On a Data Assimilation Method coupling Kalman Filtering, MCRE Concept and PGD Model Reduction for Real-Time Updating of Structural Mechanics Model

2016 SIAM Conference on Uncertainty Quantification

Basile Marchand\textsuperscript{1}, Ludovic Chamoin\textsuperscript{1}, Christian Rey\textsuperscript{2}

\textsuperscript{1} LMT/ENS Cachan/CNRS/Paris-Saclay University, France
\textsuperscript{2} SAFRAN, Research and Technology Center, France

April 5-8, 2016
DDDAS\(^1\) paradigm: a continuous exchange between

the physical system
and
its numerical model

Objectives:

Identification process

- for time dependent systems/parameters
- fast resolution
- robust even if highly corrupted data
In this work

Objectives:

Identification process

- for time dependent systems/parameters
- fast resolution
- robust even if highly corrupted data

Tools:

Kalman filter for evolution aspect
modified Constitutive Relation Error for robustness
offline/online process based on Proper Generalized Decomposition
Basics on Kalman Filtering

Proposed Approach

Numerical Results

Conclusion
Data assimilation

Dynamical system:

\[
\begin{align*}
\mathbf{u}^{(k+1)} &= \mathcal{M}^{(k)} \mathbf{u}^{(k)} + \mathbf{e}_u^{(k)} \\
\mathbf{s}^{(k)} &= \mathcal{H}^{(k)} \mathbf{u}^{(k)} + \mathbf{e}_s^{(k)}
\end{align*}
\]
Dynamical system:
\[
\begin{align*}
\mathbf{u}^{(k+1)} &= \mathcal{M}^{(k)} \mathbf{u}^{(k)} + \mathbf{e}_u^{(k)} \\
\mathbf{s}^{(k)} &= \mathcal{H}^{(k)} \mathbf{u}^{(k)} + \mathbf{e}_s^{(k)}
\end{align*}
\]

Bayes theorem:
\[
\pi \left( \mathbf{u}^{(k)} \big| \mathbf{s}^{(k)} \right) = \frac{\pi \left( \mathbf{s}^{(k)} \big| \mathbf{u}^{(k)} \right) \pi \left( \mathbf{u}^{(k)} \big| \mathbf{s}^{(0:k-1)} \right)}{\pi \left( \mathbf{s}^{(k)} \big| \mathbf{s}^{(0:k-1)} \right)}
\]

under the following hypothesis:
- State $\mathbf{u}^{(k)}$ is a Markov process,
- Observations $\mathbf{s}^{(k)}$ are statistically independent of state history.
Kalman filter is a bayesian filter combined with Maximum a Posteriori method in the case of Gaussian probability density functions.

Kalman filter is a bayesian filter combined with Maximum a Posteriori method in the case of Gaussian probability density functions.

Two main steps:

Kalman filter is a bayesian filter combined with Maximum a Posteriori method in the case of Gaussian probability density functions. Two main steps:

(a) Prediction step where is realized a priori estimation $u^{(k+\frac{1}{2})}$ of state system
Introduction
Basics on Kalman Filtering
Proposed Approach
Numerical Results
Conclusion

Principle

Kalman filter\(^2\) is a bayesian filter combined with Maximum a Posteriori method in the case of Gaussian probability density functions.

Two main steps:

(a) Prediction step where is realized a priori estimation \( \mathbf{u}^{(k+\frac{1}{2})} \) of state system

(b) Assimilation step where is realized a posteriori estimation \( \mathbf{u}_a \) using observations data

Inverse problems formulation

Kalman filter is a very well-known method to solve inverse problems.\(^3\)

---

Kalman filter is a very well-known method to solve inverse problems.\(^3\)

**Principle**: Introduce model parameters vector \( \xi \in \mathbb{R}^{np} \)

No *a priori* knowledge \( \rightarrow \) stationarity hypothesis:

\[
\frac{\partial \xi}{\partial t} \approx 0 \quad \Rightarrow \quad \xi^{(k+1)} = \xi^{(k)} + e^{(k)}
\]

---

Inverse problems formulation

Kalman filter is a very well-known method to solve inverse problems\(^3\)

**Principle**: Introduce model parameters vector \(\xi \in \mathbb{R}^{np}\)

no \textit{a priori} knowledge \(\rightarrow\) stationarity hypothesis:

\[
\frac{\partial \xi}{\partial t} \approx 0 \quad \Rightarrow \quad \xi^{(k+1)} = \xi^{(k)} + e^{(k)}
\]

Kalman filter is a very well-known method to solve inverse problems.\(^3\)

**Principle**: Introduce model parameters vector \( \xi \in \mathbb{R}^{np} \)

no *a priori* knowledge \(\rightarrow\) stationarity hypothesis:

\[
\frac{\partial \xi}{\partial t} \simeq 0 \quad \Rightarrow \quad \xi^{(k+1)} = \xi^{(k)} + e^{(k)}
\]

Two formulations

### Joint Kalman Filter

\[
\begin{align*}
\bar{u}^{(k+1)} &= \bar{M}^{(k)} \bar{u}^{(k)} + e^{(k)}_M \\
\bar{s}^{(k)} &= \bar{H}^{(k)} \bar{u}^{(k)} + e^{(k)}_s
\end{align*}
\]

### Dual Kalman filter

\[
\begin{align*}
\xi^{(k+1)} &= \xi^{(k)} + e^{(k)}_\xi \\
\bar{s}^{(k)} &= \bar{H}^{(k)} u^{(k)}(\xi^{(k)}) + e^{(k)}_s
\end{align*}
\]

---

Kalman filter is a very well-known method to solve inverse problems\textsuperscript{3}

**Principle**: Introduce model parameters vector $\xi \in \mathbb{R}^{np}$

no *a priori* knowledge $\rightarrow$ stationarity hypothesis:

$$\frac{\partial \xi}{\partial t} \simeq 0 \quad \Rightarrow \quad \xi^{(k+1)} = \xi^{(k)} + e^{(k)}_\xi$$

**Two formulations**

**Joint Kalman Filter**

$$\begin{cases}
\bar{u}^{(k+1)} = \bar{M}^{(k)} \bar{u}^{(k)} + \bar{e}^{(k)}_M \\
\bar{s}^{(k)} = \bar{H}^{(k)} \bar{u}^{(k)} + \bar{e}_s^{(k)}
\end{cases}$$

**Dual Kalman filter**

$$\begin{cases}
\xi^{(k+1)} = \xi^{(k)} + e^{(k)}_\xi \\
\bar{s}^{(k)} = H^{(k)} u^{(k)}(\xi^{(k)}) + e_s^{(k)}
\end{cases}$$

---

Kalman filter is a very well-known method to solve inverse problems.  

Principle: Introduce model parameters vector $\xi \in \mathbb{R}^{np}$  

no a priori knowledge $\rightarrow$ stationarity hypothesis:  

$$ \frac{\partial \xi}{\partial t} \simeq 0 \quad \Rightarrow \quad \xi^{(k+1)} = \xi^{(k)} + e_{\xi}^{(k)} $$

Two formulations

Joint Kalman Filter

$$ \begin{align*}
\{ \bar{u}^{(k+1)} &= \bar{M}^{(k)} \bar{u}^{(k)} + \bar{e}_{\bar{M}}^{(k)} \\
\bar{s}^{(k)} &= \bar{H}^{(k)} \bar{u}^{(k)} + e_{\bar{s}}^{(k)} \}
\end{align*} $$

Dual Kalman Filter

$$ \begin{align*}
\{ \xi^{(k+1)} &= \xi^{(k)} + e_{\xi}^{(k)} \\
\bar{s}^{(k)} &= \bar{H}^{(k)} \bar{u}^{(k)}(\xi^{(k)}) + e_{\bar{s}}^{(k)} \}
\end{align*} $$

computed with another Kalman filter

Resolution schemes: UKF vs EKF

The problem:

- Gaussian $\mathcal{N}(\bar{x}, C_x)$
- Nonlinear operator $A$
- Gaussian $\mathcal{N}(\bar{y}, C_y)$

Two main approaches in Kalman filtering context

Resolution schemes:

- UKF
- EKF

- Sorenson and Stubberud, *Non-linear Filtering by Approximation of the a posteriori Density*, 1968

Basics on Kalman Filtering
Resolution schemes: UKF vs EKF

The problem:

Two main approaches in Kalman filtering context

- First order linearization,
  
  Extended Kalman filter

---

Resolution schemes : UKF vs EKF

The problem :

Resolution schemes: UKF vs EKF

Two main approaches in Kalman filtering context

- First order linearization,
  Extended Kalman filter \(^4\)
- Deterministic Monte-Carlo like method, Unscented Transform,
  Unscented Kalman filter \(^5\)

---

\(^4\) Sorenson and Stubberud, *Non-linear Filtering by Approximation of the a posteriori Density*, 1968

\(^5\) Julier and Uhlmann, *A new extension of the kalman filter to nonlinear systems*, 1997
Linearization vs Unscented Transform

First Order Linearization

- Linearization
  \[ \bar{y} = A(\bar{x}) \]
  \[ C_y = AC_xA^T \]

Prior

Posterior

For the same computational cost
Linearization vs Unscented Transform

First Order Linearization

$$A = \nabla_x A$$

$$\tilde{y} = A(\tilde{x})$$

$$C_y = AC_x A^T$$

Unscented Transform

\(\sigma\)-points propagation

\(\{x_i\}_{i=1,\ldots,2N+1}\)

\(\{y_i\} = A(\{x_i\})\)
Linearization vs Unscented Transform

First Order Linearization

\[ A = \nabla_x A \]
\[ \tilde{y} = A(\bar{x}) \]
\[ C_y = AC_xA^T \]

Unscented Transform

\[ \sigma\text{-points propagation} \]
\[ \{x_i\}_{i=1,\ldots,2N+1} \]
\[ \{y_i\} = A(\{x_i\}) \]

For the same computational cost
Kalman Filter based methods well-adapted for evolution problems and DDDAS paradigm
Why another approach ?

Kalman Filter based methods well-adapted for evolution problems and DDDAS paradigm

But :

methods very costly
if degrees of freedom/parameters increase
Why another approach?

Kalman Filter based methods well-adapted for evolution problems and DDDAS paradigm

But:

methods very costly if degrees of freedom/parameters increase

Identification quality strongly depends on measurement noise
Basics on Kalman Filtering

- Proposed Approach

Numerical Results

Conclusion
Principle of the method

Keep the **dual formulation**

\[
\begin{align*}
\xi^{(k+1)} &= \xi^{(k)} + e^{(k)} \\
\mathbf{s}^{(k)} &= \mathcal{H}^{(k)} \mathbf{u}^{(k)}(\xi^{(k)}) + e_s^{(k)}
\end{align*}
\]

Classically computed using a **Kalman Filter**
Principle of the method

Keep the **dual formulation**

\[
\begin{align*}
\xi^{(k+1)} &= \xi^{(k)} + e^{(k)} \\
\mathbf{s}^{(k)} &= \mathcal{H}^{(k)} u^{(k)}(\xi^{(k)}) + e^{(k)}_s
\end{align*}
\]

But use another observation operator

\[
\begin{align*}
\xi^{(k+1)} &= \xi^{(k)} + e^{(k)} \\
\mathbf{s}^{(k)} &= \mathcal{H}_m^{(k)} (\xi^{(k)}; \mathbf{s}^{(k-1:k)}) + e^{(k)}_s
\end{align*}
\]
Principle of the method

Keep the dual formulation

\[
\begin{cases}
\xi^{(k+1)} = \xi^{(k)} + e^{(k)}_\xi \\
\mathbf{s}^{(k)} = \mathcal{H}^{(k)} u^{(k)}(\xi^{(k)}) + e^{(k)}_s
\end{cases}
\]

But use another observation operator

Classically computed using a Kalman Filter

Defined from the modified Constitutive Relation Error functional

\[
\begin{cases}
\xi^{(k+1)} = \xi^{(k)} + e^{(k)}_\xi \\
\mathbf{s}^{(k)} = \mathcal{H}^{(k)}_m (\xi^{(k)} ; s^{(k-1:k)}) + e^{(k)}_s
\end{cases}
\]
MCRE framework

The idea: Weight the classical **Constitutive Relation Error** by a measurements error term

---

The idea:\n
Weight the classical **Constitutive Relation Error** by a measurements error term

Principle:\n
Primal-dual formulation based on Legendre-Fenchel inequality applied to Helmholtz free energy

---

The idea\(^6\):

Weight the classical **Constitutive Relation Error**\(^7\)
by a measurements error term

**Principle**:

Primal-dual formulation based on Legendre-Fenchel inequality
applied to Helmholtz free energy

mCRE functional for **unsteady thermal** problems:

\[ E_m(u, q; \xi) = \frac{1}{2} \int_I \int_{\Omega} (q - K \nabla u) K^{-1} (q - K \nabla u) \, dx \, dt + \frac{\delta}{2} \int_I \| \Pi u - s \|^2 \, dt \]

\[ U = \{ u \in H^1(\Omega) \otimes L^2(I_t) \mid u = u^d \text{ on } \partial \Omega_u, u = u^0 \text{ at } t = t_0 \} \]

\[ S(u) = \{ q \in [L^2(\Omega) \otimes L^2(I_t)]^d \mid q \cdot n = q^d \text{ on } \partial \Omega_q, \partial_t u + \nabla \cdot q = f \} \]

---


\(^7\) Ladevèze and Leguillon, *Error estimate procedure in the finite element method and application*, 1983
Solution is defined by:

\[ p = \arg\min_{\xi \in \mathcal{P}_{ad}} \min_{(u, q) \in \mathcal{U}_{ad} \times \mathcal{S}_{ad}} \mathcal{E}_m(u, q; \xi) \]
Solution is defined by:

\[ p = \arg\min_{\xi \in \mathcal{P}_{ad}} \min_{(u, q) \in \mathcal{U}_{ad} \times \mathcal{S}_{ad}} \mathcal{E}_m(u, q; \xi) \]

- Introduction
- Basics on Kalman Filtering
- Proposed Approach
- Numerical Results
- Conclusion
mCRE inverse problems

Solution is defined by:

\[ p = \arg\min_{\xi \in \mathcal{P}_{ad}} \min_{(u, q) \in \mathcal{U}_{ad} \times \mathcal{S}_{ad}} \mathcal{E}_m(u, q; \xi) \]

- Parameters minimization
- Gradient based methods
- Admissible fields
- Constrained minimization
Solution is defined by:

\[ p = \arg\min_{\xi \in \mathcal{P}_{ad}} \min_{(u, q) \in \mathcal{U}_{ad} \times \mathcal{S}_{ad}} \mathcal{E}_m(u, q; \xi) \]
Solution is defined by:

\[ p = \arg\min_{\xi \in P_{\text{ad}}} \min_{(u, q) \in U_{\text{ad}} \times S_{\text{ad}}} E_m(u, q; \xi) \]

- Robustness of the method with highly corrupted data
- Strong mechanical content
- Model reduction integration
The Modified Kalman Filter

\[
\begin{cases}
\xi^{(k+1)} = \xi^{(k)} + e^{(k)} \\
\bm{s}^{(k)} = \mathcal{H}_m^{(k)} \left( \xi^{(k)}, \bm{s}^{(k-1:k)} \right) + e_s^{(k)}
\end{cases}
\]
The Modified Kalman Filter

\[ \begin{align*}
\xi^{(k+1)} &= \xi^{(k)} + e^{(k)} \\
\mathbf{s}^{(k)} &= \mathcal{H}^{(k)}_{m} \left( \xi^{(k)}, \mathbf{s}^{(k-1:k)} \right) + e^{(k)}
\end{align*} \]

Two steps for \( \mathcal{H}^{(k)}_{m} \left( \xi^{(k)}, \mathbf{s}^{(k-1:k)} \right) \)
The Modified Kalman Filter

\[
\begin{align*}
\xi^{(k+1)} &= \xi^{(k)} + e^{(k)} \\
\mathbf{s}(k) &= \mathcal{H}_m^{(k)} \left( \xi^{(k)}, \mathbf{s}^{(k-1:k)} \right) + e_s^{(k)}
\end{align*}
\]

Two steps for \( \mathcal{H}_m^{(k)} \left( \xi^{(k)}, \mathbf{s}^{(k-1:k)} \right) \)

Step 1: admissible fields computation

\[
\mathbf{u}^{(k)} = G_{mCRE} \left( \xi^{(k)}, \mathbf{s}^{(k-1:k)} \right)
\]
The Modified Kalman Filter

\[
\begin{align*}
\xi^{(k+1)} &= \xi^{(k)} + e^{(k)} \\
S^{(k)} &= H_m^{(k)} \left( \xi^{(k)}, S^{(k-1:k)} \right) + e_s^{(k)}
\end{align*}
\]

Two steps for \( H_m^{(k)} \left( \xi^{(k)}, S^{(k-1:k)} \right) \)

Step 1: admissible fields computation

\[ u^{(k)} = G_{mCRE} \left( \xi^{(k)}, S^{(k-1:k)} \right) \]

Step 2: projection

Typically using boolean matrix

\[ \mathcal{H} := \Pi \]

\[ H_m(\xi^{(k)}, S^{(k)}) = \mathcal{H} \circ G_{mCRE}(\xi^{(k)}, S^{(k-1:k)}) \]
Optimization point of view

Dual Kalman filter based identification can be seen as the minimization of

\[ J(\xi) = \sum_{k=0}^{n_t} \left( s^{(k)} - H^{(k)} u^{(k)}(\xi^{(k)}) \right)^T C_s^{(k)}^{-1} \left( s^{(k)} - H^{(k)} u^{(k)}(\xi^{(k)}) \right) \]
Optimization point of view

Dual Kalman filter based identification can be seen as the minimization of

\[ J(\xi) = \sum_{k=0}^{n_t} \left( \mathbf{s}(k) - \mathbf{H}(k) \mathbf{u}(k)(\xi(k)) \right)^T \mathbf{C}_s(k)^{-1} \left( \mathbf{s}(k) - \mathbf{H}(k) \mathbf{u}(k)(\xi(k)) \right) \]

Classical

\[ \min_{\mathcal{U}} \left\| \mathbf{s}(k) - \mathbf{H}(k) \mathbf{u}(k) \right\|_{\mathbf{C}_s(k)^{-1}} \]
Dual Kalman filter based identification can be seen as the minimization of

\[ J(\xi) = \sum_{k=0}^{n_t} \left( s^{(k)} - H^{(k)} u^{(k)}(\xi^{(k)}) \right)^T C_s^{(k)-1} \left( s^{(k)} - H^{(k)} u^{(k)}(\xi^{(k)}) \right) \]

where

- \( J(\xi) \) is the optimization function.
- \( s^{(k)} \) and \( H^{(k)} \) are the observation and observation matrix, respectively.
- \( u^{(k)}(\xi^{(k)}) \) is the control input.
- \( C_s^{(k)} \) is the covariance matrix of the observation noise.

**Classical**

\[ \min_{\mathcal{U}} \| s^{(k)} - H^{(k)} u^{(k)} \|_{C_s^{(k)-1}} \]

**mCRE based**

\[ \min_{\mathcal{U} \times S} \| q - \nabla u \|_{K^{-1}, I_t^{(k)}} + \frac{\delta}{2} \| \Pi u - s \|_{I_t^{(k)}} \]
Dual Kalman filter based identification can be seen as the minimization of

\[ J(\xi) = \sum_{k=0}^{n_t} \left( s^{(k)} - H^{(k)}(\xi^{(k)}) \right)^T C_s^{(k)-1} \left( s^{(k)} - H^{(k)}(\xi^{(k)}) \right) \]

Classical

\[ \min_{\mathcal{U}} \left\| s^{(k)} - H^{(k)}u^{(k)} \right\|_{C_s^{(k)-1}} \]

Observations data strongly imposed

mCRE based

\[ \min_{\mathcal{U} \times S} \| q - \nabla u \|_{K^{-1}, I_t^{(k)}} + \frac{\delta}{2} \| \Pi u - s \|_{I_t^{(k)}} \]

Observations data weakly imposed
Admissible fields:

\[(u_{ad}, q_{ad}) = \arg\min_{(u, q) \in U_{ad} \times S_{ad}} \mathcal{E}_m(u, q; \xi^{(k)})\]
Admissible fields:

\[
(u_{ad}, q_{ad}) = \arg\min_{(u, q) \in \mathcal{U}_{ad} \times \mathcal{S}_{ad}} \mathcal{E}_m(u, q; \xi^{(k)})
\]
Admissible fields: \[
(u_{ad}, q_{ad}) = \arg\min_{(u, q) \in U_{ad} \times S_{ad}} \mathcal{E}_m(u, q; \xi^{(k)})
\]

\[\lambda \text{ lagrange multiplier field and stationarity conditions}\]
Admissible fields : 

\[(u_{ad}, q_{ad}) = \text{argmin}_{(u,q) \in U_{ad} \times S_{ad}} \mathcal{E}_m(u, q; \xi^{(k)})\]

\[\lambda \text{ lagrange multiplier field and stationarity conditions}\]

After FE discretization :

\[
\begin{bmatrix}
C & 0 \\
0 & -C
\end{bmatrix} \begin{bmatrix}
\dot{u} \\
\dot{\lambda}
\end{bmatrix} + \begin{bmatrix}
K & -K \\
\delta \Pi^T \Pi & K
\end{bmatrix} \begin{bmatrix}
u \\
\lambda
\end{bmatrix} = \begin{bmatrix}
F_{ext} \\
\delta \Pi^T s
\end{bmatrix} \quad \forall t
\]

with

\[u(\tau_k^{(0)}) = u^{(k-1)} \quad \text{and} \quad \lambda(\tau_k^{(n_s-1)}) = 0\]
Admissible fields:

\[(u_{ad}, q_{ad}) = \arg\min_{(u, q) \in U_{ad} \times S_{ad}} \mathcal{E}_m(u, q; \xi^{(k)})\]

\[\lambda\] lagrange multiplier field and stationarity conditions

After FE discretization:

\[
\begin{bmatrix}
C & 0 \\
0 & -C
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{\lambda}
\end{bmatrix}
+ \begin{bmatrix}
K & -K \\
\delta \Pi^T \Pi & K
\end{bmatrix}
\begin{bmatrix}
u \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
F_{ext} \\
\delta \Pi^T \mathbf{s}
\end{bmatrix}
\quad \forall t
\]

with

\[u(\tau_k^{(0)}) = u^{(k-1)} \quad \text{and} \quad \lambda(\tau_k^{(n_s-1)}) = 0\]

⚠️ Coupled forward-backward problem in time
Find \( u \in \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_D \) such that \( B(u, v) = L(v) \ \forall v \in \mathcal{X} \)

PGD based model reduction

Find \( u \in \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_D \) such that \( B(u, v) = L(v) \forall v \in \mathcal{X} \)

**Principle**: Low-rank tensor approximation

\[
 u \simeq u_m = \sum_{i=1}^{m} w_i^1 \otimes w_i^2 \otimes \cdots \otimes w_i^D ; \quad u_m \in \mathcal{X}_m \subset \mathcal{X}
\]

Find $\mathbf{u} \in \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_D$ such that $B(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{X}$

**Principle**: Low-rank tensor approximation

$$\mathbf{u} \simeq \mathbf{u}_m = \sum_{i=1}^{m} \mathbf{w}_i^1 \otimes \mathbf{w}_i^2 \otimes \cdots \otimes \mathbf{w}_i^D; \quad \mathbf{u}_m \in \mathcal{X}_m \subset \mathcal{X}$$

**Construction**: many strategies\(^8\); progressive Galerkin approach

---

PGD based model reduction

Find \( u \in \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_D \) such that \( B(u, v) = L(v) \ \forall v \in \mathcal{X} \)

**Principle** :
Low-rank tensor approximation

\[
u \simeq u_m = \sum_{i=1}^{m} w_i^1 \otimes w_i^2 \otimes \cdots \otimes w_i^D \ ; \ u_m \in \mathcal{X}_m \subset \mathcal{X}
\]

**Construction** : many strategies\(^8\); progressive Galerkin approach

\[u_{M-1} \text{ known}
\]

\[B_1(w^1, w^*) = L(w^*) - B_1(u_{M-1}, w^*)
\]

\[B_D(w^D, w^*) = L(w^*) - B_D(u_{M-1}, w^*)
\]

Orthogonalization and update

\[u_M = u_{M-1} + w^1 \otimes \cdots \otimes w^D
\]

\(^8\) Nouy, *A priori model reduction through Proper Generalized Decomposition for solving time-dependent partial differential equations*, 2010

---

SIAM UQ 2016 - Marchand et al April 5-8, 2016 18 / 30
Two fields problem: $u$ and $\lambda$

Two PGD decompositions *simultaneously* computed
Two fields problem: $u$ and $\lambda$

Two PGD decompositions *simultaneously* computed

Many parameters to consider as extra-coordinates
Two fields problem: $u$ and $\lambda$

- Two PGD decompositions *simultaneously* computed
- Many parameters to consider as extra-coordinates
  - space, time
  - parameters to identify $\xi$
  - observations data
  - initial condition
Two fields problem: $u$ and $\lambda$

Two PGD decompositions **simultaneously** computed

Many parameters to consider as extra-coordinates

▶ space, time
▶ parameters to identify $\xi$
▶ observations data
▶ initial condition

**Projection** into a reduced basis

\[
u_0^{(k)} = \sum_{i=0}^{n_{\text{init}}} \alpha_i \psi_i(x)
\]
Two fields problem: $u$ and $\lambda$

Two PGD decompositions *simultaneously* computed

Many parameters to consider as extra-coordinates

- space, time
- parameters to identify $\xi$
- observations data
- initial condition

Projection into a reduced basis

$$
\begin{align*}
\mathbf{u}_{0}^{(k)} &= \sum_{i=0}^{n_{\text{init}}} \alpha_{i} \psi_{i}(x) \\
\mathbf{u}_{PGD} &= \sum_{i=1}^{m} \phi_{i}^{u} \otimes \psi_{i}^{u} \otimes \chi_{j,i}^{u} \otimes \theta_{k,i}^{u} \otimes \eta_{m,i}^{u} \otimes \varphi_{q,i}^{u} \\
\mathbf{\lambda}_{PGD} &= \sum_{i=1}^{m} \phi_{i}^{\lambda} \otimes \psi_{i}^{\lambda} \otimes \chi_{j,i}^{\lambda} \otimes \theta_{k,i}^{\lambda} \otimes \eta_{m,i}^{\lambda} \otimes \varphi_{q,i}^{\lambda}
\end{align*}
$$
Two fields problem: $u$ and $\lambda$

Two PGD decompositions simultaneously computed

Many parameters to consider as extra-coordinates

- space, time
- parameters to identify $\xi$
- observations data
- initial condition

**Projection** into a reduced basis

$$u_0^{(k)} = \sum_{i=0}^{n_{\text{init}}} \alpha_i \psi_i(x)$$

$$u_{PGD} = \sum_{i=1}^{m} \phi^u_i \otimes \psi^u_i \otimes \chi^u_{j,i} \otimes \theta^u_{k,i} \otimes \eta^u_{m,i} \otimes \varphi^u_{q,i}$$

$$\lambda_{PGD} = \sum_{i=1}^{m} \phi^\lambda_i \otimes \psi^\lambda_i \otimes \chi^\lambda_{j,i} \otimes \theta^\lambda_{k,i} \otimes \eta^\lambda_{m,i} \otimes \varphi^\lambda_{q,i}$$

\[ n_p + 2 \cdot n_{\text{obs}} + n_{\text{init}} \approx 20 \]
Synthesis

○ Introduction
○ Basics on Kalman Filtering
- Proposed Approach
○ Numerical Results
○ Conclusion

- Reduced basis computation for initial condition projection
- PGD admissible fields computation

at each time step
- Project current initial condition in reduced basis
- Evaluate PGD parametric solution for set of $\sigma$-points
- Project state into observation space
- Kalman parameters update

Basics on Kalman Filtering

Proposed Approach

Numerical Results

Conclusion
Example 1

Problem setting

\[ \rho_c, \kappa \]

\[ u = u^d \]

\[ q^d(t) = ? \]

Time stepping for observation : 1000
Time stepping for identification : 100
Noise level : 20%

PGD modes
Exemple 1 : Neumann B.C. identification

Tuning parameters impact

$$\varepsilon_{MKF} = \frac{\left\| \xi_{true} - \mathbb{E}[\xi_{MKF}] \right\|_{L^2(I_t)}}{\left\| \xi_{true} \right\|_{L^2(I_t)}}$$
Example 2

Problem setting

- **Problem setting**
  - **Time stepping for observation**: 1000
  - **Time stepping for identification**: 100
  - **Noise level**: 10%

- **Sensor location**

- **Space modes**

- **Example 2**
  - **Problem setting**
  - **Time stepping for observation**: 1000
  - **Time stepping for identification**: 100
  - **Noise level**: 10%

- **Sensor location**

- **Space modes**
Exemple 2 : conductivity identification

Better accuracy and robustness

Joint Unscented Kalman Filter | Modified Kalman Filter
---|---
\(\frac{\kappa_1}{\kappa_{ref}}\) | \(\frac{\kappa_1}{\kappa_{ref}}\)
\(\frac{\kappa_2}{\kappa_{ref}}\) | \(\frac{\kappa_2}{\kappa_{ref}}\)
\(\frac{\kappa_3}{\kappa_{ref}}\) | \(\frac{\kappa_3}{\kappa_{ref}}\)
\(\frac{\kappa_4}{\kappa_{ref}}\) | \(\frac{\kappa_4}{\kappa_{ref}}\)
Example 3

Problem setting

Thermal source:

\[ f(x; x_c) = \text{sinc}^2 (\pi \| x - x_c(t) \|) \]

To include \( x_c \) as PGD's extra-coordinate

\[ f(x; x_c) \approx \sum_{i=1}^{N} F_i(x) \cdot G_i(x_c) \]

Using SVD
Exemple 3 : source localization

Time stepping for observation : 1000
Time stepping for identification : 100
Noise level : 10%

Modified Kalman Filter

Results not compared to UKF since this problem requires to solve 5000 problems at each time step with the UKF approach
Exemple 3 : source localization

PGD limits

Solution is relatively *singular* involves

Initial condition should be project on many modes

\[ n_{init} \gg 1 \]

but

\[ n_p + 2 \cdot n_{obs} + n_{init} \lesssim 20 \]
Outline

- Introduction
- Basics on Kalman Filtering
- Proposed Approach
- Numerical Results
- Conclusion

Basics on Kalman Filtering

Proposed Approach

Numerical Results

Conclusion
## Conclusion and future works

<table>
<thead>
<tr>
<th>Unscented Kalman Filter</th>
<th>Modified Kalman Filter</th>
<th>modified CRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implementation</td>
<td>Implementation</td>
<td>Robustness</td>
</tr>
<tr>
<td>Cost</td>
<td>Robustness</td>
<td>Cost</td>
</tr>
<tr>
<td>Robustness</td>
<td>Cost</td>
<td>2\times\text{Minimizations}</td>
</tr>
</tbody>
</table>

Proper Generalized Decomposition

- Introduction
- Basics on Kalman Filtering
- Proposed Approach
- Numerical Results
- Conclusion
### Conclusion and future works

#### Table: Comparison of Approaches

<table>
<thead>
<tr>
<th></th>
<th>Unscented Kalman Filter</th>
<th>Modified Kalman Filter</th>
<th>modified CRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implementation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Robustness</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Robustness</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cost</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2*Minimizations</td>
</tr>
</tbody>
</table>

- Proper Generalized Decomposition

#### Extension to field identification

- Number of parameters significantly increases
- split state and parameters meshes
- adaptive strategy

---

SIAM UQ 2016 - Marchand et al

April 5-8, 2016