem 3.20 is never true for Hardy-Orlicz spaces, so the previous theorem does not hold for $\alpha = -1$. In fact, Assertions (2), (3), (4) and (5) do not hold for $H^{\psi}(\mathbb{B}_N)$. Nevertheless we know after [11, Theorem 3.7] that every composition operator is bounded on $H^{\psi}(\mathbb{B}_N)$ whenever ψ satisfies the Δ^2 -condition. Moreover we can prove as in Theorem 3.20 that every composition operator is bounded on $H^{\psi}(\mathbb{B}_N)$ iff ψ satisfying the Δ^2 -condition, and iff the single composition operator C_{φ} is bounded on $H^{\psi}(\mathbb{B}_N)$, where φ is given by (3.10) (the estimate (3.11) also holds for $\alpha = -1$). Actually, these properties are also equivalent to the boundedness on $H^{\psi}(\mathbb{B}_N)$ of the single composition operator induced by the symbol $\widetilde{\varphi}(z) = (\theta(z), 0')$, where θ is a non-constant inner function of \mathbb{B}_N vanishing at the origin [4]. Since θ is measure-preserving as a map from $\partial \mathbb{B}_N$ to $\partial \mathbb{D}$, we get

$$\sigma_N \widetilde{\varphi}^{-1} \left(\mathcal{S} \left(e_1, h \right) \right) = \sigma_N \left(\theta^{-1} \left(\mathcal{S} \left(1, h \right) \right) \right) = h.$$

Thus $C_{\widetilde{\varphi}}$ is unbounded on $H^{\psi}(\mathbb{B}_N)$ whenever for every A > 0, there exists $(h_n)_n$ decreasing to 0 such that

$$\frac{1}{\psi\left(A\psi^{-1}\left(1/h_n^N\right)\right)} = o\left(h_n\right) \text{ as } n \to \infty$$

which is, as above, satisfied whenever ψ does not belong to the Δ^2 -class, keeping in mind that N > 1.

3.4. Compactness and order boundedness.

3.4.1. Two direct consequences of Theorem 3.20. In this paragraph we derive from Theorem 3.20 and from known results some new statements (not necessarily new results) about the compactness of composition operators on small weighted Banach spaces and small Bergman-Orlicz spaces.

First of all, up to now, no estimate of the essential norm $\|C_{\phi}\|_{e}$ of the composition operator C_{ϕ} on Bergman-Orlicz spaces is known, except when $\psi(x) = x^{p}$, $1 \leq p \leq \infty$ (see [12] and the references therein). In [7] the authors estimate $\|C_{\phi}\|_{e}$ for composition operators acting on weighted Banach spaces on certain domains in the complex plane, including the unit disc. This, together with Lemma 3.5 and Theorem 3.20, provides us with the estimates in Equation (3.12) below. We omit the proof.

Corollary 3.21. Let $\alpha > -1$, let ψ be an Orlicz function satisfying the Δ^2 -condition, and let ϕ be a holomorphic self-map of \mathbb{D} . Then

(3.12)
$$||C_{\phi}||_{e,A_{\alpha}^{\psi}} \simeq \lim_{r \to 1} \sup_{|\phi(z)| > r} \frac{\psi^{-1}(1/(1-|\phi(z)|))}{\psi^{-1}(1/(1-|z|))} \simeq \lim_{n \to \infty} \sup_{n \to \infty} \frac{||\phi^{n}||_{A_{\alpha}^{\psi}}}{||z^{n}||_{A_{\alpha}^{\psi}}}.$$

In particular C_{ϕ} is compact on $A_{\alpha}^{\psi}(\mathbb{D})$ iff

(3.13)
$$\limsup_{|z| \to 1} \frac{\psi^{-1} \left(1 / \left(1 - |\phi(z)| \right) \right)}{\psi^{-1} \left(1 / \left(1 - |z| \right) \right)} = 0$$

or iff

(3.14)
$$\limsup_{n \to \infty} \frac{\|\phi^n\|_{A^{\psi}_{\alpha}}}{\|z^n\|_{A^{\psi}_{\alpha}}} = 0.$$

Similar estimates also appear in [18, 34] where weighted composition operators acting on weighted Banach spaces of the unit disc are considered. Note that the equivalence between (3.13) and the compactness of C_{ϕ} also holds in the unit ball \mathbb{B}_{N} instead of \mathbb{D} , as a particular case of [13, Theorem 3.11]. Indeed, it is shown there that, under some mild condition, this characterization holds not only for \mathbb{B}_{N} , but also on the whole range of weighted Begman-Orlicz spaces, whenever the Orlicz functions satisfy the ∇_{0} -condition (yet the essential norm was not estimated, even in \mathbb{D}). Moreover, we believe that some part of the proof given in [7] smoothly extends to \mathbb{B}_{N} , so that it is quite likely that the first estimate in (3.12) is also true in the unit ball.

Second it was proven in [13, Theorem 3.1] that if C_{ϕ} is compact on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ for every Orlicz function ψ , then C_{ϕ} is compact on H^{∞} . Actually it is not difficult to adapt the proof of that theorem (more specifically that of Lemma 3.2 there) to show that if C_{ϕ} is compact on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ for every Orlicz function ψ satisfying the Δ^{2} -condition, then C_{ϕ} is compact on H^{∞} . We deduce from that and Theorem 3.20 the following.

Corollary 3.22. Let ϕ be a holomorphic self-map of \mathbb{B}_N . If C_{ϕ} is compact on $H_v^{\infty}(\mathbb{B}_N)$ for every weight v of the form $v = w^{\psi}$ with ψ an Orlicz function satisfying the Δ^2 -condition, then C_{ϕ} is compact on H^{∞} .

This corollary is in fact a slight refinement of [8, Corollary 3.8].

3.4.2. Compactness and order boundedness. The main result of this paragraph states as follows:

Theorem 3.23. Let $\alpha \geq -1$, let ψ be an Orlicz function satisfying the Δ^2 -condition and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. The following assertions are equivalent.

- (1) C_{ϕ} is compact on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$;
- (2) C_{ϕ} acting on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ is order bounded into M_{α}^{ψ} ;

(3) C_{ϕ} is weakly compact on $A_{\alpha}^{\psi}(\mathbb{B}_{\mathbb{N}})$.

Remark. (1) The two first equivalences were obtained in [25, Theorem 3.24] for N=1 and $\alpha=-1$.

- (2) The Δ^2 -Condition in Theorem 3.23 is certainly important for $(1)\Leftrightarrow(2)$. Indeed it was proven in [25, Theorem 4.22] that there exists an Orlicz function $\psi \in \Delta^1$, and an analytic self-map ϕ of $\mathbb D$ such that the composition operator $C_{\phi}: H^{\psi}(\mathbb D) \to H^{\psi}(\mathbb D)$ is not order bounded into $M^{\psi}(\partial \mathbb D)$, though it is compact.
- (3) Yet the Δ^2 -condition has no importance for (2) \Leftrightarrow (3) which in fact holds for a larger class of Orlicz functions, namely the Δ^0 -class, see [25, Theorem 4.21]. But this equivalence fails if $\psi \in \Delta_2 \cap \nabla_2$ because, in this case, $A_{\alpha}^{\psi}(\mathbb{B}_N)$ is reflexive and so every bounded operator in weakly compact.
- (4) The equivalence between compactness and weak compactness for a composition operator is more generally related to the duality properties of the spaces on which it acts. For instance, such equivalence was also observed in the context of weighted Banach spaces of analytic functions on the unit disc [7].

Proof of Theorem 3.23. We present it only for $\alpha > -1$, as it is again very similar if $\alpha = -1$. The proof of $(1) \Rightarrow (2) \Rightarrow (3)$ is an adaptation of the proof of Theorem 3.24 in [25]. By Proposition 3.12, we already know $(2) \Rightarrow (1)$. To prove the converse, it is enough to show, according to Theorem 3.15, that the condition

(3.15)
$$||f_a \circ \phi||_{A_{\alpha}^{\psi}(\mathbb{B}_N)} = o_{|a| \to 1} \left(\frac{1}{\psi^{-1} \left(1 / \left(1 - |a| \right)^{N + \alpha + 1} \right)} \right)$$

implies

$$\mu_{\phi}\left(C_{\alpha}\left(h\right)\right) \leq \frac{C_{A}}{\psi\left(A\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)}$$

for every A > 0, every $h \in (0, h_A)$ and some $h_A \in (0, 1)$ and $C_A > 0$. (note that we may just require the preceding to be true for any A bigger than some arbitrary positive constant). Now let us remind that we showed, at the end of the proof of Corollary 3.3, that if Condition (3.15) holds then for every $\varepsilon > 0$, there exists $h_{\varepsilon} \in (0, 1)$ such that for every $\zeta \in \mathbb{S}_N$ and every $h \in (0, h_{\varepsilon})$ we have

(3.16)
$$\mu_{\phi,\alpha}(S(\zeta,h)) \le \frac{1}{\psi\left(\frac{K}{\varepsilon}\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)}.$$

Now it is a classical geometric fact that there exists a constant M > 0, depending only on α such that for any $h \in (0,1)$, there exists $\zeta \in \mathbb{S}_N$ such that

$$\mu\left(C_{\alpha}\left(h\right)\right) \leq \frac{M}{h^{N+\alpha+1}}\mu\left(S\left(\zeta,Kh\right)\right) [37]$$

Thus we get that for every $\varepsilon > 0$, there exists $h_{\varepsilon} \in (0,1)$ such that for every $h \in (0,h_{\varepsilon})$

$$\mu\left(C_{\alpha}\left(h\right)\right) \leq \frac{M}{h^{N+\alpha+1}\psi\left(\frac{K}{\varepsilon}\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)}.$$

Since ψ satisfies the Δ^2 -condition, there exists a constant C > 0 such that for every x large enough, $(\psi(x))^2 \leq \psi(Cx)$. Then for any $\varepsilon > 0$ small such that $K/(C\varepsilon) > 1$, there exists h'_{ε} such that for every $h \in (0, h'_{\varepsilon})$, we have

$$\frac{M}{h^{N+\alpha+1}\psi\left(\frac{K}{\varepsilon}\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)} \leq \frac{M\psi\left(\frac{K}{C\varepsilon}\psi^{-1}(1/h^{N+\alpha+1})\right)}{\left[\psi\left(\frac{K}{C\varepsilon}\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)\right]^2} = \frac{M}{\psi\left(\frac{K}{C\varepsilon}\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)}.$$

We conclude by setting $A = K/(C\varepsilon)$ and choosing $h_A = \min(h_\varepsilon, h'_\varepsilon)$.

To prove $(1) \Leftrightarrow (3)$, we need only to prove $(3) \Rightarrow (1)$ and then, according to Corollary 3.3, that the weak compactness of C_{ϕ} on $A_{\alpha}^{\psi}(\mathbb{B}_{N})$ implies

(3.17)
$$||f_a \circ \phi||_{A_{\alpha}^{\psi}(\mathbb{B}_N)} = o_{|a| \to 1} \left(\frac{1}{\psi^{-1} \left(1/\left(1 - |a|\right)^{N+\alpha+1} \right)} \right).$$

But this can be done exactly in the same way as in the proof of [25, Theorem 3.20]. Indeed, by Theorem 2.3 (4), $C_{\phi}: AM_{\alpha}^{\psi}(\mathbb{B}_{N}) \to AM_{\alpha}^{\psi}(\mathbb{B}_{N})$ is also weakly compact so that we can appeal to [24, Theorem 4] and get, for any $\varepsilon > 0$, the existence of a constant $K_{\varepsilon} > 0$ such that for every $f \in AM_{\alpha}^{\psi}(\mathbb{B}_{N})$,

$$||f \circ \phi||_{A^{\psi}_{\alpha}} \leq K_{\varepsilon} ||f||_{L^{1}_{\alpha}} + \varepsilon ||f||_{A^{\psi}_{\alpha}}.$$

Taking $f = f_a$, we obtain

$$||f_a \circ \phi||_{A_{\alpha}^{\psi}} \le K_{\varepsilon} (1 - |a|)^{N + \alpha + 1} + \frac{2^{N + \alpha + 1} \varepsilon}{\psi^{-1} (1/(1 - |a|)^{N + \alpha + 1})},$$

since $||f_a||_{L^1_\alpha} = (1-|a|)^{N+\alpha+1}$ (a change a variable for $\alpha > -1$ and the Cauchy formula when $\alpha = -1$). We conclude that (3.17) holds, for $\psi(x)/x \to +\infty$ as $x \to +\infty$.

4. Some geometric conditions for the order boundedness of composition operators on Hardy-Orlicz and Bergman-Orlicz spaces

For $\zeta \in \mathbb{S}_N$ and a > 1, we recall that the Korányi approach region $\Gamma(\zeta, a)$ of angular opening a is defined by

$$\Gamma\left(\zeta,a\right) = \left\{z \in \mathbb{B}_N, \left|1 - \langle z,\zeta\rangle\right| < \frac{a}{2} \left(1 - \left|z\right|^2\right)\right\}.$$

When N = 1, Korányi approach regions reduce to non-tangential approach regions (or Stolz domains) given by

$$\Gamma\left(\zeta,a\right) = \left\{z \in \mathbb{D}, \left|\zeta - z\right| < \frac{a}{2}\left(1 - \left|z\right|^{2}\right)\right\}.$$

In [?], the author proves that if ϕ takes \mathbb{B}_N into a Korányi approach region, then C_{ϕ} is bounded or compact on $H^p(\mathbb{B}_N)$ (hence on $A^p_{\alpha}(\mathbb{B}_N)$ for any $\alpha \geq -1$) whenever the angular opening of the Korányi region is not too large. This result immediately extends to the $H^{\psi}(\mathbb{B}_N)$ or $A^{\psi}_{\alpha}(\mathbb{B}_N)$ setting for Orlicz functions ψ satisfying the Δ_2 -condition [13, Theorem 3.3].

On the opposite side we have the following.

Theorem (Theorem 3.5 of [13]). Let $N \geq 1$ and let ψ be an Orlicz function satisfying the Δ^2 -condition. Then, for every $\zeta \in \mathbb{S}_N$ and every b > 1, there exists a holomorphic self-map ϕ taking \mathbb{B}_N into $\Gamma(\zeta, b)$, such that C_{ϕ} is not compact on $H^{\psi}(\mathbb{B}_N)$ (and hence not order-bounded into $M^{\psi}(\mathbb{S}_N, \sigma_N)$). The same holds for the weighted Bergman-Orlicz spaces in dimension 1.

So, when $\psi \in \Delta^2$, the condition $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, b)$, whatever the opening b > 1, becomes non-sufficient for the compactness of C_{ϕ} on A_{α}^{ψ} , while it is trivially sufficient for its order boundedness into L_{α}^{ψ} (since in this case every composition operator is order bounded into L_{α}^{ψ} by Theorem 3.20).

The next result tells that if $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, a)$ for some $\zeta \in \mathbb{S}_N$ and a > 1 sufficiently small, then $C_{\phi} : A_{\alpha}^{\psi}(\mathbb{B}_N) \to A_{\alpha}^{\psi}(\mathbb{B}_N)$ is still order bounded into L_{α}^{ψ} when ψ satisfies the Δ^1 -condition.

Theorem 4.1. Let $\alpha \geq -1$ and let ψ be an Orlicz function.

- (1) Let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic. If $\phi(\mathbb{D})$ is contained in some Stolz domain, then $C_{\phi} : A_{\alpha}^{\psi}(\mathbb{D}) \to A_{\alpha}^{\psi}(\mathbb{D})$ is order bounded into L_{α}^{ψ} .
- (2) Let N > 1 and let $\phi : \mathbb{B}_N \to \mathbb{B}_N$ be holomorphic. We set $a_N = 1/\cos\left(\frac{\pi}{2}\frac{\alpha+2}{N+\alpha+1}\right)$.
 - (a) If $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, \gamma)$ for $\gamma < a_N$ then $C_{\phi} : A_{\alpha}^{\psi}(\mathbb{B}_N) \to A_{\alpha}^{\psi}(\mathbb{B}_N)$ is order bounded into L_{α}^{ψ} ;
 - (b) We assume that ψ satisfies the Δ^1 -condition. If $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, a_N)$, then $C_{\phi}: A_{\alpha}^{\psi}(\mathbb{B}_N) \to A_{\alpha}^{\psi}(\mathbb{B}_N)$ is order bounded into L_{α}^{ψ} .

Proof. We will assume that $\alpha > -1$, the case $\alpha = -1$ being similarly proven (to get the case $\alpha = -1$, it is essentially enough to change suitably \mathbb{B}_N into \mathbb{S}_N and v_α into σ_N , and to consider ϕ^* instead of ϕ , or $\mathcal{S}(\zeta,h)$ instead of $\mathcal{S}(\zeta,h)$. The details are left to the reader). Moreover we can assume for simplicity that all Korányi approach regions $\Gamma(\zeta,a)$ appearing in the sequel are based at the point $\zeta = e_1 = (1,0,\ldots,0) \in \mathbb{S}_N$. Indeed for every unitary maps U of \mathbb{B}_N $U\Gamma(\zeta,a) = \Gamma(U\zeta,a)$, and $C_\phi: A_\alpha^\psi(\mathbb{B}_N) \to A_\alpha^\psi(\mathbb{B}_N)$ is order bounded into L_α^ψ if and only if $C_{U\phi}: A_\alpha^\psi(\mathbb{B}_N) \to A_\alpha^\psi(\mathbb{B}_N)$ is order bounded into L_α^ψ (by (1) in Proposition 3.13).

The proof of (1) and (2)–(a) are similar: if N=1, we recall that if ϕ maps \mathbb{D} into a Stolz domain, then C_{ϕ} is Hilbert-Schmidt on $A_{\alpha}^{2}(\mathbb{D})$ [39] and then the conclusion follows by Remark 3.14 (1). If N>1 it is classical that if $\phi(\mathbb{B}_{N})\subset\Gamma(\zeta,\gamma)$ for $\gamma< a_{N}$ then C_{ϕ} is Hilbert-Schmidt on $A_{\alpha}^{2}(\mathbb{B}_{N})$ (see the remarks on Theorem 2.2 in [32], Page 246). We conclude again by Remark 3.14.

We now deal with (2)–(b). For this purpose, we need first an adaptation of [15, Lemma 6.3] to the Orlicz setting.

Lemma 4.2. Let ψ be an Orlicz function and $\alpha \geq -1$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic and assume that $\varphi(\mathbb{D}) \subset \Gamma(1,\gamma)$ for some $\gamma > 1$. There exists a constant A > 0 which depends only on $\varphi(0)$ and γ such that

$$\mu_{\varphi,\alpha}\left(S\left(1,h\right)\right) \le \frac{1}{\psi\left(A\psi^{-1}\left(1/h^{\beta(\alpha+2)}\right)\right)},$$

where
$$\beta = \frac{\pi}{2\cos^{-1}(1/\gamma)}$$
.

Proof. Since $\varphi(\mathbb{D}) \subset \Gamma(1,\gamma)$, the beginning of the proof of [15, Lemma 6.3] ensures that $\frac{1}{1-\varphi(z)} = F \circ \tau$ where $F(z) = \left(\frac{1+z}{1-z}\right)^{2b/\pi}$ with $b = \cos^{-1}(1/\gamma)$ and τ is an analytic self-map of \mathbb{D} . It follows that

$$\left\{ z \in \mathbb{D}, \, \left| 1 - \varphi \left(z \right) \right| < h \right\} = \left\{ z \in \mathbb{D}, \, \left| F \circ \tau \left(z \right) \right| > 1/h \right\}$$

$$\subseteq \left\{ z \in \mathbb{D}, \, \left| 1 - \tau \left(z \right) \right| < 2h^{\pi/(2b)} \right\}.$$

Hence, since every composition operator is bounded on every $A_{\alpha}^{\psi}(\mathbb{D})$, $\alpha \geq -1$, the measure $\mu_{\tau,\alpha} = v_{\alpha} \circ \tau^{-1}$ satisfies, by the Fact 3.1,

$$v_{\alpha}\left(\varphi^{-1}\left(S\left(1,h\right)\right)\right) \leq \frac{1}{\psi\left(A\psi^{-1}\left(1/h^{\beta(\alpha+2)}\right)\right)} \text{ (recall that } \beta = \frac{\pi}{2b}\text{)},$$

for any $h \in (0, \eta)$, some $\eta \in (0, 1)$ and some constant A > 0 depending only on α and $\varphi(0)$.

We finish the proof of (2)–(b) of Theorem 4.1, assuming that $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, a_N)$. The absorption property of the non-isotropic balls $S(\zeta, t)$, $\zeta \in \mathbb{S}_N$, ensures that there exists a constant $1 \leq C < \infty$ such that $C_{\alpha}(h) \cap \Gamma(1, a) \subset S(1, Ch)$. Thus

(4.1)
$$\mu_{\phi,\alpha}\left(C_{\alpha}\left(h\right)\right) \leq \mu_{\phi,\alpha}\left(S\left(e_{1},Ch\right)\right).$$

Let $\chi_{\phi^{-1}(S(e_1,h))}$ be the characteristic function of $\phi^{-1}(S(e_1,h))$. By integrating in polar coordinates and applying the slice integration formula we get, for any $h \in (0,1)$,

$$\mu_{\phi,\alpha}\left(S\left(e_{1},h\right)\right) = \int_{\mathbb{B}_{N}} \chi_{\phi^{-1}\left(S\left(e_{1},h\right)\right)} dv_{\alpha}$$

$$= 2N \int_{0}^{1} \int_{\mathbb{S}_{N}} \chi_{\phi^{-1}\left(S\left(e_{1},h\right)\right)}(r\zeta) d\sigma_{N}(\zeta) \left(1-r^{2}\right)^{\alpha} r^{2N-1} dr$$

$$= 2N \int_{0}^{1} \int_{\mathbb{S}_{N}} \int_{0}^{2\pi} \chi_{\phi^{-1}\left(S\left(e_{1},h\right)\right)} \left(e^{i\vartheta}r\zeta\right) \frac{d\vartheta}{2\pi} d\sigma_{N}\left(\zeta\right) \left(1-r^{2}\right)^{\alpha} r^{2N-1} dr$$

$$\leq 2N \int_{0}^{1} \int_{\mathbb{S}_{N}} \int_{0}^{2\pi} \chi_{\phi^{-1}\left(S\left(e_{1},h\right)\right)} \left(e^{i\vartheta}r\zeta\right) \frac{d\vartheta}{2\pi} d\sigma_{N}\left(\zeta\right) \left(1-r^{2}\right)^{\alpha} r dr.$$

$$(4.2)$$

Let us now denote by $\phi^{\zeta}: \mathbb{D} \to \mathbb{D}$ the holomorphic self-map of \mathbb{D} defined by $\phi^{\zeta}(z) = \phi_1(z\zeta)$ for $z \in \mathbb{D}$. Since $\phi(\mathbb{B}_N) \subset \Gamma(e_1, a_N)$ then $\phi^{\zeta}(\mathbb{D}) \subset \Gamma(1, a_N)$ and, for any $\zeta \in \mathbb{S}_N$, $\phi(z\zeta) \in S(e_1, h)$ if and only if $\phi^{\zeta}(z) \in S(1, h)$. Moreover $\phi(0) = \phi^{\zeta}(0)$ for every $\zeta \in \mathbb{S}_N$. Applying the previous lemma to ϕ^{ζ} and $\gamma = a_N$, it follows, for any $\zeta \in \mathbb{S}_N$,

$$2\int_{0}^{1} \int_{0}^{2\pi} \chi_{\phi^{-1}(S(e_{1},h))} \left(e^{i\vartheta}r\zeta\right) \frac{d\vartheta}{2\pi} \left(1-r^{2}\right)^{\alpha} r dr = v_{\alpha} \left(\left(\phi^{\zeta}\right)^{-1} \left(S\left(1,h\right)\right)\right)$$

$$\leq \frac{1}{\psi \left(A\psi^{-1}\left(1/h^{N+\alpha+1}\right)\right)}.$$

From (4.1) and (4.2) we conclude the proof of (2)–(b) using the concavity of ψ^{-1} and Theorem 3.15 (1)–(b), ψ satisfying the Δ^1 –condition.

Remark 4.3. Point (2)–(b) in the previous theorem is false if ψ satisfies the Δ_2 –condition. Indeed in this case, the order boundedness of a composition operator implies its compactness and it is known [15] (see also [13]) that there exists holomorphic self-map ϕ of \mathbb{B}_N such that $\phi(\mathbb{B}_N) \subset \Gamma(\zeta, a_N)$, though C_{ϕ} is not compact on $A_{\alpha}^{\psi}(\mathbb{B}_N)$, $\alpha \geq -1$.

In view of the previous, it is more accurate to consider known geometric conditions to the compactness of a composition operator on Hardy(-Orlicz) or Bergman(-Orlicz) spaces as conditions to its order boundedness. This leads to the following problems:

- (1) To find reasonable geometric conditions somehow similar to that involving Korányi regions but depending on the Orlicz function sufficient for the compactness of C_{ϕ} at every scale of Hardy-Orlicz (or Bergman-Orlicz) spaces.
- (2) In the classical cases H^p and A^p_{α} , to exhibit a geometric condition ensuring the compactness of C_{ϕ} but not necessarily its order boundedness. As mentioned in [15, Page 147–148], it already requires some effort to find an example of a composition operator which is compact but not Hilbert-Schmidt on $H^2(\mathbb{D})$ (i.e. not order bounded), see also [40, Section 4].
- (3) Also it would be interesting to find where in the scale of Hardy-Orlicz or Bergman-Orlicz spaces in terms of the type of growth of Orlicz functions the order boundedness of C_{ϕ} becomes no longer stronger than its compactness.

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