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| Léo Agélas. A new path to the non blow-up of incompressible flows. 2018. hal-01380349v4

HAL Id: hal-01380349

<https://hal.science/hal-01380349v4>

Preprint submitted on 22 Jan 2019

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A new path to the non blow-up of incompressible flows

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Abstract

One of the most challenging questions in fluid dynamics is whether the three-dimensional (3D) incompressible Navier-Stokes, 3D Euler and two-dimensional Quasi-Geostrophic (2D QG) equations can develop a finite-time singularity from smooth initial data. Recently, from a numerical point of view, Luo & Hou presented a class of potentially singular solutions to the Euler equations in a fluid with solid boundary Luo and Hou (2014a,b). Furthermore, in two recent papers Tao (2016a,b), Tao indicates a significant barrier to establishing global regularity for the 3D Euler and Navier-Stokes equations, in that any method for achieving this, must use the finer geometric structure of these equations. In this paper, we show that the singularity discovered by Luo & Hou which lies right on the boundary is not relevant in the case of the whole domain $\mathbb{R}^3$. We reveal also that the translation and rotation invariance present in the Euler, Navier-Stokes and 2D QG equations are the key for the non blow-up in finite time of the solutions. The translation and rotation invariance of these equations combined with the anisotropic structure of regions of high vorticity allowed to establish a new geometric non blow-up criterion which yield us to the non blow-up of the solutions in all the Kerr's numerical experiments and to show that the potential mechanism of blow-up introduced in Brenner et al. (2016) cannot lead to the blow-up in finite time of solutions of Euler equations.

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1. Introduction

The Navier-Stokes and Euler equations describe the motion of a fluid in the three-dimensional space. These fundamental equations were derived over 250 years ago by Euler and since then have played a major role in fluid dynamics. They have enriched many branches of mathematics, were involved in many areas outside mathematical activity from weather prediction to exploding supernova (see for instance the surveys [Constantin \(2007\)](#), [Bardos and Titi \(2007\)](#)) and present important open physical and mathematical problems (see [Constantin \(2007\)](#)). Regarding the 2D Quasi-Geostrophic (2D QG) equation, it appears in atmospheric studies. It describes the evolution of potential temperature u on the two dimensional boundary of a rapidly rotating half space with small Rossby and Ekman numbers, for the case of special solutions with constant potential vorticity in the interior and constant buoyancy frequency (normalized to one), where equations in the bulk are compressible Euler or Navier-Stokes equations coupled with temperature equation, continuity equation, and equation of state.

In the case of Navier-Stokes equations, for a long time ago, a global weak solution $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3))^3$ and $\nabla u \in L^2(\mathbb{R}^3 \times (0, \infty))^3$ was built by Leray [Leray \(1934\)](#). In particular, Leray introduced a notion of weak solutions for the Navier-Stokes equations, and proved that, for every given $u_0 \in L^2(\mathbb{R}^3)^3$, there exists a global weak solution $u \in L^\infty([0, +\infty[; L^2(\mathbb{R}^3))^3 \cap L^2([0, \infty[; \dot{H}^1(\mathbb{R}^3))^3$. Hopf has proved the existence of a global weak solution in the general case $\mathbb{R}^d$, $d \geq 2$, [Hopf \(1951\)](#). Meanwhile the regularity and the uniqueness of this weak solution has been known for a long time ago for the two-dimensional case (see [Ladyzhenskaya \(1969\)](#), [Lions and Prodi \(1959\)](#), [Lions \(1969\)](#), [Temam \(1977\)](#)), in the three-dimensional case the problem remains widely open in spite of great efforts made. On the uniqueness many works have been done (see [Furioli et al. \(2000\)](#), [Giga \(1983\)](#), [Monniaux \(2000\)](#), [Lions \(1960\)](#), [Gallagher and Planchon \(2002\)](#)). Concerning the regularity of weak solutions, in [Serrin \(1962\)](#), it is proved that if u is a Leray-Hopf weak solution belonging to $L^q([0, T]; L^q(\mathbb{R}^3))^3$ with $\frac{2}{q} + \frac{3}{q} \leq 1$, $2 < p < \infty$, $3 < q < \infty$, then the solution $u \in C^\infty(\mathbb{R}^3 \times]0, T])^3$. In [Von Wahl \(1986\)](#) and [Giga \(1986\)](#), it is showed that if u is a weak solution in $C([0, T]; L^3(\mathbb{R}^3))^3$, then $u \in C^\infty(\mathbb{R}^3 \times]0, T])^3$. The limit case of $L^\infty([0, T]; L^3(\mathbb{R}^3))^3$ has been solved in [Iskauriaza et al. \(2003\)](#). Other criterion regularity can also be found in [He \(2002\)](#); [Heywood \(1988\)](#); [Giga \(1983\)](#); [Kato \(1984, 1990\)](#); [Beirão da Veiga \(1995\)](#); [Chae and Choe \(1999\)](#); [Zhou \(2002\)](#); [Constantin and Fefferman \(1994\)](#).

38 In the case of Euler Equations, in the two dimension case, uniqueness and ex-
39 istence of classical solutions have been known for a long time ago (see Yudovich
40 (1963, 1995); Vishik (1999); DiPerna and Majda (1987); Ladyzhenskaya (1969)).
41 However for the full three space dimensions, little is known about smooth solu-
42 tions apart from classical short-time existence and uniqueness. Moreover, weak
43 solutions are known to be badly behaved from the point of view of Hadamard's
44 well-posedness theory (see for instance the surveys De Lellis and Székelyhidi
45 (2012); Villani (2008-2009)). Considerable efforts have been devoted to the
46 study of the regularity properties of the 3D Euler equations. The main diffi-
47 culty in the analysis lies in the presence of the nonlinear vortex stretching term
48 and the lack of a regularization mechanism. Despite these difficulties, a few
49 important partial results concerning the regularity of 3D Euler equations have
50 been obtained over the years (see Beale et al. (1984); Ponce (1985); Ferrari
51 (1993); Shirota and Yanagisawa (1993); Constantin et al. (1996); Deng et al.
52 (2005); Gibbon and Titi (2013)).

53
54 In the case of 2D QG equation, besides its direct physical significance Held et al.
55 (1995); Pedlosky (1987), the 2D QG equation has very interesting features of re-
56 semblance to the 3D Euler equation, being also an outstanding open problem of
57 the finite time blow-up issue. In particular, one can derive a necessary and suffi-
58 cient blow-up condition for the 2D QG equation similar to the well-known Beale-
59 Kato-Majda (BKM) criterion (Beale-Kato-Majda Beale et al. (1984)). More
60 precisely, the solution to the 2D QG equation (11) becomes singular at time T^*
61 if and only if $\int_0^{T^*} \|\nabla^\perp u(t)\|_{L^\infty} dt = +\infty$ (see Constantin et al. (1994)). Thus,
62 $\nabla^\perp u$ plays a role similar to the vorticity ω in the 3D Euler equations. In the
63 recent years, the 2D QG equation has been the focus of intense mathemati-
64 cal research Constantin et al. (1994); Córdoba (1998); Córdoba and Fefferman
65 (2002); Chae (2003); Ohkitani and Yamada (1997); Constantin et al. (2012);
66 Chae et al. (2012).

67
68 Unfortunately despite of considerable efforts devoted to the regularity is-
69 sue of the 3D Euler, 3D Navier-Stokes and 2D QG equations, standard scaling
70 heuristics have long indicated to the experts that the identity energy, together
71 with the harmonic analysis estimates available for the heat equation and for the
72 Euler bilinear operator, are not sufficient by themselves if one wishes to improve
73 the theory on the Cauchy problem for these equations. It seems crucial to use
74 the specific structure of the nonlinear term in these equations, as well as the
75 divergence free assumption. Indeed, some finite time blowup results have been
76 established for various Navier-Stokes type equations (see Montgomery-Smith
77 (2001); Gallagher and Paicu (2009); Li and Sinai (2008); Plecháč and Sverák
78 (2003); Katz and Pavlovic (2005)). Nevertheless, for all of these Navier-Stokes
79 type equations, the cancellation property of the Euler bilinear operator did not
80 hold and for some, the energy identity did not hold (see Montgomery-Smith
81 (2001); Gallagher and Paicu (2009); Li and Sinai (2008)).

82 However, recently it was shown also in [Tao \(2016a\)](#), a finite time blow up so-
 83 lution to an averaged three-dimensional Navier-Stokes equations of type $\partial_t u =$
 84 $\Delta u + \tilde{B}(u, u)$, where $\tilde{B}$ is an averaged version of the Euler bilinear operator
 85 B , acting also on divergence free vector fields u and obeying as B to the can-
 86 cellation property $\langle \tilde{B}(u, u), u \rangle = 0$. This result suggests that any successful
 87 method to affirmatively answer to the Existence and Smoothness problem must
 88 either use finer structure of B or else must rely crucially on some estimate
 89 or other property of the Euler bilinear operator B that is not shared by the
 90 averaged operator $\tilde{B}$. Such additional structure exists for instance, the Euler
 91 equation has a vorticity formulation involving only differential operators rather
 92 than pseudo-differential ones.

93 However, even this vorticity formulation is not a barrier to get a finite time
 94 blow up solution. Indeed, it was shown in [Tao \(2016b\)](#), finite time blow-up solu-
 95 tions in the *class* of generalised Euler equations sharing with the Euler equation
 96 its main features such as: *vorticity formulation, energy conservation, Kelvin*
 97 *circulation theorem, vorticity-vector potential formulation viewed as the Gen-*
 98 *eralised Biot-Savart, function space estimates for the vector potential operator.*
 99 Then, it seems that there is no room left to establish global regularity of solu-
 100 tions of 3D Euler equations. However, as it is mentioned in [Tao \(2016b\)](#), there
 101 are two properties of the Euler equations which are not obeyed by the gener-
 102 alised Euler equations, namely translation invariance and rotation invariance.
 103 Further, these symmetries basically determined the usual Biot-Savart law (see
 104 [Kambe \(2003a,b\)](#)) which are thus not shared by the Generalised Euler equations
 105 introduced in [Tao \(2016b\)](#). Furthermore, as it was shown in [Constantin \(1994\)](#);
 106 [Constantin and Fefferman \(1994\)](#), the use of Biot-Savart law yield to rewrite
 107 the vorticity equation in the case of Euler ($\nu = 0$) and Navier-Stokes ($\nu > 0$)
 108 equations as follows:

$$\partial_t \omega + (\mathbf{v} \cdot \nabla) \omega - \nu \Delta \omega = \alpha \omega, \quad (1)$$

109 where

$$\alpha(x, t) = \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} (\hat{y} \cdot \xi(x, t)) \det(\hat{y}, \xi(x + y, t), \xi(x, t)) |\omega(x + y, t)| \frac{dy}{|y|^3}, \quad (2)$$

110 with $\hat{y} = \frac{y}{|y|}$, $\xi = \frac{\omega}{|\omega|}$ and $\det(a, b, c)$ is the determinant of the matrix with
 111 columns a, b, c in that order. We thus notice from the expression of α that if the
 112 direction of the vorticity, ξ varies mildly within a small region around x , then
 113 the singularity of the integrand in (2) will be mild.

114 In this paper, we bring new insights which shed light on the mechanisms in-
 115 volved in the non blow-up of the solutions. We highlight through new geo-
 116 metric non blow-up criteria how the geometric regularity of the direction of
 117 vorticity combined with the anisotropic structure of the localized regions con-
 118 taining the positions where the maximum of the magnitude of the vorticity
 119 are reached, should prevent the formation of singularities. The novelty in
 120 the results of this paper lies on the use of the these two features in obtain-

ing geometric non blow-up criteria using the finer structure of the Euler bi-
 linear operator B . Up to now, many progress had been made to better take
 into account the geometrical properties and flow structures in the non blow-up
 criteria (see e.g. Constantin and Fefferman (1994); Beirão da Veiga and Berselli
 (2009); Berselli (2009); Constantin (1994); Constantin et al. (1996); Hou and Li
 (2008); Deng et al. (2005, 2006a,b)). However none of these non blow-up cri-
 teria integrated both the geometric regularity of the direction of vorticity and
 the anisotropic structure of localized regions containing the positions where the
 maximum of the magnitude of the vorticity are reached. The most advanced
 non blow-up criteria were given in Deng et al. (2005, 2006a,b) and were estab-
 lished by using the Lagrangian formulation of the vorticity equation of the 3D
 Euler and 2D QG equations.
 However the results obtained in Tao (2016b) suggest that even the most ad-
 vanced non blow-up criterion Deng et al. (2005, 2006a) do not capture the finest
 structures of the the Euler bilinear operator B since it was shown in Tao (2016b)
 that there exist generalised Euler equations sharing the same property than Euler
 equations as the Lagrangian formulation for their vorticity equations and for
 which their solutions blow up in finite time. Indeed, from Deng et al. (2005,
 2006a), one can observe that the Deng-Hou-Yu non-blowup criterion can be ap-
 plied to all the *class* of generalised Euler equations introduced in Tao (2016b).
 Then, in order to bring new insights in the investigation of whether the 3D
 incompressible Navier-Stokes, Euler and 2D QG equations can develop a finite-
 time singularity from smooth initial data, it was crucial to establish new non
 blow-up criteria which take into account the special structure of these equations
 not shared by the Generalised Euler equations.
 Then in our Theorem 7.1, under mild assumptions based on the anisotropic
 structure of regions of high vorticity, we show that the solutions of 3D Euler,
 3D Navier-Stokes and 2D QG equations cannot blow up at a finite time T^* if

$$\int_0^{T^*} \mathbf{A}_d(t) \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right) dt < \infty,$$

where the functions $\mathbf{A}_d$ and Ω satisfy:

$$\begin{aligned} \mathbf{A}_d(t) &\leq \|\nabla \xi(t)\|_\infty \\ \Omega(t) &= \frac{(T^* - t)^{-1}}{1 + \log^+((T^* - t)\|u(t)\|_\infty \mathbf{A}_0(t))} \\ \mathbf{A}_0(t) &\leq \|\nabla \xi(t)\|_\infty. \end{aligned}$$

Note that $\xi(t)$ is well defined only on $\mathcal{O}(t)$ the set of points x of $\mathbb{R}^d$ where
 $\omega(x, t) \neq 0$ and then $\|\nabla \xi(t)\|_\infty$ must be understood as $\|\nabla \xi(t)\|_{L^\infty(\mathcal{O}(t))}$.
 In the case of 3D Euler equations and 2D QG equations by using their La-
 grangian formulation, in Theorem 7.2 we go further in the non blow-up criteria
 by showing under mild assumptions based on the anisotropic structure of re-
 gions of high vorticity, that their solutions do not blow up at a finite time T^*

156 if

$$\int_0^{T^*} \mathbf{A}_d(t) dt < \infty.$$

157 These results are obtained after a fine analysis of the term α defined by (2) com-
 158 bined with some results based on the anisotropic structure of regions of high
 159 vorticity. Our analysis starts by considering at each time $t \in]0, T^*[$ the regions
 160 containing the positions where the maximum of the magnitude of the vorticity
 161 are reached and shrinking to zero as time tends to T^* the alleged time of singular-
 162 ity. More precisely, these regions are balls of radius $\rho_0(t) = O((T^* - t)\|u(t)\|_\infty)$
 163 and of center the position of a point where the maximum of the magnitude of
 164 the vorticity is reached. Inside these regions, we then consider the regions of
 165 high vorticity for which the magnitude of the vorticity is greater than some
 166 function $\Omega(t)$ such that $\Omega(t) \gtrsim \frac{(T^* - t)^{-1}}{1 + \log^+(\rho_0(t)\|\nabla \xi(t)\|_\infty)}$.

167 In our analysis, to track in time the positions where the maximum of the mag-
 168 nitude of the vorticity is reached, we had to overcome the obstruction that
 169 we do not know if there exists an isolated absolute maximum for the vorticity
 170 achieved along a smooth curve in time as it was assumed in Proposition 2.1 of
 171 Constantin et al. (1994) and also in Deng et al. (2005, 2006a,b) (which assume
 172 that the position where the maximum of vorticity is reached, is advected with
 173 the flow). Moreover, recent numerical experiments show that it is not always the
 174 case (see Kerr and Bustamante (2012), see also section 5.4.5 in Grafke (2012)).
 175 We thus overcome this difficulty by using a result of Pshenichnyi concerning
 176 directional derivatives of the function of maximum and the structure of a set of
 177 supporting functionals Pshenichny (1971).

178 Our analysis led first to the non blow-up criterion given by our Theorem 5.1,
 179 namely, the solutions of 3D Euler, 3D Navier-Stokes and 2D QG equations
 180 cannot blow up at a finite time T^* if

$$\int_0^{T^*} \mathbf{A}_d(t)\pi(t) dt < \infty, \quad (3)$$

181 where the function π is given by:

$$\pi(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{0 < R \leq \rho_0(t)} \frac{1}{R^{d-1}} \int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| dz, \quad (4)$$

182 with $\Theta(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x,t)| = \|\omega(t)\|_\infty\}$ and $\mathcal{V}(t) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d; |\omega(z,t)| \geq$
 183 $\Omega(t)\}$.

184 In our Lemma 6.1, we thus derive a straightforward estimate of the function
 185 $\pi(t)$, that is

$$\pi(t) \leq 3\|\omega(t)\|_\infty \sup_{x \in \Theta(t)} |\mathcal{V}(t) \cap B(x, \rho_0(t))|^{\frac{1}{d}}. \quad (5)$$

186 Thanks to the non blow-up criterion (3) and (5), we show the non blow-up
 187 in finite time of the solutions of Euler equations for Kerr's numerical experi-
 188 ments Kerr (2005, 1998, 1997) without additional numerical tests as it was the

case in [Deng et al. \(2005, 2006a\)](#), just by using the anisotropic structure of regions of high vorticity whose the features are described in [Kerr \(2005, 1998, 1997\)](#). Moreover, we show that the potential mechanism of blow-up introduced in [Brenner et al. \(2016\)](#) cannot lead to blow-up in finite time for Euler equations. To go further in our estimate of the function π , we use some assumptions characterizing the anisotropic structure of regions of high vorticity whose the justifications are given at the begininig of subsection 7.2 and we show in Proposition 7.1 that

$$\pi(t) \lesssim 1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right), \quad (6)$$

which yield to Theorem 7.1.

In the case of Euler equations and 2D QG equations by using their Lagrangian formulation, after a fine and sharp analysis of the expression of the function π (4) led thanks to our Lemmata 7.2, 7.3 and 7.4, in Proposition 7.2 we go further in the non blow-up criteria by showing under mild assumptions based on the anisotropic structure of regions of high vorticity, that

$$\pi(t) = O(1). \quad (7)$$

We emphasize that according the 'thickness' of the structure of regions of high vorticity that these two estimates (6) and (7) can be much better. Indeed from the analysis led in [Kuznetsov and Ruban \(2000\)](#); [Kuznetsov et al. \(2001\)](#) for the study of collapse of vortex lines and agrees with numerical experiments [Agafontsev et al. \(2015, 2017\)](#), we could expect that (see Remark 7.1)

$$\pi(t) \lesssim \Omega(t)^{-\frac{1}{2}},$$

and then obtain in this case, the non blow-up in finite time of the solutions of Euler equations if

$$\int_0^{T^*} \mathbf{A}_d(t) \Omega(t)^{-\frac{1}{2}} dt < \infty. \quad (8)$$

We point out also that our geometric non blow-up criterion reveals the role of the geometric structures of the Incompressible flows in the non blow-up in finite time of the solutions and presents the advantage to be established in an Eulerian setting in comparison with all the recent geometric non blow-up criteria [Constantin et al. \(1996\)](#); [Deng et al. \(2005, 2006a\)](#); [Constantin et al. \(1994\)](#) using the Lagrangian formulation of Incompressible Inviscid Flows, which requires much more computational effort as it is mentioned in [Grafke and Grauer \(2013\)](#) and in section 5.4.5 of [Grafke \(2012\)](#). Furthermore, due to the existence of hyperbolic-saddle singularities suggested by the generation of strong fronts in geophysical/meteorology observations (see [Constantin et al. \(1994\)](#); [Córdoba \(1998\)](#)), and antiparallel vortex line pairing observed in numerical simulations and physical experiments, it was important to take them into account in our geometric non blow-up criterion. This is performed thanks to the term $\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))$ (see (27), (28)) involved in the definition of the function $\mathbf{A}_d$ given at (39).

Then, the paper is organized as follows:

- 226 • In section 2 , we give some notations and definitions.
- 227 • In section 3, we recall some results about the local regularity of solutions
- 228 of Navier-Stokes, Euler and 2D QG equations.
- 229 • In section 4, we give the reason for which we can assume for any time t
- 230 that $\|\omega(t)\|_\infty > 0$ without loss of generality.
- 231 • In section 5, in Theorem 5.1, we establish a new geometric criterion for
- 232 the non blow-up in finite time of the solutions of 3D Navier-Stokes, 3D
- 233 Euler and 2D QG equations. We show that their solutions cannot blow up
- 234 at a finite time T^* if $\int_0^{T^*} \mathbf{A}_d(t)\pi(t) dt < \infty$, where $\mathbf{A}_d(t)$ is based on the
- 235 regularity of the direction of the vorticity ξ in regions shrinking to zero as
- 236 time tends to T^* and containing the positions where the maximum of the
- 237 magnitude of the vorticity is reached (see definition of $\mathbf{A}_d$ at (39)).
- 238 • In section 6, we show the non-blowup in finite time of the solutions of
- 239 the Euler equations in the numerical experiments considered these last
- 240 years, by using inequality (5) about the function π (4) and the anisotropic
- 241 structure of regions of high vorticity described in Kerr (2005, 1998, 1997).
- 242 • In section 7, we show the estimates (5), (6) and (7) concerning the function
- 243 π defined by (4), and obtain new non blow-up criteria in Theorems 7.1
- 244 and 7.2.

245 Let us now introduce the 3D Navier-Stokes and Euler equations given by,

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \\ \nabla \cdot u = 0, \end{cases} \quad (9)$$

246 in which $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}$ and

247 $\nu \geq 0$ ($\nu = 0$ corresponds to the Euler equations) denote respectively the

248 unknown velocity field, the scalar pressure function of the fluid at the point

249 $(x, t) \in \mathbb{R}^3 \times [0, \infty[$ and the viscosity of the fluid,

250

251 with initial conditions,

$$u(x, 0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3, \quad (10)$$

252 where the initial data u_0 is a divergence free vector field on $\mathbb{R}^3$.

253

254 Regarding the 2D QG equation in $\mathbb{R}^2$, it is given by

$$\begin{cases} \frac{\partial u}{\partial t} + v \cdot \nabla u = 0, \\ v = \nabla^\perp (-\Delta)^{-\frac{1}{2}} u, \end{cases} \quad (11)$$

255 with initial data,

$$u(x, 0) = u_0. \quad (12)$$

Here $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. For v we have also the following representation

$$v = R^\perp u, \quad (13)$$

where we have used the notation, $R^\perp u = (-R_2 u, R_1 u)$ with $R_j, j = 1, 2$, for the 2D Riesz transform defined by (see e.g. [Stein \(1970\)](#))

$$R_j(u)(x, t) = \frac{1}{2\pi} P.V \int_{\mathbb{R}^2} \frac{(x_j - y_j)}{|x - y|^3} u(y, t) dy.$$

2. Some notations and definitions

In this section, we assume that $d \in \mathbb{N}$, $d \geq 2$.

For any vector $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we denote by $|x|$ the euclidean norm of

x given by $|x| = \sqrt{\sum_{i=1}^d |x_i|^2}$. For any $y \in \mathbb{R}^d$, $y \neq 0$, we denote by $\hat{y}$ the unit

vector $\hat{y} = \frac{y}{|y|}$. For any m -dimensional subset A of $\mathbb{R}^d$, $1 \leq m \leq d$, we denote

by $|A|$ its measure. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of real square matrices of size

d . We denote by Id the identity matrix of $\mathcal{M}(\mathbb{R}^d)$. For any vector field v defined

from $\mathbb{R}^d$ to $\mathbb{R}^d$, we denote by ∇v the gradient matrix of v , the matrix of $\mathcal{M}(\mathbb{R}^d)$

with ij -component, $\frac{\partial v_i}{\partial x_j}$ for all $1 \leq i, j \leq d$. For any real a , we denote by a^+

the real defined by $a^+ \stackrel{\text{def}}{=} \max(a, 0)$. For any function φ defined on $\mathbb{R}^d \times [0, +\infty[$,

for all $t \geq 0$, we denote by $\varphi(t)$ the function defined on $\mathbb{R}^d$ by $x \mapsto \varphi(x, t)$. We

denote by $C_c^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact

support in $\mathbb{R}^d$. We denote by BC the class of bounded and continuous functions

and by BC^m the class of bounded and m times continuously derivable functions.

For any $R > 0$ and $x_0 \in \mathbb{R}^d$, we denote by $B(x_0, R)$, the ball of $\mathbb{R}^d$ of center x_0

and radius R . For any $R > 0$, we denote by B_R , the ball of $\mathbb{R}^d$ of center 0 and

radius R .

We denote by div the differential operator given by, $\text{div} = \sum_{i=1}^d \frac{\partial}{\partial x_i}$.

We denote $A \lesssim B$, $B \gtrsim A$ or $A = O(B)$ the estimate $A \leq cB$ where $c > 0$ is an

absolute constant. If we need c to depend on a parameter, we shall indicate this

by subscripts, thus for instance $A \lesssim_s B$ denotes the estimate $A \leq c_s B$ for some c_s

depending on s . We use $A \sim B$ as shorthand for $A \lesssim B \lesssim A$.

For any $f \in L^p(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)^d$ or $L^p(\mathbb{R}^d)^{d \times d}$) with $1 \leq p \leq +\infty$, we denote by $\|f\|_p$ and $\|f\|_{L^p}$, the L^p -norm of f .

We denote by $H^s(\mathbb{R}^d)$ the Sobolev space $J^{-s}L^2(\mathbb{R}^d)$ where $J = (1 - \Delta)^{\frac{1}{2}}$. We

denote by $H_\sigma^s(\mathbb{R}^3)$ the Sobolev space $H_\sigma^s(\mathbb{R}^3) \stackrel{\text{def}}{=} \{\psi \in H^s(\mathbb{R}^3)^3 : \text{div} \psi = 0\}$. In

order to unify our notations with the two dimensional case 2D QG, we denote

by $H_\sigma^s(\mathbb{R}^2)$ the Sobolev space $H^s(\mathbb{R}^2)$.

We denote by $\mathbb{P}$ the well-known 3D matrix Leray's projection operator with components,

$$\mathbb{P}_{i,j} = \delta_{i,j} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Delta^{-1} = \delta_{i,j} - R_j R_k, \quad (14)$$

where R_j are the Riesz transform given by $R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}} = \frac{1}{4\pi} \frac{x_j}{|x|^4} \star$ (see Stein (1970) for more details), Δ^{-1} is the inverse of Laplace operator given by $\Delta^{-1} = -\frac{1}{4\pi|x|} \star$, with $\star$ the convolution operator.

3. Local regularity of the solutions

In this section, we deal with the main result on local regularity of 3D Navier-Stokes and Euler equations in its general form. By introducing $\mathbb{P}$ the matrix Leray operator, Euler equations (9)-(10) can be re-written as follows,

$$\frac{\partial u}{\partial t} + \mathbb{P}(u \cdot \nabla)u = 0, \quad (15)$$

with initial conditions,

$$u(0) = u_0. \quad (16)$$

For u solution of (15)-(16), $\omega = \nabla \times u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu \Delta \omega = 0, \quad (17)$$

with initial conditions,

$$\omega(0) = \omega_0,$$

where $\omega_0 = \nabla \times u_0$ is the vorticity of u_0 .

In the case of 2D QG equation, we get for u solution of (11), $\omega = \nabla^\perp u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega - (\omega \cdot \nabla)v = 0, \quad (18)$$

with initial conditions,

$$\omega(0) = \omega_0,$$

where $\omega_0 = \nabla^\perp u_0$ is the vorticity of u_0 .

In the region where $|\omega| > 0$, we define ξ the direction of the vorticity by $\xi = \frac{\omega}{|\omega|}$.

3.1. Local regularity for 3D Navier-Stokes or 3D Euler equations

Assuming $u_0 \in H_\sigma^r(\mathbb{R}^3)$ with $r > \frac{5}{2}$, thanks to Theorem 3.5 in [Kato and Ponce \(1988\)](#), Theorem 1 in [Bourguignon and Brezis \(1974\)](#) (see also Theorem I in [Kato and Ponce \(1986\)](#) and the results obtained in [Beale et al. \(1984\)](#)), we deduce that there exists a time $T > 0$ such that there exists a unique strong solution $u \in C([0, T[, H_\sigma^r(\mathbb{R}^3)) \cap C^1([0, T[, H_\sigma^{r-2}(\mathbb{R}^3))$ to the Navier-Stokes or Euler equations (15)-(16) and the energy equality holds for u , that means for all $t \in [0, T[$,

$$\|u(t)\|_2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2 ds = \|u_0\|_2. \quad (19)$$

Moreover, if $u \notin C([0, T], H_\sigma^r(\mathbb{R}^3))$, then we get (see [Beale et al. \(1984\)](#); [Kato and Ponce \(1988\)](#); [Kozono and Taniuchi \(2000\)](#)),

$$\int_0^T \|\omega(t)\|_\infty dt = +\infty. \quad (20)$$

Notice thanks to Remark 3.7 in [Kato and Ponce \(1988\)](#), in the case of Euler equations, we get in addition that $u \in C^1([0, T[, H_\sigma^{r-1}(\mathbb{R}^3))$. We retrieve the pressure p from the velocity u with the formula,

$$p = -\Delta^{-1} \operatorname{div}((u \cdot \nabla)u).$$

Furthermore, we get the local estimate (21). Indeed, thanks to remark 4.4 in [Kato and Ponce \(1988\)](#), we get

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ with } t_0 < t < T, \quad (21)$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

3.2. Local regularity for 2D QG equation

This subsection is devoted to the local well-posedness of the 2D QG equation with a characterization of the maximal time existence of strong solutions. By using the same arguments as the proof of Proposition 4.2 in [Agélas \(2016\)](#), we get that the H^s -norm of u is controlled by the integral in time of the maximum magnitude of the vorticity of u . A such Proposition has been proved in [Constantin et al. \(1994\)](#) for any integer $s \geq 3$, but here we extend this result to all real $s > 2$. This improvement is obtained by using the logarithmic Sobolev inequality proved in [Kozono and Taniuchi \(2000\)](#); [Kato and Ponce \(1988\)](#) which requires only that $s > 2$ instead of using the one proved in [Beale et al. \(1984\)](#) as it is the case in [Constantin et al. \(1994\)](#) and which requires integer $s \geq 3$. Then by using the same arguments as the proof of Proposition 4.3 in [Agélas \(2016\)](#), we get the following result which gives an improvement in comparison with Theorem 2.1 in [Constantin et al. \(1994\)](#):

Assuming $u_0 \in H^r(\mathbb{R}^2)$ with $r > 2$, we get that there exists a time $T > 0$ such that there exists a unique strong solution $u \in C([0, T[, H^r(\mathbb{R}^2))$ to the

2D QG equation (11)-(12) and the energy equality holds for u , that means for all $p \in [2, \infty]$ and $t \in [0, T[$,

$$\|u(t)\|_p = \|u_0\|_p. \quad (22)$$

Moreover, if $u \notin C([0, T], H^r(\mathbb{R}^2))$, then

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \quad (23)$$

Owing to $u \in C([0, T[, H^r(\mathbb{R}^2))$ and thanks to Lemma X4 in Kato and Ponce (1988), from 2D QG (11), we get $u \in C^1([0, T[, H^{r-1}(\mathbb{R}^2))$.

Similarly as in (21), we have

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ for } t_0 < t < T, \quad (24)$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

4. Assumption on the maximum vorticity

Let $d \in \{2, 3\}$, $r > \frac{d}{2} + 1$ and $u_0 \in H_\sigma^r(\mathbb{R}^d)$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler or 2D QG equations (9)-(10) in the class

$$u \in C([0, T^*[, H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*[, H^{r-2}(\mathbb{R}^d)).$$

Thanks to the results of the section 3, a such time T^* exists.

In this paper, we are concerned with the non blowup in finite time of the solutions u at times such T^* . Then, without loss of generality, in the whole of this paper, we consider only times of existence T^* such that for all $t \in [0, T^*[$,

$$\|\omega(t)\|_\infty > 0. \quad (25)$$

Indeed, let us assume that there exists $t_0 \in [0, T^*[$ such that $\|\omega(t_0)\|_\infty = 0$.

In the case of 2D QG equations (11), we get that $\omega(t_0) \equiv 0$ and then $\nabla u(t_0) \equiv 0$. Since $x \mapsto u(t_0, x)$ vanishes at infinity, then we get $u(t_0) \equiv 0$. Then by using inequality (24) concerning the local regularity, we deduce that $u(t) \equiv 0$ for all $t \in [t_0, T^*[$ and no blowup can occur at the time T^* .

By following step by step the proof of Lemma 4 given in Deng et al. (2005) but keeping the term $\|u(t)\|_{L^2(\mathbb{R}^d)}$ after using the Cauchy-Schwarz inequality, we obtain for all $t \in [0, T^*[$,

$$\begin{aligned} \|u(t)\|_\infty &\lesssim \|u(t)\|_2^{\frac{2}{d+2}} \|\omega(t)\|_\infty^{\frac{d}{d+2}} \\ &\leq \|u_0\|_2^{\frac{2}{d+2}} \|\omega(t)\|_\infty^{\frac{d}{d+2}}, \end{aligned} \quad (26)$$

where we have used (19) for the last inequality. Then thanks to (26) used with $d = 3$, we obtain that $\|u(t_0)\|_\infty \equiv 0$ which implies that $u(t_0) \equiv 0$. Then by using the inequality (21) of local regularity, we deduce $u(t) \equiv 0$ for all $t \in [t_0, T^*[$ and thus no blowup can occur at the time T^* .

5. Geometric properties for non blow-up of the solutions

Historically, non blow-up criteria for the incompressible Euler equations and 2D QG equations commonly focus on global features of the flow, such as norms of the velocity or the vorticity fields. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy for overcoming such shortcomings was established by focusing more on geometrical properties and flow structures (see e.g. [Constantin et al. \(1996\)](#); [Cordoba and Fefferman \(2001\)](#)), such as vortex tubes or vortex lines.

In particular, in [Constantin et al. \(1996, 1994\)](#) the authors showed that local geometric regularity of the unit vorticity vector can lead to depletion of the vortex stretching. They prove that if there is up to time T an $O(1)$ region in which the vorticity vector is smoothly directed, i.e., the maximum norm of $\nabla \xi$ (here $\xi = \frac{\omega}{|\omega|}$, ω the vorticity) in this region is L^2 integrable in time from 0 to T , and the maximum norm of velocity in some $O(1)$ neighbourhood of this region is uniformly bounded in time, then no blow-up can occur in this region up to time T .

However, this theorem dealt with $O(1)$ regions in which the vorticity vector is assumed to have some regularity, while in numerical computations, the regions that have such regularity and contain maximum vorticity are all shrinking with time (see [Kerr \(1993, 1995, 1997, 1998\)](#); [Pelz \(1997, 2001\)](#)).

Inspired by the work of [Constantin et al. \(1996, 1994\)](#), in [Deng et al. \(2005, 2006a,b\)](#) the authors showed that geometric regularity of Lagrangian vortex filaments, even in an extremely localized region containing the maximum of vorticity which may shrink with time, can lead to depletion of the nonlinear vortex stretching, thus avoiding finite time singularity formation of the 3D Euler equations and 2D QG equations.

However, all the recent geometric constraints for non blow-up criteria of Euler and 2D QG equations based on local geometric regularity of Lagrangian vortex filaments [Deng et al. \(2005, 2006a,b\)](#) make the assumption that the position where the maximum of vorticity is reached, is advected with the flow, however it is not always the case, as described in [Kerr and Bustamante \(2012\)](#) (see also section 5.4.5 of [Grafke \(2012\)](#)).

Then in our Theorem 5.1, we establish in an Eulerian setting a new geometric non blow-up criterion for the Navier-Stokes, Euler and 2D QG equations based on the regularity of the direction of the vorticity in extremely localized regions containing the positions where the maximum of the magnitude of the vorticity are reached and shrinking to zero as time increase to some T^* the alleged time of singularity. Our Eulerian geometric non blow-up criterion should give also new impetus to the numerical experiments due to their ease of implementation in comparison with Lagrangian geometric non blow-up criteria (see [Grafke and Grauer \(2013\)](#), see also section 5.4.5 of [Grafke \(2012\)](#)). Moreover our geometric non blow-up criterion is also valid for the Navier-Stokes equations that is not the case for the existing geometric non blow-up criteria obtained in [Constantin et al. \(1996\)](#); [Deng et al. \(2005, 2006a,b\)](#) based on a Lagrangian for-

409 mulation of Incompressible Inviscid Flows.

410 To obtain our Theorem 5.1, we begin with Lemma 5.1.

411 **Lemma 5.1.** *Let $d \in \mathbb{N}^*$, $T > 0$ and $f \in C([0, T]; BC(\mathbb{R}^d))$ such that*
 412 *$\inf_{t \in [0, T]} \|f(t)\|_\infty > 0$ and for any $t \in [0, T]$, $|f(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$. Then*
 413 *there exists $R > 0$ such that for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x, t)|$.*

414 *Proof.* We set $a = \inf_{t \in [0, T]} \|f(t)\|_\infty > 0$. Since $t \mapsto f(t)$ is a continuous function
 415 from the compact $[0, T]$ into the metric space $L^\infty(\mathbb{R}^d)$ then it is uniformly
 416 continuous. Hence, there exists $N \in \mathbb{N}^*$ such that for all $t, t' \in [0, T]$, $|t - t'| \leq \frac{T}{N}$
 417 we have $\|f(t) - f(t')\|_\infty \leq \frac{a}{4}$. We introduce the subdivision $\{t_i\}_{i \in \llbracket 0, N \rrbracket}$ of
 418 $[0, T]$ defined by $t_i = i \frac{T}{N}$ for $i \in \llbracket 0, N \rrbracket$. Since for any $t \in [0, T]$, $|f(x, t)| \rightarrow 0$
 419 as $|x| \rightarrow +\infty$, then for each $i \in \llbracket 0, N \rrbracket$, there exists $R_i > 0$ such that for all
 420 $|x| \geq R_i$, $|f(x, t_i)| \leq \frac{a}{4}$. We set $R = \max_{i \in \llbracket 0, N \rrbracket} R_i$. Let $t \in [0, T]$ then there exists
 421 $j \in \llbracket 0, N \rrbracket$ such that $|t - t_j| \leq \frac{T}{N}$ and hence for all $|x| \geq R \geq R_j$, we have
 422 $|f(x, t)| \leq |f(x, t) - f(x, t_j)| + |f(x, t_j)| \leq \frac{a}{2} \leq \frac{\|f(t)\|_\infty}{2}$. Then, we infer that
 423 for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x, t)|$, which concludes the proof. $\square$

424 Before to prove Theorem 5.1, we need to introduce the following function
 425 $\mathbf{D}_d$ defined from $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}$ with $d \in \{2, 3\}$ as follows: for $d = 3$,

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3) \text{Det}(a_1, a_2, a_3).$$

426 The Det in $\mathbf{D}_d$ is the determinant of the matrix whose columns are the three
 427 unit column vectors a_1, a_2, a_3 . We observe that $\text{Det}(a_1, a_2, a_3) = a_1 \cdot (a_2 \times a_3)$,
 428 then, we get

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3) a_1 \cdot (a_2 \times a_3). \quad (27)$$

429 and for $d = 2$,

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3^\perp) (a_2 \cdot a_3^\perp), \quad (28)$$

430 where for any $z = (z_1, z_2) \in \mathbb{R}^2$, $z^\perp = (-z_2, z_1)$. We can notice that for
 431 $d \in \{2, 3\}$ the function $\mathbf{D}_d$ is linear from its second variable.

432 From (27) and (28) we get $\mathbf{D}_d(a_1, a_3, a_3) = 0$ then we deduce that for any
 433 $a_1, a_2, a_3 \in B(0, 1)$,

$$|\mathbf{D}_d(a_1, a_2, a_3)| \leq |a_2 - a_3|, \quad (29)$$

434 and we get also

$$|\mathbf{D}_d(a_1, a_2, a_3)| \leq 1. \quad (30)$$

435 Now, we turn to the proof of our Theorem.

436 **Theorem 5.1.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be*
 437 *such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D*
 438 *Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)). \quad (31)$$

439 Let ρ_0 be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by

$$\rho_0(t) \stackrel{\text{def}}{=} 36(T^* - t)\|u(t)\|_\infty. \quad (32)$$

440 Let $\mathbf{A}_0$ be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by:

$$\mathbf{A}_0(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho_0(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|}, \quad (33)$$

441 where for any $t \in [0, T^*[$

$$\Theta(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x, t)| = \|\omega(t)\|_\infty\}. \quad (34)$$

442 Let Ω be the function defined from $[0, T^*[$ to $]0, +\infty[$ by:

$$\Omega(t) \stackrel{\text{def}}{=} \frac{(T^* - t)^{-1}}{8(1 + \log^+(4\rho_0(t)\mathbf{A}_0(t)))}. \quad (35)$$

443 We introduce also the set of high vorticity regions defined for all $t \in [0, T^*[$ by

$$\mathcal{V}(t) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d; |\omega(z, t)| \geq \Omega(t)\}. \quad (36)$$

444 Let π be the function defined from $[0, T^*[$ to $]0, +\infty[$, for all $t \in [0, T^*[$ by

$$\pi(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{0 < R \leq \rho_0(t)} \frac{1}{R^{d-1}} \int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz. \quad (37)$$

445 Let ρ be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by

$$\rho(t) \stackrel{\text{def}}{=} 4(d+1)c_d(T^* - t)\pi(t), \quad (38)$$

446 where $c_d = \frac{3}{4\pi}$ if $d = 3$, $c_d = \frac{1}{2\pi}$ else.

447 Let $\mathbf{A}_d$ be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by

$$\mathbf{A}_d(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|}. \quad (39)$$

448 Then if there exists $t_1 \in [0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t)\pi(t) < +\infty, \quad (40)$$

449 then the solution u cannot blowup at the finite time T^* .

450 Moreover, we have for all $t \in [0, T^*[$ and $x \in \Theta(t)$,

$$\nabla|\omega|(x, t) = 0 \text{ and } \nabla \cdot \xi(x, t) = 0.$$

451 *Proof.* Let $0 < T < T^*$. We want first to apply Lemma 5.1 to the function ω ,
 452 then we check that the hypotheses of the Lemma are satisfied.
 453 Since $u \in C([0, T]; H^r(\mathbb{R}^d)) \cap C^1([0, T]; H^{r-2}(\mathbb{R}^d))$, then we infer that $\omega \in$
 454 $C([0, T]; H^{r-1}(\mathbb{R}^d)) \cap C^1([0, T]; H^{r-3}(\mathbb{R}^d))$. Thanks to the Sobolev embedding
 455 $H^s(\mathbb{R}^d) \hookrightarrow BC^m(\mathbb{R}^d)$ for $s > \frac{d}{2} + m$, $m \in \mathbb{N}$ and since $r > \frac{d}{2} + 3$ we deduce
 456 that $\omega \in C([0, T]; BC^2(\mathbb{R}^d)) \cap C^1([0, T]; BC(\mathbb{R}^d))$. Thanks to (25), we get
 457 that $\inf_{t \in [0, T]} \|\omega(t)\|_\infty > 0$. Moreover, since $\omega \in C([0, T]; H^{r-1}(\mathbb{R}^d))$ with $r >$
 458 $\frac{d}{2} + 3$, we have for any $t \in [0, T]$, $|\omega(x, t)| \rightarrow 0$ as $|x| \rightarrow +\infty$, the proof follows
 459 immediately by using the density of $C_0^\infty(\mathbb{R}^d)$ in $H^{r-1}(\mathbb{R}^d)$ and the Sobolev
 embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $r > \frac{d}{2} + 3$.
 460 Then thanks to Lemma 5.1, there exists $R > 0$ such that for all $t \in [0, T]$,
 461 $\|\omega(t)\|_\infty = \sup_{x \in B_R} |\omega(x, t)|$. Then for all $t \in [0, T]$, the set $\Theta(t)$ defined by (34)
 462 can be rewritten as follows:
 463

$$\Theta(t) = \{x \in B_R; |\omega(x, t)| = \|\omega(t)\|_\infty\}. \quad (41)$$

464 We introduce the direction of the vorticity $\xi = \frac{\omega}{|\omega|}$ defined on the non empty
 465 open set $\mathcal{O} \stackrel{\text{def}}{=} \{(x, t) \in \mathbb{R}^d \times [0, T]; |\omega(x, t)| > 0\}$.
 466 We set $\mathbf{v} = u$ in the case of 3D Navier-Stokes or 3D Euler equations and
 467 $\mathbf{v} = R^\perp u$ with $\nu = 0$ in the case of 2D QG equation.
 468 Then by multiplying (17) or (18) by ξ , we get that for all $(x, t) \in \mathcal{O}$,

$$\begin{aligned}
 \frac{\partial |\omega|}{\partial t}(x, t) + \mathbf{v}(x, t) \cdot \nabla |\omega|(x, t) &= -(\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) \\
 &\quad - \nu \Delta |\omega|(x, t) + \nu |\omega(x, t)| |\nabla \xi(x, t)|^2 = 0.
 \end{aligned} \quad (42)$$

469 We introduce the function φ defined for all $t \in [0, T]$ by

$$\varphi(t) \stackrel{\text{def}}{=} \sup_{x \in B_R} |\omega(x, t)|$$

470 and we search the expression of its derivative. For this, we use the main Theorem
 471 obtained in Pshenichny (1971) or Theorem 1 in Borisenko and Minchenko (1992)
 472 after verifying that the hypotheses of the Theorem are satisfied.
 473 Since $\omega \in C([0, T]; BC^2(\mathbb{R}^d)) \cap C^1([0, T]; BC(\mathbb{R}^d))$, then we deduce that $|\omega| \in$
 474 $BC(\mathcal{O})$, $\frac{\partial |\omega|}{\partial t} \in BC(\mathcal{O})$ and $\nabla^2 |\omega| \in BC(\mathcal{O})$. Since for any $t \in [0, T]$, $\Theta(t) \subset$
 475 $\mathcal{O} \times \{t\}$, then, thanks to the results obtained in Pshenichny (1971) (see also
 476 Theorem 1 in Borisenko and Minchenko (1992)), by using also (41) we obtain
 477 the expression of the derivative of φ given for any $t \in [0, T]$ by,

$$\varphi'(t) = \sup_{x \in \Theta(t)} \frac{\partial |\omega|}{\partial t}(x, t). \quad (43)$$

Further for all $x \in \Theta(t) \subset B_R$, we have $|\omega(x, t)| = \varphi(t) = \|\omega(t)\|_\infty$, we thus infer that

$$\nabla|\omega|(x, t) = 0 \text{ and } \Delta|\omega|(x, t) \leq 0. \quad (44)$$

Therefore, we have for all $x \in \Theta(t)$,

$$\begin{aligned} \frac{\partial|\omega|}{\partial t}(x, t) &= \frac{\partial|\omega|}{\partial t}(x, t) + \mathbf{v}(x, t) \cdot \nabla|\omega|(x, t) \\ &= (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t) + \nu\Delta|\omega|(x, t) - \nu|\omega(x, t)| |\nabla\xi(x, t)|^2 \\ &\leq (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t), \end{aligned} \quad (45)$$

where we have used (42) for the second equality and (44) for the last inequality. We can notice that we get equality for (45) in the case of 3D Euler or 2D QG equations, since for these equations we have not the terms $\nu\Delta|\omega|(x, t)$ and $\nu|\omega(x, t)| |\nabla\xi(x, t)|^2$.

Then using (45), from (43), we obtain,

$$\varphi'(t) \leq \sup_{x \in \Theta(t)} (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t),$$

which means that

$$\frac{d}{dt} \|\omega(t)\|_\infty \leq \sup_{x \in \Theta(t)} (\omega(x, t) \cdot \nabla)\mathbf{v}(x, t) \cdot \xi(x, t), \quad (46)$$

where equality holds in the case of 3D Euler or 2D QG equations. We use now the function α introduced in Constantin (1994); Constantin and Fefferman (1994) for the 3D Navier-Stokes or 3D Euler equations and in Constantin et al. (1994) for the 2D QG equation, defined for all $(x, t) \in \mathcal{O}$ by,

$$\alpha(x, t) = c_d P.V. \int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t)) |\omega(x+y, t)| \frac{dy}{|y|^d}, \quad (47)$$

where $\hat{y} = \frac{y}{|y|}$ and in the case of 3D Navier-Stokes or 3D Euler equations for which $d = 3$, $c_d = \frac{3}{4\pi}$ and in the case of 2D QG equation for which $d = 2$, $c_d = \frac{1}{2\pi}$. We use the fact that $|\omega(x+y, t)|\xi(x+y, t) = \omega(x+y, t)$ and the fact that $\mathbf{D}_d$ is linear in comparison with its second variable, to rewrite (47) as follows:

$$\alpha(x, t) = c_d P.V. \int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \omega(x+y, t), \xi(x, t)) \frac{dy}{|y|^d}. \quad (48)$$

By using the Biot-Savart law (see Chorin and Marsden (1993)) for which in the case of Euler and Navier-Stokes equations, we have

$$\mathbf{v}(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y) dy,$$

498 and in the case of 2D QG equations, we get an equivalent formula

$$\mathbf{v}(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|y|} \omega(x + y, t) dy,$$

499 we deduce as in [Constantin \(1994\)](#); [Constantin and Fefferman \(1994\)](#) and [Constantin et al. \(1994\)](#) that for all $(x, t) \in \mathcal{O}$

$$(\omega(x, t) \cdot \nabla) \mathbf{v}(x, t) \cdot \xi(x, t) = \alpha(x, t) |\omega(x, t)|.$$

501 Therefore, from (46), we deduce that for all $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt} \|\omega(t)\|_\infty &\leq \sup_{x \in \Theta(t)} \alpha(x, t) |\omega(x, t)| \\ &= \left(\sup_{x \in \Theta(t)} (\alpha(x, t)) \right) \|\omega(t)\|_\infty, \end{aligned} \quad (49)$$

502 where we have used the fact that for all $x \in \Theta(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty$. Let us
503 estimate now $\alpha(x, t)$ for any $t \in [0, T]$ and $x \in \Theta(t)$. For this purpose, let us
504 take $t \in [0, T]$ and $x \in \Theta(t)$, then we decompose the term $\alpha(x, t)$ as the sum of
505 three terms,

$$\alpha(x, t) = I_1 + I_2 + I_3 \quad (50)$$

506 where,

$$I_1 = c_d \int_{B(0, \min(\rho(t), \rho_0(t)))} \mathbf{D}_d(\hat{y}, \omega(x + y, t), \xi(x, t)) \frac{dy}{|y|^d}, \quad (51)$$

507

$$I_2 = c_d \int_{B(0, \rho_0(t)) \cap B(0, \min(\rho(t), \rho_0(t)))^c} \mathbf{D}_d(\hat{y}, \omega(x + y, t), \xi(x, t)) \frac{dy}{|y|^d} \quad (52)$$

508 and

$$I_3 = c_d \int_{B(0, \rho_0(t))^c} \mathbf{D}_d(\hat{y}, \omega(x + y, t), \xi(x, t)) \frac{dy}{|y|^d}. \quad (53)$$

509 Then, we estimate the three terms I_1, I_2 and I_3 . For the term I_1 , from (51) we
510 get

$$\begin{aligned} I_1 &= c_d \int_{B(0, \min(\rho(t), \rho_0(t)))} \frac{\mathbf{D}_d(\hat{y}, \xi(x + y, t), \xi(x, t))}{|y|} |\omega(x + y, t)| \frac{dy}{|y|^{d-1}} \\ &\leq c_d \mathbf{A}_d(t) \int_{B(0, \min(\rho(t), \rho_0(t)))} \frac{|\omega(x + y, t)|}{|y|^{d-1}} dy \\ &= c_d \mathbf{A}_d(t) \int_{B(x, \min(\rho(t), \rho_0(t))) \cap \mathcal{V}(t)^c} \frac{|\omega(z, t)|}{|x - z|^{d-1}} dz \\ &+ c_d \mathbf{A}_d(t) \int_{B(x, \min(\rho(t), \rho_0(t))) \cap \mathcal{V}(t)} \frac{|\omega(z, t)|}{|x - z|^{d-1}} dz \\ &\leq c_d \mathbf{A}_d(t) \Omega(t) \int_{B(0, \rho(t))} \frac{dy}{|y|^{d-1}} + c_d \mathbf{A}_d(t) \int_{B(x, \min(\rho(t), \rho_0(t))) \cap \mathcal{V}(t)} \frac{|\omega(z, t)|}{|z - x|^{d-1}} dz. \end{aligned}$$

511 Furthermore, we have

$$\int_{B(0, \rho(t))} \frac{dy}{|y|^{d-1}} = |B(0, 1)| \rho(t).$$

512 Therefore, by using the fact that $c_d |B(0, 1)| \leq 1$ (since $|B(0, 1)| = \frac{4\pi}{3}$ for $d = 3$
 513 and $|B(0, 1)| = \pi$ for $d = 2$), we deduce

$$I_1 \leq \mathbf{A}_d(t) \Omega(t) \rho(t) + \mathbf{A}_d(t) I_{1,1}, \quad (54)$$

514 where

$$I_{1,1} \stackrel{\text{def}}{=} c_d \int_{B(x, \min(\rho(t), \rho_0(t))) \cap \mathcal{V}(t)} \frac{|\omega(z, t)|}{|z - x|^{d-1}} dz. \quad (55)$$

515 Let $\varepsilon(t) \stackrel{\text{def}}{=} \frac{\pi(t)}{\|\omega(t)\|_\infty}$, then we have

$$I_{1,1} \leq c_d \int_{B(x, \varepsilon(t))} \frac{|\omega(z, t)|}{|z - x|^{d-1}} dz + c_d I_{1,2}, \quad (56)$$

516 with

$$I_{1,2} \stackrel{\text{def}}{=} \int_{B(x, \varepsilon(t))^c \cap B(x, \min(\rho(t), \rho_0(t))) \cap \mathcal{V}(t)} \frac{|\omega(z, t)|}{|z - x|^{d-1}} dz. \quad (57)$$

517 On one hand, we have

$$\begin{aligned} c_d \int_{B(x, \varepsilon(t))} \frac{|\omega(z, t)|}{|z - x|^{d-1}} dz &\leq c_d \|\omega(t)\|_\infty \int_{B(x, \varepsilon(t))} \frac{dz}{|z - x|^{d-1}} \\ &= c_d |B(0, 1)| \|\omega(t)\|_\infty \varepsilon(t) \\ &\leq \pi(t). \end{aligned} \quad (58)$$

518 On the other hand, we estimate $I_{1,2}$.

519 If $\mathcal{V}(t) = \emptyset$ or $\min(\rho(t), \rho_0(t)) \leq \varepsilon(t)$ then from (57) we infer that $I_{1,2} = 0$.
 520 Let us assume now that $\mathcal{V}(t) \neq \emptyset$ and $\min(\rho(t), \rho_0(t)) > \varepsilon(t)$. Then, from (57)
 521 we get

$$\begin{aligned} I_{1,2} &= \int_{\varepsilon(t)}^{\min(\rho(t), \rho_0(t))} \left(\int_{\partial B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| \frac{d\gamma(z)}{|z - x|^{d-1}} \right) dR \\ &= \int_{\varepsilon(t)}^{\min(\rho(t), \rho_0(t))} \frac{1}{R^{d-1}} \left(\int_{\partial B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| d\gamma(z) \right) dR. \end{aligned} \quad (59)$$

522 We introduce the function F_t defined from $[0, \rho_0(t)]$ to $[0, +\infty[$ for all $0 \leq R \leq$
 523 $\rho_0(t)$ by

$$F_t(R) \stackrel{\text{def}}{=} \int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz. \quad (60)$$

524 We observe that $\frac{dF_t}{dR}(R) = \int_{\partial B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| d\gamma(z)$, then from (59) we get
 525 $I_{1,2} = \int_{\varepsilon(t)}^{\min(\rho(t), \rho_0(t))} \frac{1}{R^{d-1}} \frac{dF_t}{dR}(R) dR$ and by using an integration by parts we
 526 obtain

$$I_{1,2} = \frac{F_t(\min(\rho(t), \rho_0(t)))}{\min(\rho(t), \rho_0(t))^{d-1}} - \frac{F_t(\varepsilon(t))}{\varepsilon(t)^{d-1}} + (d-1) \int_{\varepsilon(t)}^{\min(\rho(t), \rho_0(t))} \frac{F_t(R)}{R^d} dR. \quad (61)$$

527 From (60), we notice that $\pi(t) = \sup_{0 < R \leq \rho_0(t)} \frac{F_t(R)}{R^{d-1}}$, then from (61) we deduce

$$\begin{aligned} I_{1,2} &\leq \pi(t) + (d-1)\pi(t) \int_{\varepsilon(t)}^{\min(\rho(t), \rho_0(t))} \frac{dR}{R} \\ &= \pi(t) \left(1 + (d-1) \log \left(\frac{\min(\rho(t), \rho_0(t))}{\varepsilon(t)} \right) \right). \end{aligned}$$

528 Therefore whatever the case we get

$$I_{1,2} \leq \pi(t) \left(1 + (d-1) \log^+ \left(\frac{\min(\rho(t), \rho_0(t))}{\varepsilon(t)} \right) \right). \quad (62)$$

529 Since $\min(\rho(t), \rho_0(t)) \leq \rho(t)$ and thanks to (38) we get

$$\begin{aligned} \frac{\min(\rho(t), \rho_0(t))}{\varepsilon(t)} &\leq 4(d+1)c_d(T^* - t)\|\omega(t)\|_\infty \\ &\leq 4(T^* - t)\|\omega(t)\|_\infty. \end{aligned}$$

530 Therefore from (62) we infer

$$I_{1,2} \leq \pi(t) \left(1 + (d-1) \log^+(4(T^* - t)\|\omega(t)\|_\infty) \right). \quad (63)$$

531 Thanks to (55)-(58) and (63), we get

$$I_{1,1} \leq \pi(t) + c_d\pi(t)(1 + (d-1) \log^+(4(T^* - t)\|\omega(t)\|_\infty)). \quad (64)$$

532 Then thanks to (64), from (54) we obtain

$$I_1 \leq \mathbf{A}_d(t)(\Omega(t)\rho(t) + \pi(t) + c_d\pi(t)(1 + (d-1) \log^+(4(T^* - t)\|\omega(t)\|_\infty))). \quad (65)$$

533 By using the definition of the functions ρ and Ω , for which we have $\Omega(t)\rho(t) \leq$
 534 $4c_d(d+1)\pi(t)$, from (65) we deduce

$$\begin{aligned} I_1 &\leq \mathbf{A}_d(t)\pi(t)(4c_d(d+1) + 1 + c_d(1 + (d-1) \log^+(4(T^* - t)\|\omega(t)\|_\infty))) \\ &\leq 6\mathbf{A}_d(t)\pi(t)(1 + \log^+(4(T^* - t)\|\omega(t)\|_\infty)). \end{aligned} \quad (66)$$

535 For the term I_2 given by (52), after using the change of variables $z = x + y$,
 536 we decompose I_2 as the sum of two terms $I_{2,1}$ and $I_{2,2}$ defined by

$$I_{2,1} = c_d \int_{B(x, \rho_0(t)) \cap B(x, \min(\rho(t), \rho_0(t)))^c \cap \mathcal{V}(t)^c} \mathbf{D}_d(\widehat{z-x}, \omega(z, t), \xi(x, t)) \frac{dz}{|z-x|^d} \quad (67)$$

537 and

$$I_{2,2} = c_d \int_{B(x, \rho_0(t)) \cap B(x, \min(\rho(t), \rho_0(t)))^c \cap \mathcal{V}(t)} \mathbf{D}_d(\widehat{z-x}, \omega(z, t), \xi(x, t)) \frac{dz}{|z-x|^d}. \quad (68)$$

538 Let us estimate the terms $I_{2,1}$ and $I_{2,2}$. For this purpose, we introduce the
 539 function γ defined for all $t \in [0, T^*[$ by $\gamma(t) = \min\left(\frac{1}{4A_0(t)\Omega(t)(T^*-t)}, \rho_0(t)\right)$.
 540 From (67), we observe

$$\begin{aligned} I_{2,1} &\leq c_d \int_{B(x, \rho_0(t)) \cap \mathcal{V}(t)^c} \mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+ \frac{|\omega(z, t)|}{|z-x|^d} dz \\ &\leq c_d \Omega(t) \int_{B(x, \rho_0(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz \\ &= c_d \Omega(t) \int_{B(x, \gamma(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz \\ &\quad + c_d \Omega(t) \int_{B(x, \rho_0(t)) \cap B(x, \gamma(t))^c} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz. \end{aligned} \quad (69)$$

541 Furthermore, on one hand, by (33) we have for all $z \in B(x, \gamma(t))$,

$$\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+ \leq A_0(t)|z-x|,$$

542 and hence, we obtain

$$\begin{aligned} \Omega(t) \int_{B(x, \gamma(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz &\leq A_0(t)\Omega(t) \int_{B(x, \gamma(t))} \frac{dz}{|z-x|^{d-1}} \\ &= |B(0, 1)| A_0(t)\Omega(t)\gamma(t) \\ &\leq \frac{|B(0, 1)|}{4(T^*-t)}, \end{aligned} \quad (70)$$

543 where we have used the definition of the function γ . On the other hand, since

544 $|\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))| \leq 1$, we get

$$\begin{aligned}
& \int_{B(x, \rho_0(t)) \cap B(x, \gamma(t))^c} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z, t), \xi(x, t))^+}{|z-x|^d} dz \\
& \leq \int_{B(x, \rho_0(t)) \cap B(x, \gamma(t))^c} \frac{dz}{|z-x|^d} \\
& = |B(0, 1)| \int_{\gamma(t)}^{\rho_0(t)} \frac{ds}{s} \\
& = |B(0, 1)| \log \left(\frac{\rho_0(t)}{\gamma(t)} \right) \\
& = |B(0, 1)| \log^+ (4\rho_0(t)A_0(t)\Omega(t)(T^* - t)), \tag{71}
\end{aligned}$$

545 where we have used again the definition of the function γ . Owing to (70) and
546 (71), from (69), we deduce

$$I_{2,1} \leq c_d |B(0, 1)| \left(\frac{1}{4(T^* - t)} + \Omega(t) \log^+ (4\rho_0(t)A_0(t)\Omega(t)(T^* - t)) \right). \tag{72}$$

547 For the term $I_{2,2}$, if $\mathcal{V}(t) = \emptyset$ or if $\rho(t) \geq \rho_0(t)$ then from (68) we infer that
548 $I_{2,2} = 0$. Let us assume now that $\mathcal{V}(t) \neq \emptyset$ and $\rho(t) < \rho_0(t)$. Then, from (68)
549 we get

$$\begin{aligned}
I_{2,2} &= c_d \int_{\rho(t)}^{\rho_0(t)} \left(\int_{\partial B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| \frac{d\gamma(z)}{|z-x|^d} \right) dR \\
&= c_d \int_{\rho(t)}^{\rho_0(t)} \frac{1}{R^d} \left(\int_{\partial B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| d\gamma(z) \right) dR. \tag{73}
\end{aligned}$$

550 By using (60) and an integration by parts we obtain

$$I_{2,2} = c_d \left(\frac{F_t(\rho_0(t))}{\rho_0(t)^d} - \frac{F_t(\rho(t))}{\rho(t)^d} + d \int_{\rho(t)}^{\rho_0(t)} \frac{F_t(R)}{R^{d+1}} dR \right). \tag{74}$$

551 Since $\pi(t) = \sup_{0 < R \leq \rho_0(t)} \frac{F_t(R)}{R^{d-1}}$, then from (74) we deduce

$$\begin{aligned}
I_{2,2} &\leq c_d \left(\frac{\pi(t)}{\rho_0(t)} + d\pi(t) \int_{\rho(t)}^{\rho_0(t)} \frac{dR}{R^2} \right) \\
&\leq c_d(d+1) \frac{\pi(t)}{\rho(t)},
\end{aligned}$$

552 where we have used the fact that $\rho(t) < \rho_0(t)$. Therefore whatever the case, we
553 obtain that

$$I_{2,2} \leq c_d(d+1) \frac{\pi(t)}{\rho(t)}. \tag{75}$$

554 By using (38) the definition of the function ρ , from (75) we deduce

$$I_{2,2} \leq \frac{1}{4(T^* - t)}. \tag{76}$$

Owing to (72) and (76), since also that $I_2 = I_{2,1} + I_{2,2}$ and $c_d|B(0,1)| \leq 1$ we thus obtain

$$I_2 \leq \frac{1}{2(T^* - t)} + \Omega(t) \log^+ (4\rho_0(t)A_0(t)\Omega(t)(T^* - t)). \quad (77)$$

For the term I_3 given by (53), to obtain a precise non blowup criterion for 3D the Navier-Stokes, 3D Euler and 2D QG equations that could be used easily in numerical experiments, it was important to explicit the constant involved in the estimate of the term I_3 . For this purpose, we deal first with the case of the 3D the Navier-Stokes and 3D Euler equations, then after we consider the case of the 2D QG equations.

In the case of the 3D Euler or 3D Navier-Stokes equations for which $d = 3$, we get $\mathbf{D}_3(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)) \det(\hat{y}, \omega(x+y, t), \xi(x, t))$. Since $\det(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\xi(x, t) \times \hat{y}) \cdot \omega(x+y, t)$ and $\omega(x+y, t) = \nabla_y \times u(x+y, t)$, we deduce

$$\mathbf{D}_3(\hat{y}, \omega(x+y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y}) \cdot \nabla_y \times u(x+y, t).$$

Then, after using an integration by parts, from (53), we deduce,

$$\begin{aligned} I_3 &= c_3 \int_{B(0, \rho_0(t))^c} \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \right) \cdot u(x+y, t) dy \\ &\quad - c_3 \int_{\partial B(0, \rho_0(t))} \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \right) \cdot \hat{y} \times u(x+y, t) d\gamma(y). \end{aligned} \quad (78)$$

After setting $\psi(y) \equiv \frac{(\hat{y} \cdot \xi(x, t))}{|y|^3}$ and $\mathbf{V}(y) \equiv (\xi(x, t) \times \hat{y})$, by using the following vectorial identity $\nabla \times (\psi \mathbf{V}) = \nabla \psi \times \mathbf{V} + (\nabla \times \mathbf{V})\psi$, we obtain after elementary computations, that for all $y \neq 0$,

$$\left| \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \right) \right| \leq \left| \nabla \left(\frac{\hat{y}}{|y|^3} \right) \right| + \frac{|\nabla_y \times (\xi(x, t) \times \hat{y})|}{|y|^3}.$$

We have $\left| \nabla \left(\frac{\hat{y}}{|y|^3} \right) \right| \leq \frac{3}{|y|^4}$. Furthermore, we have $\nabla_y \times (\xi(x, t) \times \hat{y}) = (\nabla_y \cdot \hat{y})\xi(x, t) - (\xi(x, t) \cdot \nabla_y)\hat{y}$ and then we deduce $|\nabla_y \times (\xi(x, t) \times \hat{y})| \leq |\nabla \cdot \hat{y}| + |\nabla \hat{y}| \leq \frac{3}{|y|}$. After gathering these results, we obtain that for all $y \neq 0$,

$$\left| \nabla_y \times \left(\frac{(\hat{y} \cdot \xi(x, t))(\xi(x, t) \times \hat{y})}{|y|^3} \right) \right| \leq \frac{6}{|y|^4}.$$

Therefore, from (78) we obtain

$$I_3 \leq 6c_3 \int_{B(0, \rho_0(t))^c} \frac{|u(x+y, t)|}{|y|^4} dy + c_3 \int_{\partial B(0, \rho_0(t))} \frac{|u(x+y, t)|}{|y|^3} d\gamma(y). \quad (79)$$

575 **In the case of 2D QG equations** for which $d = 2$, we get $\mathbf{D}_2(\hat{y}, \omega(x +$
576 $y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)^\perp)(\omega(x + y, t) \cdot \xi(x, t)^\perp)$. Since $\omega(x + y, t) = \nabla_y^\perp u(x + y, t)$,
577 we deduce

$$\mathbf{D}_2(\hat{y}, \omega(x + y, t), \xi(x, t)) = (\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp \cdot \nabla_y^\perp u(x + y, t).$$

578 Then, after using an integration by parts, from (53), we deduce,

$$\begin{aligned} I_3 = & -c_2 \int_{B(0, \rho_0(t))^c} \nabla_y^\perp \cdot \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \right) u(x + y, t) dy \\ & + c_2 \int_{\partial B(0, \rho_0(t))} \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \right) \cdot y^\perp u(x + y, t). \end{aligned} \quad (80)$$

579 After setting $\psi(y) \equiv \frac{(\hat{y} \cdot \xi(x, t)^\perp)}{|y|^2}$ and $\mathbf{V}(y) \equiv \xi(x, t)^\perp$, by using the following
580 vectorial identity $\text{curl}(\psi \mathbf{V}) = \nabla^\perp \psi \cdot \mathbf{V} + \psi \text{curl} \mathbf{V}$, we obtain after elementary
581 computations, that for all $y \neq 0$,

$$\begin{aligned} \left| \nabla_y^\perp \left(\frac{(\hat{y} \cdot \xi(x, t)^\perp) \xi(x, t)^\perp}{|y|^2} \right) \right| & \leq \left| \nabla \left(\frac{\hat{y}}{|y|^2} \right) \right| \\ & \leq \frac{2}{|y|^3}. \end{aligned}$$

582 Therefore, from (80) we obtain

$$I_3 \leq 2c_2 \int_{B(0, \rho_0(t))^c} \frac{|u(x + y, t)|}{|y|^3} dy + c_2 \int_{\partial B(0, \rho_0(t))} \frac{|u(x + y, t)|}{|y|^2} d\gamma(y). \quad (81)$$

583 Therefore, whatever the case considered, 3D Navier-Stokes, 3D Euler or 2D QG
584 equations, from (79) and (81) we get

$$I_3 \leq d(d-1)c_d \int_{B(0, \rho_0(t))^c} \frac{|u(x + y, t)|}{|y|^{d+1}} dy + c_d \int_{\partial B(0, \rho_0(t))} \frac{|u(x + y, t)|}{|y|^d} d\gamma(y). \quad (82)$$

585 Then from (82), we obtain

$$\begin{aligned} I_3 & \leq c_d \|u(t)\|_\infty \left(d(d-1) \int_{B(0, \rho_0(t))^c} \frac{dy}{|y|^{d+1}} + \int_{\partial B(0, \rho_0(t))} \frac{d\gamma(y)}{|y|^d} \right) \\ & = c_d \|u(t)\|_\infty \left(d(d-1) |B(0, 1)| \int_{\rho_0(t)}^{+\infty} \frac{ds}{s^2} + \frac{|\partial B(0, 1)|}{\rho_0(t)} \right) \\ & \leq 9 \frac{\|u(t)\|_\infty}{\rho_0(t)}. \end{aligned} \quad (83)$$

586 Then, owing to (66), (77) and (83), from (50) we deduce that for any $t \in [0, T]$
587 and $x \in \Theta(t)$,

$$\begin{aligned} \alpha(x, t) & \leq 6\mathbf{A}_d(t) \pi(t) (1 + \log^+(4(T^* - t) \|\omega(t)\|_\infty)) \\ & \quad + \frac{1}{2(T^* - t)} + \Omega(t) \log^+(4\rho_0(t) A_0(t) \Omega(t) (T^* - t)) + 9 \frac{\|u(t)\|_\infty}{\rho_0(t)}. \end{aligned} \quad (84)$$

By using (32), we deduce that for any $t \in [0, T]$ and $x \in \Theta(t)$,

$$\begin{aligned} \alpha(x, t) &\leq 6\mathbf{A}_d(t)\pi(t)(1 + \log^+(4(T^* - t)\|\omega(t)\|_\infty)) \\ &\quad + \frac{3}{4(T^* - t)} + \Omega(t) \log^+(4\rho_0(t)A_0(t)\Omega(t)(T^* - t)). \end{aligned} \quad (85)$$

Furthermore, thanks to (35) we get $(T^* - t)\Omega(t) \log^+(4\rho_0(t)\mathbf{A}_0(t)) \leq \frac{1}{8}$ and $(T^* - t)\Omega(t) < 1$ which implies that $\log^+((T^* - t)\Omega(t)) = 0$ and hence we obtain that for all $t \in [0, T^*]$,

$$\begin{aligned} (T^* - t)\Omega(t) \log^+(4\rho_0(t)\mathbf{A}_0(t)\Omega(t)(T^* - t)) &\leq (T^* - t)\Omega(t)(\log^+(4\rho_0(t)\mathbf{A}_0(t)) + \log^+(\Omega(t)(T^* - t))) \\ &= (T^* - t)\Omega(t) \log^+(4\rho_0(t)\mathbf{A}_0(t)) \\ &\leq \frac{1}{8}. \end{aligned}$$

Therefore from (85), we deduce that for any $t \in [0, T]$ and $x \in \Theta(t)$,

$$\alpha(x, t) \leq 6\mathbf{A}_d(t)\pi(t)(1 + \log^+(4(T^* - t)\|\omega(t)\|_\infty)) + \frac{7}{8(T^* - t)}. \quad (86)$$

Then from (49) we deduce

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \left(6\mathbf{A}_d(t)\pi(t)(1 + \log^+(4(T^* - t)\|\omega(t)\|_\infty)) + \frac{7}{8(T^* - t)} \right) \|\omega(t)\|_\infty, \quad (87)$$

which is valid for all $t \in [0, T]$ and $T < T^*$ and then inequality (87) is valid for all $t \in [0, T^*]$. Let $t_0 \in [0, T^*]$ such that

$$4(T^* - t_0) \leq 1. \quad (88)$$

Then we get that for all $t \in [t_0, T^*]$, $4(T^* - t) \leq 1$ and hence

$$\begin{aligned} \log^+(4(T^* - t)\|\omega(t)\|_\infty) &\leq \log^+(4(T^* - t)) + \log^+\|\omega(t)\|_\infty \\ &= \log^+\|\omega(t)\|_\infty. \end{aligned} \quad (89)$$

Owing to (89), from (87) we deduce that for all $t \in [t_0, T^*]$

$$\frac{d}{dt}\|\omega(t)\|_\infty \leq \left(6\mathbf{A}_d(t)\pi(t)(1 + \log^+\|\omega(t)\|_\infty) + \frac{7}{8(T^* - t)} \right) \|\omega(t)\|_\infty. \quad (90)$$

Thanks to Grönwall inequality, from (90) we deduce that for all $t \in [t_0, T^*]$,

$$\begin{aligned} \|\omega(t)\|_\infty &\leq \|\omega(t_0)\|_\infty e^{\int_{t_0}^t \left(6\mathbf{A}_d(\tau)\pi(\tau)(1 + \log^+\|\omega(\tau)\|_\infty) + \frac{7}{8(T^* - \tau)} \right) d\tau} \\ &= \left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \|\omega(t_0)\|_\infty e^{\int_{t_0}^t 6\mathbf{A}_d(\tau)\pi(\tau)(1 + \log^+\|\omega(\tau)\|_\infty) d\tau}. \end{aligned} \quad (91)$$

599 Since the function $z \mapsto \log^+(z)$ is non-decreasing on $]0, +\infty[$, then after applying
600 the function $1 + \log^+$ to the inequality (91) and using the fact that $\log^+(ab) \leq$
601 $\log^+ a + \log^+ b$, we thus obtain that for all $t \in [t_0, T^*[$

$$\begin{aligned} 1 + \log^+ \|\omega(t)\|_\infty &\leq 1 + \log \left(\left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \right) + \log^+ \|\omega(t_0)\|_\infty \\ &\quad + 6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) (1 + \log^+ \|\omega(\tau)\|_\infty) d\tau. \end{aligned} \quad (92)$$

602 Since the function $t \mapsto \log \left(\left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \right)$ is increasing over $[t_0, T^*[$ then thanks
603 to Gronwall Lemma, by (92) we deduce that for all $t \in [t_0, T^*[$,

$$1 + \log^+ \|\omega(t)\|_\infty \leq \left(\log \left(\left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \right) + 1 + \log^+ \|\omega(t_0)\|_\infty \right) e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau}. \quad (93)$$

604 Since $\log \|\omega(t)\|_\infty \leq 1 + \log^+ \|\omega(t)\|_\infty$, then from (93) we infer that for all
605 $t \in [t_0, T^*[$,

$$\begin{aligned} \|\omega(t)\|_\infty &\leq \exp \left(\left(\log \left(\left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8}} \right) + 1 + \log^+ \|\omega(t_0)\|_\infty \right) e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau} \right) \\ &= \exp \left((1 + \log^+ \|\omega(t_0)\|_\infty) e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau} \right) \left(\frac{T^* - t_0}{T^* - t} \right)^{\frac{7}{8} e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau}}. \end{aligned} \quad (94)$$

606 Let us assume that there exists $t_1 \in [0, T^*[$ such that $\int_{t_1}^{T^*} \mathbf{A}_d(\tau) \pi(\tau) d\tau <$
607 $+\infty$. Then, in addition of (88), we choose $t_0 \in [t_1, T^*[$ such that $M_{t_0} \stackrel{\text{def}}{=} \int_{t_0}^{T^*} \mathbf{A}_d(\tau) \pi(\tau) d\tau < \frac{1}{6} \log \left(\frac{8}{7} \right)$. We thus get for all $t \in [t_0, T^*[$,

$$\frac{7}{8} e^{6 \int_{t_0}^t \mathbf{A}_d(\tau) \pi(\tau) d\tau} \leq \frac{7}{8} e^{6 M_{t_0}} < 1.$$

609 Therefore with $\eta_{t_0} \stackrel{\text{def}}{=} \frac{7}{8} e^{6 M_{t_0}}$, from (94) we deduce that for all $t \in [t_0, T^*[$

$$\|\omega(t)\|_\infty \leq \exp \left((1 + \log^+ \|\omega(t_0)\|_\infty) e^{6 M_{t_0}} \right) \left(\frac{T^* - t_0}{T^* - t} \right)^{\eta_{t_0}}. \quad (95)$$

610 Since $\eta_{t_0} < 1$, from (95) we thus deduce that $\int_{t_0}^{T^*} \|\omega(t)\|_\infty dt < +\infty$. Since $u \in$
611 $C([0, T^*]; H_\sigma^r(\mathbb{R}^d))$ and thanks to the Sobolev embedding $H^r(\mathbb{R}^d) \hookrightarrow BC^3(\mathbb{R}^d)$
612 due to $r > \frac{d}{2} + 3$, we infer that $\omega \in C([0, T^*]; BC^2(\mathbb{R}^d))$ which implies that
613 $\int_0^{t_0} \|\omega(t)\|_\infty dt < +\infty$. Therefore we deduce that $\int_0^{T^*} \|\omega(t)\|_\infty dt < +\infty$.
614 If u blows up at the finite time T^* then thanks to (20) and (23) we have

615 $\int_0^{T^*} \|\omega(t)\|_\infty dt = +\infty$ which leads to a contradiction. Then, we deduce that u
616 cannot blow up at the time T^* which concludes the first part of proof.
617 Thanks to (44), we have already $\nabla|\omega|(x, t) = 0$. Since $\nabla \cdot \omega = 0$ and $\omega = |\omega|\xi$,
618 then we get

$$0 = \nabla \cdot \omega = |\omega| \nabla \cdot \xi + \xi \cdot \nabla |\omega|. \quad (96)$$

619 However, for all $x \in \Theta(t)$, $|\omega(x, t)| = \|\omega(t)\|_\infty > 0$ and from (44), we have
620 $\nabla|\omega|(x, t) = 0$. Therefore, from (96), we deduce that for all $t \in [0, T]$ and
621 $x \in \Theta(t)$,

$$\nabla \cdot \xi(x, t) = 0, \quad (97)$$

622 which completes the proof. $\square$

623 6. No blow up in finite time for numerical experiments

624 In this section, we show the non-blowup in finite time of the solutions of the
625 3D Euler equations in the numerical experiments considered these last years.

626 First, we emphasize that the singularity discovered in Luo and Hou (2014b)
627 which lies right on the boundary is not relevant in the case of the whole do-
628 main $\mathbb{R}^3$. Indeed recently, the authors found a convincing numerical evidence
629 for a singular solution to the Euler equations in a fluid with periodic boundary
630 condition along the axial direction and no-flow boundary condition on the solid
631 wall Luo and Hou (2014b) (see also Luo and Hou (2014a)), for which the point
632 of the potential singularity, which is also the point of the maximum vorticity,
633 is always located at the solid boundary. However thanks to Theorem 5.1, we
634 deduce that such singularity can not exist in the whole domain $\mathbb{R}^3$. Indeed, in
635 the whole domain of $\mathbb{R}^3$ at any point of the maximum vorticity, $q_0 \in \mathbb{R}^3$, thanks
636 to Theorem 5.1 we get $\nabla|\omega|(q_0, t) = 0$ for any time t before the alleged time of
637 singularity T^* , then this result combined with the fact that the vorticity ω is a
638 divergence-free vector field, yields to get $\nabla \cdot \xi(q_0, t) = 0$ in Theorem 5.1. However
639 in Luo and Hou (2014b), the presence of a solid boundary and the fact that q_0
640 the point of the maximum vorticity is always located on the solid boundary, pre-
641 vent to get $\nabla|\omega|(q_0, t) = 0$ and this allows to get $\nabla \cdot \xi(q_0, t) \sim (T^* - t)^{-2.9165} \neq 0$
642 as it is observed in their numerical test. This latter is the main element used to
643 invalidate the Deng-Hou-Yu non-blowup criterion Deng et al. (2005, 2006a).
644

645 There have been many computational attempts to find finite-time singulari-
646 ties of the 3D Euler and Navier-Stokes equations: see, e.g Melander and Hussain
647 (1989); Kerr and Hussain (1989); Pumir and Siggia (1990); Kerr (1993); Grauer et al.
648 (1998); Hou and Li (2006); Kerr (2005). One example that has been studied ex-
649 tensively in these numerical investigations is the interaction of two perturbed an-
650 tiparallel vortex tubes. All the subsequent calculations assumed an anti-parallel
651 geometry, for which there are two symmetry planes. One in $y - z$ is between the
652 vortices and was called the 'dividing plane'. The other in $x - z$ is at the position
653 of maximum perturbation and was called the 'symmetry plane'. The difficulty
654 faced in each computational attempts cited was to find a better initial condition

within this geometry (see [Kerr \(2006\)](#)). From these computational attempts, a numerical controversy takes place around the question to know whether or not there is finite-time blow-up of the solutions of Euler equations (see [Kerr \(2006\)](#)).

In this section, we propose an answer to this controversy by using our Theorem 5.1. By using the anisotropic structure of regions of high vorticity described in [Kerr \(2005, 1998\)](#), we show straightforward thanks to our Theorem 5.1 that the solutions of Euler equations cannot blow up in finite time in these numerical experiments [Kerr \(2005, 1998, 1997\)](#).

For this purpose, we give a first bound of the function π defined by (37) in the following Lemma. The bound given in Lemma 6.1 of the function π is not a sharp bound but obtained without assumptions.

Lemma 6.1. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

Under the definitions (32)-(37) in the Theorem 5.1, we have the following estimate: for all $t \in [0, T^*[$

$$\pi(t) \leq 3\|\omega(t)\|_\infty \sup_{x \in \Theta(t)} |\mathcal{V}(t) \cap B(x, \rho_0(t))|^{\frac{1}{d}}.$$

Proof. For any $t \in [0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$, we get

$$\frac{1}{R^{d-1}} \int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz \leq \frac{1}{R^{d-1}} \|\omega(t)\|_\infty |\mathcal{V}(t) \cap B(x, R)|.$$

Furthermore, we have

$$\begin{aligned} |\mathcal{V}(t) \cap B(x, R)| &= |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}} |\mathcal{V}(t) \cap B(x, R)|^{\frac{d-1}{d}} \\ &\leq |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}} |B(x, R)|^{\frac{d-1}{d}} \\ &= |B(0, 1)|^{\frac{d-1}{d}} |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}} R^{d-1} \\ &\leq 3 |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}} R^{d-1} \end{aligned}$$

where we have used the fact that $|B(0, 1)| = \left(\frac{4\pi}{3}\right)^{\frac{2}{3}}$ if $d = 3$ or $|B(0, 1)| = \pi^{\frac{1}{2}}$ if $d = 2$.

Then, we deduce for that any $t \in [0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$,

$$\frac{1}{R^{d-1}} \int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz \leq 3\|\omega(t)\|_\infty |\mathcal{V}(t) \cap B(x, R)|^{\frac{1}{d}}. \quad (98)$$

Owing to (98), we thus conclude the proof. $\square$

677 Now, we can show straightforward thanks to Theorem 5.1 and Lemma 6.1
 678 that the solutions of Euler equations cannot blow up in finite time in the nu-
 679 merical experiments Kerr (2005, 1998, 1997).
 680 For this purpose, we recall that in the numerical experiments Kerr (2005, 1998),
 681 the author show that the blow-up rates at some time T^* the alleged time of sin-
 682 gularity, to be considered for $\|\omega(t)\|_\infty$, $\|u(t)\|_\infty$ and $\|\nabla\xi(t)\|_\infty$ in Kerr (2005,
 683 1998) are

$$\|\omega(t)\|_\infty \sim (T^* - t)^{-1}, \|u(t)\|_\infty \sim (T^* - t)^{-\frac{1}{2}} \text{ and } \|\nabla\xi(t)\|_\infty \sim (T^* - t)^{-\frac{1}{2}}, \quad (99)$$

684 for time $t \in [t_0, T^*[$ with $t_0 \in [0, T^*[$ sufficiently close to T^* . Moreover for time
 685 $t \in [t_0, T^*[$ with t_0 sufficiently close to T^* , the author showed that the support
 686 of the maximum vorticity

$$\mathcal{E}(t) = \{x \in \mathbb{R}^3, |\omega(x, t)| \sim \|\omega(t)\|_\infty\}$$

687 is characterized by two length scales $(T^* - t)$ and $(T^* - t)^{\frac{1}{2}}$ and its volume is
 688 bounded by

$$|\mathcal{E}(t)| \lesssim (T^* - t)^2. \quad (100)$$

689 Thanks to the blow-up rates (99), from (32) we get that for all $t \in [t_0, T^*[$

$$\rho_0(t) \sim (T^* - t)^{\frac{1}{2}}.$$

690 From (33), thanks again to (99) we have that for all $t \in [t_0, T^*[$

$$\begin{aligned} \mathbf{A}_0(t) &\leq \sup_{x \in \Theta(t)} \|\nabla\xi(t)\|_{L^\infty(B(x, \rho_0(t)))} \\ &\lesssim (T^* - t)^{-\frac{1}{2}}. \end{aligned}$$

691 Then we deduce that for all $t \in [t_0, T^*[$

$$\mathbf{A}_0(t)\rho_0(t) \lesssim 1. \quad (101)$$

692 Therefore thanks to (101), from the definition (35) of the function Ω , we deduce
 693 for all $t \in [t_0, T^*[$

$$\Omega(t) \gtrsim (T^* - t)^{-1}. \quad (102)$$

694 Owing to (102) and since for all $t \in [t_0, T^*[$, $\|\omega(t)\|_\infty \sim (T^* - t)^{-1}$ thanks to
 695 (99), for the set $\mathcal{V}(t)$ defined by (36) we deduce that for all $t \in [t_0, T^*[$

$$\mathcal{V}(t) = \{x \in \mathbb{R}^3, |\omega(x, t)| \sim \|\omega(t)\|_\infty\}.$$

696 Then thanks to (100) we get that for all $t \in [t_0, T^*[$

$$|\mathcal{V}(t)| \lesssim (T^* - t)^2. \quad (103)$$

697 Thanks to Lemma 6.1, inequality (103) and the fact that $\|\omega(t)\|_\infty \sim (T^* - t)^{-1}$
 698 thanks to (99), for the function π defined by (37), we get that for all $t \in [t_0, T^*[$
 699

$$\pi(t) \lesssim (T^* - t)^{-\frac{1}{3}}. \quad (104)$$

Furthermore, thanks to (99), for the function $\mathbf{A}_3$ defined by (39) for $d = 3$, we get that for all $t \in [t_0, T^*[$

$$\mathbf{A}_3(t) \leq \|\nabla \xi(t)\|_\infty \lesssim (T^* - t)^{-\frac{1}{2}}. \quad (105)$$

Owing to (104) and (105) we deduce that for all $t \in [t_0, T^*[$

$$\mathbf{A}_3(t)\pi(t) \lesssim (T^* - t)^{-\frac{5}{6}}. \quad (106)$$

Then, we deduce

$$\int_{t_0}^{T^*} \mathbf{A}_3(t)\pi(t) \lesssim (T^* - t_0)^{\frac{1}{6}} < +\infty. \quad (107)$$

Therefore, thanks to (107) and Theorem 5.1, we deduce that the solutions of the Euler equations considered for the numerical experiments Kerr (2005, 1998) cannot blow-up in finite time at the alleged time of singularity T^* . If one considers the plausible scenario of blow up proposed in Brenner et al. (2016), one observed that we get also the blow-up rates (99) and the estimate (100) (see (Hormoz and Brenner, 2012, section 4)), hence the potential mechanism proposed for the blow-up in finite time of solutions of Euler equations in Brenner et al. (2016) cannot in fact lead to the blow-up in finite time of the solutions of Euler equations.

7. Toward the non blowup in finite time of the solutions

In this section, under mild assumptions deriving from the structure of the regions of high vorticity, we obtain the non-blowup in finite time at some time T^* of the solutions of 2D QG, 3D Euler and 3D Navier-Stokes equations in the case where

$$\|\nabla \xi(t)\|_\infty \sim (T^* - t)^{-\gamma_\xi}, \quad 0 \leq \gamma_\xi < 1.$$

In the previous section, we have outlined that the estimate obtained in Lemma 6.1 for the function π defined by (37) is not sharp, then in the subsection 7.2 we propose a better estimate for the function π and go further in the non blow-up criteria. However, before to deal with new non blow-up criteria in the subsection 7.2 we need to introduce in the subsection 7.1 the Lagrangian flow map X and the definitions of vortex lines and vortex tubes in order to justify the assumption (121) used in Proposition 7.1 and for their use in Lemmata 7.2, 7.3, 7.4 and in the Proposition 7.2.

7.1. Lagrangian flow map, vortex lines and vortex tubes

Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 2$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-1}(\mathbb{R}^d)). \quad (108)$$

731 Solutions in this class exist thanks to section 3. We set $\mathbf{v} = u$ in the case of 3D
 732 Euler equations and $\mathbf{v} = R^\perp u$ in the case of 2D QG equation.
 733 Owing to $u \in C([0, T^*[, H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*[, H^{r-1}(\mathbb{R}^d))$ with $r > \frac{d}{2} + 2$
 734 and thanks to the L^2 -boundedness of the Riesz transforms, we infer that
 735 $\mathbf{v} \in C([0, T^*[, H^r(\mathbb{R}^d)) \cap C^1([0, T^*[, H^{r-1}(\mathbb{R}^d))$ with $r > \frac{d}{2} + 2$. Then by using
 736 the Sobolev embedding, $H^m(\mathbb{R}^d) \hookrightarrow BC^n(\mathbb{R}^d)$ with $n = [m - \frac{d}{2}]$ and $m > \frac{d}{2}$, we
 737 deduce that for any $0 < T < T^*$,

$$u \in BC^1(\mathbb{R}^d \times [0, T]), \quad \nabla u \in C([0, T^*[; BC^1(\mathbb{R}^d)) \quad (109)$$

738 and

$$\mathbf{v} \in BC^1(\mathbb{R}^d \times [0, T]), \quad \nabla \mathbf{v} \in C([0, T^*[; BC^1(\mathbb{R}^d)). \quad (110)$$

739 In Proposition 7.1, in the case of 3D Euler equations and 2D QG equations, we
 740 give an estimate of the function π defined by (37). For this purpose, we need to
 741 give the definition of a vortex line and recall some results about the Lagrangian
 742 flow map.

743 We thus introduce the flow map $X(\alpha, \tau, t)$ the particle path that passes by
 744 $\alpha \in \mathbb{R}^d$ at time $\tau \in [0, T^*[$. That is for $\tau \in [0, T^*[$ fixed, $X(\alpha, \tau, t)$ solves on
 745 $[0, T^*[$ (see chapter 4 in Majda and Bertozzi (2002) for more details on the flow
 746 map X)

$$\begin{aligned} \frac{\partial X(\alpha, \tau, t)}{\partial t} &= \mathbf{v}(X(\alpha, \tau, t), t), \\ X(\alpha, \tau, \tau) &= \alpha \in \mathbb{R}^d, \end{aligned} \quad (111)$$

747 Thanks to Cauchy-Lipschitz Theorem (see Theorems 2.2 and 2.13 in Teschl
 748 (2012)), for any $\alpha \in \mathbb{R}^d$ and $\tau \in [0, T^*[$ thanks to (110) we get that there
 749 exists a unique solution $X(\alpha, \tau, \cdot) \in C^1([0, T^*[$) to equation (111). For all
 750 $t \in [0, T^*[$ and $\tau \in [0, T^*[$, the map $X(\cdot, \tau, t)$ defined by equation (111) is a
 751 volume preserving C^1 -diffeomorphism from $\mathbb{R}^d$ on itself. Indeed thanks to (110)
 752 and the Theorems 2.2, 2.10 and 2.13 in Teschl (2012), we deduce that for any
 753 $t \in [0, T^*[$ and $\tau \in [0, T^*[$, $X(\cdot, \tau, t)$ is a C^1 -diffeomorphism from $\mathbb{R}^d$ on itself
 754 with inverse $X(\cdot, t, \tau)$, we notice also $X \in C^1(\mathbb{R}^d \times [0, T^*[\times [0, T^*[$). Moreover
 755 for any $t \in [0, T^*[$ and $\tau \in [0, T^*[$, $X(\cdot, \tau, t)$ is a volume preserving mapping
 756 thanks to Proposition 1.4 in Majda and Bertozzi (2002), for which we get

$$\det(\nabla_\alpha X(\alpha, \tau, t)) = 1. \quad (112)$$

757 Furthermore, we have for all $\tau \in [0, T^*[$, $t \in [0, T^*[$ and $\alpha \in \mathbb{R}^d$ (see Proposition
 758 1.8 in Majda and Bertozzi (2002) or Proposition page 24 in Chorin and Marsden
 759 (1993) for Euler equations and see Deng et al. (2006b) for 2D QG equation),

$$\omega(X(\alpha, \tau, t), t) = \nabla_\alpha X(\alpha, \tau, t) \omega(\alpha, \tau). \quad (113)$$

760

761 Recall that a vortex line in a fluid is an arc on an integral curve of the vorticity
 762 $\omega(x, t)$ for fixed t , and a vortex tube is a tubular neighborhood in $\mathbb{R}^d$, $d \in \{2,$

arising as a union of vortex lines. In what follows, we give a parametrization of
 vortex lines and vortex tubes.
 We set $\mathcal{O} \stackrel{\text{def}}{=} \{(x, \tau') \in \mathbb{R}^d \times [0, T^*]; |\omega(x, \tau')| > 0\}$ and for any $t \in [0, T^*[,$
 $\mathcal{O}(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d; |\omega(x, t)| > 0\}$. From (25), we get that for any $t \in [0, T^*[, \mathcal{O}(t)$
 is nonempty. Thanks to (109), we get that ω is continuous on $\mathbb{R}^d \times [0, T^*[,$ and
 then we deduce that $\mathcal{O}$ is an open subset of $\mathbb{R}^d \times [0, T^*[,$ and also that for all
 $t \in [0, T^*[, \mathcal{O}(t)$ is an open subset of $\mathbb{R}^d$. Notice thanks again to (109) that ξ and
 $\nabla \xi$ are continuous on $\mathcal{O}$. Then, we get that for all $t \in [0, T^*[, \xi(\cdot, t) \in C^1(\mathcal{O}(t))$.
 Then, for all $t \in [0, T^*[,$ and $\alpha \in \mathcal{O}(t)$, we denote by $\mathbf{x}_t(\alpha, \cdot) : \mathfrak{J}_{\alpha, t} \rightarrow \mathbb{R}^d$ the
 vortex line that passes through α at the time t and defined by the ordinary
 differential equation:

$$\begin{aligned}
 \frac{\partial \mathbf{x}_t(\alpha, s)}{\partial s} &= \xi(\mathbf{x}_t(\alpha, s), t), \\
 \mathbf{x}_t(\alpha, 0) &= \alpha.
 \end{aligned} \tag{114}$$

The set $\mathfrak{J}_{\alpha, t} \subset \mathbb{R}$ not reduced to $\{0\}$ denotes the maximal interval of existence of
 the unique solution $\mathbf{x}_t(\alpha, \cdot)$ of (114), this is ensured thanks to Cauchy-Lipschitz
 Theorem (see e.g Theorems 2.2 and 2.13 in Teschl (2012)). For any $t \in [0, T^*[,$
 we introduce $U_t = \{(\alpha, s) \in \mathcal{O}(t) \times \mathbb{R}; \mathbf{x}_t(\alpha, s) \in \mathcal{O}(t)\}$ the set of definition
 of the function $\mathbf{x}_t$. For any $t \in [0, T^*[,$ since $\xi(\cdot, t) \in C^1(\mathcal{O}(t))$, then from
 Theorem 2.9 in Teschl (2012), we get that $\mathbf{x}_t$ is continuous on U_t . We notice
 that $U_t = \mathbf{x}_t^{-1}(\mathcal{O}(t))$ and hence we obtain that U_t is an open subset of $\mathcal{O}(t) \times \mathbb{R}$.
 From Theorem 2.10 in Teschl (2012), we obtain that

$$\mathbf{x}_t \in C^1(U_t). \tag{115}$$

Any vortex tube $\mathfrak{T}$ at a time $t \in [0, T^*[,$ can be defined as $\mathfrak{T} \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in$
 $\mathcal{A}_0, s \in I_{\alpha, t} \subset \mathfrak{J}_{\alpha, t}\}$ where $\mathcal{A}_0$ is a bounded smooth surface (resp. curve) of
 $\mathbb{R}^3$ (resp. of $\mathbb{R}^2$) and for each $\alpha \in \mathcal{A}_0$, $I_{\alpha, t}$ is an interval of $\mathbb{R}$ containing 0.

7.2. Anisotropic structure for the improvement of non blow-up criteria

In this subsection, in Proposition 7.1 we propose to show that the function
 $\pi(t)$ defined by (37) involved Theorem 5.1 is bounded by $C \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right)$
 by using assumptions related to the anisotropic scaling in the collapse of regions
 of high vorticity containing the positions of the maximum vorticity. In Propo-
 sition 7.2, in the case of the Euler and 2D QG equations, we improve logarith-
 mically the result obtained in Proposition 7.1 by showing that the function π is
 bounded.
 These results come from the special feature of the structure of regions of high
 vorticity surrounding the peak of vorticity $\{y \in \mathbb{R}^d; |\omega(y, t)| \gtrsim \|\omega(t)\|_\infty\}$ ob-
 served in the numerical experiments Kerr (1998, 2005); Agafontsev et al. (2015,
 2017) and from analytical models (Agafontsev et al., 2017, section 3),
 (Majda and Bertozzi, 2002, sections 1.4 and 1.5), namely they are pancake-
 like structure characterized by two length scales whose one of it is bounded by
 $O \left(\frac{1}{\|\omega(t)\|_\infty} \right)$ and plays the role of the thickness of the pancake-like structure.

800 This suggests that for any $t \in [t_0, T^*[$ with $t_0 \in [0, T^*[$, $x \in \Theta(t)$, $\lambda \geq \Omega(t)$ the
 801 set $\{y \in \mathbb{R}^d; |\omega(y, t)| \geq \lambda\}$ may be characterized by three length scales whose
 802 one of them is of order $\frac{1}{\lambda}$, where Ω and Θ are respectively defined by (35) and
 803 (34). Since for any $0 < R \leq \rho_0(t)$ the set $\mathcal{V}_{\lambda, R}(t) \stackrel{\text{def}}{=} \{y \in B(x, R); |\omega(y, t)| \geq$
 804 $\lambda\} \subset B(x, R)$, we thus expect that the set $\mathcal{V}_{\lambda, R}(t)$ may have two of its length
 805 scales of order R and the third one of order $\frac{1}{\lambda}$. Then we expect that for for any
 806 $t \in [t_0, T^*[$, $x \in \Theta(t)$, $\lambda \geq \Omega(t)$ and $0 < R \leq \rho_0(t)$,

$$|\mathcal{V}_{\lambda, R}(t)| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}. \quad (116)$$

807 In Lemma 7.1 we give an argument in favour of the assumption (116) in the
 808 case of 3D Euler equations and 2D QG equation by using their Lagrangian
 809 structure. In Lemma 7.1, the property (P1) expresses the fact that we expect
 810 the length of any segment of a vortex line included in the structure $\mathcal{V}_{\lambda, R}(t)$
 811 is bounded by $O(R)$, since $\mathcal{V}_{\lambda, R}(t) \subset B(x, R)$. Furthermore property (P2)
 812 expresses the pancake structure of regions of high vorticity observed in numerical
 813 experiments. Indeed for the case of 3D Euler equations, if one assumes that
 814 the set $\mathcal{V}_{\lambda, R}^0(t) \stackrel{\text{def}}{=} X^{-1}(\mathcal{V}_{\lambda, R}(t), t_0, t)$ is characterized by three length scales
 815 $\ell_1^0, \ell_2^0, \ell_3^0$ associated to three main directions orthogonal between them pairwise,
 816 then we should have for one of these length scales $\ell_1^0 \lesssim \ell_{\lambda, R}^0(t)$ or $\ell_2^0 \lesssim \ell_{\lambda, R}^0(t)$ or
 817 $\ell_3^0 \lesssim \ell_{\lambda, R}^0(t)$ (117). Let us say that $\ell_1^0 \lesssim \ell_{\lambda, R}^0(t)$. Assuming that during the time
 818 between t_0 and $t \in [t_0, T^*[$ the set $\mathcal{V}_{\lambda, R}^0(t)$ becomes a pancake-shaped structure,
 819 we thus expect that $\ell_2^0 \lesssim R$ or $\ell_3^0 \lesssim R$ since $\mathcal{V}_{\lambda, R}(t) \subset B(x, R)$. Let us say that
 820 $\ell_2^0 \lesssim R$. For the last length scale ℓ_3^0 , we just expect that $\ell_3^0 = O(1)$.
 821 In the case of 3D Euler equations, we thus expect that $|\mathcal{V}_{\lambda, R}^0(t_0)| \lesssim \ell_{\lambda, R}^0(t)R$.
 822 In the case of 2D QG equations, we will have only the two length scales ℓ_1^0 and
 823 ℓ_3^0 and then we expect that $|\mathcal{V}_{\lambda, R}^0(t_0)| \lesssim \ell_{\lambda, R}^0(t)$. Then Lemma 7.1 gives an
 824 explanation of assumption (116).

825 **Lemma 7.1.** *Under the definitions (31)-(35) in the Theorem 5.1, we assume*
 826 *that there exists $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $0 < R \leq \rho_0(t)$*
 827 *and $\lambda \geq \Omega(t)$ the sets $\mathcal{V}_{\lambda, R}(t) \stackrel{\text{def}}{=} \{y \in B(x, R); |\omega(y, t)| \geq \lambda\}$ and $\mathcal{V}_{\lambda, R}^0(t) \stackrel{\text{def}}{=}$*
 828 *$X^{-1}(\mathcal{V}_{\lambda, R}(t), t_0, t)$ satisfy:*

829 (P1) $|L_t \cap \mathcal{V}_{\lambda, R}(t)| \lesssim R$ for any vortex line L_t at time t ,

830 (P2) $|\mathcal{V}_{\lambda, R}^0(t)| \lesssim \ell_{\lambda, R}^0(t)R^{d-2}$ where

$$\ell_{\lambda, R}^0(t) \stackrel{\text{def}}{=} \sup_{L_{t_0} \subset \mathcal{T}(t_0)} |L_{t_0} \cap \mathcal{V}_{\lambda, R}^0(t)| \quad (117)$$

831 and $\mathcal{T}(t_0)$ denotes the set of all vortex lines L_{t_0} at time t_0 .

832 Then, we get that for any $t \in [t_0, T^*[$

$$|\mathcal{V}_{\lambda, R}(t)| \lesssim \|\omega(t_0)\|_\infty \frac{R^{d-1}}{\lambda}.$$

833 *Proof.* Let us take $t \in [t_0, T^*[$. If $|\mathcal{V}_{\lambda,R}(t)| = 0$ the proof follows immedi-
 834 ately. Then we assume that $|\mathcal{V}_{\lambda,R}(t)| > 0$. We have $|\mathcal{V}_{\lambda,R}(t)| = |\mathcal{V}_{\lambda,R}^0(t)|$ since
 835 $X(\cdot, t_0, t)$ is a volume preserving C^1 -diffeomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$, then we
 836 get $|\mathcal{V}_{\lambda,R}^0(t)| > 0$ and Property (P2) yields to

$$|\mathcal{V}_{\lambda,R}(t)| \lesssim \ell_{\lambda,R}^0(t) R^{d-2}. \quad (118)$$

837 Furthermore for any vortex line L_{t_0} at time t_0 , we notice $X(L_{t_0} \cap \mathcal{V}_{\lambda,R}^0(t), t_0, t) =$
 838 $L_t \cap \mathcal{V}_{\lambda,R}(t)$ where L_t is the vortex line at time t defined by $L_t = X(L_{t_0}, t_0, t)$.
 839 Then, for any vortex line L_{t_0} at time t_0 such that $|L_{t_0} \cap \mathcal{V}_{\lambda,R}^0(t)| > 0$, by using
 840 the equation just below of (3.12) in Deng et al. (2005) and the definition of the
 841 set $\mathcal{V}_{\lambda,R}(t)$, we obtain

$$\frac{|L_t \cap \mathcal{V}_{\lambda,R}(t)|}{|L_{t_0} \cap \mathcal{V}_{\lambda,R}^0(t)|} \geq \frac{\lambda}{\|\omega(t_0)\|_\infty}. \quad (119)$$

842 By using Property (P1), for any vortex line L_{t_0} at time t_0 such that $|L_{t_0} \cap$
 843 $\mathcal{V}_{\lambda,R}^0(t)| > 0$, we deduce from (119) that $\frac{R}{|L_{t_0} \cap \mathcal{V}_{\lambda,R}^0(t)|} \gtrsim \frac{\lambda}{\|\omega(t_0)\|_\infty}$ which
 844 implies

$$\frac{R}{\ell_{\lambda,R}^0(t)} \gtrsim \frac{\lambda}{\|\omega(t_0)\|_\infty}. \quad (120)$$

845 Therefore by using (120), from (118) we thus infer $|\mathcal{V}_{\lambda,R}(t)| \lesssim \|\omega(t_0)\|_\infty \frac{R^{d-1}}{\lambda}$,
 846 which concludes the proof. $\square$

847 Thanks to assumption (116), we obtain the following Proposition 7.1.

848 **Proposition 7.1.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be*
 849 *such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D*
 850 *Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

851 *Under the definitions (32)-(37) in the Theorem 5.1, we assume that there exists*
 852 *$t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$, we get that for all $\lambda \geq \Omega(t)$*
 853 *and $0 < R \leq \rho_0(t)$*

$$|\{y \in B(x, R); |\omega(y, t)| \geq \lambda\}| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}. \quad (121)$$

854 *Then we get that for all $t \in [t_0, T^*[$*

$$\pi(t) \lesssim_{t_0} 1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right).$$

855 *Proof.* We have for all $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$

$$\begin{aligned}
\int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| dz &= \int_{B(x,R) \cap \mathcal{V}(t)} \int_0^{|\omega(z,t)|} d\lambda dz \\
&= \Omega(t) |B(x,R) \cap \mathcal{V}(t)| + \int_{B(x,R) \cap \mathcal{V}(t)} \int_{\Omega(t)}^{|\omega(z,t)|} d\lambda dz \\
&= \Omega(t) |B(x,R) \cap \mathcal{V}(t)| + \int_{\{z \in B(x,R), \Omega(t) < \lambda < |\omega(z,t)|\}} d\lambda dz \\
&= \Omega(t) |B(x,R) \cap \mathcal{V}(t)| \\
&\quad + \int_{[\Omega(t), \|\omega(t)\|_\infty]} |\{z \in B(x,R); |\omega(z,t)| > \lambda\}| d\lambda,
\end{aligned}$$

856 where we have used Fubini-Tonelli Theorem. Thanks to (121) we deduce that
857 for all $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$

$$\begin{aligned}
\int_{B(x,R) \cap \mathcal{V}(t)} |\omega(z,t)| dz &\lesssim_{t_0} R^{d-1} \left(1 + \int_{[\Omega(t), \|\omega(t)\|_\infty]} \frac{d\lambda}{\lambda} \right) \\
&= R^{d-1} \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right).
\end{aligned} \tag{122}$$

858 Owing to (122), we thus conclude the proof. $\square$

859 **Remark 7.1.** The analysis led in [Kuznetsov and Ruban \(2000\)](#); [Kuznetsov et al. \(2001\)](#) for the study of collapse of vortex lines and agrees with numerical exper-
860 iments [Agafontsev et al. \(2015, 2017\)](#) suggests that the thickness of the regions
861 of high vorticity $\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}$ is $\frac{1}{\lambda^{\frac{3}{2}}}$ and since these regions are
862 included in the ball $B(x,R)$, we expect that $|\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}| \lesssim_{t_0}$
863 $\frac{R^{d-1}}{\lambda^{\frac{3}{2}}}$. Then under this assumption and by using the same arguments as previ-
864 ously, we obtain

$$\pi(t) \lesssim \Omega(t)^{-\frac{1}{2}}.$$

866 Thanks to Theorem 5.1 and Proposition 7.1 we obtain Theorem 7.1.

867 **Theorem 7.1.** Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be
868 such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D
869 Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

870 Under the definitions (32)-(35) in the Theorem 5.1, we assume that there exists
871 $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$, we get that for all $\lambda \geq \Omega(t)$
872 and $0 < R \leq \rho_0(t)$

$$|\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}.$$

873 Then if there exists $t_1 \in [t_0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t) \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right) dt < +\infty,$$

874 then the solution u cannot blowup at the finite time T^* where

$$\begin{aligned} \mathbf{A}_d(t) &\stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|} \\ \rho(t) &= O \left((T^* - t) \left(1 + \log^+ \left(\frac{\|\omega(t)\|_\infty}{\Omega(t)} \right) \right) \right). \end{aligned}$$

875 **Remark 7.2.** Under the considerations of Remark 7.1, the non blow-up of the
876 solutions of Euler equations is obtained if there exists $t_1 \in [t_0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t) \Omega(t)^{-\frac{1}{2}} dt < +\infty.$$

877 Now, in the case of Euler equations and 2D QG equations by using their
878 Lagrangian formulation, after a fine and sharp analysis of the expression of π
879 (37) we go further in the non blow-up criteria by showing in Proposition 7.2
880 under mild assumptions based on the anisotropic structure of regions of high
881 vorticity, that $\pi(t) = O(1)$. For this purpose, we need the Lemmata 7.2, 7.3
882 and 7.4.

883 **Lemma 7.2.** Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such
884 that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler
885 equations (9)-(10) or 2D QG equations (11)-(12) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

886 Let $\mathcal{A}_0$ be a smooth surface of $\mathbb{R}^3$ with boundary if $d = 3$ or a curve of $\mathbb{R}^2$ if
887 $d = 2$. For any $t_0 \in [0, T^*[$, let $\mathcal{A}(t_0, t) \stackrel{\text{def}}{=} X(\mathcal{A}_0, t_0, t)$ be the evolution of $\mathcal{A}_0$
888 through the flow map X from the time t_0 to t , for any $t \in [0, T^*[$. For any
889 $t_0 \in [0, T^*[$ and $t \in [0, T^*[$, let

$$\Gamma(t_0, t) \stackrel{\text{def}}{=} \int_{\mathcal{A}(t_0, t)} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y),$$

890 where $\mathbf{n}_t(\cdot)$ denotes a unit normal vector of $\mathcal{A}(t_0, t)$.

891 Then for any $t_0 \in [0, T^*[$ we get that $\Gamma(t_0, \cdot)$ is constant over $[0, T^*[$, that is to
892 say for all $t \in [t_0, T^*[$,

$$\Gamma(t_0, t) = \Gamma(t_0, t_0).$$

893 *Proof.* Thanks to Lemma 5 and Remark 3 of Schmidt and Schulz (2010) (see
894 also (4.9) chapter 9 in Delfour and Zolésio (2011)) and Lemma 7 of Schmidt and Schulz
895 (2010), we infer that for any $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\int_{\mathcal{A}(t_0, t)} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) = \int_{\mathcal{A}_0} |\omega(X(\alpha, t_0, t), t) \cdot (\nabla X(\alpha, t_0, t))^{-T} \mathbf{n}(\alpha)| d\sigma(\alpha),$$

where $\mathbf{n}$ is a unit normal vector of $\mathcal{A}_0$. Thanks to (113), we infer that for any $\alpha \in \mathcal{A}_0$, $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\omega(X(\alpha, t_0, t), t) \cdot (\nabla X(\alpha, t_0, t))^{-T} \mathbf{n}(\alpha) = \omega(\alpha, t_0) \cdot \mathbf{n}(\alpha).$$

Therefore, we thus obtain that for any $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\int_{\mathcal{A}(t_0, t)} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) = \int_{\mathcal{A}_0} |\omega(\alpha, t_0) \cdot \mathbf{n}(\alpha)| d\sigma(\alpha),$$

which concludes the proof. $\square$

Lemma 7.3. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

Let $t \in [0, T^*[$ and $\mathfrak{T}_t$ a vortex tube at this time. Let $\mathcal{A}_t$ and $\mathcal{B}_t$ be two connected smooth orientable surfaces of $\mathbb{R}^3$ (resp. curves of $\mathbb{R}^2$ if $d = 2$) with boundary such their boundary encircle the vortex tube $\mathfrak{T}_t$ and such that any vortex line of the vortex tube $\mathfrak{T}_t$ intersects both $\mathcal{A}_t$ and $\mathcal{B}_t$ once each of them. Then we get

$$\int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) = \int_{\mathcal{B}_t} |\omega(y, t) \cdot \tilde{\mathbf{n}}_t(y)| d\tilde{\sigma}(y),$$

where $\mathbf{n}_t$ and $\tilde{\mathbf{n}}_t$ are respectively the unit normal vector varying smoothly on the surfaces (resp. curves if $d = 2$) $\mathcal{A}_t$ and $\mathcal{B}_t$, oriented to be outward to the portion of the tube $\mathfrak{T}_t$ delimited by $\mathcal{A}_t$ and $\mathcal{B}_t$.

Proof. For any $x \in \mathcal{A}_t$, we denote by $\mathfrak{T}_t(x)$ the vortex line passing through x at time t , we get $\mathfrak{T}_t(x) \subset \mathfrak{T}_t$ and there exists a unique $y_{x,t} \in \mathcal{B}_t$ such that $\mathfrak{T}_t(x) \cap \mathcal{B}_t = \{y_{x,t}\}$. We thus introduce the function Φ_t defined from $\mathcal{A}_t$ to $\mathcal{B}_t$ for all $x \in \mathcal{A}_t$ by $\Phi_t(x) = y_{x,t}$. Since any vortex line of the vortex tube $\mathfrak{T}_t$ intersects both $\mathcal{A}_t$ and $\mathcal{B}_t$ once each of them and since also $\mathcal{A}_t$ and $\mathcal{B}_t$ are smooth surfaces (smooth curves if $d = 2$), we infer thanks also to (115) that the function Φ_t is a homeomorphism from $\mathcal{A}_t$ to $\mathcal{B}_t$.

We introduce the pairwise disjoint subsets of $\mathcal{A}_t$, namely $\mathcal{A}_t^+ \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y, t) \cdot \mathbf{n}_t(y) > 0\}$, $\mathcal{A}_t^- \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y, t) \cdot \mathbf{n}_t(y) < 0\}$ and $\mathcal{A}_t^0 \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y, t) \cdot \mathbf{n}_t(y) = 0\}$. By the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow BC^{m_r}(\mathbb{R}^d)$, $m_r = [r - 1 - \frac{d}{2}] \geq 2$, we get $\omega(t) \in BC^{m_r}(\mathbb{R}^d)$. Then $\mathcal{A}_t^+$ and $\mathcal{A}_t^-$ are open subsets of the surface (curve if $d = 2$) $\mathcal{A}_t$ and thus they are also smooth surfaces (smooth

923 curves if $d = 2$). On one hand, we have

$$\begin{aligned}
\int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) &= \int_{\mathcal{A}_t^+} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) + \int_{\mathcal{A}_t^-} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) \\
&\quad + \int_{\mathcal{A}_t^0} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) \\
&= \int_{\mathcal{A}_t^+} \omega(y, t) \cdot \mathbf{n}_t(y) d\sigma(y) - \int_{\mathcal{A}_t^-} \omega(y, t) \cdot \mathbf{n}_t(y) d\sigma(y),
\end{aligned} \tag{123}$$

924 where we have used the definition of the sets $\mathcal{A}_t^+$, $\mathcal{A}_t^-$ and $\mathcal{A}_t^0$. Since Φ_t is a
925 homeomorphism from $\mathcal{A}_t$ to $\mathcal{B}_t$, we get that $\Phi_t(\mathcal{A}_t^+) \subset \mathcal{B}_t$, $\Phi_t(\mathcal{A}_t^-) \subset \mathcal{B}_t$ and
926 $\Phi_t(\mathcal{A}_t^+) \cap \Phi_t(\mathcal{A}_t^-) = \emptyset$. On the other hand, thanks to Helmholtz's first vortex
927 Theorem (see e.g. (Wu, 2018, chapter 2)), we have

$$\int_{\mathcal{A}_t^+} \omega(y, t) \cdot \mathbf{n}_t(y) d\sigma(y) = - \int_{\Phi_t(\mathcal{A}_t^+)} \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) d\tilde{\sigma}(y), \tag{124}$$

928 and

$$\int_{\mathcal{A}_t^-} \omega(y, t) \cdot \mathbf{n}_t(y) d\sigma(y) = - \int_{\Phi_t(\mathcal{A}_t^-)} \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) d\tilde{\sigma}(y). \tag{125}$$

929 Then owing to (124) and (125), from (123) we deduce

$$\begin{aligned}
\int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) &= - \int_{\Phi_t(\mathcal{A}_t^+)} \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) d\tilde{\sigma}(y) \\
&\quad + \int_{\Phi_t(\mathcal{A}_t^-)} \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) d\tilde{\sigma}(y),
\end{aligned}$$

930 which implies

$$\int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y) \leq \int_{\mathcal{B}_t} |\omega(y, t) \cdot \tilde{\mathbf{n}}_t(y)| d\tilde{\sigma}(y). \tag{126}$$

931 It remains to show that

$$\int_{\mathcal{B}_t} |\omega(y, t) \cdot \tilde{\mathbf{n}}_t(y)| d\tilde{\sigma}(y) \leq \int_{\mathcal{A}_t} |\omega(y, t) \cdot \mathbf{n}_t(y)| d\sigma(y). \tag{127}$$

932 By introducing the pairwise disjoint subsets of $\mathcal{B}_t$, namely $\mathcal{B}_t^+ \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y, t) \cdot$
933 $\tilde{\mathbf{n}}_t(y) > 0\}$, $\mathcal{B}_t^- \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y, t) \cdot \tilde{\mathbf{n}}_t(y) < 0\}$ and $\mathcal{B}_t^0 \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y, t) \cdot$
934 $\tilde{\mathbf{n}}_t(y) = 0\}$ and using the fact that Φ_t^{-1} is a homeomorphism from $\mathcal{B}_t$ to $\mathcal{A}_t$,
935 we deduce with the same arguments used to get (126), inequality (127). Then,
936 owing to (126) and (127) we conclude the proof. $\square$

937 Before to turn to the proof of Lemma 7.4, Proposition 7.2 and Theorem 7.2,
938 we need to introduce some definitions. Let $r > \frac{d}{2} + 3$ and $T^* > 0$ be such

that there exists an unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

For any $t \in [0, T^*]$ and any vortex tube $\mathfrak{T}_t$ at time t , we define by $\mathcal{S}(\mathfrak{T}_t)$ the set of the connected smooth orientable surfaces of $\mathbb{R}^3$ (curves of $\mathbb{R}^2$ if $d = 2$) with boundary that is intersected only once by any vortex line of $\mathfrak{T}_t$ and such that their boundary encircle the vortex tube $\mathfrak{T}_t$. We define also the function $\Gamma_{\mathfrak{T}_t}$ defined from $\mathcal{S}(\mathfrak{T}_t)$ to $[0, +\infty[$ for all $\mathcal{A} \in \mathcal{S}(\mathfrak{T}_t)$ by

$$\Gamma_{\mathfrak{T}_t}(\mathcal{A}) \stackrel{\text{def}}{=} \int_{\mathcal{A}} |\omega(y, t) \cdot \mathbf{n}(y)| d\sigma(y). \quad (128)$$

Thanks to Lemma 7.3, we deduce that for any $t \in [0, T^*]$ and any vortex tube $\mathfrak{T}_t$ at time t

$$\Gamma_{\mathfrak{T}_t} \text{ is constant over } \mathcal{S}(\mathfrak{T}_t). \quad (129)$$

Owing to (129), for any $t \in [0, T^*]$ and any vortex tube $\mathfrak{T}_t$ at time t , we define $\Gamma_{\text{abs}}(\mathfrak{T}_t)$ that we call the absolute strength of the vortex tube $\mathfrak{T}_t$ by

$$\Gamma_{\text{abs}}(\mathfrak{T}_t) \stackrel{\text{def}}{=} \Gamma_{\mathfrak{T}_t}(\mathcal{A}_0), \quad (130)$$

with $\mathcal{A}_0$ an arbitrary element of $\mathcal{S}(\mathfrak{T}_t)$. As vortex tube moves with the fluid characterized by the flow map X (thanks to Helmholtz's first vortex Theorem), then for any vortex tube $\mathfrak{T}_t$ at a time $t \in [0, T^*]$, we deduce that $X(\mathfrak{T}_t, t, \tau)$ is a vortex tube at time τ for any $\tau \in [0, T^*]$.

Thanks to Lemma 7.2, we infer that for any $t \in [0, T^*]$ and any vortex tube $\mathfrak{T}_t$ at time t ,

$$\Gamma_{\text{abs}}(\mathfrak{T}_t) = \Gamma_{\text{abs}}(X(\mathfrak{T}_t, t, \tau)) \text{ for any } \tau \in [0, T^*], \quad (131)$$

which means that the absolute strength of any vortex tube $\mathfrak{T}_t$ at a time $t \in [0, T^*]$ moving with the fluid does not change with the time.

Lemma 7.4. *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

Let $t \in [0, T^*]$ and $\mathfrak{T}_t$ a vortex tube at time t defined by $\mathfrak{T}_t \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_t, s \in J_t\}$ with $\mathcal{A}_t$ a connected smooth orientable surface of $\mathbb{R}^3$ (curve of $\mathbb{R}^2$ if $d = 2$) with boundary and J_t an interval of $\mathbb{R}$ containing 0 such that

$$\bullet \quad J_t \subset \bigcap_{\alpha \in \mathcal{A}_t} \mathfrak{J}_{\alpha, t},$$

any vortex line of the tube $\mathfrak{T}_t$ intersects $\mathcal{A}_t$ only once, i.e

$$\forall \beta \in \mathcal{A}_t, \quad \{\mathbf{x}_t(\beta, s); s \in \mathfrak{J}_{\beta, t}\} \cap \mathcal{A}_t = \{\beta\}. \quad (132)$$

967 Then we have

$$\int_{\mathfrak{T}_t} |\omega(z, t)| dz = |J_t| \Gamma_{\text{abs}}(\mathfrak{T}_t).$$

968 *Proof.* If $J_t = 0$ then the result follows immediately. Therefore we assume that
 969 $J_t \neq \{0\}$. For any $s \in J_t$, we define the smooth surface of $\mathbb{R}^3$ (curve of $\mathbb{R}^2$ if
 970 $d = 2$) with boundary,

$$\mathcal{A}_t(s) \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_t\}.$$

971 Due to the definition of the vortex tube $\mathfrak{T}_t$, we get that for any $s \in J_t$, the
 972 boundary of $\mathcal{A}_t(s)$ encircles the vortex tube $\mathfrak{T}_t$. Thanks to (114), we get

$$\int_{\mathfrak{T}_t} |\omega(z, t)| dz = \int_{s \in J_t} \int_{\mathcal{A}_t(s)} |\omega(y, t)| |\mathbf{n}_t(s) \cdot \xi(y, t)| d\sigma(\alpha) ds,$$

973 where $\mathbf{n}_t(s)$ is a unit normal vector of $\mathcal{A}_t(s)$. Since $\omega(y, t) = |\omega(y, t)|\xi(y, t)$ then
 974 we obtain

$$\begin{aligned} \int_{\mathfrak{T}_t} |\omega(z, t)| dz &= \int_{s \in J_t} \int_{\mathcal{A}_t(s)} |\omega(y, t) \cdot \mathbf{n}_t(s)| d\sigma(\alpha) ds \\ &= \int_{s \in J_t} \Gamma_{\mathfrak{T}_t}(\mathcal{A}_t(s)) ds. \end{aligned} \quad (133)$$

975 We show now that for any $s_0 \in J_t$, any vortex line of the vortex tube $\mathfrak{T}_t$
 976 intersects $\mathcal{A}_t(s_0)$ only once. For this purpose, let $\alpha_1 \in \mathcal{A}_t(s_0)$. Thanks to
 977 Cauchy-Lipschitz Theorem (see e.g Theorem 2.2 in [Teschl \(2012\)](#)) used for (114),
 978 we deduce that there exists a unique $\beta_1 \in \mathcal{A}_t$ such that $\alpha_1 = \mathbf{x}_t(\beta_1, s_0)$.
 979 Suppose for a contradiction that

$$\{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1, t}\} \cap \mathcal{A}_t(s_0) \neq \{\alpha_1\}.$$

980 Then there exists $\alpha_2 \neq \alpha_1$ such that $\alpha_2 \in \{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1, t}\} \cap \mathcal{A}_t(s_0)$. There-
 981 fore we get that $\alpha_2 = \mathbf{x}_t(\beta_1, s_2)$ with $s_2 \in \mathfrak{J}_{\beta_1, t}$, $s_2 \neq s_0$ since $\alpha_2 \neq \alpha_1$. We
 982 get also that there exists a unique $\beta_2 \in \mathcal{A}_t$ such that $\alpha_2 = \mathbf{x}_t(\beta_2, s_0)$ where
 983 $\beta_2 \neq \beta_1$ since $\alpha_2 \neq \alpha_1$. We thus infer that

$$\mathbf{x}_t(\beta_1, s_2) = \mathbf{x}_t(\beta_2, s_0). \quad (134)$$

984 By the maximality of $\mathbf{x}_t$, from (134) we infer that $s_2 - s_0 \in \mathfrak{J}_{\beta_1, t}$ and $\beta_2 =$
 985 $\mathbf{x}_t(\beta_1, s_2 - s_0)$ which implies $\{\beta_1, \beta_2\} \subset \{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1, t}\} \cap \mathcal{A}_t$. This latter
 986 contradicts (132). Therefore, we deduce that

$$\{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1, t}\} \cap \mathcal{A}_t(s_0) = \{\alpha_1\}.$$

987 This means that the vortex line of the vortex tube $\mathfrak{T}_t$ passing through $\alpha_1 \in$
 988 $\mathcal{A}_t(s_0)$ intersects $\mathcal{A}_t(s_0)$ only once, which matches to our desired result. Then
 989 we get that for any $s \in J_t$, $\mathcal{A}_t(s) \in \mathcal{S}(\mathfrak{T}_t)$ and hence thanks to (129) and (130),
 990 from (133) we obtain $\int_{\mathfrak{T}_t} |\omega(z, t)| dz = |J_t| \Gamma_{\text{abs}}(\mathfrak{T}_t)$. Then we conclude the
 991 proof. $\square$

992 **Proposition 7.2.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be*
 993 *such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D*
 994 *Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

995 *Under the definitions (32)-(37) in the Theorem 5.1, we assume that there exists*
 996 *$t_0 \in [0, T^*]$ such that for any $t \in [t_0, T^*]$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$ there*
 997 *exists a vortex tube $\mathfrak{T}_{x,t}^R$ defined by $\mathfrak{T}_{x,t}^R \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_{x,t}^R, s \in I_{x,t}^R\}$ with*
 998 *$\mathcal{A}_{x,t}^R$ a connected smooth orientable surface of $\mathbb{R}^3$ (curve of $\mathbb{R}^2$ if $d = 2$) and $I_{x,t}^R$*
 999 *an interval of $\mathbb{R}$ containing 0 such that:*

1000 (P1) $\mathcal{V}(t) \cap B(x, R) \subset \mathfrak{T}_{x,t}^R.$

1001 (P2) any vortex line of the tube $\mathfrak{T}_{x,t}^R$ intersects $\mathcal{A}_{x,t}^R$ only once.

1002 (P3) $|I_{x,t}^R| \lesssim R$

1003 (P4) $\Gamma_{\text{abs}}(\mathfrak{T}_{x,t}^R) \lesssim \overline{\mathbf{v}}(t_0) R^{d-2}$ where $\overline{\mathbf{v}}(t_0) > 0$ is a real which depend only on
 1004 t_0 (and have the characteristic of a velocity).

1005 Then we get that for all $t \in [t_0, T^*]$

$$\pi(t) \lesssim \overline{\mathbf{v}}(t_0).$$

1006 *Proof.* Let $t \in [t_0, T^*]$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$. Thanks to property (P1)
 1007 we have

$$\int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz \leq \int_{\mathfrak{T}_{x,t}^R} |\omega(z, t)| dz. \quad (135)$$

1008 Furthermore, thanks to property (P2) and Lemma 7.4 we get

$$\int_{\mathfrak{T}_{x,t}^R} |\omega(z, t)| dz = |I_{x,t}^R| \Gamma_{\text{abs}}(\mathfrak{T}_{x,t}^R). \quad (136)$$

1009 Thanks to the properties (P3) and (P4), from (136) we deduce

$$\int_{\mathfrak{T}_{x,t}^R} |\omega(z, t)| dz \lesssim R^{d-1} \overline{\mathbf{v}}(t_0). \quad (137)$$

1010 Owing to (137), from (135) we infer

$$\int_{B(x, R) \cap \mathcal{V}(t)} |\omega(z, t)| dz \lesssim R^{d-1} \overline{\mathbf{v}}(t_0). \quad (138)$$

1011 From the definition (37) of the function π , thanks to (138) we thus deduce that
 1012 for all $t \in [t_0, T^*]$

$$\pi(t) \lesssim \overline{\mathbf{v}}(t_0),$$

1013 which concludes the proof. $\square$

1014 In the two following Remarks, we give explicit values for $\overline{\mathbf{v}}(t_0)$.

1015 **Remark 7.3.** *In the case of 2D QG equation for which $d = 2$, we have that for*
 1016 *all $t \in [t_0, T^*[,$*

$$\pi(t) \lesssim \|u_0\|_\infty,$$

1017 *if we replace the hypothesis (P4) by the assumption that the real-valued function*
 1018 *$\omega(\cdot, t) \cdot \mathbf{n}$ keeps a constant sign over $\mathcal{A}_{x,t}^R$, where $\mathbf{n}$ is a unit normal vector*
 1019 *varying smoothly on $\mathcal{A}_{x,t}^R$.*

1020 *Indeed in this case, we get $\int_{\mathcal{A}_{x,t}^R} |\omega(\alpha, t) \cdot \mathbf{n}(\alpha)| d\alpha = \left| \int_{\mathcal{A}_{x,t}^R} \omega(\alpha, t) \cdot \mathbf{n}(\alpha) d\alpha \right|$*
 1021 *and furthermore thanks to Stokes Theorem we have $\int_{\mathcal{A}_{x,t}^R} \omega(\alpha, t) \cdot \mathbf{n}(\alpha) d\alpha =$*
 1022 *$u(\alpha_2, t) - u(\alpha_1, t)$ where α_2 and α_1 are the two endpoints of the line segment*
 1023 *$\mathcal{A}_{x,t}^R$. We thus infer $\int_{\mathcal{A}_{x,t}^R} |\omega(\alpha, t) \cdot \mathbf{n}(\alpha)| d\alpha \leq 2\|u(t)\|_\infty = 2\|u_0\|_\infty$ thanks to*
 1024 *(22) and then we take $\overline{\mathbf{v}}(t_0) = \|u_0\|_\infty$. Then with the properties (P1)-(P3), we*
 1025 *thus obtain that for all $t \in [t_0, T^*[, \pi(t) \lesssim \|u_0\|_\infty$.*

1026 **Remark 7.4.** *For any $t \in [t_0, T^*[, x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$ let us assume*
 1027 *that there exists $t_1 \in [0, t_0]$ depending on t, x and R such that for the vortex tube*
 1028 *$\mathfrak{T}_{x,t_1}^R \stackrel{\text{def}}{=} X(\mathfrak{T}_{x,t}^R, t, t_1)$ at time t_1 we have*

$$\inf_{\mathcal{A} \in \mathcal{S}(\mathfrak{T}_{x,t_1}^R)} |\mathcal{A}| \lesssim R^{d-2},$$

1029 *then Property (P4) holds with $\overline{\mathbf{v}}(t_0) = \|\omega\|_{L^\infty(\mathbb{R}^d \times [0, t_0])}$. Indeed thanks to (131)*
 1030 *we have*

$$\Gamma_{\text{abs}}(\mathfrak{T}_{x,t}^R) = \Gamma_{\text{abs}}(\mathfrak{T}_{x,t_1}^R).$$

1031 *Furthermore, thanks to (129) and (130), we deduce that*

$$\begin{aligned} \Gamma_{\text{abs}}(\mathfrak{T}_{x,t_1}^R) &= \inf_{\mathcal{A} \in \mathcal{S}(\mathfrak{T}_{x,t_1}^R)} \int_{\mathcal{A}} |\omega(y, t_1) \cdot \mathbf{n}(y)| d\sigma(y) \\ &\leq \|\omega(t_1)\|_\infty \inf_{\mathcal{A} \in \mathcal{S}(\mathfrak{T}_{x,t_1}^R)} |\mathcal{A}| \\ &\lesssim \|\omega\|_{L^\infty(\mathbb{R}^d \times [0, t_0])} R^{d-2}. \end{aligned}$$

1032 *Therefore, we deduce that $\Gamma_{\text{abs}}(\mathfrak{T}_{x,t}^R) \lesssim \|\omega\|_{L^\infty(\mathbb{R}^d \times [0, t_0])} R^{d-2}$ which matches*
 1033 *with Property (P4) for $\overline{\mathbf{v}}(t_0) = \|\omega\|_{L^\infty(\mathbb{R}^d \times [0, t_0])}$.*

1034 Then thanks to Theorem 5.1 and Proposition 7.2 we deduce Theorem 7.2.

1035 **Theorem 7.2.** *Let $d \in \{2, 3\}$, $u_0 \in H_\sigma^r(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be*
 1036 *such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D*
 1037 *Euler equations (9)-(10) or 2D QG equations (11)-(12) in the class*

$$u \in C([0, T^*]; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0, T^*]; H^{r-2}(\mathbb{R}^d)).$$

Under the definitions (32)-(36) in the Theorem 5.1, we assume that there exists $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$ there exists a vortex tube $\mathfrak{T}_{x,t}^R$ defined by $\mathfrak{T}_{x,t}^R \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_{x,t}^R, s \in I_{x,t}^R\}$ with $\mathcal{A}_{x,t}^R$ a connected smooth orientable surface of $\mathbb{R}^3$ (curve of $\mathbb{R}^2$ if $d = 2$) and $I_{x,t}^R$ an interval of $\mathbb{R}$ containing 0 such that:

$$(P1) \quad \mathcal{V}(t) \cap B(x, R) \subset \mathfrak{T}_{x,t}^R.$$

$$(P2) \quad |I_{x,t}^R| \lesssim R$$

$$(P3) \quad \text{any vortex line of the tube } \mathfrak{T}_{x,t}^R \text{ intersects } \mathcal{A}_{x,t}^R \text{ only once.}$$

$$(P4) \quad \Gamma_{\text{abs}}(\mathfrak{T}_{x,t}^R) \lesssim \overline{\mathbf{v}}(t_0) R^{d-2} \text{ where } \overline{\mathbf{v}}(t_0) > 0 \text{ is a real depending only on } t_0 \text{ (and have the characteristic of a velocity).}$$

Then if there exists $t_1 \in [t_0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t) dt < +\infty,$$

then the solution u cannot blowup at the finite time T^* with

$$\mathbf{A}_d(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y, t), \xi(x, t))^+}{|y|}$$

$$\rho(t) = O((T^* - t)\overline{\mathbf{v}}(t_0)).$$

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