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Global regularity of solutions of equation modeling epitaxy thin film growth in $\mathbb{R}^d$, $d = 1, 2$

Léo Agélas *

August 7, 2014

Abstract

We show existence and uniqueness of global strong solutions for any initial data $u_0 \in H^s(\mathbb{R}^d)$, with $d \in \{1, 2\}$, $s \geq 3$, of the general equation of surface growth models arising in the context of epitaxy thin films in the presence of the coarsening process, density variations and the Ehrlich-Schwoebel effect. Up to now, the problem of existence and smoothness of global solutions of such equations remains open in $\mathbb{R}^d$, $d \in \{1, 2\}$. In this article, we show that taking into account of the main physical phenomena and a better approximation of terms related to them in the mathematical model, lead to a kind of "cancellation" of nonlinear terms between them in some spaces and from this, we obtain existence and uniqueness of global strong solutions for such equations in $\mathbb{R}^d$, $d \in \{1, 2\}$.

1 Introduction

The formation and spatio-temporal evolution of interfaces by deposition processes are ubiquitous phenomena in nature (see [11]). Such a surface growth can be observed on macroscopic scales, e.g during the aggregation of snow flakes or the heap formation as consequence of the downpour of granular material. A deposition processes of greater technological importance than snowfall takes place during the growth of thin films by molecular beam epitaxy (MBE), a technology used to manufacture computer chips and other semiconductor devices, indispensable in today's technological world. Other applications requiring thin films include solar cells, mechanical coatings, and, more recently, microelectromechanical systems and microfluidic devices. Growth conditions have a profound effect on the morphological quality of films [28] and has recently received increasing interest in materials science. A major reason for this interest is that compositions like $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ (YBCO) are expected to be high-temperature super-conducting and could be used in the design of semi-conductors. The complex process of building up a thin film layer on a substrate by chemical vapor deposition has now given rise to several descriptions and simulations by atomistic as well as by continuum models (see [18],[23] for an extensive survey of the corresponding literature). One of the outstanding challenges is to understand these growth processes qualitatively and quantitatively, so that control laws can be formulated which optimize certain film properties, e.g., flatness, conductivity.

In consequence, the mathematical models for the study of surface growth and the experiments done thus improving these models in terms of physics has attracted a lot of attention in recent years, one can see for example the reviews in [6], [11], [26], [17], [19]-[22].

In MBE, the height h describing the local position of the moving surface obeys a conservation law,

$$\partial_t h(x, t) = -\nabla \cdot \mathbf{J}(\nabla h(x, t)) + \eta(x, t), \quad (1)$$

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where $\mathbf{J}(\nabla h)$ is the surface current depending on the macroscopic gradient ∇h of the film surface, η is the shot noise due to fluctuations of the incoming particle beam, and the height is measured in a comoving frame of reference. From [26] and [28] (see also the results obtained in [19]-[22]), the natural generalization of the differential equation modeling epitaxial thin film growth takes the form,

$$\partial_t h + \nu_1 \Delta h + \nu_2 \Delta^2 h - \nu_3 \nabla \cdot (|\nabla h|^2 \nabla h) + \nu_4 \Delta |\nabla h|^2 = \nu_5 |\nabla h|^2 + \eta, \quad (2)$$

with initial conditions,

$$h(x, 0) = h_0(x), \quad (3)$$

on $\Omega = \mathbb{R}^d$ with solutions vanishing at infinity as $|x| \rightarrow \infty$ or $\Omega = \mathbb{R}^d / \mathbb{Z}^d$, with periodic boundary conditions and in this case we require in addition that h_0 is a periodic scalar function of period one, $\nu_1 \geq 0$, $\nu_2 > 0$, $\nu_3 \geq 0$, $\nu_4 \geq 0$, $\nu_5 \geq 0$. In [26], from equation with j_α just above equation (4.2), by using a more precise linearization, we notice that the term $\nabla \cdot (|\nabla h|^2 \nabla h)$ appears simultaneously with the term Δh . In terms of physical interpretation, the term Δh denotes the diffusion due to evaporation-condensation and the term $\nabla \cdot (|\nabla h|^2 \nabla h)$ denotes the (upward) hopping of atoms, they model together the Ehrlich-Schwoebel effect (see [24], [16], [9]). Let us precise the reasons for which we assume $\nu_3 > 0$, since this sign is completely critical for the existence theory set forth in this paper. From [24] and [16], the form of the contribution of the kinetic surface current due to the Ehrlich-Schwoebel effect is given as follows $\mathbf{J}_s(\mathbf{m}) = D_s \mathbf{m} f(m^2)$, where $\mathbf{m} \in \mathbb{R}^d$, $D_s > 0$, $m = |\mathbf{m}|$ and f a univariate function. According to physical arguments, several forms for the function f have been given, as a simple analytic form, Johnson *et al* proposed $f(x) = \frac{1}{1 + l_d x}$, $l_d > 0$ (see [12], see also [16]). In [24], a more general form for the function f is proposed to model the current for a structure with, e.g, cubic symmetry with, $f(x) = (1 - x)/[(1 - x)^2 + l_d^2 x]$ replaced then by $f(x) = 1 - x$ to be in agreement with the Lifshitz-Slyozov growth law (see [24] for more details on the choice of $f(x) = 1 - x$). Then, the expression $\mathbf{J}_s(\mathbf{m}) = D_s \mathbf{m}(1 - m^2)$ models the surface current due to the Ehrlich-Schwoebel effect for a structure with cubic symmetry (we can refer also to Section Introduction in [16] and references therein).

Therefore, the positivity of coefficient ν_3 derives from the expression of the surface current $\mathbf{J}_s$ (we can also refer to [9] to deduce that $\nu_3 > 0$). In a typical step-flow or layer-by-layer epitaxial growth of thin films, adatoms-atoms that are adsorbed onto the surface but have not yet become part of the crystal, diffuse on a terrace and likely hit a terrace boundary. In order to stick to the boundary from an upper terrace, an adatom must overcome a higher energy barrier, the Ehrlich-Schwoebel barrier (see [16] and references therein). This asymmetry in attachment and detachment of adatoms to and from terrace boundaries has many important consequences: it induces an uphill current which in general destabilizes nominal surfaces, but stabilizes vicinal surfaces (see [16] and references therein). The term $\Delta^2 h$ denotes the capillarity-driven surface diffusion, the term $\Delta |\nabla h|^2$ is related to the equilibration of the inhomogeneous concentration of the diffusing particles on the surface (known as the coarsening process), the term $|\nabla h|^2$ is related to the density variations (see [19]-[22], for more details on physical interpretations of these terms).

The existence of global weak solutions in dimension $d = 1$ on bounded domains has been studied in [3]. Winkler and Stein [25] used Rothe's method to verify the existence of a global weak solution for $\nu_3 = \nu_5 = 0$, this result has been recently extended by Winkler [27] to the two-dimensional case, using energy type estimates for $\int e^h dx$.

A crucial open problem for Equation (2) when $\nu_3 = 0$ and $\nu_4 \neq 0$ or $\nu_5 \neq 0$ is the fact that existence and uniqueness of global strong solutions is not known (see [5] and references therein) in the two dimensional case. Even, in the one dimensional case, for Equation (2) when $\nu_3 = 0$ and $\nu_4 \neq 0$, the question of global regularity is still open (see [4] and references therein). In view of the quadratic growth of the nonlinear terms $\Delta^2 |\nabla h|^2$ and $|\nabla h|^2$, it is a priori not clear whether such solutions can be extended to exist for all times, or if finite-time blow-up phenomena may occur.

However, in the case where $\nu_3 \neq 0$, $\nu_4 = 0$ and $\nu_5 = 0$, uniqueness and regularity of global solutions is established (see [15]). The main difficulties for treating problem (2) are caused by the nonlinearity terms

and the lack of a maximum principle. Due to its nonlinear parts, there are more difficulties in establishing the global existence of strong solutions.

In this paper, our main result is the proof of existence and uniqueness of global strong solutions of Equation (2) only under the condition that $\nu_2\nu_3 > \nu_4^2$. In our analysis, the presence of the nonlinear term $\nabla \cdot (|\nabla h|^2 \nabla h)$ is crucial since it allows to control the coarsening process expressed by the nonlinear term $\Delta|\nabla h|^2$ and the density variations expressed by the nonlinear term $|\nabla h|^2$.

For simplicity of presentation, we neglect the noise function η in Equation (2).

In the periodic case, by choosing the initial data h_0 as a periodic function of period one, we have chosen to consider periodic solutions of period one. The general case with period $L > 0$ can be obtained from the case of period one by rescaling h the periodic solution of period L of Equation (2) on $\mathbb{R}^d/L\mathbb{Z}^d \times [0, T]$ as follows $h(x, t) = u(\frac{x}{L}, \frac{t}{L^4})$, then u is a periodic solution of period one of Equation (2) on $\Omega \times [0, \frac{T}{L^4}]$ with ν_1 and ν_5 respectively replaced by $L^2\nu_1$, $L^2\nu_5$. Then, in the periodic case, thanks to this rescaling, the results obtained for the case of period one are strictly the same for the case with period $L > 0$.

This paper is organized as follows. In Section 2, we give some notations and introduce some Sobolev spaces. In Section 3, under the condition that $\nu_2\nu_3 > \nu_4^2$, we prove existence and uniqueness of global strong solutions for initial data, h_0 sufficiently regular, in our case $h_0 \in H^s$, $s \geq 3$. In view of non-regular interfaces observed in subsections 4.7.3-4.7.5 of [11], it was natural to study the weak solutions of problem (2), therefore in Section 4, we introduce the notion of weak solution and prove existence, uniqueness of global weak solutions for initial data in L^2 . In Section 5, we show smoothness of weak solutions up to time $t = 0$ for initial data in $H^{\frac{d}{2}}$.

2 Some notations

We denote $A \lesssim B$, the estimate $A \leq CB$ where $C > 0$ is a absolute constant. We use ∂_i to denote the derivative with respect to the i^{th} spatial coordinate x_i . We denote by D^2f the hessian matrix of the scalar field f that is to say, $\{\partial_i\partial_j f\}_{1 \leq i, j \leq d}$. Given an absolutely integrable function $f \in L^1(\mathbb{R}^3)$, we define the Fourier transform $\hat{f} : \mathbb{R}^d \mapsto \mathbb{C}$ by the formula,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx,$$

and extend it to tempered distributions. For a function f which is periodic with period 1, and thus representable as a function on the torus $\mathbb{R}^d/\mathbb{Z}^d$, we define the discrete Fourier transform $\hat{f} : \mathbb{Z}^d \mapsto \mathbb{C}$ by the formula,

$$\hat{f}(k) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot k} f(x) dx,$$

when f is absolutely integrable on $\mathbb{R}^d/\mathbb{Z}^d$, and extend this to more general distributions on $\mathbb{R}^d/\mathbb{Z}^d$.

In $\mathbb{R}^d$ and for $s \in \mathbb{R}$, we define the Sobolev norm $\|f\|_{H^s(\mathbb{R}^d)}$ of a tempered distribution $f : \mathbb{R}^d \mapsto \mathbb{R}$ by,

$$\|f\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

and then we denote by $H^s(\mathbb{R}^d)$ the space of tempered distributions with finite $H^s(\mathbb{R}^d)$ norm, which matches when s is a non negative integer with the classical Sobolev space $H^k(\mathbb{R}^d)$, $k \in \mathbb{N}$. For $s > -\frac{d}{2}$, we also define the homogeneous Sobolev norm,

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

and then we denote by $\dot{H}^s(\mathbb{R}^d)$ the space of tempered distributions with finite $\dot{H}^s(\mathbb{R}^d)$ norm. Similarly, on the torus $\mathbb{R}^d/\mathbb{Z}^d$ and $s \in \mathbb{R}$, we define the Sobolev norm $\|f\|_{H^s(\mathbb{R}^d/\mathbb{Z}^d)}$ of a tempered distribution $f : \mathbb{R}^d/\mathbb{Z}^d \mapsto \mathbb{R}$ by,

$$\|f\|_{H^s(\mathbb{R}^d/\mathbb{Z}^d)} = \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\hat{u}(k)|^2 \right)^{\frac{1}{2}},$$

and then we denote by $H^s(\mathbb{R}^d/\mathbb{Z}^d)$ the space of tempered distributions with finite $H^s(\mathbb{R}^d/\mathbb{Z}^d)$ norm. On the torus $\mathbb{R}^d/\mathbb{Z}^d$, for $s > -\frac{d}{2}$, we also define the homogeneous Sobolev norm,

$$\|f\|_{\dot{H}^s(\mathbb{R}^d/\mathbb{Z}^d)} = \left(\sum_{k \in \mathbb{Z}^d} |k|^{2s} |\hat{u}(k)|^2 \right)^{\frac{1}{2}},$$

and then we denote by $\dot{H}^s(\mathbb{R}^d/\mathbb{Z}^d)$ the space of tempered distributions with finite $\dot{H}^s(\mathbb{R}^d/\mathbb{Z}^d)$ norm. We use the Fourier transform to define the fractional Laplacian operator $(-\Delta)^\alpha$, $0 \leq \alpha \leq 1$ on $\mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d$. On $\mathbb{R}^d$, we define it as follows,

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).$$

On $\mathbb{R}^d/\mathbb{Z}^d$, we define it as follows,

$$\widehat{(-\Delta)^\alpha f}(k) = |k|^{2\alpha} \hat{f}(k).$$

3 Existence and uniqueness of global strong solutions

Before to prove our main Theorem in this section, we begin by Lemma 3.1 which gives a priori energy estimates and Proposition 3.1 which deals with local existence and uniqueness of strong solution of Equation (2) with a characterization of the maximal time existence.

Lemma 3.1 *Let $h_0 \in H^s(\Omega)$, $s \geq 0$. If $h \in C([0, T]; H^s(\Omega))$ with $\int_0^T \|\nabla h(\tau)\|_{L^\infty}^4 d\tau < \infty$ is a solution of the system of Equations (2)-(3). We have for all $t \in [0, T]$,*

$$\|h(t)\|_{H^s}^2 + \nu_2 \int_0^t \|\Delta h(\tau)\|_{H^s}^2 d\tau \leq \|h_0\|_{H^s}^2 e^{\int_0^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau}, \quad (4)$$

where $0 < \beta \lesssim \frac{\nu_1^2}{\nu_2}$ and $0 < \gamma \lesssim \left(\nu_5 + \frac{\nu_3^2}{\nu_2} + \frac{\nu_4^4}{\nu_2} + \frac{\nu_5^2}{\nu_2} \right)$.

PROOF. We take the inner product in $H^s(\Omega)$ of Equation (2) with h , use integrations by parts to obtain,

$$\frac{1}{2} \frac{d}{dt} \|h\|_{H^s}^2 + \nu_2 \|\Delta h\|_{H^s}^2 = \nu_1 \|\nabla h\|_{H^s}^2 + \nu_3 \langle \nabla \cdot (|\nabla h|^2 \nabla h), h \rangle_{H^s} - \nu_4 \langle |\nabla h|^2, \Delta h \rangle_{H^s} + \nu_5 \langle |\nabla h|^2, h \rangle_{H^s}. \quad (5)$$

In what follows, the terms c_i , $i \in \llbracket 1, 5 \rrbracket$ are constant, furthermore, we will use Cauchy-Schwarz inequality, Young inequalities and the following inequalities (the first one is obtained after using an integration by parts and Cauchy-Schwarz inequality, the last one is proved in [13], [8]), for all $u, v \in L^\infty(\Omega) \cap H^s(\Omega)$,

$$\begin{aligned} \|\nabla u\|_{H^s} &\leq \|u\|_{H^s}^{\frac{1}{2}} \|\Delta u\|_{H^s}^{\frac{1}{2}}, \\ \|uv\|_{H^s} &\lesssim (\|u\|_{L^\infty} \|v\|_{H^s} + \|u\|_{H^s} \|v\|_{L^\infty}). \end{aligned}$$

For the first term at the right hand side of Equation (5), we have,

$$\begin{aligned}\nu_1 \|\nabla h\|_{H^s}^2 &\leq \nu_1 \|h\|_{H^s} \|\Delta h\|_{H^s} \\ &\leq c_1 \frac{\nu_1^2}{\nu_2} \|h\|_{H^s}^2 + \frac{\nu_2}{8} \|\Delta h\|_{H^s}^2.\end{aligned}\tag{6}$$

For the second term at the right hand side of Equation (5), we get,

$$\begin{aligned}\nu_3 |\langle \nabla \cdot (|\nabla h|^2 \nabla h), h \rangle_{H^s}| &= \nu_3 |\langle |\nabla h|^2 \nabla h, \nabla h \rangle_{H^s}| \\ &\leq \nu_3 \| |\nabla h|^2 \nabla h \|_{H^s} \|\nabla h\|_{H^s} \\ &\lesssim \nu_3 (\| |\nabla h|^2 \|_{L^\infty} \|\nabla h\|_{H^s} + \| |\nabla h|^2 \|_{H^s} \|\nabla h\|_{L^\infty}) \|\nabla h\|_{H^s} \\ &\lesssim \nu_3 \|\nabla h\|_{L^\infty}^2 \|\nabla h\|_{H^s}^2 \\ &\lesssim \nu_3 \|\nabla h\|_{L^\infty}^2 \|h\|_{H^s} \|\Delta h\|_{H^s} \\ &\leq c_2 \frac{\nu_3^2}{\nu_2} \|\nabla h\|_{L^\infty}^4 \|h\|_{H^s}^2 + \frac{\nu_2}{8} \|\Delta h\|_{H^s}^2.\end{aligned}\tag{7}$$

For the third term at the right hand side of Equation (5), we get,

$$\begin{aligned}\nu_4 |\langle |\nabla h|^2, \Delta h \rangle_{H^s}| &\leq \nu_4 \|\nabla h\|_{H^s}^2 \|\Delta h\|_{H^s} \\ &\lesssim \nu_4 \|\nabla h\|_{L^\infty} \|\nabla h\|_{H^s} \|\Delta h\|_{H^s} \\ &\lesssim \nu_4 \|\nabla h\|_{L^\infty} \|h\|_{H^s}^{\frac{1}{2}} \|\Delta h\|_{H^s}^{\frac{3}{2}} \\ &\leq c_3 \frac{\nu_4^4}{\nu_2^3} \|\nabla h\|_{L^\infty}^4 \|h\|_{H^s}^2 + \frac{\nu_2}{8} \|\Delta h\|_{H^s}^2.\end{aligned}\tag{8}$$

For the last term at the right hand side of Equation (5), we have,

$$\begin{aligned}\nu_5 |\langle |\nabla h|^2, h \rangle_{H^s}| &\leq \nu_5 \| |\nabla h|^2 \|_{H^s} \|h\|_{H^s} \\ &\lesssim \nu_5 \|\nabla h\|_{L^\infty} \|\nabla h\|_{H^s} \|h\|_{H^s} \\ &\lesssim \nu_5 \|\nabla h\|_{L^\infty} \|h\|_{H^s}^{\frac{3}{2}} \|\Delta h\|_{H^s}^{\frac{1}{2}} \\ &\leq c_4 \frac{\nu_5^{\frac{4}{3}}}{\nu_2^{\frac{1}{3}}} \|\nabla h\|_{L^\infty}^{\frac{4}{3}} \|h\|_{H^s}^2 + \frac{\nu_2}{8} \|\Delta h\|_{H^s}^2 \\ &\leq c_4 (\nu_5 + \frac{\nu_5^2}{\nu_2} \|\nabla h\|_{L^\infty}^4) \|h\|_{H^s}^2 + \frac{\nu_2}{8} \|\Delta h\|_{H^s}^2.\end{aligned}\tag{9}$$

Then, using Inequalities (6)-(9), from (5), we deduce,

$$\frac{1}{2} \frac{d}{dt} \|h\|_{H^s}^2 + \frac{\nu_2}{2} \|\Delta h\|_{H^s}^2 \leq c_5 \left(\frac{\nu_1^2}{\nu_2} + \left(\nu_5 + \frac{\nu_3^2}{\nu_2} + \frac{\nu_4^4}{\nu_2} + \frac{\nu_5^2}{\nu_2} \right) \|\nabla h\|_{L^\infty}^4 \right) \|h\|_{H^s}^2.\tag{10}$$

Then, thanks to Gronwall inequality, we obtain for all $t \in [0, T]$,

$$\|h(t)\|_{H^s}^2 \leq \|h_0\|_{H^s}^2 e^{\int_0^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau},\tag{11}$$

where $\beta = 2c_5 \frac{\nu_1^2}{\nu_2}$ and $\gamma = 2c_5 \left(\nu_5 + \frac{\nu_3^2}{\nu_2} + \frac{\nu_4^4}{\nu_2} + \frac{\nu_5^2}{\nu_2} \right)$.

By integrating inequality (10) over $[0, t]$ with $t \in [0, T]$ and using (11), we deduce that for all $t \in [0, T]$,

$$\|h(t)\|_{H^s}^2 + \nu_2 \int_0^t \|\Delta h(\tau)\|_{H^s}^2 d\tau \leq \|h_0\|_{H^s}^2 e^{\int_0^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau},$$

which concludes the proof.

□

Proposition 3.1 *Let $h_0 \in H^r(\Omega)$ with $r \geq 3$. Then there exists a maximal time of existence $T^* > 0$ such that there exists a unique solution $h \in C([0, T^*]; H^r(\Omega))$ of the system of Equations (2)-(3). Moreover if $T^* < \infty$, then*

$$\int_0^{T^*} \|\nabla h(\tau)\|_{L^\infty}^4 d\tau = \infty.\tag{12}$$

PROOF. For this, we use some results which deal with existence, uniqueness, regularity of solutions for nonlinear evolution equations of the form $\partial_t u = Au + f(u)$, more precisely, we use Proposition 2.1 in [2] with $X = H^{r-3}(\Omega)$ for our real Banach space, $A = -\nu_2 \Delta^2$ for our generator of holomorphic semigroup $T(t) = e^{tA}$ of bounded linear operators on X and f our locally Lipschitz continuous function on $X_\alpha = H^r(\Omega)$ with $\alpha = \frac{3}{4}$, defined by,

$$f(h) = \nu_1 \Delta h + \nu_3 \nabla \cdot (|\nabla h|^2 \nabla h) - \nu_4 \Delta |\nabla h|^2 + \nu_5 |\nabla h|^2.$$

Indeed, thanks to the following inequality, we have for all $f, g \in H^s \cap L^\infty \times H^s \cap L^\infty$, $s \geq 0$ (see [13], [8]),

$$\|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty},$$

and the Sobolev embedding $H^3(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ valid since $\Omega = \mathbb{R}^d/\mathbb{Z}^d$ or $\mathbb{R}^d$ where $d = 1, 2$, we deduce that f is locally Lipschitz continuous on H^r , since for all $(u, v) \in H^r \times H^r$, we have,

$$\begin{aligned} \|\Delta u - \Delta v\|_X &\leq \|u - v\|_{H^r}, \\ \||\nabla u|^2 - |\nabla v|^2\|_X &= \|(\nabla u - \nabla v) \cdot (\nabla u + \nabla v)\|_X \\ &\lesssim \|\nabla(u - v)\|_{L^\infty} (\|\nabla u\|_X + \|\nabla v\|_X) + \|\nabla(u - v)\|_X (\|\nabla u\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \\ &\lesssim \|\nabla(u - v)\|_{H^r} (\|\nabla u\|_{H^r} + \|\nabla v\|_{H^r}) \\ \|\nabla \cdot (|\nabla u|^2 \nabla u) - \nabla \cdot (|\nabla v|^2 \nabla v)\|_X &\leq \| |\nabla u|^2 \nabla u - |\nabla v|^2 \nabla v \|_{\dot{H}^{r-2}} \\ &\leq \| |\nabla u|^2 (\nabla u - \nabla v) \|_{\dot{H}^{r-2}} + \| (|\nabla u|^2 - |\nabla v|^2) \nabla v \|_{\dot{H}^{r-2}} \\ &= \|\nabla u \cdot \nabla u (\nabla u - \nabla v)\|_{\dot{H}^{r-2}} + \|(\nabla u - \nabla v) \cdot (\nabla u + \nabla v) \nabla v\|_{\dot{H}^{r-2}} \\ &\lesssim (\|u\|_{H^r} + \|v\|_{H^r})^2 \|u - v\|_{H^r}, \end{aligned}$$

We have also,

$$\begin{aligned} \|\Delta |\nabla u|^2 - \Delta |\nabla v|^2\|_X &\leq 2\| |\nabla u|^2 - |\nabla v|^2 \|_{H^{r-1}} \\ &= 2\| \nabla(u - v) \cdot \nabla(u + v) \|_{H^{r-1}} \\ &\lesssim \|u - v\|_{H^r} \|u + v\|_{H^r}, \end{aligned}$$

Therefore, we obtain,

$$\|f(u) - f(v)\|_X \lesssim (1 + \|u\|_{H^r} + \|v\|_{H^r})^2 \|u - v\|_{H^r},$$

which proves that f is well locally Lipschitz continuous on H^r . Then, we deduce thanks to Proposition 2.1 combined with Theorem 3.1 in [2], that there exists a maximal time $T^* > 0$ such that there exists an unique solution $h \in C([0, T^*]; H^r(\Omega))$ of the system of Equations (2)-(3). Moreover if $T^* < \infty$ then $\limsup_{t \rightarrow T^*} \|h(t)\|_{H^r} = \infty$.

It remains to prove (12). For this, let us assume that $T^* < \infty$, then we get,

$$\limsup_{t \rightarrow T^*} \|h(t)\|_{H^r} = \infty. \quad (13)$$

Since $H^3(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ (valid since $\Omega = \mathbb{R}^d/\mathbb{Z}^d$ or $\mathbb{R}^d$ where $d = 1, 2$), then Inequality (4) from Lemma 3.1 holds, therefore we have for all $t \in [0, T^*]$,

$$\|h(t)\|_{H^r}^2 + \nu_2 \int_0^t \|\Delta h(\tau)\|_{H^r}^2 d\tau \leq \|h_0\|_{H^r}^2 e^{\int_0^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau}, \quad (14)$$

where $\beta > 0, \gamma > 0$ are real depending only on r and $\nu_i, i \in \llbracket 1, 5 \rrbracket$.

If $\int_0^{T^*} \|\nabla h(s)\|_{L^\infty}^4 ds < \infty$ and since $T^* < \infty$, then from (14), we deduce that $\limsup_{t \rightarrow T^*} \|h(t)\|_{H^r} < \infty$

which leads to a contradiction with (13), then we infer that $\int_0^{T^*} \|\nabla h(\tau)\|_{L^\infty}^4 d\tau = \infty$, which concludes

the proof.

□

Now, we turn to the proof of our Theorem.

Theorem 3.1 *Let $h_0 \in H^s(\Omega)$ with $s \geq 3$ and $\nu_2\nu_3 > \nu_4^2$. Then there exists a unique global solution $h \in C([0, \infty[; H^s(\Omega))$ of the system of Equations (2)-(3). Moreover for all $t \geq 0$, we have for all $0 \leq \alpha \leq 1$,*

$$\|h(t)\|_{H^\alpha}^2 + \frac{\nu_2}{4} \int_0^t \|h(\tau)\|_{H^{\alpha+2}}^2 d\tau + \frac{2(\nu_2\nu_3 - \nu_4^2)}{3\nu_2} \int_0^t \|\nabla h(\tau)\|_{H^\alpha}^2 d\tau \leq \|h_0\|_{H^\alpha}^2 e^{2\left(\frac{\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}, \quad (15)$$

and we get also,

$$\int_0^t \|\nabla h(\tau)\|_{L^\infty}^4 d\tau \lesssim \frac{1}{\nu_2} \|h_0\|_{H^{\frac{d}{2}}}^4 e^{4\left(\frac{\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \quad (16)$$

PROOF. Thanks to Proposition 3.1, there exists a maximal time of existence $T^* > 0$ such that there exists a unique solution $h \in C([0, T^*]; H^s(\Omega))$ of the system of Equations (2)-(3). Moreover if $T^* < \infty$, then

$$\int_0^{T^*} \|\nabla h(\tau)\|_{L^\infty}^4 d\tau = \infty. \quad (17)$$

Let us assume that $T^* < \infty$. Let $0 \leq \alpha \leq 1$, by dotting Equation (2) with $(-\Delta)^\alpha h$ in $L^2(\Omega)$ and using integrations by parts, we obtain,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2 &+ \nu_2 \|(-\Delta)^{1+\frac{\alpha}{2}} h\|_{L^2}^2 + \nu_3 \int_{\Omega} |\nabla h|^2 \nabla h \cdot \nabla (-\Delta)^\alpha h \\ &= \nu_1 \|\nabla (-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2 + \nu_4 \int_{\Omega} |\nabla h|^2 (-\Delta)^{1+\alpha} h + \nu_5 \int_{\Omega} |\nabla h|^2 (-\Delta)^\alpha h. \end{aligned} \quad (18)$$

Since the operator ∇ commutes with the operator $(-\Delta)^\alpha$, then we have $\nabla(-\Delta)^\alpha h = (-\Delta)^\alpha \nabla h$ and thanks to Theorem 1 in [7], we have also $2\nabla h \cdot (-\Delta)^\alpha \nabla h \geq (-\Delta)^\alpha |\nabla h|^2$, therefore we deduce,

$$\begin{aligned} \int_{\Omega} |\nabla h|^2 \nabla h \cdot \nabla (-\Delta)^\alpha h &\geq \frac{1}{2} \int_{\Omega} |\nabla h|^2 (-\Delta)^\alpha |\nabla h|^2 \\ &= \frac{1}{2} \int_{\Omega} ((-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2)^2, \end{aligned} \quad (19)$$

where we have used one integration by parts. Using again integrations by parts, we get,

$$\int_{\Omega} |\nabla h|^2 (-\Delta)^{1+\alpha} h = \int_{\Omega} (-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2 (-\Delta)^{1+\frac{\alpha}{2}} h, \quad (20)$$

and

$$\int_{\Omega} |\nabla h|^2 (-\Delta)^\alpha h = \int_{\Omega} (-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2 (-\Delta)^{\frac{\alpha}{2}} h. \quad (21)$$

Thanks to (19)-(21), from (18), we deduce,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2 &+ \nu_2 \|(-\Delta)^{1+\frac{\alpha}{2}} h\|_{L^2}^2 + \frac{\nu_3}{2} \|(-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2\|_{L^2}^2 \\ &\leq \nu_1 \|\nabla (-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2 + \nu_4 \int_{\Omega} (-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2 (-\Delta)^{1+\frac{\alpha}{2}} h + \nu_5 \int_{\Omega} (-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2 (-\Delta)^{\frac{\alpha}{2}} h. \end{aligned} \quad (22)$$

Thanks to Young inequality, we get,

$$\nu_5 \int_{\Omega} (-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2 (-\Delta)^{\frac{\alpha}{2}} h \leq \frac{\nu_3}{6} \|(-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2\|_{L^2}^2 + \frac{3\nu_5^2}{2\nu_3} \|(-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2 \quad (23)$$

and we have also,

$$\nu_4 \int_{\Omega} (-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2 (-\Delta)^{1+\frac{\alpha}{2}} h \leq \frac{\nu_4^2}{3\nu_2} \|(-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2\|_{L^2}^2 + \frac{3\nu_2}{4} \|(-\Delta)^{1+\frac{\alpha}{2}} h\|_{L^2}^2. \quad (24)$$

Thanks to Interpolation inequality, we have,

$$\begin{aligned} \nu_1 \|\nabla (-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2 &\leq \nu_1 \|(-\Delta)^{\frac{\alpha}{2}} h\|_{L^2} \|(-\Delta)^{1+\frac{\alpha}{2}} h\|_{L^2} \\ &\leq \frac{2\nu_1^2}{\nu_2} \|(-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2 + \frac{\nu_2}{8} \|(-\Delta)^{1+\frac{\alpha}{2}} h\|_{L^2}^2. \end{aligned} \quad (25)$$

Using (23)-(25), from (22), we deduce,

$$\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2 + \frac{\nu_2}{8} \|(-\Delta)^{1+\frac{\alpha}{2}} h\|_{L^2}^2 + \frac{\nu_3\nu_2 - \nu_4^2}{3\nu_2} \|(-\Delta)^{\frac{\alpha}{2}} |\nabla h|^2\|_{L^2}^2 \leq \left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3} \right) \|(-\Delta)^{\frac{\alpha}{2}} h\|_{L^2}^2, \quad (26)$$

which can be re-written as,

$$\frac{1}{2} \frac{d}{dt} \|h\|_{\dot{H}^\alpha}^2 + \frac{\nu_2}{8} \|h\|_{\dot{H}^{\alpha+2}}^2 + \frac{\nu_3\nu_2 - \nu_4^2}{3\nu_2} \|\nabla h\|_{\dot{H}^\alpha}^2 \leq \left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3} \right) \|h\|_{\dot{H}^\alpha}^2. \quad (27)$$

We recall that $\nu_3\nu_2 \geq \nu_4^2$, then thanks to Gronwall inequality, we obtain for all $t \in [0, T^*[$,

$$\|h(t)\|_{\dot{H}^\alpha}^2 \leq \|h_0\|_{\dot{H}^\alpha}^2 e^{2\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \quad (28)$$

We integrate Inequality (27) over $[0, t]$ with $t \in]0, T^*[$ to obtain,

$$\begin{aligned} \|h(t)\|_{\dot{H}^\alpha}^2 + \frac{\nu_2}{4} \int_0^t \|h(\tau)\|_{\dot{H}^{\alpha+2}}^2 d\tau &+ \frac{2(\nu_3\nu_2 - \nu_4^2)}{3\nu_2} \int_0^t \|\nabla h(\tau)\|_{\dot{H}^\alpha}^2 d\tau \\ &\leq \|h_0\|_{\dot{H}^\alpha}^2 + 2 \left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3} \right) \int_0^t \|h(\tau)\|_{\dot{H}^\alpha}^2 d\tau \\ &\leq \|h_0\|_{\dot{H}^\alpha}^2 e^{2\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \end{aligned} \quad (29)$$

where for the last inequality, we have used Inequality (28).

Thanks to an Interpolation inequality, we have $\|\nabla h\|_{L^\infty} \lesssim \|h\|_{\dot{H}^{\frac{d}{2}}}^{\frac{1}{2}} \|h\|_{\dot{H}^{2+\frac{d}{2}}}^{\frac{1}{2}}$ (notice, in the periodic case, this inequality is valid since $\int_{\Omega} \nabla h = 0$). Therefore, we have for all $t \in [0, T^*[$,

$$\int_0^t \|\nabla h(\tau)\|_{L^\infty}^4 d\tau \lesssim \sup_{0 \leq \tau \leq t} \|h(\tau)\|_{\dot{H}^{\frac{d}{2}}}^2 \int_0^t \|h(\tau)\|_{\dot{H}^{2+\frac{d}{2}}}^2 d\tau. \quad (30)$$

Thanks to Inequality (29) used with $\alpha = \frac{d}{2}$, from (30), we deduce that for all $t \in [0, T^*[$,

$$\int_0^t \|\nabla h(\tau)\|_{L^\infty}^4 d\tau \lesssim \frac{4}{\nu_2} \|h_0\|_{\dot{H}^{\frac{d}{2}}}^4 e^{4\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \quad (31)$$

Therefore, from (31) and since T^* is finite, we deduce,

$$\int_0^{T^*} \|\nabla h(\tau)\|_{L^\infty}^4 d\tau \lesssim \frac{4}{\nu_2} \|h_0\|_{\dot{H}^{\frac{d}{2}}}^4 e^{4\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)T^*} < \infty,$$

which leads to a contradiction with (17), then we deduce that $T^* = \infty$ and Inequalities (29) and (31) hold for all $t \geq 0$, which concludes the proof.

□

4 Existence and uniqueness of global weak solutions

The assumption that initial data h_0 is in H^s , $0 \leq s \leq 1$ is more natural than the one that h_0 is in H^s , $s \geq 3$, this assumption is motivated by the non-regular interfaces observed in the subsections 4.7.3-4.7.5 of [11]. Therefore, in this section, in Lemma 4.1, we study existence of weak solutions by constructing smooth approximate solutions obtained by regularizing the initial data and applying Theorem 3.1. Then, in Proposition 4.1, we establish uniqueness of weak solutions by showing that any weak solution are strongly continuous in L^2 .

Let us introduce the notion of weak solutions.

For any $T > 0$, we introduce the Sobolev space $E_T = \{u \in L^\infty([0, T]; L^2(\Omega)); \Delta u \in L^2([0, T] \times \Omega); \nabla u \in L^4([0, T] \times \Omega)\}$ equipped with the norm $\|\cdot\|_{E_T}$ defined for all $u \in E_T$ by,

$$\|u\|_{E_T} = \max(\|u\|_{L^\infty([0, T]; L^2(\Omega))}, \|\Delta u\|_{L^2([0, T] \times \Omega)}, \|\nabla u\|_{L^4([0, T] \times \Omega)}).$$

We introduce the notion of weak solution.

Definition 4.1 For any $T > 0$, h is said a weak solution of the system of Equations (2)-(3), if and only if $h \in E_T$ and for all $\varphi \in C_c^\infty([0, T] \times \Omega)$,

$$\int_0^T \int_\Omega h \frac{\partial \varphi}{\partial t} - \nu_1 \Delta h \varphi - \nu_2 \Delta h \Delta \varphi - \nu_3 |\nabla h|^2 \nabla h \cdot \nabla \varphi - \nu_4 |\nabla h|^2 \Delta \varphi + \nu_5 |\nabla h|^2 \varphi = - \int_\Omega h_0 \varphi(\cdot, 0). \quad (32)$$

Now, we can give the proof of existence of weak solutions.

Lemma 4.1 Let $s \geq 0$, $h_0 \in H^s(\Omega)$ and $\nu_2 \nu_3 > \nu_4^2$, then for any $T > 0$, there exists a weak solution $h \in L^\infty([0, T]; H^s(\Omega)); \Delta h \in L^2([0, T]; H^s(\Omega)); |\nabla h|^2 \in L^2([0, T]; H^s(\Omega))$ of the system of Equations (2)-(3). Moreover for all $t \in [0, T]$, we have for all $0 \leq r \leq \min(1, s)$,

$$\|h(t)\|_{\dot{H}^r}^2 + \frac{\nu_2}{4} \int_0^t \|h(\tau)\|_{\dot{H}^{2+r}}^2 d\tau + \frac{2(\nu_2 \nu_3 - \nu_4^2)}{3\nu_2} \int_0^t \|\nabla h(\tau)\|^2_{\dot{H}^r} d\tau \leq \|h_0\|_{\dot{H}^r}^2 e^{2\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \quad (33)$$

PROOF. Using a Faedo-Galerkin approximation, we construct $h_0^n \in C^\infty(\Omega) \cap H^m(\Omega)$, for all $m \geq 0$ (periodic of period one if h_0 is periodic of period one) such that for all $0 \leq r \leq s$, $\|h_0^n\|_{\dot{H}^r} \leq \|h_0\|_{\dot{H}^r}$ and $\|h_0^n - h_0\|_{H^s(\Omega)} \rightarrow 0$. Then thanks to Theorem 3.1, we deduce that there exists a unique global solution $h^n \in C([0, \infty]; H^m(\Omega))$ for all $m \geq 3$ of Equation (2) with initial data h_0^n , then h^n satisfies (32) with h_0 replaced by h_0^n . Moreover for all $t \geq 0$, we have for all $0 \leq r \leq \min(1, s)$,

$$\|h^n(t)\|_{\dot{H}^r}^2 + \frac{\nu_2}{4} \int_0^t \|h^n(\tau)\|_{\dot{H}^{2+r}}^2 d\tau + \frac{2(\nu_2 \nu_3 - \nu_4^2)}{3\nu_2} \int_0^t \|\nabla h^n(\tau)\|^2_{\dot{H}^r} d\tau \leq \|h_0\|_{\dot{H}^r}^2 e^{2\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \quad (34)$$

Therefore, for any $T > 0$, thanks to (34), up to a subsequence, h^n converges weakly in E_T^s to some $h \in E_T^s$, where $E_T^s = \{v \in L^\infty([0, T]; H^s(\Omega)); \Delta v \in L^2([0, T]; H^s(\Omega)); |\nabla v|^2 \in L^2([0, T]; H^s(\Omega))\}$. Moreover, we have, for all $t \in [0, T]$ and for all $0 \leq r \leq \min(1, s)$,

$$\|h(t)\|_{\dot{H}^r}^2 + \frac{\nu_2}{4} \int_0^t \|h(\tau)\|_{\dot{H}^{2+r}}^2 d\tau + \frac{2(\nu_2 \nu_3 - \nu_4^2)}{3\nu_2} \int_0^t \|\nabla h(\tau)\|^2_{\dot{H}^r} d\tau \leq \|h_0\|_{\dot{H}^r}^2 e^{2\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \quad (35)$$

We use a version of Friedrich's Lemma : For any bounded subset $\mathcal{O}$ of $\mathbb{R}^d$, and any $\epsilon > 0$, there exists an integer $N(\mathcal{O}, \epsilon) > 0$ and functions $\{\omega_1, \omega_2, \dots, \omega_N\}$ in $L^\infty(\mathcal{O})$, such that,

$$\|u\|_{L^2(\mathcal{O})}^2 \leq \sum_{k=1}^N \langle u, \omega_k \rangle_{L^2(\mathcal{O})}^2 + \epsilon \|\nabla u\|_{L^2(\mathcal{O})}^2 \text{ for all } u \in H^1(\mathcal{O}). \quad (36)$$

Up to a subsequence, if we apply Inequality (36) with $u = h^n - h$ and after with $u = \partial_i h^n - \partial_i h$ for each $i \in [1, d]$ and using the weakly convergence in E_T^s , inequalities (34), (35) with $r = 0$, we deduce up

to a subsequence that h^n converges to h strongly in $L^2([0, T]; H^1(\mathcal{O}))$ for any bounded subset $\mathcal{O}$ of $\mathbb{R}^d$. Then, using again inequalities (34), (35) with $r = 0$, up to a subsequence, we pass to the limit as $n \rightarrow \infty$ in Equation (32) satisfied by h^n for the initial data h_0^n and we deduce that h is a weak solution of the system of Equations (2)-(3), which concludes the proof.

□

For this section, in the following Proposition, we finish with the proof of uniqueness of weak solutions.

Proposition 4.1 *Let $h_0 \in L^2(\Omega)$ and $\nu_2\nu_3 > \nu_4^2$, then there exists an unique weak solution h of the system of Equations (2)-(3), moreover $h \in C([0, +\infty[, L^2(\Omega))$.*

PROOF. Let $T > 0$. Let $u_0 \in L^2(\Omega)$ and $v_0 \in L^2(\Omega)$. Let us assume that u and v are two weak solutions of Equation (2) respectively for the initial data u_0 and v_0 .

We consider $w = u - v \in E_T$ and from (32), we write Equation satisfied by w for all $\varphi \in C_c^\infty([0, T[\times \Omega)$, in other words,

$$\begin{aligned} \int_0^T \int_\Omega w \frac{\partial \varphi}{\partial t} - \nu_1 \Delta w \varphi - \nu_2 \Delta w \Delta \varphi &= -\nu_3(|\nabla u|^2 \nabla u - |\nabla v|^2 \nabla v) \cdot \nabla \varphi \\ &\quad - \nu_4(|\nabla u|^2 - |\nabla v|^2) \Delta \varphi + \nu_5(|\nabla u|^2 - |\nabla v|^2) \varphi = - \int_\Omega w(0) \varphi(0). \end{aligned} \quad (37)$$

Using the same arguments as Lemma 2.1 in [10], from Equation (37), we infer that for all $0 \leq s < t \leq T$ and for all $\varphi \in C_c^\infty([0, T[\times \Omega)$,

$$\begin{aligned} \int_s^t \int_\Omega w \frac{\partial \varphi}{\partial t} - \nu_1 \Delta w \varphi - \nu_2 \Delta w \Delta \varphi &= -\nu_3(|\nabla u|^2 \nabla u - |\nabla v|^2 \nabla v) \cdot \nabla \varphi \\ &\quad - \nu_4(|\nabla u|^2 - |\nabla v|^2) \Delta \varphi + \nu_5(|\nabla u|^2 - |\nabla v|^2) \varphi \\ &= \int_\Omega w(t) \varphi(t) - \int_\Omega w(s) \varphi(s). \end{aligned} \quad (38)$$

Let us fix $s < t \leq T$ and let $\varepsilon > 0$ such that $t - s > \varepsilon$. We introduce j_ε an even, positive, infinitely differentiable function with support in $] - \varepsilon, \varepsilon[$ and $\int_{-\infty}^\infty j_\varepsilon(\tau) d\tau = 1$. We introduce the mollifier w_ε of w defined by, for all $\tau \in [0, t]$,

$$w_\varepsilon(\tau) = \int_s^t j_\varepsilon(\tau - \sigma) w(\sigma) d\sigma.$$

Since $w \in E_T$, we take w_ε as test function in (38) instead of φ and using the same arguments as Theorem 4.1 in [10] and taking after the limit as $\varepsilon \rightarrow 0$ in (38), we deduce,

$$\begin{aligned} \int_s^t \int_\Omega -\nu_1 \Delta w w - \nu_2 (\Delta w)^2 &= -\nu_3(|\nabla u|^2 \nabla u - |\nabla v|^2 \nabla v) \cdot \nabla w - \nu_4(|\nabla u|^2 - |\nabla v|^2) \Delta w + \nu_5(|\nabla u|^2 - |\nabla v|^2) w \\ &= \frac{1}{2} \int_\Omega w(t)^2 - \frac{1}{2} \int_\Omega w(s)^2. \end{aligned} \quad (39)$$

If we take $v_0 = 0$ and $v = 0$, from Equation (39), we get,

$$\frac{1}{2} \int_\Omega u(t)^2 - \frac{1}{2} \int_\Omega u(s)^2 = \int_s^t \int_\Omega -\nu_1 \Delta u u - \nu_2 (\Delta u)^2 - \nu_3 |\nabla u|^4 - \nu_4 |\nabla u|^2 \Delta u + \nu_5 |\nabla u|^2 u. \quad (40)$$

Thanks to Cauchy-Schwarz inequality, we obtain,

$$\begin{aligned} \frac{1}{2} \left| \int_\Omega u(t)^2 - \int_\Omega u(s)^2 \right| &\leq \int_s^t \nu_1 \|\Delta u(\tau)\|_{L^2} \|u(\tau)\|_{L^2} + \nu_2 \|\Delta u(\tau)\|_{L^2}^2 + \nu_3 \|\nabla u(\tau)\|_{L^4}^4 \\ &\quad + \int_s^t \nu_4 \|\nabla u(\tau)\|_{L^4}^2 \|\Delta u(\tau)\|_{L^2} + \nu_5 \|\nabla u(\tau)\|_{L^4}^2 \|u(\tau)\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leq \nu_1 \|u\|_{E_T}^2 \sqrt{t-s} + \nu_2 \int_s^t \|\Delta u(\tau)\|_{L^2}^2 + \nu_3 \int_s^t \|\nabla u(\tau)\|_{L^4}^4 \\
&+ \nu_4 \left(\int_s^t \|\nabla u(\tau)\|_{L^4}^4 \right)^{\frac{1}{2}} \left(\int_s^t \|\Delta u(\tau)\|_{L^2}^2 \right)^{\frac{1}{2}} + \nu_5 \|u\|_{E_T}^2 \sqrt{t-s}.
\end{aligned}$$

Since $u \in E_T$, then from inequality just above, we deduce that $u \in C([0, T]; L^2(\Omega))$.

This means that for any $u_0 \in L^2(\Omega)$ and $T > 0$, if u is a weak solution of Equation (2) for the initial data u_0 then $u \in C([0, T]; L^2(\Omega))$.

Thanks to Lemma 4.1, there exists h a weak solution of Equation (2) for the initial data h_0 . Therefore, we deduce that $h \in C([0, T]; L^2(\Omega))$.

If there exists g another weak solution of Equation (2) for the initial data h_0 , we infer also $g \in C([0, T]; L^2(\Omega))$. We use Equation (39) with $u = h$ and $v = g$. Furthermore, using Young inequalities, we have,

$$\begin{aligned}
\nu_1 |\Delta w w| &\leq \frac{\nu_2}{8} (\Delta w)^2 + \frac{2\nu_1^2}{\nu_2} w^2 \\
\nu_4 (|\nabla h|^2 - |\nabla g|^2) \Delta w &\leq \frac{3\nu_2}{4} (\Delta w)^2 + \frac{\nu_4^2}{3\nu_2} (|\nabla h|^2 - |\nabla g|^2)^2 \\
\nu_5 (|\nabla h|^2 - |\nabla g|^2) w &\leq \frac{\nu_3}{6} (|\nabla h|^2 - |\nabla g|^2)^2 + \frac{3\nu_5^2}{2\nu_3} w^2.
\end{aligned} \tag{41}$$

We notice also,

$$\begin{aligned}
\nu_3 (|\nabla h|^2 \nabla h - |\nabla g|^2 \nabla g) \cdot (\nabla h - \nabla g) &= \nu_3 (|\nabla h|^4 + |\nabla g|^4 - (|\nabla h|^2 + |\nabla g|^2) \nabla h \cdot \nabla g) \\
&= \nu_3 \left(\frac{1}{2} (|\nabla h|^2 + |\nabla g|^2) |\nabla h - \nabla g|^2 + (|\nabla h|^2 - |\nabla g|^2)^2 \right) \\
&\geq \frac{\nu_3}{2} (|\nabla h|^2 - |\nabla g|^2)^2.
\end{aligned} \tag{42}$$

Thanks to Inequalities (41) and (42), from (39) used with $s = 0$, we deduce,

$$\frac{1}{2} \int_{\Omega} w(t)^2 - \frac{1}{2} \int_{\Omega} w(0)^2 + \frac{\nu_2}{8} \int_0^t \int_{\Omega} (\Delta w)^2 + \frac{\nu_2 \nu_3 - \nu_4^2}{3\nu_2} \int_0^t \int_{\Omega} (|\nabla h|^2 - |\nabla g|^2)^2 \leq \left(\frac{3\nu_5^2}{2\nu_3} + \frac{2\nu_1^2}{\nu_2} \right) \int_0^t \int_{\Omega} w^2. \tag{43}$$

Since $\nu_2 \nu_3 > \nu_4^2$, we infer for all $t \in [0, T]$,

$$\frac{1}{2} \|w(t)\|_{L^2}^2 - \frac{1}{2} \|w(0)\|_{L^2}^2 \leq \left(\frac{3\nu_5^2}{2\nu_3} + \frac{2\nu_1^2}{\nu_2} \right) \int_0^t \|w(\tau)\|_{L^2}^2 d\tau.$$

Thanks to Gronwall inequality, we infer for all $t \in [0, T]$,

$$\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 e^{\left(\frac{3\nu_5^2}{\nu_3} + \frac{4\nu_1^2}{\nu_2} \right) t}. \tag{44}$$

Since $w(0) = h(0) - g(0) = h_0 - h_0 = 0$, then from (44), we deduce that for all $t \in [0, T]$, $w(t) = 0$, which implies that $h(t) = g(t)$, therefore h is the unique weak solution in E_T . Due to existence and uniqueness of weak solutions for all $T > 0$, we conclude the proof.

□

5 Smoothness of weak solutions

In this section, we deal with the regularity of weak solutions, we show that weak solutions are smooth up to the initial time as soon as the initial data is in $H^{\frac{d}{2}}$.

Proposition 5.1 *Let $h_0 \in H^{\frac{d}{2}}(\Omega)$ and $\nu_2 \nu_3 > \nu_4^2$, then there exists an unique weak solution $h \in L^\infty([0, \infty[; H^{\frac{d}{2}}(\Omega))$; $\Delta h \in L^2([0, \infty[; H^{\frac{d}{2}}(\Omega))$; $|\nabla h|^2 \in L^2([0, \infty[; H^{\frac{d}{2}}(\Omega))$ of the system of Equations (2)-(3). Moreover $h \in C^\infty([0, +\infty[\times \Omega)$.*

PROOF. Thanks to Lemma 4.1 and Propostion 4.1, we deduce that there exists a unique weak solution $h \in L^\infty([0, \infty[; H^{\frac{d}{2}}(\Omega))$; $\Delta h \in L^2([0, \infty[; H^{\frac{d}{2}}(\Omega))$; $|\nabla h|^2 \in L^2([0, \infty[; H^{\frac{d}{2}}(\Omega))$ of the system of Equations (2)-(3). Moreover, from (35), we deduce that for all $t \geq 0$,

$$\|h(t)\|_{H^{\frac{d}{2}}}^2 + \nu_2 \int_0^t \|\Delta h(\tau)\|_{H^{\frac{d}{2}}}^2 d\tau \lesssim \|h_0\|_{H^{\frac{d}{2}}}^2 e^{2\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}, \quad (45)$$

which implies that,

$$\|h(t)\|_{H^{\frac{d}{2}}}^2 + \int_0^t \|h(\tau)\|_{H^{2+\frac{d}{2}}}^2 d\tau \lesssim \left(1 + \frac{1}{\nu_2} + t\right) \|h_0\|_{H^{\frac{d}{2}}}^2 e^{2\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \quad (46)$$

Using the same arguments as Inequality (31), we get for all $t \geq 0$,

$$\int_0^t \|\nabla h(\tau)\|_{L^\infty}^4 d\tau \lesssim \frac{4}{\nu_2} \|h_0\|_{H^{\frac{d}{2}}}^4 e^{4\left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t}. \quad (47)$$

We begin by some a priori estimates. Let $\epsilon > 0$. Let $k \in \mathbb{N}$ and $\epsilon_k = \epsilon(1 - 2^{-k})$. Thanks to Lemma 3.1 used with $s = 2k + 2 + \frac{d}{2}$, we get for all $\epsilon_k \leq s < \epsilon_{k+1} \leq t$,

$$\|h(t)\|_{H^{2k+2+\frac{d}{2}}}^2 + \nu_2 \int_s^t \|\Delta h(\tau)\|_{H^{2k+2+\frac{d}{2}}}^2 d\tau \leq \|h(s)\|_{H^{2k+2+\frac{d}{2}}}^2 e^{\int_s^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau}, \quad (48)$$

which implies that,

$$\|h(t)\|_{H^{2k+2+\frac{d}{2}}}^2 + \int_s^t \|h(\tau)\|_{H^{2k+4+\frac{d}{2}}}^2 d\tau \lesssim \left(1 + \frac{1}{\nu_2} + t\right) \|h(s)\|_{H^{2k+2+\frac{d}{2}}}^2 e^{\int_s^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau}, \quad (49)$$

where $\beta > 0$ and $\gamma > 0$ are real depending only on $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$. We integrate inequality (49) over $s \in [\epsilon_k, \epsilon_{k+1}[$, to obtain for all $t \geq \epsilon_{k+1}$,

$$\|h(t)\|_{H^{2k+2+\frac{d}{2}}}^2 + \int_{\epsilon_{k+1}}^t \|h(\tau)\|_{H^{2k+4+\frac{d}{2}}}^2 d\tau \leq \frac{2^{k+1}}{\epsilon} C_k \left(1 + \frac{1}{\nu_2} + t\right) e^{\int_0^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau} \int_{\epsilon_k}^t \|h(s)\|_{H^{2k+2+\frac{d}{2}}}^2 ds, \quad (50)$$

where $C_k > 0$ is a constant depending only on k . We set

$$U_k(t) = \int_{\epsilon_k}^t \|h(\tau)\|_{H^{2k+2+\frac{d}{2}}}^2 d\tau, \quad (51)$$

then from (50), we have,

$$U_{k+1}(t) \leq \frac{2^{k+1}}{\epsilon} C_k \left(1 + \frac{1}{\nu_2} + t\right) e^{\int_0^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau} U_k(t),$$

which implies for all $k \in \mathbb{N}$, $k \geq 1$,

$$U_k(t) \leq 2^{\frac{k(k+1)}{2}} \alpha_k \left(\left(1 + \frac{1}{\nu_2} + t\right) e^{\int_0^t (\beta + \gamma \|\nabla h(\tau)\|_{L^\infty}^4) d\tau} \right)^k U_0(t), \quad (52)$$

where $\alpha_k = \prod_{i=0}^{k-1} C_i$. From (50), thanks to (51), (52), (46) and (47), we deduce that for all $k \in \mathbb{N}$ and for all $t \geq \epsilon$,

$$\|h(t)\|_{H^{2k+\frac{d}{2}}}^2 + \int_\epsilon^t \|h(\tau)\|_{H^{2k+2+\frac{d}{2}}}^2 d\tau \leq C, \quad (53)$$

where $C > 0$ is a real depending continuously only on $k, \frac{1}{\epsilon}, \beta, \gamma, \|h_0\|_{H^{\frac{d}{2}}}^2, \left(\frac{2\nu_1^2}{\nu_2} + \frac{3\nu_5^2}{2\nu_3}\right)t$.

Inequality (53) is then justified by using the same arguments in Lemma 4.1 combined with the uniqueness of weak solutions of Equation (2).

Then, thanks to inequality (53) and using Equation (2), we deduce that $h \in C^\infty([\epsilon, +\infty[\times\Omega)$, which concludes the proof.

□

Remark 5.1 *If we do not take into account the density variations and the diffusion due to evaporation-condensation in the model equation (2) which means $\nu_1 = 0$ and $\nu_5 = 0$, then we notice that the real C obtained in (53) does not depend on the time t and therefore infinite time blow-up can not occur.*

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