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Index of projective elliptic operators

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Abstract

Mathai, Melrose, and Singer introduced the notion of projective elliptic operators on manifolds equipped with an Azumaya bundle. In this note we compute the equivariant index of transversally elliptic operators that are the pullback of projective elliptic operators on the trivialization of the Azumaya bundle. It encompasses the fractional index formula of projective elliptic operator by Mathai-Melrose-Singer.

1 Introduction

In [6, 7, 8, 9], R. Melrose, V. Mathai and I. M. Singer studied the notion of projective elliptic operators and the corresponding index problem. There were interested to the situation where a compact manifold M carries an Azumaya bundle $\mathcal{A} \rightarrow M$ of rank $N \times N$. In this setting we have a PU_N -principal bundle $P_{\mathcal{A}} \rightarrow M$ which corresponds to the trivialization of \mathcal{A} ,

$$\mathcal{A} \simeq P_{\mathcal{A}} \times_{\mathrm{PU}_N} \mathrm{End}(\mathbb{C}^N),$$

and one works with the central extension $1 \rightarrow \Gamma_N \rightarrow \mathrm{SU}_N \rightarrow \mathrm{PU}_N \rightarrow 1$.

An \mathcal{A} -projective elliptic operator on M can be understood as an elliptic operator on the orbifold $P_{\mathcal{A}}/\mathrm{SU}_N \simeq M$. Thanks to the work of Atiyah-Singer [1] and Kawasaki [5], a good way to think to an elliptic operator D on the orbifold $P_{\mathcal{A}}/\mathrm{SU}_N$ is to view it, after pulling back by the projection $P_{\mathcal{A}} \rightarrow M$, as a SU_N -transversally elliptic operators \tilde{D} on $P_{\mathcal{A}}$.

In [7], the authors define the analytical index, $\mathrm{Indice}_a(D)$, of an \mathcal{A} -projective elliptic operator D on M , and they prove that one has an Atiyah-Singer type formula for this invariant which shows in particular that $\mathrm{Indice}_a(D)$ is a rational number.

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In [8], the authors explained the link between the analytical index of D and the equivariant index of the transversally elliptic operator \tilde{D} , denoted $\text{Indices}_{\text{SU}_N}(\tilde{D})$. Recall that $\text{Indices}_{\text{SU}_N}(\tilde{D})$ can be viewed as an invariant distribution on SU_N and Atiyah-Singer theory tell us that $\text{Indices}_{\text{SU}_N}(\tilde{D})$ is supported on the subgroup $\Gamma_N \subset \text{SU}_N$. The main result of [8] is that

$$(1) \quad \text{Indice}_a(D) = \left\langle \text{Indices}_{\text{SU}_N}(\tilde{D}), \varphi \right\rangle_{\text{SU}_N}$$

when φ is a smooth function on SU_N which is constantly equal to 1 around the identity element and has small enough support.

The above formula gives “the coefficient of the Dirac function” in the distribution $\text{Indices}_{\text{SU}_N}(\tilde{D})$. The main purpose of the present note is to compute the “higher order derivatives of the Dirac function” in the distribution $\text{Indices}_{\text{SU}_N}(\tilde{D})$ (see Theorem 4.1). Our computation generalizes the previous result obtained by Yamashita [11] in the case of the projective Dirac operator.

2 Index of transversally elliptic operators

We consider the following setting :

1. a G -principal bundle $\pi : P \rightarrow M$, where G is a compact Lie group,
2. a central extension $1 \rightarrow \Gamma \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ where Γ is finite.

We are interested to the \tilde{G} -equivariant pseudo-differential operators on P that are transversally elliptic [1]. We are here in a very particular situation: the subgroup Γ acts trivially on P and the group G acts freely on P , so the set $\mathbf{T}_{\tilde{G}}^*P = \mathbf{T}_G^*P$ formed by the covectors orthogonal to the orbits is a sub-bundle of \mathbf{T}^*P , and the quotient by G defines a principal bundle $\mathbf{T}_G^*P \rightarrow \mathbf{T}^*M$.

The index of a \tilde{G} -transversally elliptic operator on P , says \tilde{D} , depends uniquely of the class defined by its principal symbol $\sigma(\tilde{D})$ in the group $\mathbf{K}_{\tilde{G}}^0(\mathbf{T}_G^*P)$ (see [1]). Hence the analytical index defines a morphism

$$\text{Indice}_{\tilde{G}} : \mathbf{K}_{\tilde{G}}^0(\mathbf{T}_G^*P) \longrightarrow \mathcal{C}^{-\infty}(\tilde{G})^{\text{Ad}},$$

and our aim is to compute it by cohomological formulas. Our main tool will be delocalized formulas obtained by Berline-Paradan-Vergne in [3, 4, 10].

A general property of the equivariant index states that for any class $\sigma \in \mathbf{K}_{\tilde{G}}^0(\mathbf{T}_G^*P)$ the distribution $\text{Indice}_{\tilde{G}}(\sigma)$ is supported on elements $\tilde{g} \in \tilde{G}$

such that $P^{\mathfrak{g}} \neq \emptyset$. In our case, such elements belong to the finite subgroup Γ , so we have a decomposition

$$\text{Indice}_{\tilde{G}}(\sigma) = \sum_{\gamma \in \Gamma} \mathcal{Q}_{\gamma}(\sigma)$$

where $\mathcal{Q}_{\gamma}(\sigma)$ is an Ad-invariant distribution on \tilde{G} supported at the central element γ . Recall that the envelopping algebra $U(\mathfrak{g}) = U(\tilde{\mathfrak{g}})$ is canonically identified with the algebra of distributions on \tilde{G} supported at the identity. So the center $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ corresponds to the Ad-invariant distributions on \tilde{G} supported at the identity.

Let $\delta_{\gamma} \in \mathcal{C}^{-\infty}(\tilde{G})^{\text{Ad}}$ be the Dirac distribution supported at the central element $\gamma \in \Gamma$. The convolution map

$$T \in Z(\mathfrak{g}) \longmapsto T \star \delta_{\gamma}$$

defines a bijection between $Z(\mathfrak{g})$ and the Ad-invariant distributions on \tilde{G} supported at γ . As the invariant distribution $\mathcal{Q}_{\gamma}(\sigma)$ is supported at γ , there exists an element $T_{\gamma}(\sigma) \in Z(\mathfrak{g})$ such that $\mathcal{Q}_{\gamma}(\sigma) = T_{\gamma}(\sigma) \star \delta_{\gamma}$.

The exponential map $\exp : \mathfrak{g} \rightarrow G$ defines a linear isomorphism¹

$$\exp_* : S(\mathfrak{g})^G \rightarrow Z(\mathfrak{g})$$

where $S(\mathfrak{g})^G$ is viewed as the algebra of Ad-invariant distributions on \mathfrak{g} supported at 0. Our main purpose is to give a cohomological formula for the element $\exp_*^{-1}(T_{\gamma}(\sigma)) \in S(\mathfrak{g})^G$.

3 Chern-Weil morphism and twisted Chern characters

Let E_1, \dots, E_r be a basis of \mathfrak{g} , and let $\theta = \sum_{i=1}^r \theta_i \otimes E_i \in (\mathcal{A}^1(P) \otimes \mathfrak{g})^G$ be a connection on the principal bundle $P \rightarrow M$. Its curvature $\Theta = \sum_{i=1}^r \Theta_i \otimes E_i$ belongs to $(\mathcal{A}_{hor}^+(P) \otimes \mathfrak{g})^G$, where $\mathcal{A}_{hor}^+(P)$ is the algebra of horizontal forms of even degree on P . A polynomial function $P(X) = P(X_1, \dots, X_r)$ on \mathfrak{g} can be evaluated at Θ : the corresponding element $P(\Theta) = P(\Theta_1, \dots, \Theta_r)$ is a closed differential form of $\mathcal{A}_{bas}^+(P) \simeq \mathcal{A}^+(M)$ when P is G -invariant. We can define $\varphi(\Theta)$ for any smooth function $\varphi : \mathfrak{g} \rightarrow \mathbb{C}$, by using its Taylor series at 0. If we denote $H_{dR}(M)$ the de Rham cohomology of M , the

¹It is not a morphism of algebras !

map $\varphi \in \mathcal{C}^\infty(\mathfrak{g})^{\text{Ad}} \mapsto \varphi(\Theta) \in H_{dR}(M)$ is the Chern-Weil morphism (it is independent of the choice of the connection).

Another way to look at the Chern-Weil morphism is to consider the exponential $e^\Theta \in (\mathcal{A}_{hor}^+(P) \otimes S(\mathfrak{g}))^G$. If we view the polynomial algebra $S(\mathfrak{g})$ as the algebra of distributions on \mathfrak{g} supported at 0, we see that

$$\varphi(\Theta) := \langle e^\Theta, \varphi \rangle_{\mathfrak{g}}.$$

Note that $\varphi(\Theta) = 1$ if φ is equal to 1 in a neighborhood of 0.

Definition 3.1. For any closed form α on \mathbf{T}^*M with compact support, the expression $\int_{\mathbf{T}^*M} \alpha \wedge e^\Theta$ defines an element of $S(\mathfrak{g})^G$ through the relation

$$\left\langle \int_{\mathbf{T}^*M} \alpha \wedge e^\Theta, \varphi \right\rangle_{\mathfrak{g}} := \int_{\mathbf{T}^*M} \alpha \wedge \bar{\varphi}(\Theta),$$

that holds for any $\varphi \in \mathcal{C}^\infty(\mathfrak{g})$. Here $\bar{\varphi} = \int_G g \cdot \varphi dg$ is the average of φ relatively to the Haar measure on G of volume 1, and the forms $\bar{\varphi}(\Theta) \in \mathcal{A}^+(M)$ are also considered as forms on \mathbf{T}^*M . Note that $\int_{\mathbf{T}^*M} \alpha \wedge e^\Theta \in S(\mathfrak{g})^G$ is independent of the choice of the connection.

Now we explain how are constructed the twisted Chern characters $\text{Ch}_\gamma(\sigma)$ of a \tilde{G} -transversally elliptic symbol on P .

Let $\sigma \in \mathcal{C}^\infty(\mathbf{T}^*P, \text{hom}(p^*\mathcal{E}^+, p^*\mathcal{E}^-))$ be a \tilde{G} -transversally elliptic symbol: here \mathcal{E}^\pm are \tilde{G} -complex vector bundles on P and $p : \mathbf{T}^*P \rightarrow P$ is the projection. By definition the set

$$\mathcal{K}_\sigma := \{(x, \xi) \in \mathbf{T}_G^*P \mid \sigma(x, \xi) : \mathcal{E}_x^- \rightarrow \mathcal{E}_x^+ \text{ is not invertible}\}$$

is compact.

We start with a \tilde{G} -equivariant connection $\nabla^{\mathcal{E}^+}$ on the vector bundle $\mathcal{E}^+ \rightarrow P$. The pull-back $\nabla^{p^*\mathcal{E}^+} := p^*\nabla^{\mathcal{E}^+}$ is then a connection on $p^*\mathcal{E}^+$ viewed as a vector bundle on the manifold \mathbf{T}_G^*P . Since \mathcal{K}_σ is compact, we can define on the vector bundle $p^*\mathcal{E}^- \rightarrow \mathbf{T}_G^*P$ a connection $\nabla^{p^*\mathcal{E}^-}$ such that the following relation

$$(2) \quad \nabla^{p^*\mathcal{E}^-} = \sigma \circ \nabla^{p^*\mathcal{E}^+} \circ \sigma^{-1}$$

holds outside a compact subset of \mathbf{T}_G^*P .

Let $R^+(X), R^-(X), X \in \mathfrak{g}$ be the equivariant curvatures of the connections $\nabla^{p^*\mathcal{E}^+}$ and $\nabla^{p^*\mathcal{E}^-}$ (see [2], section 7). We consider the equivariant Chern character, twisted by the central element $\gamma \in \Gamma$:

$$\text{Ch}_\gamma^{\tilde{G}}(\sigma)(X) := \text{Tr} \left(\gamma^{\mathcal{E}^+} e^{R^+(X)} \right) - \text{Tr} \left(\gamma^{\mathcal{E}^-} e^{R^-(X)} \right), \quad X \in \mathfrak{g}.$$

In this formula, $\gamma^{\mathcal{E}^\pm}$ denotes the linear action of γ on the fibers of the bundles \mathcal{E}^\pm . Thanks to (2), the closed equivariant form $\text{Ch}_\gamma^{\tilde{G}}(\sigma)$ has a compact support on \mathbf{T}_G^*P . It defines a class, still denoted $\text{Ch}_\gamma^{\tilde{G}}(\sigma)$, in the equivariant cohomology group with compact support $H_{\tilde{G},c}^\infty(\mathbf{T}_G^*P)$. Since the finite subgroup Γ acts trivially on P , we have a canonical isomorphism between $H_{\tilde{G},c}^\infty(\mathbf{T}_G^*P)$ and $H_{G,c}^\infty(\mathbf{T}_G^*P)$.

Definition 3.2. Let $H_{dR,c}(\mathbf{T}^*M)$ be the de Rham cohomology of \mathbf{T}^*M with compact support. The twisted Chern characters $\text{Ch}_\gamma(\sigma) \in H_{dR,c}(\mathbf{T}^*M)$ is defined as the image of $\text{Ch}_\gamma^{\tilde{G}}(\sigma)$ under the Chern-Weil isomorphism

$$H_{G,c}^\infty(\mathbf{T}_G^*P) \rightarrow H_{dR,c}(\mathbf{T}^*M)$$

that is associated to the principal G -bundle $\mathbf{T}_G^*P \rightarrow \mathbf{T}^*M$.

4 Main result

Let R_M is the curvature of the tangent bundle $\mathbf{T}M \rightarrow M$. The $\widehat{\text{A}}$ -class of M is normalized as follows

$$\widehat{\text{A}}(M) = \det^{1/2} \left(\frac{R_M}{e^{\frac{R_M}{2}} - e^{-\frac{R_M}{2}}} \right) \in H_{dR}(M).$$

The following Theorem is the main result of this note.

Theorem 4.1. Let $\sigma \in \mathbf{K}_{\tilde{G}}^0(\mathbf{T}_G^*P)$. We have $\text{Indice}_{\tilde{G}}(\sigma) = \sum_{\gamma \in \Gamma} T_\gamma(\sigma) \star \delta_\gamma$ where

$$T_\gamma(\sigma) = (2i\pi)^{-\dim M} \exp_* \left(\int_{\mathbf{T}^*M} \widehat{\text{A}}(M)^2 \wedge \text{Ch}_\gamma(\sigma) \wedge e^\Theta \right).$$

Here $\text{Ch}_\gamma(\sigma)$ is the twisted Chern character (see Definition 3.2).

Corollary 4.2. We have

$$\langle \text{Indice}_{\tilde{G}}(\sigma), \varphi \rangle_{\tilde{G}} = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\text{A}}(M)^2 \wedge \text{Ch}_\gamma(\sigma)$$

if $\varphi \in \mathcal{C}^\infty(\tilde{G})$ is a function constantly equal to 1 around γ and has small enough support.

The proof of Theorem 4.1 follows from a direct application of the delocalized formulas obtained by Berline-Paradan-Vergne in [3, 4, 10]. Here we just explain how one can obtain easily this result when $\tilde{G} = G$. In this setting the map $\pi : P \rightarrow M$ defines an isomorphism $\sigma \mapsto \pi^*\sigma, \mathbf{K}^0(\mathbf{T}^*M) \rightarrow \mathbf{K}_G^0(\mathbf{T}_G^*P)$ at the level of \mathbf{K} -groups, and the distributions $\text{Indice}_G(\pi^*\sigma)$ are supported only at the identity element.

For any irreducible representation V_λ (parametrized by $\lambda \in \hat{G}$) we denote χ_λ its character and \mathcal{V}_λ the complex vector bundles $P \times_G V_\lambda \rightarrow M$. Theorem 3.1 of Atiyah in [1] tells us that

$$\langle \text{Indice}_G(\pi^*\sigma), \chi_\lambda \rangle_G = \text{Indice}(\sigma \otimes \mathcal{V}_\lambda) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \hat{A}(M)^2 \wedge \text{Ch}(\sigma) \wedge \text{Ch}(\mathcal{V}_\lambda).$$

The Chern character of the vector bundle \mathcal{V}_λ is defined by the Chern-Weil morphism: $\text{Ch}(\mathcal{V}_\lambda) = \langle e^\Theta, \exp^*(\chi_\lambda) \rangle_{\mathfrak{g}}$. Then

$$\langle \text{Indice}_G(\pi^*\sigma), \chi_\lambda \rangle_G = (2i\pi)^{-\dim M} \left\langle \int_{\mathbf{T}^*M} \hat{A}(M)^2 \wedge \text{Ch}(\sigma) \wedge e^\Theta, \exp^*(\chi_\lambda) \right\rangle_{\mathfrak{g}}$$

for any $\lambda \in \hat{G}$. We can conclude that

$$\text{Indice}_G(\pi^*\sigma) = (2i\pi)^{-\dim M} \exp_* \left(\int_{\mathbf{T}^*M} \hat{A}(M)^2 \wedge \text{Ch}(\sigma) \wedge e^\Theta \right) \star \delta_e.$$

If we consider the group $\hat{\Gamma}$ of characters of the finite abelian group Γ , we can decompose a \tilde{G} -transversally elliptic symbol $\sigma \in \mathcal{C}^\infty(\mathbf{T}^*P, \text{hom}(p^*\mathcal{E}^+, p^*\mathcal{E}^-))$ as $\sigma = \bigoplus_{\chi \in \hat{\Gamma}} \sigma_\chi$, where $\sigma_\chi \in \mathcal{C}^\infty(\mathbf{T}^*P, \text{hom}(p^*\mathcal{E}_\chi^+, p^*\mathcal{E}_\chi^-))$ is a \tilde{G} -transversally elliptic symbol on P . Here we take \mathcal{E}_χ^\pm as the subbundle of \mathcal{E}^\pm where Γ acts through the character χ . From Definition 3.2, it is obvious that the twisted Chern character $\text{Ch}_\gamma(\sigma)$ admits the decomposition

$$\text{Ch}_\gamma(\sigma) = \sum_{\chi \in \hat{\Gamma}} \langle \chi, \gamma \rangle \text{Ch}_e(\sigma_\chi),$$

where $\langle -, - \rangle : \hat{\Gamma} \times \Gamma \rightarrow \mathbb{C}$ is the duality bracket.

We have a reformulation of Theorem 4.1.

Theorem 4.3. *Let $\sigma \in \mathbf{K}_{\tilde{G}}^0(\mathbf{T}_G^*P)$ with decomposition $\sigma = \bigoplus_{\chi \in \hat{\Gamma}} \sigma_\chi$. We have*

$$\text{Indice}_{\tilde{G}}(\sigma) = \sum_{(\chi, \gamma) \in \hat{\Gamma} \times \Gamma} \langle \chi, \gamma \rangle T_e(\sigma_\chi) \star \delta_\gamma$$

where $T_e(\sigma_\chi) = (2i\pi)^{-\dim M} \exp_* \left(\int_{\mathbf{T}^*M} \hat{A}(M)^2 \wedge \text{Ch}_e(\sigma_\chi) \wedge e^\Theta \right)$.

5 Projective elliptic operators

Let us come back to the setting of an \mathcal{A} -projective elliptic operator D on M . Let \tilde{D} be its pullback on $P_{\mathcal{A}}$: it is a SU_N -transversally elliptic operator, and we denote $\sigma(\tilde{D}) \in \mathbf{K}_{\mathrm{SU}_N}^0(\mathbf{T}_{\mathrm{PU}_N}^* P_{\mathcal{A}})$ its principal symbol. Here the action of the center $\Gamma_N \subset \mathrm{SU}_N$ on $\sigma(\tilde{D})$ is prescribed by the canonical character $\chi_o : \Gamma_N \rightarrow \mathbb{C}$. In other words, $\sigma(\tilde{D}) = \sigma(\tilde{D})_{\chi_o}$.

In this case Theorem 4.3 gives

Proposition 5.1. *Let D be an \mathcal{A} -projective elliptic operator on M . Let \tilde{D} be its pullback on $P_{\mathcal{A}}$. We have the following relation in $\mathcal{C}^{-\infty}(\mathrm{SU}_N)^{\mathrm{Ad}}$,*

$$\mathrm{Indices}_{\mathrm{SU}_N}(\tilde{D}) = T_D \star \left(\sum_{z \in \Gamma_N} \langle \chi_o, z \rangle \delta_z \right),$$

where $T_D \in Z(\mathfrak{su}_N)$ is defined by the relation

$$T_D = (2i\pi)^{-\dim M} \exp_* \left(\int_{\mathbf{T}^* M} \hat{\mathbf{A}}(M)^2 \wedge \mathrm{Ch}_e(\sigma(\tilde{D})) \wedge e^{\Theta} \right).$$

Here Θ is the curvature of the PU_N -principal bundle $P_{\mathcal{A}} \rightarrow M$.

We finish this note by considering the particular case of the projective Dirac operator. Let M be a compact Riemannian manifold of dimension $2n$ that is supposed oriented: here we work with the exemple of Azumaya bundle defined by the Clifford bundle $\mathcal{A} := \mathrm{Cl}(\mathbf{T}M)_{\mathbb{C}}$. In this context R. Melrose, V. Mathai and I. M. Singer showed that we have a natural projective Dirac operator \not{D}_M^{pr} on M [7].

In [11], Yamashita considers the pullback, denoted \not{D}_M^{su} , of the projective Dirac operator \not{D}_M^{pr} relatively to the PU_N -principal bundle $P_{\mathcal{A}} \rightarrow M$ (here $N = 2^n$). The operator \not{D}_M^{su} is SU_N -transversally elliptic, and Yamashita computes the distribution on SU_N defined by its equivariant index. The expression he obtains is similar to the one of Proposition 5.1, modulo a small mistake : the terms $\langle \chi_o, z \rangle$ are missing in his formula (see Corollary 6 in [11]).

Instead of working with the PU_N principal bundle $P_{\mathcal{A}} \rightarrow M$, we can work with the SO_{2n} -principal bundle $P_{\mathrm{SO}} \rightarrow M$ of oriented orthonormal frames, since it defines also a trivialization of $\mathrm{Cl}(\mathbf{T}M)_{\mathbb{C}}$. The pull-back of \not{D}_M^{pr} relatively to the projection $\pi : P_{\mathrm{SO}} \rightarrow M$ is a Spin_{2n} -transversally elliptic operator on P_{SO} , that we denote $\not{D}_M^{\mathrm{Spin}}$.

We recall the definition of its principal symbol $\sigma(\not\partial_M^{\text{Spin}})$. The kernel of $\mathbf{T}\pi : \mathbf{T}P \rightarrow \mathbf{T}M$ is the trivial sub-bundle isomorphic to $\mathfrak{so}_{2n} \times P$. Let $\mathbf{T}_{\text{SO}}^*P \subset \mathbf{T}^*P$ be the orthogonal of $\mathfrak{so}_{2n} \times P$. The bundle $\mathbf{T}_{\text{SO}}^*P$ admits an SO_{2n} -equivariant trivialisation $\alpha : \mathbb{R}^{2n} \times P \rightarrow \mathbf{T}_{\text{SO}}^*P$ defined as follows. For $f \in P$, the map $f : \mathbb{R}^{2n} \rightarrow \mathbf{T}_{\pi(f)}M$ is orthogonal and $(\mathbf{T}\pi|_f)^* : \mathbf{T}_{\pi(f)}^*M \rightarrow \mathbf{T}_{\text{SO}}^*P|_f$ is an isomorphism. Hence the map

$$\alpha_f := \mathbf{T}\pi^*|_f \circ (f^*)^{-1} : \mathbb{R}^{2n} \simeq (\mathbb{R}^{2n})^* \longrightarrow \mathbf{T}_{\text{SO}}^*P|_f$$

is a linear isomorphism. We see then that $P \times \mathbb{R}^{2n} \rightarrow \mathbf{T}_{\text{SO}}^*P, (f, x) \mapsto \alpha_f(x)$ is an SO_{2n} -equivariant trivialization.

Let $S_{2n} = S_{2n}^+ \oplus S_{2n}^-$ be the spinor representation. We denote $\text{cl} : \mathbb{R}^{2n} \rightarrow \text{End}(S_{2n})$ the clifford action (which is Spin_{2n} -equivariant). The symbol $\sigma(\not\partial_M^{\text{Spin}}) : \mathbf{T}_{\text{SO}}^*P \rightarrow \text{hom}(S_{2n}^+, S_{2n}^-)$ is defined by the relations

$$\sigma(\not\partial_M^{\text{Spin}})|_{(f,v)} = \text{cl}(\tilde{v}) : S_{2n}^+ \rightarrow S_{2n}^-, \quad v \in \mathbf{T}_{\text{SO}}^*P|_f,$$

where $\tilde{v} = \alpha_f^{-1}(v)$.

Proposition 5.2. *We have the following equality in $\mathcal{C}^{-\infty}(\text{Spin}_{2n})^{\text{Ad}}$:*

$$\text{Indices}_{\text{Spin}_{2n}}(\not\partial_M^{\text{Spin}}) = T_M \star \delta_1 - T_M \star \delta_{-1}$$

where $T_M \in Z(\mathfrak{so}_{2n})$ is defined by the relation

$$(3) \quad T_M = (2i\pi)^{-n} \exp_* \left(\int_M \widehat{\mathbf{A}}(M) \wedge e^\Theta \right).$$

Here Θ is the curvature of the SO_{2n} -principal bundle $P_{\text{SO}} \rightarrow M$.

Proposition 5.2 follows from Theorem 4.3 as the Chern class $\text{Ch}(\not\partial_M^{\text{Spin}})$ satisfies the classical relation $\text{Ch}_e(\sigma(\not\partial_M^{\text{Spin}})) = (2i\pi)^n \widehat{\mathbf{A}}(M)^{-1} \wedge \text{Thom}(\mathbf{T}^*M)$.

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