Low-energy spectrum of Toeplitz operators with a miniwell

Alix Deleporte

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Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France

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Abstract

We study the concentration properties of low-energy states for quantum systems in the semiclassical limit, in the setting of Toeplitz operators, which include quantum spin systems as a large class of examples. We establish tools proper to Toeplitz quantization to give concentration properties under weak conditions. In addition, we build up symplectic normal forms in two particular settings, including a generalisation of Helffer-Sjöstrand miniwells, in order to prove asymptotics for the ground state and estimates on the number of low-lying eigenvalues.

1 Introduction

1.1 Quantum selection

The computation of ground states for quantum systems is an ubiquitous problem of great difficulty in the non-integrable case, such as antiferromagnetic models on lattices in several dimensions. On those systems, approaches in the large spin limit are commonly used [23, 17, 21, 8], in an effort to reduce the problem to the study of the minimal set of the classical energy. A general procedure of *semiclassical order by disorder* was proposed by Douçot and Simon [11], in situations where this classical minimal set is not discrete.

In the mathematical setting of Schrödinger operators in the semiclassical limit, a general study of ground state properties was done by Helffer and Sjöstrand [13, 14], including situations where the minimal set of the potential is a smooth submanifold. The classical phase space of spin systems, a product of spheres, is compact. In particular, spin systems are neither Schrödinger operators nor given by Weyl quantization. However, spin operators are example of Toeplitz operators, which allows to understand the large spin limit as a semiclassical limit. In a previous article [9], we studied semiclassical concentration of ground states

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in the context of semiclassical Toeplitz operators, when the minimal set of the classical energy (or symbol) is a finite set of non-degenerate points, with results analogous to the Schrödinger case [13].

In frustrated antiferromagnetic spin systems, such as on the Kagome lattice, the minimal set of the classical energy does not form a smooth submanifold. The goal of this article is to not only to extend the degenerate case [14] to Toeplitz quantization, but also to generalise the geometrical conditions on the zero set of the classical energy.

We prove several results of *quantum selection*: not all points of classical phase space where the energy is minimal are equivalent for quantum systems; and the semiclassical quantum ground state localises only on a subset of the classical minimal set. To do so, on one hand we develop techniques which are proper to Toeplitz quantization; on the other hand we prove new symplectic normal forms which are also useful in the context of pseudodifferential calculus.

Theorem A requires a weak condition on the symbol near its zero set, and applies in particular to symbols which are analytic near zero, which is the case of all physical examples of interest. It is done in the spirit of Melin's inequality (see [19], or [15], Thm 22.2.3). Theorem B is more precise and applies in a particular setting which generalises the miniwells of [14]. Theorem C treats a degenerate case where the symbol is minimal on a set with a singular point. Theorem D analyses the relative role of regular and singular points in the low-energy Weyl law.

1.2 Main results

In order to state the main theorems we need to introduce Toeplitz quantization, the criterion under which localisation takes place, and what localisation means in this context.

Toeplitz quantization takes place on quantizable Kähler manifolds [6], which are complex manifolds with a symplectic and a Riemannian structure. Let M be such a manifold and let $h \in C^\infty(M, \mathbb{R})$. Through Toeplitz quantization we associate to this function a sequence of self-adjoint operators $(T_N(h))_{N \geq 1}$ acting on a sequence of Hilbert spaces $(H_N)_{N \geq 1}$ (see Definitions 2.1 and 2.10). The semiclassical limit is $N \to +\infty$. In this article we are interested in the spectrum and eigenvectors of $T_N(h)$ for N large.

Suppose $\min(h) = 0$. The selection criterion consists in a continuous function μ (see Definition 2.16). This function is defined on $\{h = 0\}$, and depends on the Hessian of h. It captures the effects of order N^{-1} on the low-energy spectrum of $T_N(h)$. For each point x such that $h(x) = 0$, we call $\mu(x)$ the *Melin value* at x.

The Hilbert spaces H_N consist of L^2 functions on a bundle over M (with projection π). In particular, for any Borel set $B \subset M$ and function $u \in H_N$, the microlocal mass of u on B is directly defined (as $\int_{\pi^{-1}(B)} |u|^2 \mathrm{d}Vol$), in contrast with Weyl quantization where this needs some work; see Definition 2.5.

Theorem A. *Let M be a compact Kähler quantizable manifold and $h \in C^\infty(M, \mathbb{R})$. Suppose that $\min(h) = 0$. Let μ the function associating to each point where h vanishes the Melin value at this point. Let $\mu_{\min} = \min(\mu(x), x \in M, h(x) = 0)$. Suppose that there exist $C > 0$ and $\alpha > 0$ such that, for every $t \geq 0$, one has*

$$\mathrm{dist}_{Hausdorff}(\{h \leq t\}, \{h = 0\}) \leq Ct^\alpha.$$

Then there exist $C > 0$ and $\epsilon > 0$ such that, for every $N \geq 1$, one has

$$|\min \mathrm{Sp}(T_N(h)) - N^{-1}\mu_{\min}| \leq CN^{-1-\epsilon}.$$

Here $\mathrm{Sp}(T_N(h))$ denotes the spectrum of $T_N(h)$.

Let $((\lambda_N, u_N))_{N \geq 1}$ be a sequence of eigenpairs of $(T_N(h))_{N \geq 1}$. If $\|u_N\|_{H_N} = 1$ and $\lambda_N = N^{-1}\mu_{min} + o(N^{-1})$, then for any open set U at positive distance from

$$\{x \in M, h(x) = 0, \mu(x) = \mu_{\min}\},$$

as $N \to +\infty$ there holds

$$\int_{\pi^{-1}(U)} |u_N|^2 \mathrm{d}Vol = O(N^{-\infty}).$$

Theorem A already appears in previous work [9], under the much stronger hypothesis that $\{h = 0\}$ is a finite set of regular critical points.

Theorem B. *Under the hypotheses of Theorem A, suppose that the function μ reaches its non-degenerate minimum on a unique point P_0. Suppose further that, in a neighbourhood of P_0, the set $\{h = 0\}$ is an isotropic submanifold of M, on which h has non-degenerate transverse Hessian matrix.*

Then for any sequence $(u_N)_{N \geq 1}$ of unit eigenfunctions corresponding to the first eigenvalue of $T_N(h)$, for any $\epsilon > 0$, one has

$$\int_{\left\{dist(\pi(y), P_0) > N^{-\frac{1}{4}+\epsilon}\right\}} |u_N(y)|^2 \mathrm{d}Vol = O(N^{-\infty}).$$

Moreover, the first eigenvalue is simple and the spectral gap is of order $N^{-\frac{3}{2}}$. There is a full expansion of the first eigenvalue and eigenvector in powers of $N^{-\frac{1}{4}}$.

Following Helffer-Sjöstrand [14], we will call P_0 a *miniwell* for h.

Under the conditions of Theorem B, the first eigenvector concentrates rapidly on $\{h = 0\}$, and the speed of concentration towards the point which minimises μ is much slower. In particular this state is more and more squeezed as N increases.

Theorem C. *Under the hypotheses of Theorem A, suppose that the function μ reaches its minimum on a unique point P_0 at which there is a simple crossing (see Definition 6.1).*

Then for any sequence $(u_N)_{N \geq 1}$ of unit eigenfunctions corresponding to the first eigenvalue of $T_N(h)$, for any $\epsilon > 0$, one has

$$\int_{\left\{dist(\pi(y), P_0) > N^{-\frac{1}{3}+\epsilon}\right\}} |u_N(y)|^2 \mathrm{d}Vol = O(N^{-\infty}).$$

Moreover, the first eigenvalue is simple and the spectral gap is of order $N^{-\frac{4}{3}}$. There is a full expansion of the first eigenvalue and eigenvector in powers of $N^{-\frac{1}{6}}$.

An example of symbol with a *simple crossing*, with dimensions $(1, 1)$, is the following function on $\mathbb{R}^4$:

$$h : (q_1, q_2, p_1, p_2) \mapsto p_1^2 + p_2^2 + q_1^2 q_2^2,$$

which reaches its minimum on the transverse union of two manifolds, $\mathbb{R} \times \{0,0,0\}$ and $\{0\} \times \mathbb{R} \times \{0,0\}$, intersecting at one point.

As in the case of Theorem B, the first eigenvector is more and more squeezed as $N \to +\infty$. Note that the speed of convergence, and the powers of N involved in the expansions, differ between the two cases.

The question now arises of the inverse spectral problem in our setting: given the high N spectrum of a Toeplitz operator, is one able to distinguish the geometry of the set on which the Melin value μ is minimal?

Theorem D. *Let $h \in C^\infty(M, \mathbb{R})$ with $\min(h) = 0$. There exist $0 < c \leq C$, $\epsilon > 0$ and $N_0 \geq 0$ such that the following is true. Let $\mu_{\min}$ be the infimum of the Melin value, and $N \geq N_0$.*

  A. *Any eigenfunction of $T_N(h)$ associated with an eigenvalue in the spectral window $[0, \mu_{\min} N^{-1} + \epsilon N^{-1}]$ is localised on a small neighbourhood of the set of minimal Melin value.*

  B. *For each regular miniwell with Melin value $\mu_{\min}$ and dimension r, for each sequence (Λ_N) with*
$$N^{-\frac{1}{2}+\epsilon} \leq \Lambda_N \leq \epsilon,$$
  in the spectral window $[0, N^{-1}(\mu_{\min} + \Lambda_N)]$, the number of orthogonal almost eigenfunctions of $T_N(h)$ supported on a small neighbourhood of the miniwell belongs to the interval
$$\left[c(N^{\frac{1}{2}}\Lambda_N)^r, C(N^{\frac{1}{2}}\Lambda_N)^r \right].$$

  C. *For each simple crossing with Melin value $\mu_{\min}$ and dimensions (r, r), for each sequence (Λ_N) with*
$$N^{-\frac{1}{3}+\epsilon} \leq \Lambda_N \leq \epsilon,$$
  in the spectral window $[0, N^{-1}(\mu_{\min} + \Lambda_N)]$, the number of orthogonal almost eigenfunctions of $T_N(h)$ supported on a small neighbourhood of the crossing point belongs to the interval
$$\left[c(N^{\frac{1}{3}}\Lambda_N)^{\frac{3r}{2}} \log(N^{\frac{1}{3}}\Lambda_N), C(N^{\frac{1}{3}}\Lambda_N)^{\frac{3r}{2}} \log(N^{\frac{1}{3}}\Lambda_N) \right].$$

The notion of dimension of a miniwell and a simple crossing can be found in Definition 7.1. In Theorem D, case A is a generalisation of Theorem A. Cases B and C apply respectively in the settings of Theorems B and C.

Remark 1.1. If $\Lambda_N < N^{-\epsilon}$, then there are more eigenvalues near a miniwell than near a crossing point (the ratio is of order $N^{\frac{\epsilon}{2}}$). If we look at eigenvalues in such windows, then a miniwell of dimension r not only "hides" miniwells of smaller dimensions, but also crossing points of dimensions up to and including (r, r).

If $\Lambda_N > \frac{\epsilon}{2}$, then there are more eigenvalues near a crossing point than near a miniwell (the ratio is of order $\log(N)$). In these windows, crossing points hide miniwells of dimension smaller or equal.

In particular, this proves that the spectral inverse problem allows, not only to recover the value of $\mu_{\min}$, but also to determine the largest dimensions of the miniwells or crossing points achieving $\mu_{\min}$, and to tell whether there are only miniwells, only crossing points, or both.

Theorem D also allows to study low-temperature quantum states for a model on which there is a competition between a regular point and a crossing point with the same μ. It shows a transition from temperature ranges similar to N^{-1}, for which the Gibbs measure concentrates on the crossing point, and temperature ranges of order $N^{-1-\epsilon}$, for which this measure concentrates on the regular point.

In this work only rapid decay estimates are obtained: quantities are controlled modulo an $O(N^{-\infty})$ error as $N \to +\infty$. The natural question of exponential decay [14] requires refined estimates on the Szegő kernel which are currently unknown for general compact Kähler manifolds.

The study of the function μ on examples of spin systems requires the full diagonalisation of matrices which size grows with the number of sites. Few theoretical results are known in this setting (see Section 8). The general conjecture is that μ should reach a minimum on planar configurations; up to now this is only supported by numerical evidence and the fact that planar configurations are local minima for μ.

1.3 Application to spin systems

One of the main physical motivations for this study, discussed in detail in Section 8, is the mathematical foundation of *quantum selection* in the context of spin systems. The search for materials with a non-conventional magnetic behaviour led experimental and theoretical physicists to consider frustrated antiferromagnetic spin systems, such as pyrochlore or the Kagome lattice. *Order by disorder* approaches in the large spin limit are commonly used in the physics literature, and the subprincipal effects presumably select a very small subset of configurations [11, 23, 17, 21, 8].

Spin systems are particular cases of Toeplitz operators. In such systems the base manifold is a product of 2-spheres. Let $G = (V, E)$ be a finite graph and $M = (\mathbb{S}^2)^{\times |V|}$. At each vertex of the graph one associates a unit vector in $\mathbb{R}^3$. Let us consider the $3|V|$ real functions associating to a given vertex $i \in V$, the coordinates x_i, y_i, z_i of the associated unit vector e_i. The symplectic structure on M is such that $\{x_i, y_j\} = \delta_{ij} z_i$; two similar identities hold by cyclic permutation. We introduce the *antiferromagnetic Heisenberg* symbol:

$$h : \begin{array}{ccc} M & \mapsto & \mathbb{R} \\ (e_i)_{i \in V} & \mapsto & \sum\limits_{(i,j) \in E} x_i x_j + y_i y_j + z_i z_j. \end{array}$$

The classical minimum of this function corresponds to situations where the sum of the scalar products between neighbouring vectors is the smallest. If G is bipartite, this minimum is reached in situations where neighbouring vectors are opposite. In *frustrated systems*, this is not possible. If for instance three vertices in the graph are linked with each other, then not all of them can be opposite to the other ones. This is the case of the Kagome lattice, and the Husimi tree, considered in [11] and depicted in Figure 1.
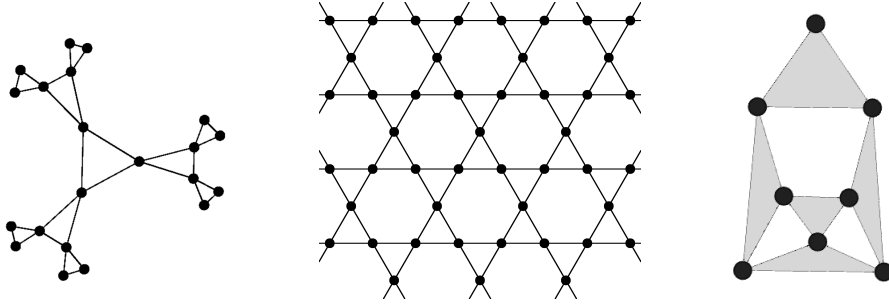
Figure 1: Main examples: a piece of the Husimi tree (left), and the Kagome lattice (middle). On the right, a graph made of 5 triangles on which the symbol cannot reach $-15/2$.

We will consider a class of graphs *made of triangles*. A finite connected graph $G = (E, V)$ is *made of triangles* when there is a partition $V = \bigsqcup_{i \in J} V_i$ where, for every i, V_i contains three edges that link together three vertices; in addition, we ask that the degree at any vertex does not exceed 4 (and is hence equal to either 2 or 4). We will call the V_i's the triangles of the graph.

Finite subgraphs of the Kagome lattice and the Husimi tree of Figure 1 are made of triangles. In general, from a 3-regular finite graph $G = (V, E)$, one can build an associated graph made of triangles $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ which is the *edge graph* of G: the set of vertices is $\widetilde{V} = E$ and two elements of $\widetilde{V}$ are adjacent in $\widetilde{G}$ when they are adjacent as edges of G (i.e. when they share a common vertex). In this case the triangles of $\widetilde{G}$ correspond to the vertices of G. The Kagome lattice is thus associated with the hexagonal lattice, and the infinite Husimi tree with the 3-regular tree.

The presence of the "frustration" by triangles leads to a large degeneracy of the classical minimal set. Indeed, h is minimal if, on each triangle V_i, the sum of the three spins at the vertices of the triangle is zero (so that these elements of $\mathbb{S}^2$ must form a great equilateral triangle). This is not always possible as the example to the right of Figure 1 shows. Those configurations exist on subsets of the Husimi tree and the Kagome lattice, and are highly degenerate: on the Husimi tree, once the spins on a triangle are chosen, there is an $\mathbb{S}^1$ degeneracy for each of its children; the set of minimal configurations is an isotropic torus whose dimension grows linearly with the number of triangles. On the Kagome lattice, the set of these configurations does not form a smooth submanifold, hence the need for Theorems A and C. It is currently unknown which minimal points of h achieve $\mu_{\min}$.

Proposition 1.2. *For a loop of 6 triangles (the basic element of the Kagome lattice), the minimal set is not a smooth manifold.*

For a loop of 4 triangles, the minimal set is the direct product of $SO(3)$ and the union of three circles, two of each having transverse intersection at exactly one point. Planar configurations are local minima for μ.

The proof is presented in Section 8.

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1.4 Outline

In Section 2 we recall the necessary material on Toeplitz operators, including a universality lemma proved in [9], and quantum maps as developed in [7]. In this section we also define and study the Melin value.

Section 3 contains the main tool in the proof of Theorem A, which is a Toeplitz version of the Melin estimates [19, 15]. We give a global and a local version of these estimates, and use them to prove pseudolocality of the resolvent at a distance $\geq \varepsilon N^{-1}$ of the spectrum, for every $\varepsilon > 0$.

Section 4 concludes the proof of Theorem A, based on the Melin estimate.

Sections 5 and 6 respectively contain the proofs of Theorems B and C, following the same strategy. We first find a convenient symplectic normal form, then use quantum maps to reduce the problem to a normal form, then we use a particular perturbative argument to exhibit, for every $k \in \mathbb{N}$, an approximate eigenvector up to $O(N^{-k})$.

In Section 7 we prove Theorem D, using estimates developed in the previous sections, and especially Proposition 5.9 and Proposition 6.16.

Section 8 contains a discussion on frustrated spin systems.

2 Toeplitz operators

Toeplitz operators are a generalisation of the Bargmann-Fock point of view on the quantum harmonic oscillator [2]. They realise a *quantization* on some symplectic manifolds, and are a particular case of geometric quantization [16, 24]. Another particular case of geometric quantization is Weyl quantization which leads to pseudodifferential operators. Toeplitz operators were first studied from a microlocal point of view [4, 5], and the study of the Szegő kernel was further motivated by geometrical applications [10, 26]. Here we directly use the semi-classical point of view developed in [22, 6, 18].

In this Section we recall the properties of Toeplitz operators, and we refer to earlier work on the topic [9, 3, 18, 6, 22, 25, 2] for the proofs of the exposed facts.

2.1 Hardy spaces and the Szegő projector

If a symplectic manifold (phase space) M has a complex structure, the idea behind Berezin-Toeplitz operators is to consider quantum states as holomorphic functions. If M is compact, holomorphic functions on M are all constant, so that one needs to consider sections of a convenient line bundle over M or, by duality, holomorphic functions on a dual line bundle.

Let M be a Kähler manifold of dimension n, with symplectic form ω. If the Chern class of $\omega/2\pi$ is integral, there exists a Hermitian holomorphic line bundle (L, h) over M, with curvature $-i\omega$ ([25], pp. 158-162).

Let (L^*, h^*) be the dual line bundle of L, with dual metric. Let D be the unit ball of L^*, that is:
$$\{D = (m, v) \in L^*, \|v\|_{h^*} < 1\}.$$

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The boundary of D is denoted by X. It is a circle bundle over M, with projection π and an $\mathbb{S}^1$ action

$$
\begin{aligned}
r_\theta : \quad X &\mapsto \quad X \\
(m, v) &\mapsto (m, e^{i\theta} v).
\end{aligned}
$$

X inherits a Riemannian structure from L^* so that $L^2(X)$ is well-defined. We are interested in the equivariant Hardy spaces on X, defined as follows:

Definition 2.1.

- The *Hardy space* $H(X)$ is the closure in $L^2(X)$ of

$$
\{ f|_X,\ f \in C^\infty(D \cup X),\ f \text{ holomorphic in } D \}.
$$

- The Szegő projector S is the orthogonal projection from $L^2(X)$ onto $H(X)$.

- Let $N \in \mathbb{N}$. The *equivariant Hardy space* $H_N(X)$ is:

$$
H_N(X) = \{ f \in H(X),\ \forall (x, \theta) \in X \times \mathbb{S}^1,\ f(r_\theta x) = e^{iN\theta} f(x) \}.
$$

- The equivariant Szegő projector S_N is the orthogonal projection from $L^2(X)$ onto $H_N(X)$.

Throughout this paper, we will work with the sequence of spaces $(H_N(X))_{N \in \mathbb{N}}$. If M is compact, then the spaces $H_N(X)$ are finite-dimensional spaces of smooth functions. (Note, however, that this dimension grows polynomially with N.) Hence, the Szegő projector has a Schwartz kernel, that we will also denote by S_N.

Example 2.2 (The sphere)**.** The sphere $\mathbb{S}^2$ has a canonical Kähler structure as $(\mathbb{CP}^1, \omega_{FS})$, which is quantizable. Here D is the unit ball in $\mathbb{C}^2$, blown up at zero, and $X = \mathbb{S}^3$. One recovers the usual $\mathbb{S}^1$ free action on $\mathbb{S}^3$ with quotient $\mathbb{S}^2$.

Here, $H_N(X)$ is the space of homogeneous polynomials of two complex variables, of degree N, with Hilbert structure the scalar product of the restriction to X of these polynomials. A natural Hilbert basis corresponds to the normalized monomials

$$
(z_1, z_2) \mapsto \sqrt{\frac{N+1}{\pi} \binom{N}{k}}\, z_1^k z_2^{N-k}.
$$

In particular, the Szegő projector has kernel

$$
S_N^{\mathbb{CP}_1} : (z, w) \mapsto \frac{N+1}{\pi} (z \cdot \overline{w})^N.
$$

Example 2.3 ($\mathbb{C}^n$)**.** Another important example (though non compact) is the case $M = \mathbb{C}^n$, with standard Kähler form. As $\mathbb{C}^n$ is contractile, the bundle L is trivial, but the metric h is not. The curvature condition yields:

$$
(L, h) = \left(\mathbb{C}_z^n \times \mathbb{C}_v,\ e^{|z|^2} |v|^2 \right).
$$

This leads to the following identification [2]:

$$H_N(X) \simeq B_N := L^2(\mathbb{C}^n) \cap \left\{ z \mapsto e^{-\frac{N}{2}|z|^2} f(z), \ f \text{ is an entire function} \right\}.$$

The space B_N is a closed subspace of $L^2(\mathbb{C}^n)$. The orthogonal projector Π_N from $L^2(\mathbb{C}^n)$ to B_N admits as Schwartz kernel the function

$$\Pi_N : (z, w) \mapsto \left(\frac{N}{\pi} \right)^n \exp\left(-\frac{1}{2}N|z - w|^2 + iN\Im(z \cdot \overline{w}) \right).$$

As the case $M = \mathbb{C}^n$ is of particular interest, we will keep separate notations for the Szegő kernel in this case, which will always be denoted by Π_N.

The sequence of kernels $(\Pi_N)_{N \geq 1}$ is rapidly decreasing outside the diagonal set. This key property also holds in the case of a compact Kähler manifold:

Proposition 2.4 ([18], prop 4.1, or [6, 22, 3]). *Let M be a compact Kähler manifold, and $(S_N)_{N \geq 1}$ be the sequence of Szegő projectors of Definition 2.1. Let $\delta \in [0, 1/2)$. For every $k \geq 0$ there exists C such that, for every $N \geq 1$, for every $x, y \in X$ such that $\operatorname{dist}(\pi(x), \pi(y)) \geq N^{-\delta}$, one has*

$$|S_N(x, y)| \leq C N^{-k}.$$

This roughly means that, though the operators S_N are non-local, the "interaction range" shrinks with N.

In the spirit of Proposition 2.4, we define what it means for a sequence of functions in $H_N(X)$ to be localised on a set.

Definition 2.5. Let $u = (u_N)_{N \geq 1}$ be a sequence of unit elements of $L^2(X)$. Let $\mathrm{d}Vol$ denote the Liouville volume form on M. For every N, the probability measure $|u_N|^2 \mathrm{d}Vol \otimes \mathrm{d}\theta$ is well-defined on X, and we call μ_N the push-forward of this measure on M.

Let moreover $Z \subset M$ be compact. We will say that the sequence u localises on Z when, for every open set $U \subset M$ at positive distance from Z, one has, as $N \to +\infty$:

$$\mu_N(U) = O(N^{-\infty}).$$

A corollary of this definition is that, if a sequence $(u_N)_{N \geq 1}$ localises on a set Z, then so does the sequence $(S_N u_N)_{N \geq 1}$.

Remark 2.6. Elements in the Hardy space are functions on the whole phase space. Hence, Definition 2.5 corresponds to *micro*localisation for elements of $L^2(\mathbb{R}^n)$.

To complete Proposition 2.4, we have to describe how S_N acts on sequences of functions concentrated on a point. For this we need a convenient choice of coordinates.

Let $P_0 \in M$. The real tangent space $T_{P_0}M$ carries a Euclidian structure and an almost complex structure coming from the Kähler structure on M. Then, we can (non-uniquely) identify $T_{P_0}M$ with $\mathbb{C}^n$ endowed with the standard metric.

The map $\exp_M : T_{P_0}M \mapsto M$ on the Riemannian manifold M, together with the identification $\mathbb{C}^n \simeq T_{P_0}M$, leads to the notion of normal coordinates:

Definition 2.7. Let U be a neighbourhood of 0 in $\mathbb{C}^n$ and V be a neighbourhood of a point P_0 in M.

A smooth diffeomorphism $\rho : U \times \mathbb{S}^1 \to \pi^{-1}(V)$ is said to be a *normal map* or a map of *normal coordinates* if it satisfies the following conditions:

- $\forall (z, v) \in U \times \mathbb{S}^1$, $\forall \theta \in \mathbb{R}$, $\rho(z, ve^{i\theta}) = r_\theta \rho(z, v)$

- Identifying $\mathbb{C}^n$ with $T_{P_0} M$ as previously, one has:

$$\forall (z, v) \in U \times \mathbb{S}^1, \ \pi(\rho(z, v)) = \exp_M(z).$$

Remark 2.8. The choice of a normal map around a point P_0 reflects the choice of an identification of $\mathbb{C}^n$ with $T_{P_0}(M)$ and a point over P_0 in X. Hence, if ρ_1 and ρ_2 are two normal maps around the same point P_0, then $\rho_1^{-1} \circ \rho_2 \in U(n) \times U(1)$.

Using Definition 2.7, one can compare, for N large, the Szegő kernel S_N with the flat case Π_N. For this, we push by ρ the Bargmann kernel and multiply by the correct factor in the fibre to obtain an equivariant kernel on X:

$$\rho^* \Pi_N(\rho(z, \theta), \rho(w, \phi)) := e^{iN(\theta - \phi)} \Pi_N(z, w).$$

By convention, $\rho^* \Pi_N$ is zero outside $\pi^{-1}(V)^2$.

Proposition 2.9 ([9]). *Let $P_0 \in M$, and ρ a normal map around P_0. For every $\epsilon > 0$ there exists $\delta \in (0, 1/2)$ and $C > 0$ such that for every $N \in \mathbb{N}$, for every $u \in L^2(X)$, if the support of u lies inside $\rho(B(0, N^{-\delta}) \times \mathbb{S}^1)$, then*

$$\|(S_N - \rho^* \Pi_N)u\|_{L^2} < CN^{-\frac{1}{2} + \epsilon} \|u\|_{L^2}.$$

In a sense, Proposition 2.9 states that the operator S_N asymptotically looks like Π_N on small scales. The proof can be found in a previous paper [9], as a consequence of previously known results on the asymptotical behaviour of the Schwartz kernel of S_N near the diagonal set [22, 18, 6].

2.2 Toeplitz operators

Definition 2.10. Let M be a Kähler manifold, with equivariant Szegő projectors $(S_N)_{N \geq 1}$. Let $h \in C^\infty(M)$ be a smooth function on M. For all $N \geq 1$, the Toeplitz operator $T_N(h) : H_N(X) \to H_N(X)$ associated with the symbol h is defined as

$$T_N(h) = S_N h S_N.$$

In this work we investigate the spectral properties of the operators $T_N(f)$, for fixed f and $N \to +\infty$.

Example 2.11 (Spin operators). Let us continue from Example 2.2. The sphere $\mathbb{S}^2$ is naturally a submanifold of $\mathbb{R}^3$; as such, there are three coordinate functions $(x, y, z) : \mathbb{S}^2 \mapsto \mathbb{R}^3$. They are closed under Poisson brackets: one has $\{x, y\} = z$ and two similar identities by cyclic permutation.

In the Hilbert basis given by the normalized monomials, the associated Toeplitz operators $T_N(x), T_N(y), T_N(z)$ are, up to a factor $\frac{N}{N+2}$, the usual spin matrices with spin $\frac{N}{2}$.

2.2.1 Toeplitz operators on $\mathbb{C}^n$ and the Melin value

The manifold $\mathbb{C}^n$ is not compact. Let us release the condition that the symbol is bounded. This defines Toeplitz operators as unbounded operators on B_N.

Toeplitz operators on $\mathbb{C}^n$ whose symbols are semipositive definite quadratic forms play an important role in this work. If Q is a quadratic form on $\mathbb{R}^{2n}$ identified with $\mathbb{C}^n$, then $T_N(Q)$ is essentially self-adjoint. This operator is related to the Weyl quantization $Op_W^\hbar(Q)$ with semi-classical parameter $\hbar = N^{-1}$. In fact, $T_N(Q)$ is conjugated, via the Bargmann transform $\mathcal{B}_N$ [2], with the operator

$$Op_W^{N^{-1}}(Q) + \frac{N^{-1}}{4} \operatorname{tr}(Q).$$

If Q is semi-definite positive, then it takes non-negative values as a function on $\mathbb{R}^{2n}$, hence $T_N(Q) \geq 0$ for all $N \geq 0$ since, for $u \in B_N$, one has

$$\langle u, \Pi_N Q \Pi_N u \rangle = \langle u, Qu \rangle \geq 0.$$

The infimum of the spectrum of $T_N(Q)$ is of utmost interest, since it leads to the notion of Melin value. As Q is 2-homogeneous, and the Bargmann spaces are identified with each other through a scaling, one has $T_N(Q) \sim N^{-1} T_1(Q)$, and in particular

$$\inf(\operatorname{Sp}(T_N(Q))) = N^{-1} \inf(\operatorname{Sp}(T_1(Q))).$$

Definition 2.12. Let Q be a semi-definite positive quadratic form on $\mathbb{R}^{2n}$, identified with $\mathbb{C}^n$.

We denote by $\mu(Q)$ the *Melin value* of Q, defined by

$$\mu(Q) := \inf\left(\operatorname{Sp}(T_1(Q))\right).$$

Given $Q \geq 0$, how can one compute $\mu(Q)$? As stated above, it depends first on the trace of Q (which is easy to compute), and second on the infimum of the spectrum of $Op_W^1(Q)$. This second part is invariant through a symplectic change of variables, and the problem reduces to a *symplectic diagonalisation* of Q (see Proposition 5.1). As an example:

Example 2.13. Let $\alpha, \beta \geq 0$. Then

$$\mu\left((x,y) \mapsto \alpha x^2 + \beta y^2\right) = \frac{1}{4}(2\sqrt{\alpha\beta} + \alpha + \beta).$$

The function μ itself is *not* invariant under symplectomorphisms (for example, in the previous example it does not only depend on $\alpha\beta$). However, it is invariant under unitary changes of variables.

The regularity of the map μ will be useful in the proof of Theorem A:

Proposition 2.14 ([19])**.** *The function $Q \mapsto \mu(Q)$ is Hölder continuous with exponent $\frac{1}{2n}$ on the set of semi-definite positive quadratic forms of dimension $2n$.*

If Q is definite positive, then $T_N(Q)$ has compact resolvent, and the first eigenvalue is simple.

2.2.2 Toeplitz operators on compact manifolds

When the base manifold M is compact and h is real-valued, for fixed N the operator $T_N(h)$ is a symmetric operator on a finite-dimensional space. In this setting, we will speak freely about eigenvalues and eigenfunctions of Toeplitz operators.

The composition of two Toeplitz operators can be written, in the general case, as a formal series of Toeplitz operators [6], that is:

$$T_N(f)T_N(g) = T_N(fg) + N^{-1}T_N(C_1(f,g)) + N^{-2}T_N(C_2(f,g)) + \ldots.$$

The composition properties of formal series of Toeplitz operators lead to the following property, which appears in previous work [9], and which is an important first step towards the study of the low-lying eigenvalues.

Proposition 2.15. *Let M be a compact Kähler manifold and h a real non-negative smooth function on M.*

Let $u = (u_N)_{N\geq 1}$ be a sequence of unit elements in the Hardy spaces such that, for every N, one has

$$T_N(h)u_N = \lambda_N u_N,$$

with $\lambda_N = O(N^{-1})$.

Then the sequence u localises on $\{h = 0\}$. More precisely, for every $\epsilon > 0$, if

$$Z_N = \{m \in M, h(m) \geq N^{-1+\epsilon}\},$$

one has, as $N \to +\infty$,

$$\int_{\pi^{-1}(Z_N)} |u(x)|^2 \mathrm{d}Vol = O(N^{-\infty}).$$

On a point where h is minimal, one can pull-back Definition 2.12 by normal coordinates of Definition 2.7:

Definition 2.16. Let $h \in \mathbb{C}^\infty(M, \mathbb{R}^+)$. Let $P_0 \in M$ be such that $h(P_0) = 0$. Let ρ be a normal map around P_0; the function $h \circ \rho$ is well-defined and non-negative on a neighbourhood of 0 in $\mathbb{C}^n$, and the image of 0 is 0. Hence, there exists a semi-definite positive quadratic form Q such that

$$h \circ \rho(x) = Q(x) + O(|x|^3).$$

We define the Melin value $\mu(P_0)$ as $\mu(Q)$.

Remark 2.17. A different choice of normal coordinates corresponds to a $U(n)$ change of variables for Q, under which μ is invariant. Hence, $\mu(P_0)$ does not depend on the choice of normal coordinates.

The function $P_0 \mapsto \mu(P_0)$ is $\frac{1}{2n}$-Hölder continuous as a composition of the smooth function $P_0 \mapsto Q$ and the Hölder continuous function $Q \mapsto \mu$.

2.3 Quantum maps

To a (local) symplectomorphism between Kähler manifolds, one can associate an almost unitary (local) transformation on the Hardy spaces, such that, at first order, the Toeplitz quantizations on both sides are related by the symplectic change of variables in the symbols [7]:

Proposition 2.18. *Let $\sigma : (M, x) \mapsto (N, y)$ be a local symplectomorphism between two quantizable Kähler manifolds.*

Let U be a small open set around x. Then there exists, for every N, a linear map $\mathfrak{S}_N : H_N(M, L) \mapsto H_N(N, K)$ and a sequence of differential operators $(L_j)_{j \geq 1}$, such that, for any sequence $(u_N)_{N \geq 1}$ of sections which are $O(N^{-\infty})$ outside of U, and for any symbol $a \in C^\infty(N)$, one has:

$$\|\mathfrak{S}_N u_N\|_{L^2} = \|u_N\|_{L^2} + O(N^{-\infty})$$

$$\mathfrak{S}_N^{-1} T_N(a) \mathfrak{S}_N u_N = T_N \left(a \circ \sigma + \sum_{k=1}^\infty N^{-i} L_j(a \circ \sigma) \right) u_N + O(N^{-\infty}).$$

Moreover, for every $j \geq 1$, the differential operator L_j is of degree $2j$.

As a preliminary lemma for Sections 5 and 6, let us show that quantum maps preserve concentration speed:

Lemma 2.19. *Let $\sigma : (M, x) \mapsto (N, y)$ a local symplectomorphism between two quantizable Kähler manifolds.*

Let $0 < \delta < \frac{1}{2}$ and let $(u_N)_{N \in \mathbb{N}}$ a sequence of unit elements in the Hardy spaces $H_N(M)$ such that

$$\int_{\left\{ \operatorname{dist}(\pi(y), x) \leq N^{-\frac{1}{2}+\delta} \right\}} |u_N(y)|^2 = O(N^{-\infty}).$$

Then

$$\int_{\left\{ \operatorname{dist}(\pi'(y), \sigma(x)) \leq N^{-\frac{1}{2}+\delta} \right\}} |\mathfrak{S} u_N(y)|^2 = O(N^{-\infty}).$$

Proof. Let us observe that the condition on (u_N) is equivalent to the following: for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$\langle u_N, T_N(\operatorname{dist}(\cdot, x)^{2k}) u_N \rangle \leq C_k N^{-k(1+2\delta)}.$$

Let us prove, by induction on k, the estimate

$$\langle \mathfrak{S}_N u_N, T_N(\operatorname{dist}(\cdot, \sigma(x))^{2k}) \mathfrak{S}_N u_N \rangle \leq \tilde{C}_k N^{-k(1+2\delta)}.$$

The case $k = 0$ is clearly true since $\mathfrak{S}_N$ is an almost unitary operator when acting on functions localised near x.

Let us now apply Proposition 2.18 with $a = \operatorname{dist}(\cdot, x)^{2k}$, stopping the expansion at order k.

For $j \leq k$, the error terms are controlled:

$$\left| N^{-j} L_j(a \circ \sigma) \right| \leq N^{-j} C_{j,k} \operatorname{dist}(\cdot, \sigma(x))^{2(k-j)}$$

Hence, by induction,

$$\langle \mathfrak{S}_N u_N, T_N(\operatorname{dist}(\cdot, \sigma(x))^{2k}) \mathfrak{S}_N u_N \rangle$$

$$\leq \sum_{j=0}^{2k} C_{j,k} \langle u_N, T_N(\operatorname{dist}(\cdot, x)^{2(k-j)}) u_N \rangle + O(N^{-k}) = O(N^{k(-1+2\delta)}).$$

This ends the proof. $\qquad\square$

3   Resolvent estimates

We begin this section with a technical Lemma, which associates to a function on a Riemannian manifold a covering of the manifold with small open sets such that, on the intersections of the open sets, the function is not relatively larger than elsewhere.

3.1   A cutting lemma

Lemma 3.1. *Let Y be a compact Riemannian manifold. There exist two constants $C > 0$ and $a_0 > 0$ such that, for every positive integrable function f on Y, for every $0 < a < a_0$ and $t \in (0,1)$, there exists a finite family $(U_j)_{j \in J}$ of open subsets covering Y with the following properties:*

$$\forall j \in J, \operatorname{diam}(U_j) < a$$

$$\forall j \in J, \operatorname{dist}\left(Y \setminus U_j, Y \setminus \bigcup_{i \neq j} U_i\right) \geq ta$$

$$\sum_{i \neq j} \int_{U_i \cap U_j} f \leq Ct \int_Y f.$$

Proof.   Let $m \in \mathbb{N}$ be such that there exists a smooth embedding of differential manifolds from Y to $\mathbb{R}^m$, and let Φ be such an embedding. Φ may not preserve the Riemannian structure, so let c_1 be such that, for any $\xi \in TY$, one has

$$c_1 \|\Phi^* \xi\| \leq \|\xi\|.$$

We now let $L > 0$ be such that any hypercube H in $\mathbb{R}^m$ of side $2/L$ is such that $\operatorname{diam}(\Phi^{-1}(H)) < a$. Since Φ^{-1} is uniformly Lipschitz continuous, then if a is small enough one has $aL \leq C_1$ for some C_1 depending only on Y.

We then prove the claim with $C = \frac{2mC_1}{c_1}$.

Let $1 \leq k \leq m$, and let Φ_k denote the k-th component of Φ. The function Φ_k is continuous from Y onto a segment of $\mathbb{R}$. Without loss of generality this segment is $[0,1]$. Let g_k denote the integral of f along the level sets of Φ_k. The function g_k is a positive integrable function on $[0,1]$. Let $t' > 0$ be the inverse of an integer, and $0 \leq \ell \leq L - 1$. In the interval $[\ell/L, (\ell+1)/L]$, there exists a subinterval I, of length t'/L, such that

$$\int_I g_k \leq t' \int_{\ell/L}^{(\ell+1)/L} g_k. \qquad (1)$$

Indeed, one can cut the interval $[\ell/L, (\ell+1)/L]$ into $1/t'$ intervals of size t'/L. If none of these intervals was verifying (1), then the total integral would be strictly greater than itself.

Let $x_{k,\ell}$ denote the centre of such an interval. Then, let

$$V_{k,0} = \left[0, x_{k,0} + \frac{t'}{2L}\right)$$

$$V_{k,\ell} = \left(x_{k,\ell-1} - \frac{t'}{2L}, x_{k,\ell} + \frac{t'}{2L}\right) \text{ for } 1 \leq \ell \leq L$$

$$V_{k,L+1} = \left(x_{k,L} - \frac{t'}{2L}, 1\right].$$

Each open set $V_{k,l}$ has a length smaller than $2/L$. The overlap of two consecutive sets has a length t'/L, and the sum over k of the integrals on the overlaps is less than $t' \int_0^1 g_k = t' \int_Y f$.

Now let ν denote a polyindex $(\nu_k)_{1 \leq k \leq m}$, with $\nu_k \leq L+1$ for every k. Define

$$U_\nu = \Phi^{-1}\left(V_{1,\nu_1} \times V_{2,\nu_2} \times \ldots \times V_{m,\nu_m}\right).$$

Then the family $(U_\nu)_\nu$ covers Y. For every polyindex ν one has $\operatorname{diam} U_\nu \leq a$ since U_ν is the pull-back of an open set contained in a hypercube of side $2/L$. Moreover, one has

$$\operatorname{dist}\left(Y \setminus U_\nu, Y \setminus \bigcup_{\nu' \neq \nu} U_{\nu'}\right) \geq \frac{c_1 t'}{L}.$$

To conclude, observe that

$$\sum_{\nu \neq \nu'} \int_{U_\nu \cap U_{\nu'}} f = \sum_{k=1}^m \sum_{\ell=0}^L \int_{V_{k,\ell} \cap V_{k,\ell+1}} g_k \leq mt' \int_Y f.$$

It only remains to choose t' conveniently. The fraction $t\frac{aL}{c_1}$ may not be the inverse of an integer; however the inverse of some integer lies in $[\frac{aL}{2c_1}, \frac{aL}{c_1}]$. This allow us to conclude. □

Remark 3.2. In the previous Lemma, the number of elements of J is bounded by a polynomial in a that depends only on the geometry of Y.

3.2 Melin estimate

The following Proposition is a Berezin-Toeplitz version of the well-known Melin estimate for pseudodifferential estimates. It requires a weak condition on the speed of growth of the symbol near its zero set.

Proposition 3.3 (Melin estimate). *Let $h \in C^\infty(M, \mathbb{R}^+)$ with $\min(h) = 0$. Let*

$$\mu_{\min} = \min_{h(x)=0}(\mu(x)).$$

Suppose there exist $C > 0$ and $\alpha > 0$ such that, for every $t \geq 0$, one has

$$\operatorname{dist}_H(\{h \leq t\}, \{h = 0\}) \leq Ct^\alpha.$$

Then there exist $\varepsilon > 0$, N_0 and $C' > 0$ such that, for every $N \geq N_0$, one has

$$\min \operatorname{Sp}(T_N(h)) \geq \mu_{\min} N^{-1} - C' N^{-1-\varepsilon}.$$

Proof. We begin with a local result: There exist $\delta_0, \delta_1, \varepsilon$ small enough and N_0 such that, for $N \geq N_0$, for every $x \in M$ with $h(x) < N^{-1+\delta_1}$, for every u supported on $B(x, N^{-\frac{1}{2}+\delta_0}) \times \mathbb{S}^1$, one has

$$\langle S_N u, h S_N u \rangle \geq (\mu_{\min} N^{-1} - C N^{-1-\varepsilon}) \|u\|^2.$$

Indeed, the following holds by hypothesis:

$$\operatorname{dist}(x, \{h = 0\}) \leq C N^{\alpha(-1+\delta_1)}.$$

In particular,

$$\operatorname{Hess}(h)(x) \geq -C N^{\alpha(-1+\delta_1)}.$$

The following perturbation of h is convex on $B(x, N^{-\frac{1}{2}+\delta_0})$:

$$\tilde{h} : y \mapsto h(y) + C N^{\max(\alpha(-1+\delta_1), -1/2+\delta_0)} \operatorname{dist}(y, x)^2.$$

If now $u \in L^2(X)$ is supported on $B(x, N^{-\frac{1}{2}+\delta_0}) \times \mathbb{S}^1$, one has

$$\left| \langle S_N u, (h - \tilde{h}) S_N u \rangle \right| \leq C N^{-1+2\delta_0 + \max(\alpha(-1+\delta_1), -1/2+\delta_0)}.$$

As $\sqrt{h}$ is Lipschitz-continuous, one has

$$\sup \left(\sqrt{h(y)}, \operatorname{dist}(y, x) < 2N^{-\frac{1}{2}+\delta_0} \right) < C N^{\frac{-1+\delta_1}{2}} + C N^{-\frac{1}{2}+\delta_0}.$$

Hence,

$$\sup \left(h(y), \operatorname{dist}(x, y) < 2N^{-\frac{1}{2}+\delta_0} \right) < C N^{-1+\max(\delta_1, 2\delta_0)}.$$

Recall from Proposition 2.9 that, for δ small enough, for every $x \in M$ with associated normal map ρ, for every u with support inside $\rho(B(0, N^{-\frac{1}{2}+\delta}) \times \mathbb{S}^1)$, one has

$$\|(S_N - \rho^* \Pi_N) u\|_{L^2} < C N^{-\frac{1}{4}}.$$

Hence, if $\delta_0 < \delta$, then, for N large enough,

$$\left| \langle (S_N - \Pi_N^*) u, \tilde{h} S_N u \rangle \right| \leq C N^{-\frac{1}{4}} N^{-1+\max(\delta_1, 2\delta_0)}$$

$$\left| \langle \Pi_N^* u, \tilde{h} (S_N - \Pi_N^*) u \rangle \right| \leq C N^{-\frac{1}{4}} N^{-1+\max(\delta_1, 2\delta_0)}.$$

If the function $\tilde{h}$ reaches its minimum on $B(x, N^{-\frac{1}{2}+\delta_0})$ at an interior point x' and if Q denotes half of the Hessian matrix of $\tilde{h}$ at x', then

$$\left| \langle \Pi_N u_*, \tilde{h}_* - Q, \Pi_N u_* \rangle \right| \leq C N^{-\frac{3}{2}+3\delta_0}.$$

Here, the subscript $_*$ denotes the pull-back by the normal map.

If $\tilde{h}$ reaches its minimum at a boundary point x', then if L denotes the differential of $\tilde{h}$ at x' one has, by convexity, for all $y \in B(x, N^{-\frac{1}{2}+\delta_0})$,

$$L(y - x') \geq 0.$$

In particular,

$$\langle \Pi_N^* u, \tilde{h} \Pi_N^* u \rangle \geq \langle \Pi_N^* u, (\tilde{h} - L) \Pi_N^* u \rangle.$$

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Then $y \mapsto \tilde{h}(y) - L(y, x')$ has a critical point at x'. If Q denotes again half of the Hessian matrix of $\tilde{h}$ at x', then

$$\left| \langle \Pi_N u_*, \tilde{h}_* - L - Q, \Pi_N u_* \rangle \right| \le C N^{-\frac{3}{2} + 3\delta_0}.$$

Since in any case $\operatorname{dist}(x', \{h = 0\}) \le C N^{\max(\alpha(-1+\delta_1), -\frac{1}{2} + \delta_0)}$, the matrix Q is $C N^\varepsilon$-close to half of the Hessian matrix of h at a zero point (recall we only added $C N^{\alpha(-1+\delta_1)} I$ to the Hessian matrix of h at x.)

The Melin value μ is Hölder-continuous with exponent $(2n)^{-1}$ on the set of semi-positive quadratic forms [19], hence

$$\mu(Q) \le \mu_{\min} + C N^\varepsilon.$$

To conclude, for ε small enough depending on $M, \alpha, \delta_0, \delta_1$, one has

$$\langle S_N u, h S_N u \rangle \ge N^{-1} \mu_{\min} + C N^{-1+\varepsilon}.$$

From this local estimate, we deduce a global estimate using Lemma 3.1 proved previously, and general localisation estimates proved in [9].

Indeed, let $(u_N)_{N \ge 1}$ be a sequence of normalised eigenfunctions for $T_N(h)$ with minimal eigenvalue. Either the associated sequence of eigenvalues is not $O(N^{-1})$, in which case the proposition clearly holds, or it is, in which case, by Proposition 2.15, u_N is $O(N^{-\infty})$ outside $\{h \le N^{-1+\delta_1}\}$ for every $\delta_1 > 0$.

We now invoke Lemma 3.1 with the following data:

- $Y = M$.

- $f = |u_N|^2$.

- $a = N^{-\frac{1}{2} + \delta_0}$.

- $t = N^{-\frac{\delta_0}{2}}$.

The Lemma yields a sequence of coverings $(U_{j,N})_{j \in J_N, N \in \mathbb{N}}$. The proof also yields a sequence of coverings by slightly smaller open sets $(U'_{j,N})$, with

- $U'_{j,N} \subset U_{j,N}$.

- $d(M \setminus U_{j,N}, U'_{j,N}) \ge \frac{1}{2} N^{-\frac{1-\delta_0}{2}}$.

Let $(\chi_{j,N})_{j \in J_N, N \in \mathbb{N}}$ be a partition of unity associated with $(U'_{j,N})_{j \in J_N, N \in \mathbb{N}}$. Then, for every N, for every $j \ne k \in J_N$, the integral

$$\langle S_N \chi_{j,N} u_N, h S_N \chi_{k,N} u_N \rangle$$

is $O(N^{-\infty})$ outside $(U_{j,N} \cap U_{k,N})^3$. Moreover, $S_N \chi_{k,N} u_N$ is $O(N^{-\infty})$ outside $\{h \ge N^{-1+\delta_1}\}$ (because of Proposition 2.15). Hence,

$$|\langle S_N \chi_{j,N} u_N, h S_N \chi_{k,N} u_N \rangle| \le C N^{-1+\delta_1} \||u_N|^2\|_{L^1(U_{j,N} \cap U_{k,N})} + O(N^{-\infty}).$$

Hence, by Lemma 3.1, for every N, one has

$$\sum_{j \ne k \in J_N} |\langle \chi_{j,N} u_N, T_N(h) \chi_{k,N} u_N \rangle| \le C N^{-1+\delta_1} N^{-\frac{\delta_0}{2}} \|u_N\|_{L^2}^2 + O(N^{-\infty}).$$

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(Recall $|J_N|$ has polynomial growth in N.)

On the other hand, the following holds:

$$\sum_{j \in J_N} \langle \chi_{j,N} u_N, T_N(h) \chi_{j,N} u_N \rangle \geq (\mu_{\min} N^{-1} - C N^{-1-\epsilon}) \sum_{j \in J_N} \|S_N \chi_{j,N} u_N\|_{L^2}^2.$$

Since $S_N \chi_{j,N} u_N$ and $S_N \chi_{k,N} u_N$ are almost orthogonal for $j \neq k$, one has

$$\sum_{j \in J_N} \|S_N \chi_{j,N} u_N\|_{L^2}^2 \geq (1 - C N^{-\epsilon}) \|u_N\|_{L^2}^2.$$

Then, choosing $\delta_1 < \frac{\delta_0}{2}$ allows us to conclude:

$$\langle u_N, T_N(h) u_N \rangle \geq N^{-1} (\mu_{\min} - C N^{-\epsilon}) \|u_N\|_{L^2}^2.$$

$\square$

Note that, in the last proof, it is essential that we know beforehand that u_N is $O(N^{-\infty})$ on $\{h \geq N^{-1+\delta}\}$ for every $\delta > 0$. This was achieved by picking u_N as the unique minimizer of $\langle u, T_N(h) u \rangle$ under $\|u\| = 1$, in which case u_N is an eigenfunction of $T_N(h)$.

Remark 3.4. Proposition 3.3 only relies on elementary properties of the Szegő kernel and Toeplitz operators (that is, Propositions 2.4 and 2.9). As such, it extends readily to more general contexts of quantizations, such as Spin^c-Dirac [18] (up to a modification in the definition of $\mu_{\min}$).

The condition of polynomial growth near zero is satisfied for every analytic symbol, in particular, it is true for spin systems considered in Section 8.

3.3 Pseudo-locality of the resolvent

Proposition 3.5. *Let h and $\mu_{\min}$ be as in Proposition 3.3 and $\epsilon_0 > 0$ small enough, depending only on M. Then, for every $0 \leq \epsilon < \epsilon_0$ and every $c > 0$, the operator $T_N(h - N^{-1}\mu_{\min} + cN^{-1-\epsilon})$ is invertible (as a positive definite operator on a finite-dimensional space). Its inverse R is pseudo-local: if a and b are smooth functions with $\operatorname{supp}(a) \cap \operatorname{supp}(b) = \emptyset$, then*

$$T_N(a) R T_N(b) = O_{L^2 \to L^2}(N^{-\infty}).$$

Proof. The proposition may be reformulated this way: if $U \subset\subset V$ are two open sets in M and a sequence $(u_N)_{N \geq 1}$ of normalised states is such that $T_N(h - N^{-1}\mu_{\min} + cN^{-1-\epsilon})u_N = O_{L^2}(N^{-\infty})$ on V, then we wish to prove that $u_N = O_{L^2}(N^{-\infty})$ on U. Here

$$\operatorname{supp}(a) \subset\subset U \subset\subset V \subset\subset (M \setminus \operatorname{supp}(b)).$$

We first remark that, for every δ, and for every $U \subset\subset V_1 \subset\subset V$, the following holds:

$$\int_V \overline{u} \, T_N(h) u \geq C N^{-1+\delta} \int_{V_1 \cap \{h \geq N^{-1-\delta}\}} |u|^2.$$

Hence, u is $O(N^{-\infty})$ on $V_1 \cap \{h \geq N^{-1-\delta}\}$ for every δ.

We are now able to repeat the global part of the proof of Proposition 3.3 by cutting a neighbourhood of U into small pieces, hence the claim. $\square$

4 Proof of Theorem A

4.1 Estimate of the first eigenvalue

Proposition 4.1. *Let $h \geq 0$ with $\min(h) = 0$ and let $\mu_{\min}$ be as in Proposition 3.3. Then there exists $\epsilon > 0$ such that*

$$\inf Sp(T_N(h)) \leq N^{-1}\mu_{\min} + N^{-1-\epsilon}.$$

Proof. Let $P_0 \in M$ achieve the minimal value $\mu_{\min}$, let ρ be a normal map around P_0, and let $\delta > 0$ and $C > 0$ be such that, for every N, for every u supported on $B(P_0, N^{-\frac{1}{2}+\delta}) \times \mathbb{S}^1$, one has $\|(S_N - \rho^*\Pi_N)u\| \leq CN^{-\frac{1}{4}}$. Without loss of generality $\delta < \frac{1}{8}$.

Pick $\alpha \geq 2\delta$, and let Q denote half of the Hessian of h at P_0. Then, since the function $Q \mapsto \mu(Q)$ is Hölder continuous with exponent $\frac{1}{2n}$ [19], one has

$$\mu(Q + N^{-\alpha}|\cdot|^2) \leq \mu(Q) + CN^{-\frac{\alpha}{2n}}.$$

Let v_N denote a normalised ground state of $T_N(Q+N^{-\alpha}|\cdot|^2)$, then v_N is $O(N^{-\infty})$ outside $B(0, N^{-\frac{1}{2}+\delta})$. Then $\langle \rho^* v_N, T_N(h)\rho^* v_N \rangle = \mu(Q+N^{-\alpha}|\cdot|^2)+O(N^{-\frac{5}{4}+2\delta}) \leq \mu_{\min}N^{-1} + O(N^{-1-\epsilon})$ for some $\epsilon > 0$. $\qquad\square$

4.2 End of the proof

We can now conclude the proof of Theorem A. Let $a \in C^\infty(M)$ be supported away from the set of points achieving $\mu_{\min}$. Let $\tilde{h} \in C^\infty(M)$ be such that $\tilde{h} = h$ on the support of a, and such that $\mu_{\min}(\tilde{h}) > \mu_{\min}(h)$. Then $T_N(\tilde{h} - N^{-1}\mu_{\min}(h))$ is invertible because of the Melin estimate of Proposition 3.3. Its inverse R is pseudolocal, with norm $O(N)$; in particular, for every integer k,

$$T_N(a) = T_N(h - N^{-1}\mu_{\min}(h))^k R^k T_N(a) + O(N^{-\infty}).$$

If u_N is a sequence of unit ground states of $T_N(h)$, then by Propositions 3.3 and 4.1 there holds

$$|\langle u_N, T_N(h - N^{-1}\mu_{\min}(h))u_N \rangle| \leq CN^{1+\epsilon}.$$

Hence,

$$\langle u_N, T_N(a)u \rangle \leq C^k N^{-k-k\epsilon}\langle u, R^k T_N(a)u_N. \rangle$$

In particular, for every integer k, one has

$$\langle u_N, T_N(a)u_N \rangle = O(N^{-k\epsilon}),$$

which concludes the proof of Theorem A.

5 The regular case

In this Section we prove Theorem B, and establish the necessary material for the Weyl asymptotics of Section 7.

We first study a problem of symplectic geometry, which consists in finding a normal form for a non-negative function vanishing at order 2 on an isotropic submanifold. Then, we apply a Quantum Map to find an expansion of the first eigenvalue and eigenfunction.

5.1 A convenient chart

We begin with the following fact:

Proposition 5.1. *Let us endow $\mathbb{R}^{2n}$ with the canonical symplectic structure.*

Let $Q : \mathbb{R}^d \mapsto S_{2n}^+(\mathbb{R})$ be a smooth d-parameter family of semi-positive quadratic forms. Suppose rank $Q = r$ is a constant function and suppose that, for every $t \in \mathbb{R}^d$, the space $\ker(Q(t))$ is isotropic. In particular, $\ker Q$ is a smooth map into the set of isotropic subspaces of $\mathbb{R}^{2n}$.

Then there is a smooth family $(e_1, f_1, \ldots, e_n, f_n) : \mathbb{R}^d \mapsto (\mathbb{R}^n)^n$ of symplectic bases, and smooth functions $\lambda_i : \mathbb{R}^d \mapsto \mathbb{R}_+^*$, $r + 1 \leq i \leq n$, such that

$$Q\left(\sum_{i=1}^n q_i e_i + p_i f_i\right) = \sum_{i=1}^r p_i^2 + \sum_{i=r+1}^n \lambda_i(p_i^2 + q_i^2).$$

In particular, under the conditions above, the function $\mu(Q)$ is smooth.

In the study of the Hamiltonian dynamics related to Q, the vectors $f_i, 1 \leq i \leq r$ are called *slow modes*. They correspond to the motion of a free particle. The vectors $(e_i, f_i), r + 1 \leq i \leq n$ are called *fast modes* and correspond to harmonic oscillations; the associated values $\lambda_i, r + 1 \leq i \leq n$ are the *symplectic eigenvalues* of Q.

Here, the *zero modes* (that is, the kernel of Q) are supposed to form an isotropic subspace; this can be generalised, but $\ker Q$ must have a constant symplectic rank in order for such a result to apply.

Proof. Let us construct a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of $\mathbb{R}^{2n}$, depending smoothly on the parameters, on which the quadratic form Q is diagonal, with the desired diagonal terms. We proceed by partial induction: if Q is degenerate, we construct the first pair (e_1, f_1) with $e_1 \in \operatorname{Ker} Q$, hence the reduction to Q' on $\mathbb{R}^{2(n-1)}$ with $\dim \operatorname{Ker} Q' = r - 1$. If Q is non-degenerate, we use a standard construction of the full basis in one step.

Suppose $r > 0$. Pick $e_1 \in \ker Q$ smoothly depending on the parameters. The quadratic form Q is degenerate, but it is a well-known fact that it has no co-isotropic subspaces: if a subspace F is such that

$$\{e \in \mathbb{R}^{2n}, \forall f \in F, Q(e + f) = Q(e) + Q(f)\} \subset F,$$

then $F = \mathbb{R}^{2n}$.

Hence, with $F = \{z \in \mathbb{R}^{2n}, \langle z, Je_1 \rangle = 0\}$ denoting the symplectic orthogonal of e_1, there exists f_1 such that:

$$\langle e_1, Jf_1 \rangle = 1$$
$$\forall z \in F, Q(z + f_1) = Q(z) + Q(f_1).$$

The vector f_1 again depends smoothly on the parameters. As $\lambda = Q(f_1)$ is far from zero on compact sets (recall that $\ker Q$ is a continuous family of isotropic subspaces), changing e_1 into $\sqrt{\lambda}e_1$ and f_1 into $f_1/\sqrt{\lambda}$ yields two smooth vectors with the supplementary condition that $Q(f_1) = 1$.

If one can find a smooth symplectic basis $(e_2, \ldots, e_n, f_2, \ldots, f_n)$ of the symplectic orthogonal of $\operatorname{Span}(e_1, f_1)$, which diagonalises the restriction of Q with

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diagonal values as above, then completing this basis with e_1 and f_1 concludes the proof.

If $r = 0$, let M be the matrix of Q in the (symplectic) canonical basis. Then $M^{\frac{1}{2}}$ is a smooth family of symmetric matrices, so that $M^{\frac{1}{2}}JM^{\frac{1}{2}}$ is a smooth family of antisymmetric matrices, where J is the matrix of the standard symplectic form in the canonical basis. Hence, there is a smooth family U of orthogonal matrices, and a smooth family D of positive diagonal matrices, such that

$$U^T M^{\frac{1}{2}} J M^{\frac{1}{2}} U = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

In particular, with $A = \begin{pmatrix} D^{\frac{1}{2}} & 0 \\ 0 & D^{\frac{1}{2}} \end{pmatrix}$, one has

$$(AU^T M^{\frac{1}{2}} J) M (-J M^{\frac{1}{2}} U A) = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},$$

and

$$(AU^T M^{\frac{1}{2}} J) J (-J M^{\frac{1}{2}} U A) = J.$$

Hence, the desired symplectic matrix is $AU^T M^{\frac{1}{2}} J$, which depends smoothly on the parameters. This concludes the proof. $\square$

Recall the following well-known application of Moser's principle:

Proposition 5.2. *Let S be a symplectic manifold and $Z \subset S$ be a smooth d-dimensional submanifold of constant symplectic rank. Then, in a neighbourhood (in S) of any point in Z, there is a symplectomorphism ρ onto a neighbourhood 0 of $\mathbb{R}^{2n}$, such that $\rho(Z)$ is a piece of linear subspace.*

Using the two previous Propositions, we will prove the normal form for mini-wells on isotropic submanifolds:

Proposition 5.3. *Let h be a smooth non-negative function on M, which vanishes on an isotropic manifold Z of dimension r with everywhere non-degenerate transverse Hessian.*

Near any point of Z, there is a symplectomorphism ρ into $\mathbb{R}^{2r}_{q,p} \times \mathbb{R}^{2(n-r)}_{x,\xi}$, a smooth function Q_S from $\mathbb{R}^r$ into the set of positive quadratic forms of dimension r, and $n-r$ smooth positive functions $(\lambda_i)_{1 \le i \le n-r}$ such that:

$$h \circ \rho^{-1} = \sum_{i=1}^{n-r} \lambda_i(q)(x_i^2 + \xi_i^2) + Q_S(q)(p) + O_{(x,\xi,p) \to 0}(|x|^3 + |\xi|^3 + |p|^3).$$

In particular, Z is mapped into $\{(p, x, \xi) = 0\}$.

Proof. Let $P_0 \in Z$, and let U be a small neighbourhood of P_0 in M. Let us use Proposition 5.1 with set of parameters $Z \cap U$ and quadratic form $\mathrm{Hess}(h)$, which is semi-positive definite along $Z \cap U$, with kernel of constant symplectic rank.

This yields, at each point of Z in a neighbourhood of P_0, a family of $2n$ vector fields which form a symplectic basis:

$$\mathcal{B} = (Q_1, \ldots, Q_r, P_1, \ldots, P_r, X_1, \ldots, X_{n-r}, \Xi_1, \ldots, \Xi_{n-r}),$$

such that $\mathrm{Span}(Q_1, \ldots, Q_r) = TZ$. In the general setting, this does not give a symplectic change of variables under which the quadratic form is diagonal along the whole zero set (indeed, $Q_1, \ldots, Q_r$ are prescribed by the $2n - r$ other vector fields, and do not commute in general). However, one can separate the slow variables and the fast variables (first step), then diagonalise the fast variables (second step).

First step: Let us define the distribution $\mathcal{F}$ on $Z \cap U$ as follows: for $x \in Z \cap U$,

$$\mathcal{F}_x = \mathrm{Span}(Q_1, \ldots, Q_r, P_1, \ldots, P_r)(x).$$

Then $T(Z \cap U) \subset \mathcal{F}$. In particular, there is a piece S of symplectic submanifold of M, containing $Z \cap U$, and tangent to $\mathcal{F}$ on $Z \cap U$.

Using Proposition 5.2, we let ϕ_0 be a symplectomorphism from a neighbourhood of P_0 in M into a neighbourhood of 0 in $\mathbb{R}^{2r} \times \mathbb{R}^{2(n-r)}$, such that S is mapped into $\mathbb{R}^{2r} \times \{0\}$. Using Proposition 5.2 again, let ϕ_1 be a symplectomorphism on a neighbourhood of 0 in $\mathbb{R}^{2r}$, that maps $\phi_0(Z)$ into $\mathbb{R}^r \times \{0\}$. Then the map $\tilde{\phi}_1$ acting on $\mathbb{R}^{2r} \times \mathbb{R}^{2(n-r)}$ by

$$\tilde{\phi}_1(p, q, x, \xi) = (\phi_1(p, q), x, \xi)$$

is a symplectomorphism. We claim that $\rho = \tilde{\phi}_1 \circ \phi_0$ separates the fast variables from the slow variables up to $O((x, \xi, p)^3)$.

Indeed, consider $D\rho$ at a point of Z. Since ρ sends Z into $\mathbb{R}^r \times \{0\}$, and S into $\mathbb{R}^{2r} \times \{0\}$, the matrix of $D\rho$, from the basis $\mathcal{B}$ to the canonical basis, is of the form:

$$D\rho = \left(\begin{array}{cc|cc} A_{qq} & 0 & 0 & 0 \\ A_{pq} & A_{pp} & 0 & 0 \\ \hline A_{xq} & A_{xp} & A_{xx} & A_{x\xi} \\ A_{\xi q} & A_{\xi p} & A_{\xi x} & A_{\xi\xi} \end{array} \right).$$

Moreover, D_ρ is symplectic, so that the bottom left part vanishes. Hence,

$$h \circ \rho^{-1} = Q_F(q)(x, \xi) + Q_S(q)(p) + O(|p|^3 + |x|^3 + |\xi|^3),$$

for some quadratic forms Q_F and Q_S.

Since h vanishes at order exactly 2 on Z, the quadratic forms Q_F and Q_S are positive definite.

Second step: It only remains to diagonalise Q_F with a symplectomorphism. In fact, this is possible without modifying Q_S. Indeed, let $\phi : (\mathbb{R}^r, 0) \mapsto Sp(2(n-r))$ be such that, for every q near zero, the matrix $\phi(q)$ realises a symplectic reduction of $Q_F(q)$, with eigenvalues $\lambda_1(q), \ldots, \lambda_{n-r}(q)$. All of this depends smoothly on q by Proposition 5.1. With J the standard complex structure matrix on $\mathbb{R}^{2(n-r)}$ and $\langle \cdot, \cdot \rangle$ its standard Euclidian norm, we define, for every $1 \leq i \leq r$, the real function

$$f_i : (q, x, \xi) \mapsto \frac{1}{2} \langle (x, \xi), (\partial_{q_i} \phi(q) J \phi^t(q))(x, \xi) \rangle.$$

We then define $f : (\mathbb{R}^{2n-r}, 0) \to \mathbb{R}^r$ as the map with components f_i in the canonical basis. Then a straightforward computation shows that the map

$$\Phi : (q, p, x, \xi) \mapsto (q, p + f, \phi(q)(x, \xi))$$

is a symplectomorphism. As $f = O_{(x,\xi)\to 0}((x,\xi)^2)$, the 2-jet of $h \circ \Phi$ at $(q, 0, 0, 0)$ is the same as the 2-jet of $h \circ ((q, p, x, \xi) \mapsto (q, p, \phi(q)(x, \xi)))$, i.e. $Q_S(q)(p) + \sum_{i=1}^{n-r} \lambda_i(q)(x_1^2 + \xi_i^2)$. This concludes the proof. $\qquad\square$

Remark 5.4. We corrected the map

$$(q, p, x, \xi) \mapsto (q, p, \phi_q(x, \xi))$$

into a symplectomorphism by only changing the second coordinate. This does not depend on the fact that ϕ_q acts linearly but relies only on $\phi_q(0, 0) = (0, 0)$.

5.2 Approximate first eigenfunction

Let us quantize, using Proposition 2.18, the symplectic map of Proposition 5.3, and conjugate with pseudodifferential operators:

Definition 5.5. For any choice $\mathfrak{S}_N$ of quantization of the map ρ of Proposition 5.3, the classical symbol $g_{\mathfrak{S}} \sim \sum N^{-i} g_i$ on a neighbourhood U of 0 in $\mathbb{R}^{2n}$ is defined as follows: for any sequence $(v_N)_{N \geq 1}$ with microsupport in a compact set of U, the following holds:

$$\mathcal{B}_N^{-1} \mathfrak{S}_N^{-1} T_N(h) \mathfrak{S}_N \mathcal{B}_N v_N = Op_W^{N^{-1}}(g_{\mathfrak{S}}) v_N + O(N^{-\infty}).$$

In what follows, we choose an arbitrary quantum map $\mathfrak{S}_N$, and we write g instead of $g_{\mathfrak{S}}$. The reason we use Weyl quantization in this subsection is because we will rely heavily on squeezing operators. The computations are much easier to follow for this formalism.

The principal and subprincipal symbols of g are explicit at the points of interest: $g_0 = h \circ \rho$ by construction, and g_1 is prescibed on $\{g_0 = 0\}$ by the Melin estimates for Weyl and Toeplitz quantizations:

Proposition 5.6. *For any q close to 0, one has*

$$g_1(q, 0, 0, 0) = \frac{1}{2} \sum_{i=1}^{n-r} \lambda_i(q) + \frac{1}{4} \operatorname{tr}(Q_S(q)).$$

Proof. From the expression of $h \circ \rho$ in Proposition 5.3, one has

$$\mu(\rho(q, 0, 0, 0)) = \sum_{i=1}^{n-r} \lambda_i(q) + \frac{1}{4} \operatorname{tr}(Q_S(q)).$$

If $h(x) = 0$ and $\delta > 0$ is small enough, the value $\mu(x)$ has the following variational characterisation:

$$\mu(x) = \lim_{N \to +\infty} \left(N \inf \left(\int_M h|u|^2, u \in H_N(X), \int_{B(x, N^{-\frac{1}{2}+\delta})} |u|^2 = 1 \right) \right)$$

This variational problem can be read via the quantum map. If

$$\int_{B(x, 2N^{-\frac{1}{2}+\delta})} |u|^2 = O(N^{-\infty}),$$

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then $\mathcal{B}_N^{-1}\mathfrak{S}_N^{-1}u$ microlocalises at speed $N^{-\frac{1}{2}+\delta}$ on $\rho^{-1}(x)$, and moreover,

$$\int_M h|u|^2 = \left\langle \mathcal{B}_N^{-1}\mathfrak{S}_N^{-1}u, Op_W^{N^{-1}}(g_0 + N^{-1}g_1)\mathcal{B}_N^{-1}\mathfrak{S}_N^{-1}u \right\rangle + O(N^{-2})\|u\|_2.$$

Now, if $x = \rho(q,0,0,0)$, the usual Melin estimate yields

$$\lim_{N\to+\infty} \left(N\inf\left(\left\langle vOp_W^{N^{-1}}(g_0)v \right\rangle, \mathfrak{S}_N\mathcal{B}_N v \text{ as above} \right) \right) = \frac{1}{2}\sum_{i=1}^{n-r}\lambda_i(q),$$

hence, $g_1(\rho(q,0,0,0))$ contains all the defect between $\mu(\rho(q,0,0,0))$ and this estimate. $\qquad\square$

Remark 5.7. In general, the subprincipal symbol is not unique after application of a quantum map. Indeed, if a is any smooth real-valued function on M then $\exp(iT_N(a))$ is a unitary operator, and composing $\mathfrak{S}_N$ with this operator changes the subprincipal term.

Proposition 5.6 shows that on the points where the principal symbol vanishes, the subprincipal symbol is in fact rigid through any such transformations.

Let us find a candidate for an approximate first eigenfunction:

Proposition 5.8. *Suppose that the function $q \mapsto \mu \circ \rho(q,0,0,0)$ has a non-degenerate minimum at $q = 0$. Let ϕ be the positive quadratic form such that $q \mapsto e^{-\phi(q)}$ is the ground state of the operator*

$$Q_S(0)(-i\nabla) + \frac{1}{2}\sum_{i,j=1}^{r} q_iq_j\frac{\partial^2}{\partial q_i\partial q_j}(\mu\circ\rho)(0,0,0,0),$$

with eigenvalue μ_2.

Then there exists a sequence of polynomials $(b_i)_{i\geq1}$, and a sequence of real numbers $(\mu_i)_{i\geq1}$, with

$$\mu_0 = \mu \circ \rho(q,0,0,0)$$
$$\mu_1 = 0$$

and μ_2 as previously, such that, for every k,

$$f_N^k : (q,x) \mapsto N^{\frac{n}{2}-\frac{r}{4}}e^{-N\frac{x^2}{2}-\sqrt{N}\phi(q)}\left(1 + \sum_{i=1}^{k}N^{-\frac{i}{4}}b_i(N^{\frac{1}{4}}q, N^{\frac{1}{2}}x)\right)$$

is an approximate eigenvector to $Op_W^{N^{-1}}(g)$, with eigenvalue

$$\lambda_N^k = N^{-1}\sum_{i=0}^{k}N^{-\frac{i}{4}}\mu_i,$$

in the sense that, for every K there exists k such that

$$\|Op_W^{N^{-1}}(g)f_N^k - \lambda_N^k f_N^k\|_{L^2} = O(N^{-K}).$$

This proposition provides an almost eigenfunction which we will show to be associated to the lowest eigenvalue (see Proposition 5.10). It is the main argument in the proof of Theorem B; the concentration speed of this eigenfunction on zero, which is $N^{-\frac{1}{4}}$, is the concentration speed of the lowest eigenvector of $T_N(h)$ on the miniwell P_0, because of Proposition 2.19.

Proof. The proof proceeds by a squeezing of $Op_W^{N^{-1}}(g)$ by a factor $N^{\frac{1}{4}}$ along the q variable.

Let
$$\tilde{g}_N = g(N^{-\frac{1}{4}}q, N^{-\frac{3}{4}}p, N^{-\frac{1}{2}}x, N^{-\frac{1}{2}}\xi).$$

Then $Op_W^{N^{-1}}(g_N)$ is conjugated with $Op_W^1(\tilde{g}_N)$ through the unitary change of variables $u \mapsto N^{\frac{n}{2}-\frac{r}{4}}u(N^{-\frac{1}{4}}q, N^{-\frac{1}{2}}x)$.

Grouping terms in a Taylor expansion of $\tilde{g}_N$ yields
$$\tilde{g}_N = N^{-1}\sum_{i=0}^{K} N^{-\frac{i}{4}}a_i(q,p,x,\xi) + O(N^{-\frac{K+5}{4}}),$$

with first terms

$$a_0 = g_1(0,0,0,0) + \sum_{i=1}^{n-r}\lambda_i(0)(x_i^2 + \xi_i^2)$$

$$a_1 = q\cdot\nabla_q\left(g_1(\cdot,0,0,0) + \sum_{i=1}^{n-r}\lambda_i(\cdot)(x_i^2 + \xi_i^2)\right)(0)$$

$$a_2 = Q_S(p) + \frac{1}{2}\operatorname{Hess}_q\left(g_1(\cdot,0,0,0) + \sum_{i=1}^{n-r}\lambda_i(\cdot)(x_i^2 + \xi_i^2)\right)(0)(q)$$
$$\quad + R_3(x,\xi) + L(x,\xi).$$

Here R_3 is a homogeneous polynomial of degree 3 and L is a linear form.

We further write $A_i = Op_W^1(a_i)$.

Recall $g_1(q,0,0,0) + \frac{1}{2}\sum\lambda_i(q) = \mu\circ\rho(q,0,0,0)$ and let ϕ be the positive quadratic form such that $e^{-\phi}$ is the ground state (up to a positive factor) of

$$Op_W^1\left(Q_S(p) + \frac{1}{2}\operatorname{Hess}(\mu\circ\rho)(0)(q)\right),$$

and let
$$u_0(q,x) = e^{-\frac{x^2}{2}-\phi(q)}.$$

We will provide a sequence of almost eigenfunctions of $Op_W^1(\tilde{g}_N)$, of the form

$$u_0(q,x)\left(1 + \sum_{i=1}^{+\infty} N^{-\frac{i}{4}}b_i(q,x)\right),$$

with approximate eigenvalue

$$N^{-1}\sum_{i=0}^{+\infty} N^{-\frac{i}{4}}\mu_i.$$

We proceed by perturbation of the dominant order A_0, which does not depend on q. Our starting point is

$$u_0 = e^{-\frac{|x|^2}{2} - \phi(q)}, \ \mu_0 = \min \mathrm{Sp}(A_0)$$

$$u_1 = 0, \ \mu_1 = 0.$$

Indeed, one has $A_0 u_0 = \mu_0 u_0$, and $A_1 u_0 = 0$ since

$$\nabla \left(g_1(\cdot, 0, 0, 0) + \frac{1}{2} \sum_{i=1}^{n-r} \lambda_i(\cdot) \right)(0) = 0,$$

so that u_0 is an approximate eigenvector for $Op^1_W(\tilde{g})$.

Let us proceed by induction. Let $k \geq 1$ and suppose that we have already built $u_0, \ldots, u_k$ and $\mu_1, \ldots, \mu_k$ which solve the eigenvalue equation at order k; suppose further that there exists $C_{k+1} \in \mathbb{R}$ such that, for every $q \in \mathbb{R}^r$,

$$\int_{\mathbb{R}^{n-r}} \overline{u_0}(x, q) \left(\sum_{i=1}^{k+1} [A_i u_{k+1-i}](q, x) - \sum_{i=1}^{k} \mu_i u_{k+1-i}(q, x) \right) dx = C_{k+1} |u_0(x, q)|^2.$$

Then one can solve the equation

$$(A_0 - \mu_0) u_{k+1} + \cdots + (A_{k+1} - \mu_{k+1}) u_0 = 0,$$

up to a multiple of $e^{-\frac{|x|^2}{2}}$ in u_{k+1}. Indeed, if we write

$$u_{k+1} = v(q) e^{-\frac{|x|^2}{2}} + w(x, q),$$

where for every $q \in \mathbb{R}^r$ one has $w(q, \cdot) \perp e^{-\frac{|\cdot|^2}{2}}$, the equation reduces to

$$(A_0 - \mu_0) w + \cdots + (A_{k+1} - \mu_{k+1}) u_0 = 0.$$

Freezing q and taking the scalar product with $x \mapsto e^{-\frac{|x|^2}{2}}$ yields

$$\lambda_{k+1} = C_{k+1}.$$

Then, with q still frozen one has $(A_0 - \mu_0) w = r.h.s$ where the r.h.s is orthogonal to the ground state of A_0, which allow us to solve for w.

If the r.h.s is u_0 times a polynomial in (q, x), then the same holds for w (in particular, for all i one has $A_i w \in L^2$ so that it makes sense to proceed with the induction).

It remains to choose v so that u_{k+1} satisfies the orthogonality constraint above, in order to be able to build the next terms.

Since $\mu_1 = 0$ and $A_1 u_0 = 0$, the terms $i = 1$ vanish so that the first integral in which u_{k+1} appears is not the next one but the one after it:

$$\int_{\mathbb{R}^{n-r}} \overline{u_0}(x, q) \left(\sum_{i=2}^{k+3} [A_i u_{k+3-i}](q, x) - \sum_{i=2}^{k+2} \mu_i u_{k+3-i}(q, x) \right) dx.$$

Hence, one wants to solve

$$\int_{\mathbb{R}^{n-r}} e^{-\frac{|x|^2}{2}}[(A_2 - \mu_2)ve^{-\frac{|\cdot|^2}{2}}](q, x) = F(q) + C_{k+3}e^{-\phi(q)},$$

with

$$F(q) = -\int_{\mathbb{R}^{n-r}} e^{-\frac{|x|^2}{2}} \left([(A_2 - \mu_2)w](x, q) + \sum_{i=3}^{k+3} [A_i u_{k+3-i}](x, q) \right.$$
$$\left. - \sum_{i=3}^{k+2} \mu_i u_{k+3-i}(x, q) \right) \mathrm{d}x.$$

The symbol a_2 decomposes into a quadratic symbol in (q, p), and an odd polynomial in (x, ξ). The latter does not contribute to the integral in the left-hand-side, and the former commutes with multiplication by $e^{-\frac{|x|^2}{2}}$, so that

$$\int_{\mathbb{R}^{n-r}} e^{-\frac{|x|^2}{2}}[(A_2 - \mu_2)ve^{-\frac{|\cdot|^2}{2}}](q, x) = C_{n-r} \left(Q_S(iD) + \frac{1}{2} \mathrm{Hess}(g_1 + \frac{1}{2}\sum_{i=1}^{n-r} \lambda_i)(q) - \mu_2 \right) v.$$

The equation on v is then

$$\left(Q_S(iD) + \frac{1}{2} \mathrm{Hess}(\mu \circ \rho)(0)(q) - \mu_2 \right) v = C_{n-r}^{-1} \left(F(q) + C_{k+3}e^{-\phi(q)} \right).$$

With

$$C_{k+1} = -\langle e^{-\phi(q)}, F(q) \rangle,$$

one has

$$F - C_{k+1}e^{-\phi} \perp e^{-\phi},$$

so that one can solve for v.

Again, if $u_0, \ldots, u_k$ and w are u_0 times a polynomial function in (x, q), then F is $e^{-\phi}$ times a polynomial function, so that the same is true for v. This concludes the construction by induction.

The estimation of the error terms stems directly from the fact that the terms u_k are polynomials time a function with Gaussian decay. Hence, this formal construction yields approximate eigenfunctions. $\square$

Before we show that the almost eigenfunction computed in Proposition 5.8 corresponds indeed to the lowest eigenvalue, let us use the quantum maps $\mathfrak{S}_N$ to obtain upper and lower bounds for $T_N(h)$, which will be useful in Section 7.

Proposition 5.9. *For $t > 0$ let A_N^{reg} the following operator on $L^2(\mathbb{R}^r)$:*

$$A_N^{reg} = Op_W^{N^{-1}} \left(|p|^2 + N^{-1}|q|^2 \right)$$

Under the conditions of Proposition 5.8, there exists $a_0 > 0$, and two constants $0 < c < C$ such that, for any N, for any $a < a_0$, for any normalized $u \in L^2(X)$ supported in $B(P_0, a) \times \mathbb{S}^1$, with $v = \mathcal{B}_N^{-1}\mathfrak{S}_N^{-1}u$, one has:

$$c\langle v, A_N^{reg}v \rangle - C\langle v, Op_W^{N^{-1}}(|N^{-\frac{1}{2}}, x, \xi|^3)v \rangle + c \left(\langle v, Op_W^{N^{-1}}(|x|^2 + |\xi|^2)v \rangle - N^{-1}\frac{n-r}{2} \right)$$
$$\leq \langle u, hu \rangle - N^{-1}\mu(P_0)$$

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In addition, the following bound holds:

$$c\langle v, A_N^{reg} v\rangle + \langle v, Op_W^{N^{-1}}(Q_F(0)(x,\xi))v\rangle - C\langle v, Op_W^{N^{-1}}(|N^{-\frac{1}{2}}, x, \xi|^3)v\rangle$$

$$- aC\left(\langle v, Op_W^{N^{-1}}(|x|^2 + |\xi|^2)v\rangle - N^{-1}\frac{n-r}{2}\right)$$

$$\leq \langle u, hu\rangle - N^{-1}\mu(P_0) + \frac{N^{-1}}{2}\sum_i \lambda_i(0)$$

$$\leq C\langle v, A_N^{reg} v\rangle + \langle v, Op_W^{N^{-1}}(Q_F(0)(x,\xi))v\rangle + C\langle v, Op_W^{N^{-1}}(|N^{-\frac{1}{2}}, x, \xi|^3)v\rangle$$

$$+ aC\left(\langle v, Op_W^{N^{-1}}(|x|^2 + |\xi|^2)v\rangle - N^{-1}\frac{n-r}{2}\right)$$

Here, $O(|x, \xi, N^{-\frac{1}{2}}|^3)$ stands for $O(|x, \xi|^3 + N^{-\frac{3}{2}})$.

Proof. Let us prove the first lower bound. As

$$g_0(q, p, x, \xi) = Q_F(q)(x, \xi) + Q_S(q)(p) + O(|p, x, \xi|^3),$$

one has first, by a lower bound on $Op_W^{N^{-1}}(Q_F(q)(x, \xi))$,

$$\langle v, Op_W^{N^{-1}}(g_0)v\rangle$$

$$\geq c\langle v, Op_W^{N^{-1}}(|p|^2)v\rangle + \frac{N^{-1}}{2}\langle v, \sum_i \lambda_i(q), v\rangle + \langle v, Op_W^{N^{-1}}(|x, \xi, N^{-\frac{1}{2}}|^3)v\rangle.$$

Let us make this bound more precise. Since $\lambda_i(0) > 0 \, \forall i$, for q small enough one has

$$Op_W^{N^{-1}}(Q_F(q)(x, \xi)) - \frac{N^{-1}}{2}\sum_i \langle v, \lambda_i(q)v\rangle \geq c Op_W^{N^{-1}}(|x|^2 + |\xi|^2) - \frac{N^{-1}}{2}c(n-r).$$

Hence

$$\langle v, Op_W^{N^{-1}}(g_0)v\rangle \geq c\langle v, Op_W^{N^{-1}}(|p|^2)v\rangle + \frac{N^{-1}}{2}\langle v, \sum_i \lambda_i(q), v\rangle$$

$$+ c\langle v, Op_W^{N^{-1}}(|x|^2 + |\xi|^2)\rangle - c\frac{N^{-1}}{2}(n-r) - C\langle v, Op_W^{N^{-1}}(|x, \xi, N^{-\frac{1}{2}}|^3)v\rangle.$$

Recall from Proposition 5.6 that $g_1 = \mu(\rho(q, 0, 0, 0)) - \frac{1}{2}\sum_i \lambda_i(q) + O(|x, p, \xi|)$. Hence,

$$\langle v, Op_W^{N^{-1}}(g)v\rangle \geq c\langle v, Op_W^{N^{-1}}(|p|^2)v\rangle + \frac{N^{-1}}{2}\langle v, \mu(\rho(q, 0, 0, 0)), v\rangle$$

$$+ c\langle v, Op_W^{N^{-1}}(|x|^2 + |\xi|^2) - c\frac{N^{-1}}{2}(n-r) - C\langle v, Op_W^{N^{-1}}(|N^{-\frac{1}{2}}, x, \xi|^3)v\rangle.$$

As $\mu(\rho(q, 0, 0, 0)) \geq \mu(P_0) + c|q|^2$, this yields the lower bound.

We now turn to the second estimate. This requires a bound on

$$\langle v, Op_W^{N^{-1}}(Q_F(q)(x, \xi) - Q_F(0)(x, \xi))v\rangle.$$

Since Q_F has been diagonalised this operator is diagonal in the Hilbert base given by the Hermite functions. Let us write

$$v = \sum_{\nu \in \mathbb{N}^{n-r}} \alpha_\nu(q) H_{\nu,N}(x),$$

Where $(H_{\nu,N})_\nu$ denote the Hilbert base of $\mathbb{R}^{n-r}$ given by the Hermite functions. Then

$$\langle v, Op_W^{N^{-1}}(Q_F(q)(x,\xi) - Q_F(0)(x,\xi))v\rangle$$
$$= \sum_\nu \langle \alpha_\nu(q), \nu \cdot (\lambda(q) - \lambda(0))\alpha_\nu(q)\rangle + \frac{1}{2}\langle v, \sum_i \lambda_i(q) - \lambda_i(0)v\rangle.$$

The key point is

$$\left|\sum_\nu \langle \alpha_\nu(q), \nu \cdot (\lambda(q) - \lambda(0))\alpha_\nu(q)\rangle\right| \leq Ca \sum_\nu \langle \alpha_\nu(q), |\nu|\alpha_\nu(q)\rangle.$$

The right-hand term is then equal to

$$Ca\left(\langle v, Op_W^{N^{-1}}(|x|^2 + |\xi|^2)v\rangle - N^{-1}\frac{n-r}{2}\right).$$

This yields the desired control since

$$g_1 = \mu(q) - \frac{1}{2}\sum_i \lambda_i(q) + O(|p,x,\xi|).$$

$\square$

5.3 Spectral gap

It only remains to show that the sequence of almost eigenfunctions given by Proposition 5.8 corresponds to the first eigenvalue of $T_N(h)$.

Proposition 5.10. *Let $h \geq 0$ be such that the minimum of the Melin value μ is only reached at one point, which is a miniwell for h.*

Let (μ_i) be the real sequence constructed in the previous proposition, and let $\lambda_{\min}$ be the first eigenvalue of $T_N(h)$.

Then

$$\lambda_{\min} \sim N^{-1}\sum_{i=0}^\infty N^{-\frac{i}{4}}\mu_i.$$

Moreover, there exists $c > 0$ such that, for every N, one has

$$\operatorname{dist}(\lambda_{\min}, Sp(T_N(h)) \setminus \{\lambda_{\min}\}) \geq cN^{-\frac{3}{2}}.$$

Proof. Let us show that any function orthogonal to the one proposed in Proposition 5.8 has an energy which is larger by at least $cN^{-\frac{3}{2}}$.

Let (v_N) be a sequence of unit vectors in $L^2(\mathbb{R}^n)$. If

$$\langle v_N, Op_W^1(\tilde{g}_N)v_N\rangle \leq N^{-1}\mu_0 + CN^{-\frac{3}{2}}$$

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for some C, then $v_N = e^{-\frac{|x|^2}{2}} w_N(q) + O(N^{-\frac{1}{2}})$, with $\|w_N\|_{L^2} = 1 + O(N^{-\frac{1}{2}})$.

If $C - \mu_2$ is strictly smaller than the spectral gap of the quadratic operator

$$Op_W^1 \left(Q_S(0)(p) + \frac{1}{2} \operatorname{Hess} \mu \circ \rho(\cdot, 0, 0, 0)(q) \right),$$

then $\langle w_N, e^{-\phi(q)} \rangle \geq a$ for some $a > 0$ independent of N, which concludes the proof. $\qquad\square$

6 A degenerate case

In this section we treat a case in which the zero set of the symbol is not a submanifold. The local hypotheses on the symbol are as follows:

Definition 6.1. Let $h \in C^\infty(M, \mathbb{R}^+)$ and $P_0 \in M$. The zero set of h is said to have a *simple crossing* at P_0 if there is an open set U containing P_0 such that:

- $\{h = 0\} \cap U = Z_1 \cup Z_2$, where Z_1 and Z_2 are two pieces of smooth isotropic submanifolds of M.

- $Z_1 \cap Z_2 = \{P_0\}$ and $T_{P_0} Z_1 \cap T_{P_0} Z_2 = \{0\}$.

- $T_{P_0} Z_1 \oplus T_{P_0} Z_2$ is isotropic.

- For $i = 1, 2$, on all of $Z_i \setminus \{P_0\}$, h vanishes at order exactly 2 on Z_i.

- There is $c > 0$ such that, for all $x \in Z_1 \cup Z_2$, one has:

$$\mu(x) - \mu(P_0) \geq c \operatorname{dist}(P_0, x).$$

The last condition may seem very strong. However, μ is typically only Lipschitz-continuous at the intersection. A typical example is

$$h(q_1, q_2, p_1, p_2) = p_1^2 + p_2^2 + q_1^2 q_2^2,$$

where along $\{q_1, 0, 0, 0\}$ one has $\mu(q_1) = |q_1| + 1$. We exclude on purpose situations like $\mu(q_1) = 1 + |q_1| - q_1 + q_1^2$, which grows like $|q_1|$ for $q_1 < 0$ but grows like q_1^2 for $q_1 > 0$.

Under the hypotheses of Definition 6.1, we first give a symplectic normal form of h near P_0, then a description of the first eigenvector and eigenvalue of $T_N(h)$.

6.1 Symplectic normal form

Let $Q \geq 0$ be a semidefinite positive quadratic form on $(\mathbb{R}^{2n}, \omega)$, and (e_i, f_i) a symplectic basis of $\mathbb{R}^{2n}$ which diagonalises Q:

$$Q \left(\sum_{i=1}^n q_i e_i + p_i f_i \right) = \sum_{i=r+1}^{r'} p_i^2 + \sum_{i=r'+1}^n \lambda_i(q_i^2 + p_i^2),$$

$$\forall i, \lambda_i \neq 0.$$

Let M denote the matrix of Q in the canonical basis. Then

$$\{\pm i\lambda_{r'+1}, \ldots, \pm i\lambda_n\} = \sigma(JM) \setminus \{0\}.$$

More precisely, if E_λ denotes the (complex) eigenspace of JM with eigenvalue λ, then

$$E_{i\lambda_j} \oplus E_{-i\lambda_j} = \operatorname{Span}_{\mathbb{C}}((e_k, f_k), k > r', \lambda_k = \lambda_j).$$

Moreover, Jordan blocks never occur for nonzero eigenvalues. Hence,

Proposition 6.2. *If $Q : \mathbb{R}^m \mapsto S_{2n}^+(\mathbb{R})$ is a smooth parameter-dependent semi-positive quadratic form on $(\mathbb{R}^{2n}, \omega)$, and if the d-th largest symplectic eigenvalue λ_{n-d+1} (with multiplicity) never crosses the $d+1$-th largest symplectic eigenvalue, then the $2d$ last vectors $(e_{n-d+1}, f_{n-d+1}, \ldots, e_n, f_n)$ of a symplectic basis diagonalising Q depend smoothly on the parameter, up to reordering of the d largest symplectic eigenvalues.*

One can build a symplectic normal form as previously, under the conditions above.

Proposition 6.3. *Let h satisfy the simple crossing conditions of Definition 6.1, and let*

$$r_1 = \dim(Z_1)$$
$$r_2 = \dim(Z_2).$$

Then there is an open set $V \subset U$, containing P_0, and a symplectic map

$$\sigma : V \mapsto \mathbb{R}^{2r_1} \times \mathbb{R}^{2r_2} \times \mathbb{R}^{2(n-r_1-r_2)}$$

such that

$$
\begin{aligned}
h \circ \sigma^{-1}&(q_1, p_1, q_2, p_2, x, \xi) \\
&= \sum_{i=1}^{n-r_1-r_2} \lambda_i(q_1, q_2)(x_i^2 + \xi_i^2) + Q_S(q_1, q_2)(p_1, p_2) \\
&\quad + \sum_{i,j=1}^{r_1} \sum_{k,l=1}^{r_2} \alpha_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} \\
&\quad + O(\|x, \xi, p_1, p_2\|^3) + O(\|q_1\|^2 \|q_2\|^2 (\|q_1\| + \|q_2\|)).
\end{aligned}
$$

Moreover, for every $(q_1, q_2) \in (\mathbb{R}^{r_1} \setminus \{0\}) \times (\mathbb{R}^{r_2} \setminus \{0\})$ small enough, the matrices given by $\left[\sum_{i,j} \alpha_{ijkl} q_{1,i} q_{1,j}\right]_{k,l}$ and $\left[\sum_{k,l} \alpha_{ijkl} q_{2,k} q_{2,l}\right]_{i,j}$ are positive.

Proof. At P_0, there are exactly $n - r_1 - r_2$ nonzero symplectic eigenvalues (with multiplicity) for the Hessian of h. Hence, in small neighbourhoods V_1 and V_2 of P_0 in Z_1 and Z_2, there is no crossing between the $r_1 + r_2$-th largest eigenvalue and the one immediatly below. From the previous lemma there is a smooth choice of symplectic eigenvectors for the $r_1 + r_2$ largest symplectic eigenvalues, which span a symplectic subbundle of $T_{Z_1 \cup Z_2} M$. The symplectic orthogonal $\mathcal{F}$ of this bundle contains $TZ_1 \cup TZ_2$ since the latter consists of zero vectors for the quadratic form.

$\mathcal{F}$ forms a distribution, which is integrable along $Z_1 \cup Z_2 \setminus P_0$ as in Proposition 5.3. Let us show that $\mathcal{F}$ is also integrable at P_0.

We take a first set of local coordinates $(q_1, q_2, ...)$ such that $Z_1 = \{q_1, 0, 0\}$ and $Z_2 = \{0, q_2, 0\}$.

Then, the restriction of the Hessian of h at $(q_1, 0, 0)$ to $\{0, q_2, 0\}$ is $O(|q_1|^2)$ as q_1 approaches zero. In particular, the distance in the Grassmannian from $\{0, q_2, 0\}$ to the distribution above at $(q_1, 0, 0)$ is $O(|q_1|^2)$. Hence, the first differential of $\mathcal{F}$ at P_0 in the directions contained in Z_1 and Z_2 are zero, so that the distribution is integrable at P_0.

Hence, there is a symplectic manifold S, containing $Z_1 \cup Z_2$, so that $T_{Z_1 \cup Z_2} S$ is the symplectic orthogonal of the bundle above.

As in the proof of Proposition 5.3, one first considers a symplectic map which

- sends Z_1 to $\mathbb{R}^{r_1} \times \{0, 0, 0, 0, 0\}$,

- sends Z_2 to $\{0, 0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\}$,

- sends S to $\mathbb{R}^{2r_1} \times \mathbb{R}^{2r_2} \times \{0, 0\}$.

Let us define
$$\hat{Z} = \mathbb{R}^{r_1} \times \{0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\}.$$

Then $\hat{Z}$ is isotropic. Along $\hat{Z}$ one wishes to find the $n - r_1 - r_2$ largest symplectic eigenpairs of the Hessian matrix of h. However, this Hessian is not necessarily semi-positive definite on $\hat{Z}$. To tackle this issue, we first formulate the span $\mathcal{D}$ of what will be the largest symplectic eigenvectors by a variatonal formulation.

We first observe that there is a symplectic subspace of $\mathbb{R}^{2n}$, of dimension $2(n - r_1 - r_2)$, on which the restriction of $\text{Hess}(h)(0)$ is definite positive.

By continuity, for $z \in \hat{Z}$ small enough, the set of subspaces of $\mathbb{R}^{2n}$ of dimension $2(n - r_1 - r_2)$, on which the restriction of $\text{Hess}(h)(z)$ is semi-definite positive, is non-empty. Let us call $G^+(z)$ this closed subset of the Grassmannian. For each element $F \in G^+(z)$, we pick a basis of F under which we denote by M_F the matrix elements of $\text{Hess}(h)(z)$ and J_F the matrix elements of the symplectic form ω. Since M_F has a square root among semidefinite positive matrices, the matrix $J_F M_F$ has purely imaginary spectrum (since it is conjugated with the antisymmetric matrix $M_F^{\frac{1}{2}} J_F M_F^{\frac{1}{2}}$). We then let $\kappa(F) = \text{dist}(0, Sp(J_F M_F))$.

We claim that maximising $\kappa(F)$ on $G^+(z)$ leads to $\mathcal{D}(z)$. We first note that $\kappa(F) = 0$ on the boundary of $G^+(z)$ (on which the matrix M_F is singular), and moreover $\kappa(F) = 0$ if F is not a symplectic subspace of R^{2n}.

For $z = 0$, the function κ has only one maximum which is positive and non-degenerate. Hence the same holds for z small since κ at z is C^2-close to κ at zero. The unique maximal point $\mathcal{D}(z)$ depends smoothly on z and is transverse to $T_z \hat{Z}$. From the expression of the differential of κ, the symplectic orthogonal of $\mathcal{D}(z)$ is also its $\text{Hess}(h)(z)$-orthogonal.

The Hessian matrix of h at z, when restricted on $\mathcal{D}(z)$, is definite positive, hence has symplectic eigenpairs which depend smoothly on z. Hence, as in Proposition 5.3 one can find a symplectic map which is identity on S and which diagonalises the fast modes along $\hat{Z}$.

Then it only remains to study the behaviour of $h \circ \sigma^{-1}$ on $\hat{Z}$, near 0. As h is non-negative and vanishes exactly on $Z_1 \cup Z_2$, one has

$$h \circ \sigma^{-1}(q_1, 0, q_2, 0, 0, 0) = O(\|q_1\|^2 \|q_2\|^2).$$

The dominant term is then of the form:

$$\sum_{i,j,k,l} \alpha_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l}.$$

The positivity conditions on the tensor α are then directly given by the fact that h vanishes at order 2 on $Z_1 \setminus \{P_0\}$ and $Z_2 \setminus \{P_0\}$. $\qquad \square$

One can easily adapt Definition 6.1 to the case of a crossing along a submanifold.

Definition 6.4 (Crossing along a submanifold). Let $h \in C^\infty(M, \mathbb{R}^+)$ and $P_0 \in M$. The zero set of h is said to *cross along a submanifold* near P_0 if there is an open set U containing P_0 such that:

- $\{h = 0\} \cap U = Z_1 \cup Z_2$, where Z_1 and Z_2 are two pieces of smooth isotropic submanifolds of M.

- $Z_1 \cap Z_2 = Z_3$ is a piece of smooth submanifold containing P_0. For each $x \in Z_3$, one has $T_x Z_3 = T_x Z_1 \cap T_x Z_2$.

- For each $x \in Z_3$, the space $T_x Z_1 + T_x Z_2$ is isotropic.

- For $i = 1, 2$, on all of $Z_i \setminus Z_3$, h vanishes at order exactly 2 on Z_i.

- There is $c > 0$ such that, for all $x \in Z_1 \cup Z_2$, one has:

$$\mu(x) - \mu(P_0) \geq c \operatorname{dist}(Z_3, x).$$

With this definition one can find a normal form as previously:

Proposition 6.5. *Let h satisfy the conditions of Definition 6.4, and let*

$$r_1 = \dim(Z_1) - \dim(Z_3)$$
$$r_2 = \dim(Z_2) - \dim(Z_3)$$
$$r_3 = \dim(Z_3).$$

Then there is an open set $V \subset U$, containing P_0, and a symplectic map

$$\sigma : V \mapsto \mathbb{R}^{2r_1} \times \mathbb{R}^{2r_2} \times \mathbb{R}^{2r_3} \times \mathbb{R}^{2(n-r_1-r_2-r_3)}$$

such that

$$h \circ \sigma^{-1}(q_1, p_1, q_2, p_2, q_3, p_3, x, \xi)$$
$$= \sum_{i=1}^{n-r_1-r_2} \lambda_i(q_1, q_2, q_3)(x_i^2 + \xi_i^2) + Q_a(q_1, q_2, q_3)(p_1, p_2) + Q_b(q_1, q_2, q_3)(p_3)$$
$$+ \sum_{i,j=1}^{r_1} \sum_{k,l=1}^{r_2} \alpha_{ijkl}(q_3) q_{1,i} q_{1,j} q_{2,k} q_{2,l}$$
$$+ O(\|x, \xi, p_1, p_2\|^3) + O(\|q_1\|^2 \|q_2\|^2 \cdot \|(q_1, q_2)\|).$$

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Moreover, for every $q_3 \in \mathbb{R}^{r_3}$ small enough, for every $(q_1, q_2) \in (\mathbb{R}^{r_1} \setminus \{0\}) \times (\mathbb{R}^{r_2} \setminus \{0\})$ small enough, the matrices given by $\left[\sum_{i,j} \alpha_{ijkl} q_{1,i} q_{1,j} \right]_{k,l}$ and $\left[\sum_{k,l} \alpha_{ijkl} q_{2,k} q_{2,l} \right]_{i,j}$ are positive.

Proof. One can repeat the proof of Proposition 6.3 with $\hat{Z}$ containing $Z_1 \cup Z_2$. This yields the desired normal form, except for Q_a and Q_b which are replaced with a more general quadratic function $Q_S(q_1, q_2, q_3)(p_1, p_2, p_3)$.

In order to separate p_3 from (p_1, p_2), we first apply Lemma 5.1 with Z_3 as parameter space, in order to find, for every q_3, a decomposition of $\mathbb{R}^{2(r_1+r_2+r_3)}$ into a sum of symplectic spaces $S \oplus S'$ with $TZ_3 \subset S$, and so that S and S' are orthogonal for $Q_S(q_1 = 0, q_2 = 0, q_3)$. There is a symplectomorphism σ sending S into $\mathbb{R}^{2r_1}$ such that $\sigma|_{Z_3} = Id$.

The map σ may distort $\{h = 0\}$, which can be flattened again on each space $\mathbb{R}^{2r_1} \times \mathbb{R}^{2r_2} \times \{q_3, 0\} \times \mathbb{R}^{2(n-r_1-r_2-r_3)}$, with smooth dependence on q_3.

Again, this leads to a symplectomorphism on a neighbourhood of zero in $\mathbb{R}^{2n}$, up to a small correction controlled by $O(|p_3|^2)$. $\qquad\square$

Remark 6.6 (More general degenerate crossings). Simple crossings (and crossings along submanifolds) are not stable by Cartesian products, which leads to a slightly more general situation (see Remark 6.7).

On the other hand, one could try to deal with symbols whose zero set form a *stratified manifold*, which are defined recursively: a stratified manifold is a union of smooth manifolds with clean intersections, such that the union of all intersections is itself a stratified manifold. The boundary of a hypercube is an instance of a stratified manifold.

In this respect, a model case for a stratified situation of degree three is

$$p_1^2 + p_2^2 + p_3^2 + q_1^2 q_2^2 q_3^2,$$

with zero set $\{p_1 = 0, p_2 = 0, p_3 = 0, q_i = 0\}$ for every $i = 1, 2, 3$.

For this operator, the ground state is rapidly decreasing at infinity [12] but this is not due to subprincipal effects. Indeed, in this setting, μ is constant along the three axes. If we add a generic transverse quadratic operator $Q_q(x, \xi)$, the subprincipal effect will dominate and has no reason to select the point $\{q = 0\}$, as opposed to the simple crossing case where an open set of symbols sharing the same minimal set have minimal Melin value at the crossing point.

6.2 Study of the model operator

As Proposition 6.3 suggests, the following operators play an important role in the study of the crossing case:

$$P = Q(iD) + \sum_{i,j=1}^{r_1} \sum_{k,l=1}^{r_2} \alpha_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} + \sum_{i=1}^{r_1} L_{1,i} q_{1,i} + \sum_{i=1}^{r_2} L_{2,i} q_{2,i},$$

acting on $L^2(\mathbb{R}^{r_1+r_2})$, where D is the differentiation operator and $Q > 0$ is a quadratic form. The linear form L will appear as an effect of the subprincipal symbol, as we will see later.

Let Q_1 and Q_2 denote the restrictions of the quadratic form Q on $\mathbb{R}^{r_1} \times \{0\}$ and $\{0\} \times \mathbb{R}^{r_2}$, respectively. Throughout this Subsection we impose the following conditions on P:

- For every (q_1, q_2), one has

$$\sum_{i,j=1}^{r_1} \sum_{i,j=1}^{r_2} \alpha_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} \geq 0.$$

- For every $q_1 \neq 0$, one has

$$Q_2(iD) + \sum_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} > \sum_{i=1}^{r_1} L_{1,i} q_{1,i},$$

- For every $q_2 \neq 0$, one has

$$Q_1(iD) + \sum_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} > \sum_{i=1}^{r_1} L_{2,i} q_{2,i},$$

Remark 6.7. These conditions are weaker than what Definition 6.1 calls for. There does not need to be a simple crossing in this case as the following example illustrates:

$$P = -\Delta + q_{1,1}^2 q_{2,1}^2 + q_{1,2}^2 q_{2,2}^2.$$

There, the zero set of the symbol is a union of four isotropic surfaces in $\mathbb{R}^8$, i.e. $\{p = 0, q_{1,i} = 0, q_{2,j} = 0\}$ for all $(i,j) \in \{1,2\}^2$.

Proposition 6.8. *Under the previous conditions, there exists $c > 0$ such that*

$$P \geq c(Q(iD) + |q|).$$

Proof. Let Q_2 be the restriction of the quadratic form Q to $\{0\} \times \mathbb{R}^{r_2}$. One has $Q \geq Q_2$, hence $Q(iD) \geq Q_2(iD)$. By hypothesis,

$$Q_2(iD) + \sum_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} > \sum_{i=1}^{r_1} L_{1,i} q_{1,i},$$

and the infimum of the spectrum of the left hand side is 1-homogeneous in q_1, so that

$$Q_2(iD) + \sum_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} \geq (1-c) \sum_{i=1}^{r_1} L_{1,i} q_{1,i} + 2c|q_1|$$

for some $c > 0$. In particular,

$$P \geq cQ(iD) + 2c|q_1|.$$

The same reasoning applies to Q_2, hence

$$2P \geq 2cQ(iD) + 2c|q_1| + 2c|q_2|,$$

which allows us to conclude. $\square$

One deduces immediately:

Proposition 6.9. *The operator P has compact resolvent. Its first eigenvalue is positive.*

We are now able to use Agmon estimates. In the particular case where Q is diagonal, the following result is contained in the Helffer-Nourrigat theory [12], see also the related results in [20].

Proposition 6.10. *Let λ_0 be the first eigenvalue of P. There exists $c > 0$ such that, if $u \in L^2(\mathbb{R}^{r_1+r_2})$, and $(C_\beta)_{\beta \in \mathbb{N}^{r_1+r_2}}$ are such that $|\partial^\beta u(q)| \leq C_\beta e^{-c|q|^{3/2}}$ for all $q \in \mathbb{R}^{r_1+r_2}, \beta \in \mathbb{N}^{r_1+r_2}$, then for any $f \in L^2(\mathbb{R}^{r_1+r_2})$ such that $(P - \lambda_0)f = u$, there exists $(C'_\beta)_{\beta \in \mathbb{N}^{r_1+r_2}} > 0$ such that $|\partial^\beta f(q)| \leq C' e^{-c|q|^{3/2}}$ for every $q \in \mathbb{R}^{r_1+r_2}, \beta \in \mathbb{N}^{r_1+r_2}$.*

Proof. With $\phi(q) = c|q|^{3/2}$, one has $Q(\vec{\nabla}\phi) \leq c'|q|$. Hence $P - \lambda_0 - Q(\vec{\nabla}\phi)$ is positive far from zero, and one can use Agmon estimates as developed in [1]. $\square$

We will also need the following two facts. Proposition 6.11 is an essential ingredient of Subsection 6.3 and Proposition 6.12 is necessary to compare the Weyl asymptotics with the regular case.

Proposition 6.11. *The first eigenvalue λ_0 of P is simple.*

Proof. This follows from an argument which is standard in the case $Q = Id$. Let $u_0 \in L^2(\mathbb{R}^{r_1+r_2})$ be such that $Pu_0 = \lambda_0 u_0$. Then u_0 is a minimizer of the Courant-Hilbert problem

$$\min_{\|u\|_{L^2}=1, \, u \in H^1} \int Q(\vec{\nabla} u) + V|u|^2.$$

The set $\{u_0 = 0\}$ has zero Lebesgue measure from a standard Unique Continuation argument. The function $|u_0|$ is then also a minimizer of this quantity, since $\vec{\nabla}|u_0| = \pm\vec{\nabla}u_0$ whenever $u_0 \neq 0$.

Then $|u_0|$ itself belongs to the eigenspace of P with value λ_0, which is (a priori) a finite-dimensional space of real analytic (complex-valued) functions. Hence, $|u_0|$ is real analytic so that $u_0 = |u_0|e^{i\theta_0}$, with θ_0 real analytic.

Now

$$\int \overline{u_0} Pu_0 = \int |u_0|(P - Q(\nabla\theta_0))|u_0| = \lambda_0 - \int Q(\nabla\theta_0)|u_0|^2.$$

As $\{|u_0| = 0\}$ has zero Lebesgue measure and $Q > 0$, the function θ_0 is constant, so that u_0 and $|u_0|$ are colinear.

To conclude, if u_0 and u_1 are two orthogonal eigenfunctions of P with eigenvalue λ_0, then $|u_0|$ and $|u_1|$ are orthogonal with each other, and both have $\mathbb{R}^{r_1+r_2}$ as support, so that either $u_0 = 0$ or $u_1 = 0$. $\square$

Proposition 6.12. *Suppose P satisfies the following two supplementary conditions:*

- *$r_1 = r_2$.*

- *For every $(q_1, q_2) \in (\mathbb{R}^{r_1} \setminus \{0\}) \times (\mathbb{R}^{r_2} \setminus \{0\})$, the matrices given by $\left[\sum_{i,j} \alpha_{ijkl} q_{1,i} q_{1,j}\right]_{k,l}$ and $\left[\sum_{k,l} \alpha_{ijkl} q_{2,k} q_{2,l}\right]_{i,j}$ are positive.*

Let $\Lambda > 0$ and let N_Λ denote the number of eigenvalues of P less than Λ (with multiplicity).

Then there are $C > c > 0$ such that, as $\Lambda \to +\infty$, one has

$$c\Lambda^{\frac{3}{2}r_1} \log(\Lambda) \le N_\Lambda \le C\Lambda^{\frac{3}{2}r_1} \log(\Lambda).$$

Proof. Under the second supplementary condition, the quartic part of the potential is greater than $c|q_1|^2|q_2|^2$ for some $c > 0$. Hence, for some $C > 0$ one has $N_\lambda \ge \tilde{N}_\lambda$, where $\tilde{N}_\Lambda$ counts the eigenvalues less than Λ of

$$-\Delta + |q_1|^2|q_2|^2 + |q_1| + |q_2|.$$

On the other hand one clearly has $P \le C(-\Delta + |q_1|^2|q_2|^2 + |q_1| + |q_2|)$ for some $C > 0$.

Thus, the problem boils down to Weyl asymptotics for the elliptic operator $-\Delta + |q_1|^2|q_2|^2 + |q_1| + |q_2|$. It suffices to control the volume of the sub-levels of its symbol:

$$\{(q_1, q_2, p_1, p_2) \in \mathbb{R}^{4r_1}, \ |p_1|^2 + |p_2|^2 + |q_1|^2|q_2|^2 + |q_1| + |q_2| \le \Lambda\}.$$

We first study

$$A_\Lambda = \{(q_1, q_2) \in \mathbb{R}^{r_1}, \ |q_1|^2|q_2|^2 + |q_1| + |q_2| \le \Lambda\}.$$

Then, decomposing A_Λ into $A_\Lambda \cap B(0, \Lambda^{\frac{r_1}{2}})$ and its complement set yields

$$Vol(A_\Lambda) \le C\Lambda^{\frac{r_1}{2}} + 2\int_{|q_1| \ge \Lambda^{1/4}} Vol\{q_2, |q_1|^2|q_2|^2 + |q_1| + |q_2| \le \Lambda\}.$$

$$\le C\Lambda^{\frac{r_1}{2}} + 2C\int_{|q_1| \ge \Lambda^{1/4}} \left(\frac{\sqrt{\Lambda - |q_1| + |q_1|^{-1}}}{|q_1|}\right)^{r_1}$$

$$\le C\Lambda^{\frac{r_1}{2}} + 2C\Lambda^{\frac{r_1}{2}} \int_{\Lambda^{-\frac{3}{4}}}^{2} \frac{1}{x} dx$$

$$\le C\Lambda^{\frac{r_1}{2}} \log(\Lambda).$$

On the other hand,

$$Vol(A_\Lambda) \ge 2\int_{|q_1| \ge \Lambda^{1/4}} Vol\{q_2, |q_1|^2|q_2|^2 + |q_1| + |q_2| \le \Lambda\}.$$

$$\ge 2c\int_{|q_1| \ge \Lambda^{1/4}} \left(\frac{\sqrt{\Lambda - |q_1| + |q_1|^{-1}}}{|q_1|}\right)^{r_1}$$

$$\ge 2c\Lambda^{\frac{r_1}{2}} \int_{\Lambda^{-\frac{3}{4}}}^{2} \frac{1}{x} dx$$

$$\ge c\Lambda^{\frac{r_1}{2}} \log(\Lambda).$$

Integrating yields

$$Vol(\{(q_1, q_2, p_1, p_2) \in \mathbb{R}^{4r_1}, \ |p_1|^2 + |p_2|^2 + |q_1|^2|q_2|^2 + |q_1| + |q_2| \le \Lambda\})$$

$$\in [c\Lambda^{\frac{3}{2}r_1} \log(\Lambda), C\Lambda^{\frac{3}{2}r_1} \log(\Lambda)],$$

hence the claim. $\qquad\square$

6.3 Approximate first eigenfunction

In this Subsection we give an expansion for the first eigenfunction and eigenvalue in a crossing case, following the same strategy as Subsection 5.2. We quantize the symplectic map of Proposition 6.3 and we use the Bargmann transform to reformulate the problem in the pseudodifferential algebra, in which we squeeze the operator. This time, the squeezing is of order $N^{\frac{1}{6}}$ along (q_1, q_2), with a concentration speed of $N^{-\frac{1}{3}+\epsilon}$ along the zero set, instead of $N^{-\frac{1}{4}+\epsilon}$ as was seen in the regular case. We then apply a perturbative argument to obtain the full expansion of the first eigenvalue and eigenvector.

Definition 6.13. For any choice $\mathfrak{S}_N$ of quantization of the map σ of Proposition 6.3, the classical symbol $g_{\mathfrak{S}} \sim \sum N^{-i} g_i$ on a neighbourhood U of 0 in $\mathbb{R}^{2n}$ is defined as follows: for any sequence $(u_N)_{N \geq 1}$ with microsupport in a compact set of U, the following holds:

$$\mathcal{B}_N^{-1} \mathfrak{S}_N^{-1} T_N(h) \mathfrak{S}_N \mathcal{B}_N u_N = Op_W^{N^{-1}}(g_{\mathfrak{S}}) u_N + O(N^{-\infty}).$$

In what follows, we choose an arbitrary quantum map $\mathfrak{S}_N$, and we write g instead of $g_{\mathfrak{S}}$.

The subprincipal part g_1 is prescribed on $Z_1 \cup Z_2$ by the local Melin estimates.

Proposition 6.14. *Along $\sigma(Z_1)$, for q_1 close to zero, one has*

$$g_1(q_1, 0, 0, 0, 0, 0) = \frac{1}{2} \sum_i \lambda_i(q_1, 0) + \frac{1}{4} \operatorname{tr}(Q_S(q_1, 0)).$$

Along $\sigma(Z_2)$, for q_2 close to zero, one has

$$g_1(0, 0, q_2, 0, 0, 0) = \frac{1}{2} \sum_i \lambda_i(0, q_2) + \frac{1}{4} \operatorname{tr}(Q_S(0, q_2)).$$

The proof is exactly the same as for Proposition 5.6.
Let us define

$$P = Q_S(0)(-iD_{q_1}, -iD_{q_2}) + \sum_{ijkl} \alpha_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l}$$

$$+ \nabla \left(\frac{1}{2} \sum_{i=1}^{n-r_1-r_2} \lambda_i + \frac{1}{4} \operatorname{tr} Q_S \right)_{q_1=q_2=0} \cdot (q_1, q_2).$$

Then P satisfies the hypotheses of Subsection 6.2 and Proposition 6.12.

Proposition 6.15. *Under the conditions of Definition 6.1, there exists $c > 0$, a sequence $(u_i) \in (\mathbb{C}[X_1, \ldots, X_{n-r_1-r_2}, L^2(\mathbb{R}^{r_1+r_2})])^{\mathbb{N}}$ and a real sequence $C_{i,\alpha,\beta}$ with*

$$\forall (i, \alpha, \beta, q) \in \mathbb{N} \times \mathbb{N}^{n-r_1-r_2} \times \mathbb{N}^{r_1+r_2} \times \mathbb{R}^{r_1+r-2}, |\partial^\beta u_{i,\alpha}(q)| \leq C_{i,\alpha,\beta} e^{-c|q|^{3/2}},$$

and a sequence $(\mu_i) \in \mathbb{R}^{\mathbb{N}}$ with $\mu_0 = \mu(P_0)$, $\mu_1 = 0$ and $\mu_2 = \min \operatorname{Sp}(P)$, so that

$$N^{\frac{n}{2} - \frac{r_1+r_2}{6}} e^{-N|x|^2/2} \sum_{i=0}^{+\infty} N^{-\frac{i}{6}} u_i(N^{\frac{1}{2}} x, N^{\frac{1}{3}} q)$$

is an $O(N^{-\infty})$-eigenfunction of $Op_W^{N^{-1}}(g)$, with eigenvalue

$$N^{-1} \sum_{i=0}^{+\infty} N^{-\frac{i}{6}} \mu_i.$$

This proposition provides an almost eigenfunction which we will show to be associated to the lowest eigenvalue (see Proposition 6.17). It is the main argument in the proof of Theorem C; the concentration speed of this eigenfunction on zero, which is $N^{-\frac{1}{3}}$, is the concentration speed of the lowest eigenvector of $T_N(h)$ on the miniwell, because of Proposition 2.19.

Proof. As announced, let us squeeze g by computing

$$\tilde{g} = g(N^{-\frac{1}{3}}q_1, N^{-\frac{2}{3}}p_1, N^{-\frac{1}{3}}q_2, N^{-\frac{2}{3}}p_2, N^{-\frac{1}{2}}x, N^{-\frac{1}{2}}\xi).$$

Grouping terms in the Taylor expansion yields, for any fixed $K \in \mathbb{N}$,

$$Op_W^1(\tilde{g}) = N^{-1} \sum_{i=0}^{K} N^{-\frac{i}{6}} Op_W^1(a_i) + O(N^{-\frac{K+7}{6}}).$$

The first terms are:

$$a_0 = \sum_{i=1}^{n-r} \lambda_i(0) \left(x_i^2 + \xi_i^2 + \frac{1}{2} \right) + \frac{1}{4} \operatorname{tr}(Q_S(0))$$

$$a_1 = 0$$

$$a_2 = \sigma(P)$$

$$a_3 = R_3(x, \xi) + L(x, \xi).$$

Here P is as above, R_3 is a degree three polynomial and L is a linear form.

With $A_i = Op_W^1(a_i)$ let us solve by induction on k the following equation, where $(u_k)_{k \in \mathbb{N}}$ is as in the claim:

$$\left(\sum N^{-\frac{i}{6}} (A_i - \mu_i) \right) \left(\sum N^{-\frac{i}{6}} u_i \right) = 0.$$

If v_0 is the (unique) ground state of P then our starting point is

$$u_0 = e^{-\frac{|x|^2}{2}} v_0, \ \mu_0 = \min \operatorname{Sp} A_0,$$

$$u_1 = 0, \ \mu_1 = 0.$$

Indeed u_0 is an almost eigenvector for $Op_W^1(\tilde{g})$, with eigenvalue $N^{-1}\mu_0 + O(N^{-\frac{4}{3}})$.

Let us start an induction at $k = 1$. Suppose we have constructed the first k terms of the expansion $u_0, \ldots, u_k$ and $\mu_0, \ldots, \mu_k$, with $u_i \perp u_0$ for every i, and suppose that, for some $C_k \in \mathbb{R}$, one has, for every $q \in \mathbb{R}^{r_1+r_2}$,

$$\int_{\mathbb{R}^{n-r_1-r_2}} \overline{u_0}(q, x) \left(\sum_{i=2}^{k+1} [A_i u_{k+1-i}](q, x) - \sum_{i=2}^{k} [\mu_i u_{k+1-i}](q, x) \right) \mathrm{d}x = C_{k+1} |v_0(q)|^2.$$

Then the eigenvalue problem yields u_{k+1} up to a function of the form $v(q)e^{-\frac{|x|^2}{2}}$. Indeed, writing $u_{k+1}(q,x) = v(q)e^{-\frac{|x|^2}{2}} + w(q,x)$, where for every q one has $w(q,\cdot) \perp e^{-\frac{|\cdot|^2}{2}}$, the eigenvalue equation is

$$(A_0 - \mu_0)u_{k+1} + (A_2 - \mu_2)u_{k-1} + \ldots + (A_{k+1} - \mu_{k+1})u_0 = 0$$

for u_{k+1} and μ_{k+1}. First $(A_0 - \mu_0)v(q)e^{-\frac{|x|^2}{2}} = 0$ so that

$$(A_0 - \mu_0)w + (A_2 - \mu_2)u_{k-1} + \ldots + (A_{k+1} - \mu_{k+1})u_0 = 0$$

By hypothesis, freezing the q variable and taking the scalar product of this equation with $x \mapsto e^{-\frac{|x|^2}{2}}$ yields $(C_{k+1} - \mu_{k+1})|v_0(q)|^2 = 0$. Let $\mu_{k+1} = C_{k+1}$. Then, for every $q \in \mathbb{R}^{r_1+r_2}$, the function

$$f_{k+1} : x \mapsto \sum_{i=2}^{k+1}[(A_i - \mu_i)u_{k+1-i}](q,x)$$

is orthogonal to $x \mapsto e^{-\frac{|x|^2}{2}}$. Hence $w = (A_0 - \mu_0)^{-1}f_{k+1}$ is well-defined and satisfies the eigenvalue equation.

Moreover, from Proposition 6.10, if by induction f_{k+1} is $e^{-\frac{|x|^2}{2}}$ times a polynomial in x, and if any derivative of any coefficient decays as fast as $e^{-c|q|^{3/2}}$, then the same is true for w.

At this point we need to check that, after the first step $k = 1$, the value μ_2 is indeed $\min \mathrm{Sp}(P)$.

If $k = 1$ then we are interested in the integral

$$\int_{\mathbb{R}^{n-r_1-r_2}} e^{-\frac{|x|^2}{2}} v_0(q)[A_2u_0](q,x)\mathrm{d}x = \min \mathrm{Sp}(P)|v_0(q)|^2,$$

since v_0 is a ground state of P. This is indeed a constant function times $|v_0(q)|^2$, so that the induction hypothesis is satisfied at the first step, and $\mu_2 = \min \mathrm{Sp}(P)$ as required.

Now recall $u_k(q,x) = v(q)e^{-\frac{|x|^2}{2}} + w(q,x)$. The eigenvalue equation in itself does not state any condition on v; however, to compute the second next order, one needs to satisfy an orthogonality condition, i.e.

$$\int_{\mathbb{R}^{n-r_1-r_2}} \overline{u_0}(q,x)\left(\sum_{i=2}^{k+3}[A_iu_{k+3-i}](q,x) - \sum_{i=2}^{k+2}[\mu_iu_{k+3-i}](q,x)\right)\mathrm{d}x = C_{k+3}|v_0(q)|^2.$$

This is equivalent to

$$\int_{\mathbb{R}^{n-r_1-r_2}} e^{-\frac{|x|^2}{2}}\left[(A_2 - \mu_2)ve^{-\frac{|x|^2}{2}}\right](x,q)\mathrm{d}x = F(q) + C_{k+3}v_0(q).$$

Now a_2 has no terms in x or ξ so the equation reduces to

$$(A_2 - \mu_2)v = F(q) + C_{k+3}v_0(q).$$

Here,

$$F(q) = \int_{\mathbb{R}^{n-r_1-r_2}} e^{-\frac{|x|^2}{2}} \left(\sum_{i=3}^{k+3} [A_i u_{k+3-i}](q,x) - \sum_{i=3}^{k+2} [\mu_i u_{k+3-i}](q,x) \right) \mathrm{d}x,$$

so that $|\partial_\beta F(q)| \leq C_\beta e^{-c|q|^{3/2}}$.

To solve this equation, one takes $C_{k+3} = -\langle v_0, F \rangle$, then the r.h.s is orthogonal to v_0, so that one can solve for v (indeed, μ_2 is a simple eigenvalue of A_2 by Proposition 6.11).

Then, by Proposition 6.10, one has, for all $\beta \in \mathbb{N}^{r_1+r_2}$, for some C_β, that $|\partial^\beta v(q)| \leq C_\beta e^{-c|q|^{3/2}}$ for all $q \in \mathbb{R}^{r_1+r_2}$. This ends the induction.

The previous considerations were formal, but the decay properties of the functions u_k imply that $A_j u_k \in L^2$ for every j and k, which concludes the proof. $\square$

Proposition 6.16. *For $t > 0$ let A_N^{cross} the following operator on $L^2(\mathbb{R}^{r_1+r_2})$:*

$$A_N^{cross} = Op_W^{N^{-1}} \left(|p|^2 + |q_1|^2 |q_2|^2 \right)$$

Under the conditions of Definition 6.1 and Proposition 6.12, there exists $a_0 > 0$, and two constants $0 < c < C$ such that, for any N, for any $a < a_0$, for any normalized $u \in L^2(X)$ supported in $B(P_0, a) \times \mathbb{S}^1$, with $v = \mathcal{B}_N^{-1} \mathfrak{S}_N^{-1} u$, one has:

$$c\langle v, A_N^{cross} v \rangle + c \left(\langle v, Op_W^{N^{-1}} (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n-r}{2} \right)$$
$$- C\langle v, Op_W^{N^{-1}} (|N^{-\frac{1}{2}}, x, \xi|^3) v \rangle - CN^{\frac{4}{3}}$$
$$\leq \langle u, hu \rangle - N^{-1} \mu(P_0)$$

In addition, the following bound holds:

$$c\langle v, A_N^{cross} v \rangle + \langle v, Op_W^{N^{-1}} (Q_F(0)(x,\xi)) v \rangle - C\langle v, Op_W^{N^{-1}} (|N^{-\frac{1}{2}}, x, \xi|^3) v \rangle$$
$$- aC \left(\langle v, Op_W^{N^{-1}} (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n-r}{2} \right) - CN^{-\frac{4}{3}}$$
$$\leq \langle u, hu \rangle - N^{-1} \mu(P_0) + \frac{N^{-1}}{2} \sum_i \lambda_i(0)$$
$$\leq C\langle v, A_N^{cross} v \rangle + \langle v, Op_W^{N^{-1}} (Q_F(0)(x,\xi)) v \rangle + C\langle v, Op_W^{N^{-1}} (|N^{-\frac{1}{2}}, x, \xi|^3) v \rangle$$
$$+ aC \left(\langle v, Op_W^{N^{-1}} (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n-r}{2} \right) + CN^{-\frac{4}{3}}.$$

Proof. The proof follows the exact same lines as for Proposition 5.9: the difficulty lies in handling the (x,ξ) terms which take a similar form as above.

The supplementary $N^{-\frac{4}{3}}$ terms are due to positivity estimates for the Weyl quantization: from $c\sigma(A_N^{cross}) \leq g_0$ we can only deduce $cA_N^{cross} \leq Op_W^{N^{-1}}(g_0) + O(N^{-\frac{4}{3}})$. $\square$

6.4 Spectral gap

As before, we show that the almost eigenfunction found previously corresponds to the first eigenvalue.

Proposition 6.17. *Let $h \geq 0$ be such that the minimum of the Melin value μ is only reached at one point, which is a simple crossing point of h.*

Let (μ_i) be the real sequence constructed in Proposition 6.15, and let λ_{min} be the first eigenvalue of $T_N(h)$.

Then

$$\lambda_{min} \sim N^{-1} \sum_{i=0}^{\infty} N^{-\frac{i}{6}} \mu_i.$$

Moreover, there exists $c > 0$ such that, for every N, one has

$$\text{dist}(\lambda_{min}), Sp(T_H(h)) \setminus \{\lambda_{min}\}) \geq cN^{-\frac{4}{3}}.$$

Proof. Let us show that any function orthogonal to the one proposed in Proposition 6.15 has an energy which is larger by at least $cN^{-\frac{4}{3}}$.

Let $(v_N)_{N \geq 1}$ be a sequence of unit vectors in $L^2(\mathbb{R}^n)$. If $\langle v_N, Op_W^1(\tilde{g}_N)v_N \rangle \leq N^{-1}\mu_0 + CN^{-\frac{4}{3}}$ for some C, then $v_N = e^{-\frac{|x|^2}{2}} w_N(q) + O(N^{-\frac{1}{3}})$, with $\|w_N\|_{L^2} = 1 + O(N^{-\frac{1}{3}})$.

If $C - \mu_2$ is strictly smaller than the spectral gap of the operator P then $\langle w_N, v_0 \rangle \geq a$ for some $a > 0$, which concludes the proof. $\square$

7 Comparative Weyl law

Definition 7.1. We will say a miniwell has *dimension r* when the dimension of the zero set of h around the miniwell is r. Similarly, we will say a crossing point has dimensions (r_1, r_2) when the dimensions of the two manifolds Z_1 and Z_2 around the point are r_1 and r_2, respectively.

Proof of theorem D.

A. The first statement stems directly from Proposition 3.5. In fact, the Subsection 4.2 concludes the proof of Theorem A using only the fact that the considered eigenvalue lies in this spectral window for every $\epsilon > 0$, hence the claim.

B. Let u a sequence of eigenfunctions of $T_N(h)$ in the spectral window above, and P_0 the miniwell of interest. The sequence u localises near P_0 so that one can apply a quantum map $\mathfrak{S}_N$.

Let $v_N = \mathcal{B}_N^{-1} \mathfrak{S}_N^{-1} u_N$. The first lower bound on Proposition 5.9 yields

$$\langle v_N, Op_W^{N^{-1}}(|x|^2 + |\xi|^2)v_N \rangle - N^{-1}\frac{n-r}{2} \leq C\Lambda_N N^{-1}.$$

From Proposition 3.5, for every $\delta > 0$, if $\epsilon > 0$ is small enough then u localises on $B(P_0, \delta)$. If δ is small enough then

$$\delta C \left(\langle v_N, Op_W^{N^{-1}}(|x|^2 + |\xi|^2)v_N \rangle - N^{-1}\frac{n-r}{2} \right) \leq \frac{N^{-1}\Lambda_N}{2}.$$

Let us prove an upper bound in the number of eigenvalues of $T_N(h)$. The second lower bound in Proposition 5.9 leads to

$$c\langle v_N, A_N^{reg}v_N \rangle + \langle v_N, Op_W^{N^{-1}}(Q_F(0)(x,\xi))v_N \rangle - \frac{N^{-1}}{2}\sum_i \lambda_i(0)$$

$$\leq \langle u_N, hu_N \rangle - N^{-1}\mu(P_0) + \frac{N^{-1}\Lambda_N}{2}.$$

For ϵ smaller than the spectral gap of $Q_F(0)(x,D)$, the left-hand side has less than $C\Lambda_N^r$ eigenvalues smaller than $\frac{3N^{-1}\Lambda_N}{2}$, hence the claim.

The lower bound proceeds along the same lines. The upper bound in Proposition 5.9 yields

$$C\langle v_N, A_N^{reg}v_N \rangle + \langle v_N, Op_W^{N^{-1}}(Q_F(0)(x,\xi))v_N \rangle - \frac{N^{-1}}{2}\sum_i \lambda_i(0)$$

$$\geq \langle u_N, hu_N \rangle - N^{-1}\mu(P_0) - \frac{N^{-1}\Lambda_N}{2}.$$

The left-hand side has always more than $c\Lambda_N^r$ eigenvalues smaller than $\frac{N^{-1}\Lambda_N}{2}$, hence the claim.

C. The proof for crossing points is the same except for the actual count of eigenvalues of the reference operator, which stems from Proposition 6.12.

$\square$

8 Examples: frustrated spin systems

In this Section, we discuss the class of examples introduced in Subsection 1.3. We first describe the minimal set in the general setting, and we prove that, for a loop of six triangles, the classical minimal set is not a smooth manifold; then we prove that the choice of the vectors on triangle "leaves" does not affect μ; to conclude we treat numerically a simple case supporting the general conjecture that μ is minimal on planar configurations.

8.1 Description of the zero set

If a graph is made of triangles $(V_i)_{i \in J}$, and if we denote by $\{u_i, v_i, w_i\}$ the three elements of $\mathbb{S}^2$ at the vertices of V_i, we write

$$h(e) = \sum_{i \in J} u_i \cdot v_i + u_i \cdot w_i + v_i \cdot w_i.$$

Moreover, for all $u, v, w \in \mathbb{S}^2$ one has

$$u \cdot v + u \cdot w + v \cdot w = \frac{1}{2}\|u + v + w\|^2 - \frac{3}{2}.$$

A way to minimize the symbol is thus to try to choose the vectors such that, for each triangle in the graph, the vectors at the vertices form a great equilateral
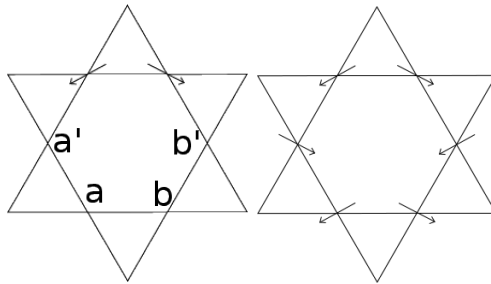
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Figure 2: On the left, a graph with 6 triangles and two prescribed vectors. On the right, a particular (planar) configuration.

triangle on $\mathbb{S}^2$ (this is equivalent to the requirement that their sum is the zero vector). As the example of the Husimi tree shows, this minimal set can be degenerate: once the vector at a vertex is chosen, there is an $\mathbb{S}^1$ degeneracy in the choice of the vectors at its children.

In the general case this solution is not always possible as can be seen on the right of Figure 1. Moreover, even if this solution is possible, the minimal set is not a submanifold, as we will see in an example.

A subset of interest of these minimal configurations consists in the case where all vectors are coplanar; this corresponds to colouring the graph with three colours. For some graphs made of triangles, there is no 3-colouring. Conversely, if the size of the graph grows the number of 3-colourings may grow exponentially fast.

A common conjecture in the physics literature is that, when applicable, the Melin value μ is always minimal only on planar configurations, except for a leaf degeneracy (see Proposition 8.2): in other terms, in the semiclassical limit, the quantum state presumably selects only planar configurations. It is unclear whether a study of the sub-subprincipal effects would discriminate further between planar configurations, but numerical evidence suggests that the quantum ground state is not distributed evenly on them at large spin.

Other selection effects tend to select the planar configurations: consider for instance the classical Gibbs measure, at a very small temperature. This measure concentrates on the points of the minimal set where the Hessian has a maximal number of zero eigenvalues (thermal selection); in this case it always corresponds to planar configurations, if any.

8.2 Irregularity of the zero set

One of the key examples of frustrated spin systems is the Kagome lattice. We restrict our study to the case of one loop of six triangles.

Proposition 8.1. *For a loop of six triangles (as in Figure 2), the minimal set is not smooth.*

Proof. The choice of the two vectors drawn on the left in Figure 2 induces a global $SO(3)$ rotation, and without loss of generality we will keep them fixed. Moreover, the position of the six inner vectors determines the position of the six outer vectors in a unique and smooth way, so we will forget about the latter.
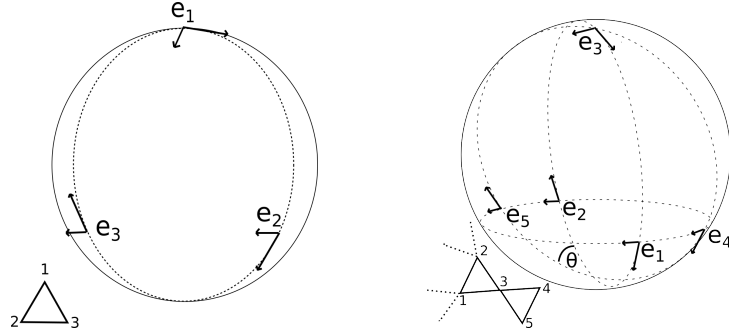
Figure 3: General minimal configuration for one triangle (left) and an triangle leaf (right) of spins, with choice of tangent coordinates.

The space of configurations of the pair (a, a') is a subset of a two-dimensional torus; indeed the choice for a' is made along a circle with center having its center on the lower-left vector, and the choice for a is similarly made along a circle with center a'. The above applies to the pair (b, b'). Hence, the set of global configurations is a subset of a four-dimensional torus: the subset on which the angle between a and b is exactly $\frac{2\pi}{3}$. This cannot be an open set of the four-dimensional torus, as every coordinate and function involved is real analytic. Hence, if this set is a smooth manifold, its dimension does not exceed three.

On the other hand, consider the particular case of Figure 2 which represents a particular configuration. From this configuration, one stays in the minimal set by moving a' along a circle with center a; one can also move along a only, or along b only, or along b' only. The set of possible smooth moves from this configuration spans a set of dimension at least four, hence the contradiction.

□

8.3 Degeneracy for triangle leaves

The simplest example of a frustrated system is a triangle with three vertices, connected with each other. In this setting the degeneracy of the minimal set (which is exactly the set of configurations such that the sum of the three vectors is zero) corresponds to a global $SO(3)$ symmetry of the problem; in this case the function μ is constant.

Consider the left part of Figure 3. The three elements e_1, e_2, e_3 lie on the same large circle. We choose the coordinate q_i along this circle and the coordinate p_i orthogonal to it. In these coordinates, the half-Hessian of the classical symbol can be written as:

$$2(p_1 + p_2 + p_3)^2 + (q_1 - q_2)^2 + (q_1 - q_3)^2 + (q_2 - q_3)^2.$$

Since this quadratic form does not depend on the positions of e_1, e_2, e_3, the function μ is constant.

In the following Proposition we consider a slightly more general situation.

Proposition 8.2. *Consider a graph with a "triangle leaf" as in the inset on the right of Figure 2. In order to find a classical minimum for such a graph, once*

all vectors except for e_4 and e_5 are chosen, then e_4 and e_5 are fixed except for a rotation of centre e_3.

The Melin value μ does not depend on this choice.

Proof. Denoting $c = \cos(\theta)$ and $s = \sin(\theta)$, and using local coordinates as in the right part of Figure 3, the 2-jet of the Hamiltonian reads, in local coordinates:

$$Q(p_1, p_2, q_1, q_2, ...) + 2(p_4 + p_5)^2 + (q_4 - q_5)^2 + q_4^2 + q_5^2 + 4q_3^2 + 4p_3^2$$
$$+ 4p_3(p_1 + p_2) - 2q_3(q_1 + q_2)$$
$$+ 4cp_3(p_4 + p_5) - 4sq_3(p_4 + p_5)$$
$$- 2cq_3(q_4 + q_5) - 2sp_3(q_4 + q_5).$$

The trace of this quadratic form does not depend on θ. Hence, in order to prove that μ does not depend on θ it is sufficient to find symplectic coordinates in which this quadratic form does not depend on θ. A first symplectic change of variables leads to:

$$Q(p_1, p_2, q_1, q_2, ...) + 4p_4^2 + q_4^2 + 3q_5^2 + 4q_3^2 + 4p_3^2$$
$$+ 4p_3(p_1 + p_2) - 2q_3(q_1 + q_2)$$
$$+ 4\sqrt{2}cp_3p_4 - 4\sqrt{2}sq_3p_4 - 2\sqrt{2}cq_3q_4 - 2\sqrt{2}sp_3q_4.$$

Let us make the following change of variables:

$$p_4 \mapsto cp_4 - s\frac{q_4}{2}$$

$$q_4 \mapsto cq_4 + 2sp_4$$

This change of variables is symplectic, and preserves $4p_4^2 + q_4^2$. The quadratic form becomes:

$$Q(p_1, p_2, q_1, q_2...) + 4p_3(p_1 + p_2) - 2q_3(q_1 + q_2)$$
$$+ 4p_4^2 + q_4^2 + 3q_5^2 + 4p_3^2 + 4q_3^2 + 8p_3p_4 - 4q_4q_3.$$

Since this quadratic form does not depend on θ, the function μ does not depend on θ. $\qquad\square$

8.4   A numerical example

The last example we treat is the case of a loop of 4 triangles. In this setting, the minimal set is not a submanifold but a union of three submanifolds, with transverse intersection. The general configuration is shown in Figure 4. The intersections correspond in fact to the case of crossing along a submanifold (see Definition 6.4), since the Hessian matrix at each minimal point can be computed explicitly. From parity properties we can deduce that the function μ reaches a *local* minimum on these crossings, however we cannot conclude that the ground state selects this set.
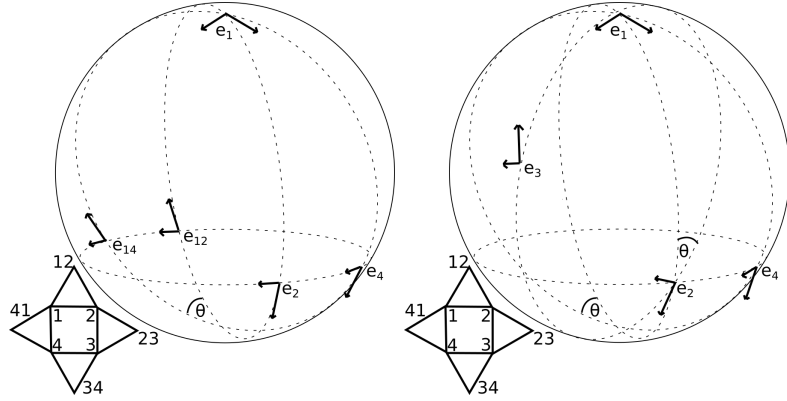
Figure 4: The two general configurations for a loop of 4 triangles. For the first configuration, $e_3 = e_1$ so $e_{12} = e_{23}$ and $e_{14} = e_{34}$. In the second configuration, for simplicity, we omitted to draw e_{12}, e_{23}, e_{34} and e_{14}.
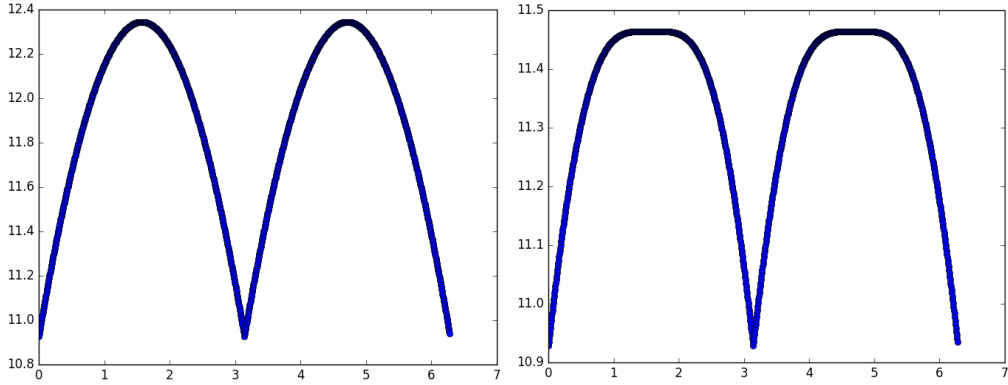

Figure 5: Numerical plot of the value of μ as a function of θ in the two situations corresponding to figure 4. In both cases the minimum is reached on planar configurations, with a minimal value $\mu \simeq 10.928$.

The quadratic form is again explicit if the local coordinates are chosen conveniently, but the computation of μ depends on an exact diagonalisation which we believe to be less explicit than in the previous case.

A numerical plot of μ as a function of θ is presented in Figure 5. Note that μ is not smooth on the crossing point, in accordance with Definition 6.4. We believe that a closed expression of μ is, in this case, rather technical to obtain. Figure 5 is a strong indication that μ is only minimal on flat configurations.

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