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Gregory A. Chechkin[a*], Tatiana P. Chechkina[b], Tudor S. Ratiu[c] and Maksim S. Romanov[a]

[a]*Department of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow 119991, Russia;* [b]*Higher Mathematics, National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), Moscow 115409, Russia;* [c]*École polytechnique fédérale de Lausanne, CH-1015, Lausanne, Switzerland*

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We study the homogenization problem for the system of equations of dynamics of a mixture of liquid crystals with random structure. We consider a simplified form of the Ericksen–Leslie equations for an incompressible medium with inhomogeneous density with random structure. Under the assumption that randomness is statistically homogeneous and ergodic, we construct the limit problem and prove almost sure convergence of solutions of the original problem to the solution of the limit (homogenized) problem.

Keywords: liquid crystals; Ericksen–Leslie equations; nematodynamics; random homogenization; director field; speed of propagation

1. Introduction

Asymptotic analysis and homogenization (cf. for example, [1–7]) are basic methods for studying mathematical problems appearing in theory of media with nontrivial microstructure. These methods allow to simplify modeling of composites, skeletons, reinforced structures, perforated materials (porous media), cell-structures, bodies with concentrated masses, stratified flow (of newtonian and nonnewtonian fluids), mixture of fluids with different viscosity (density) or mixture of fluids and gas (fluids and solid particles), and many other. One interesting example of nonnewtonian fluid is thermotropic nematic liquid crystals. System of differential equations describing nematodynamics is a nonlinear second-order Ericksen–Leslie system. Wherein, classical homogenization methods deal with periodic, locally periodic, quasiperiodic structures and mixture of liquid crystals with different parameters are naturally nonperiodic.

Main interest of this paper is to study homogenization problem for Ericksen–Leslie system with random rapidly oscillating density–function and velocity. The medium consists of alternating microscopic drops of liquid crystals of different density, i.e. mixtures of liquid crystals possessing different properties (see Figure 1).

*Corresponding author. Email: chechkin@mech.math.msu.su

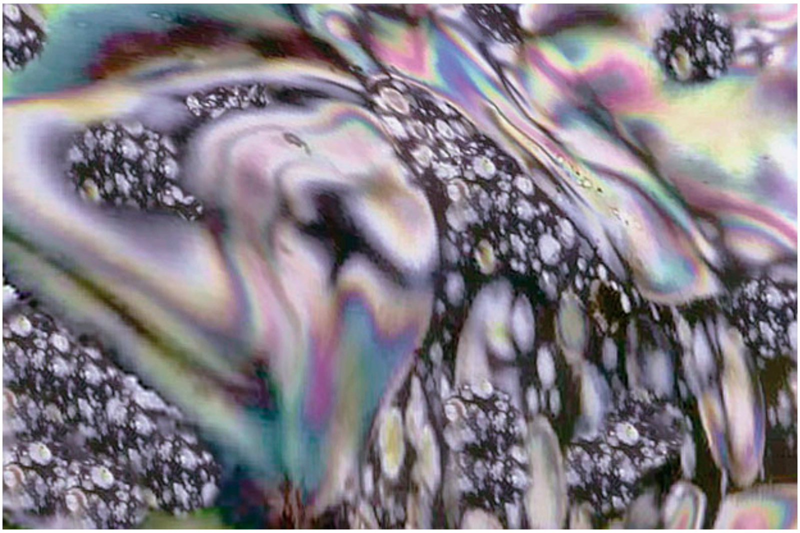

Figure 1. Microscopic structure of inhomogeneous nematic liquid crystal.

2. Weak solutions to the Ericksen–Leslie system

Assume that D is a bounded domain with smooth boundary ∂D, occupied by a liquid crystal. The Ericksen–Leslie system of equations reads as follows:

$$\begin{cases} \dot{\rho} = 0, \\ \rho \dot{\mathbf{u}}_i = \rho F_i + \sigma_{ji_{x_j}}, \\ \rho J \ddot{\mathbf{n}}_i = g_i + \pi_{ji_{x_j}}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{2.1}$$

(cf. the deduction of this system, for example, in [8]). These equations express the laws of mass conservation, linear and angular momenta, respectively, and the incompressibility constraint. Here, the constant $J > 0$ is the *moment of inertia of the molecule*

Hereinafter, 'dot' denotes the total, covariant, or material derivative

$$\dot{f} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$$

for any function f. In what follows, we use the notation $a_{i_{x_j}} := \dfrac{\partial a_i}{\partial x_j}$ for some vector (a_1, a_2, a_3). We adopt the Einstein convention of summation on repeated indices, regardless of their position.

The unknowns of the system (1.1) are the mass *density* ρ, the *Eulerian*, or *spatial velocity vector field* $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, the *director field* $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ defining the orientation of the liquid crystal molecules in space (for instance for horizontal flow, we have horizontal director for nematic calamitic and vertical for nematic discotic, see Figure 2), and the *pressure* P.

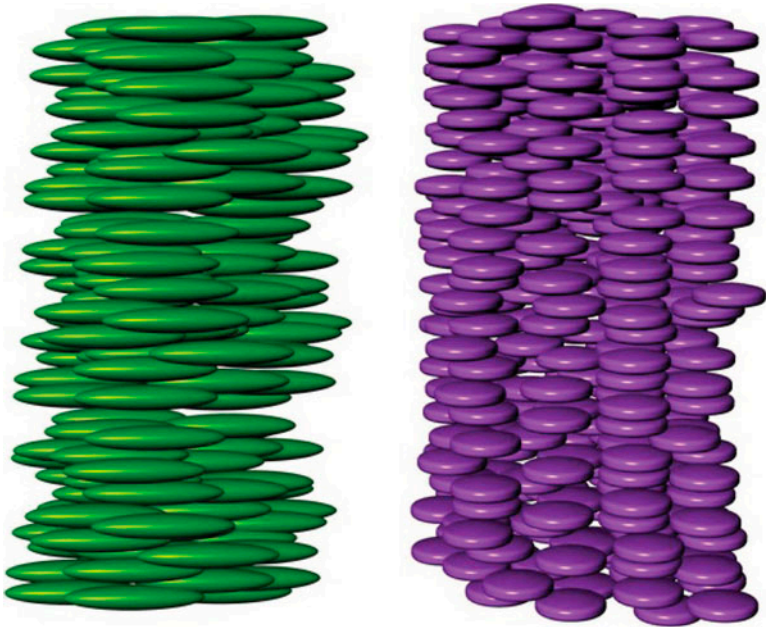

Figure 2. The structure of horizontal flow of nematic calamitic (left) and nematic discotic (right) liquid crystals.

The functions F_i stands for the external volume forces, $\mathcal{F} = \mathcal{F}(\rho, \mathbf{n}, \nabla\mathbf{n})$ is the so-called the *Oseen–Zöcher–Frank free energy* of the medium, the coefficients $\sigma_{ij}, \omega_i, g_i, \pi_{ij}$ depend on the unknowns ρ, $\mathbf{u}$, $\mathbf{n}$, P in accordance with the formulas

$$\begin{aligned}
\sigma_{ij} &= -P\delta_{ij} - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{n}_{i\,x_j}} + \widehat{\sigma}_{ji},\\
\pi_{ij} &= \beta_j\mathbf{n}_i + \rho\frac{\partial\mathcal{F}}{\partial\mathbf{n}_{i\,x_j}},\\
g_i &= \gamma\mathbf{n}_i - \beta_j\mathbf{n}_{i\,x_j} - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{n}_i} + \widehat{g}_i,\\
\widehat{\sigma}_{ji} &= \mu_1\mathbf{n}_k\mathbf{n}_p A_{kp}\mathbf{n}_i\mathbf{n}_j + \mu_2\mathbf{n}_j N_i + \mu_3\mathbf{n}_i N_j + \mu_4 A_{ij} + \mu_5\mathbf{n}_j\mathbf{n}_k A_{ki} + \mu_6\mathbf{n}_i\mathbf{n}_k A_{kj},\\
\widehat{g}_i &= \lambda_1 N_i + \lambda_2\mathbf{n}_j A_{ij};
\end{aligned} \tag{2.2}$$

moreover,

$$\begin{aligned}
&\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6 = -\mu_2 - \mu_3,\\
&N_i = \dot{\mathbf{n}}_i + \omega_{ki}\mathbf{n}_k, \quad A_{ij} = \frac{\mathbf{u}_{i\,x_j} + \mathbf{u}_{j\,x_i}}{2}, \quad \omega_{ij} = \frac{\mathbf{u}_{i\,x_j} - \mathbf{u}_{j\,x_i}}{2}.
\end{aligned}$$

The following conditions are imposed on the unknown functions:

$$\rho > 0, \tag{2.3}$$

$$|\mathbf{n}| \equiv 1. \tag{2.4}$$

It is convenient to consider, instead of the system (1.1), a modified and simplified system introduced and studied in [9] (cf. also [10,11]). First of all, we exclude the condition (2.4) and introduce, instead of (2.4), a penalty term. For this purpose, we set

$$\mathcal{F} = |\nabla \mathbf{n}|^2 + \alpha^{-1}(|\mathbf{n}|^2 - 1)^2,$$

where α is a small parameter. Introducing the penalty term, it is reasonable to put the Lagrangian multipliers γ and β_j equal to zero.

Then we assume that there are no external forces acting on the system. Moreover, since the micro-inertia constant J is sufficiently small, we also equate it to zero.

Thus, we obtain the equations

$$\begin{cases} \dot{\rho} = 0, \\ \rho \dot{\mathbf{u}}_i = \sigma_{ji\,x_j}, \quad \operatorname{div} \mathbf{u} = 0. \\ g_i + \pi_{ji\,x_j} = 0. \end{cases} \tag{2.5}$$

We supply this system of equations with the boundary conditions

$$\mathbf{u}(x,t) = 0, \quad \mathbf{n}(x,t) = \mathfrak{n}_0(x) \quad \text{for } x \in \partial D \tag{2.6}$$

and the initial conditions

$$\mathbf{u}(x,0) = \mathfrak{u}_0(x), \quad \mathbf{n}(x,0) = \mathfrak{n}_0(x), \quad \rho(x,0) = \mathfrak{p}_0(x). \tag{2.7}$$

Hereinafter, we assume that the constants λ_i and μ_i satisfy the conditions

$$\lambda_1 < 0, \quad \mu_1 > 0, \quad \mu_4 > 0, \quad \mu_5 + \mu_6 > 0, \quad (-\lambda_1)^{\frac{1}{2}}(\mu_5 + \mu_6)^{\frac{1}{2}} > \lambda_2 > 0. \tag{2.8}$$

It is easy to see that

$$\lambda_2^2 = \delta|\lambda_1|(\mu_5 + \mu_6) \tag{2.9}$$

for some $\delta < 1$.

Let us define the following functional spaces: The Sobolev space H_0^1 is the closure of the set of functions from C_0^∞ by the norm $\left(||u||_{L_2}^2 + ||\nabla u||_{L_2}^2\right)^{\frac{1}{2}}$, $Sol(D) = \{\mathbf{v} \in C_0^\infty(D) :$ $\operatorname{div} \mathbf{v} = 0\}$, $Sol(Q_T) = \{\mathbf{v} \in C^\infty(Q_T) : \ \forall t \ \mathbf{v}(t,\cdot) \in Sol(D)\}$, where $Q_T = (0,T) \times D$, $Sol_2(D)$ is the closure of $Sol(D)$ in the $L_2(D)$-norm, $Sol_2^1(D)$ is the closure of $Sol(D)$ in the $H^1(D)$-norm.

Definition 2.1 Weak solution to problem (2.5)–(2.7) is the triple $(\rho, \mathbf{u}, \mathbf{n})$, where $\rho \in L_\infty(Q_T)$, $\mathbf{u} \in L_2((0,T); Sol_2^1(D)) \cap L_\infty((0,T); Sol_2(D))$, $\mathbf{n} \in L_2((0,T); H^2(D)) \cap L_\infty((0,T); H^1(D))$, $\omega \in L_2(Q_T)$, if

(1) $(\rho, \mathbf{u}, \mathbf{n})$ satisfies the boundary and the initial conditions (2.6) and (2.7),
(2) the third equation in (2.5) holds almost everywhere,
(3) the first equation in system (2.5) holds in the sense of functionals in $L_2((0,T),$ $H^1(D))$,

(4) the second equation in (2.5) is understood as the integral identity

$$\int_{Q_T} (\rho \mathbf{u}_{i,t}\varphi^i + \rho \mathbf{u}_j \mathbf{u}_i \varphi_{i x_j})\,\mathrm{d}x\mathrm{d}t + \int_D \rho(x,T)u_i(x,T)\varphi_i(x,T)\,\mathrm{d}x$$
$$= \int_{Q_T} \sigma_{ij}\varphi_{i x_j}\,\mathrm{d}x\mathrm{d}t + \int_D \mathfrak{p}_0 \mathfrak{u}_{0i}\varphi_i\,\mathrm{d}x \qquad \forall \varphi \in Sol(Q_T). \tag{2.10}$$

Remark 2.1 In the case $\lambda_2 = 0$, the system satisfies the maximum principle, namely, if $|\mathfrak{n}_0| \leqslant 1$ on the boundary of a domain, then $|\mathbf{n}| \leqslant 1$ in the entire domain and all the integrals in the identity (2.10) are finite. In the general case, for $\mathbf{n} \in L_2((0,T); H^2(D)) \cap L_\infty((0,T); H^1(D))$ it follows that $\mathbf{n} \in L_8(Q_T)$, which guarantees the existence of integrals.

The initial conditions make sense since ρ_t, $\mathbf{u}_t$, and $\mathbf{n}_t$ are elements of $L_2((0,T); H^{-1}(D))$. The boundary conditions are understood in the sense of traces of functions.

For the Ericksen–Leslie system in the form (2.5)–(2.7), the following existence theorem holds for the weak solution.

THEOREM 2.1 *Suppose that* $\mathfrak{u}_0 \in Sol_2(D)$*,* $\mathfrak{n}_0 \in H^1(D)$*, and* $\mathfrak{n}_0\big|_{\partial D} \in H^{\frac{3}{2}}(\partial D)$*. Then there exists a weak solution to the problem* (2.5)–(2.7).

Proof The proof of the theorem can be proved similarly to [9–11] by the techniques from [12], which is based on a modified method of Galerkin approximations.

First, we consider a sequence of embedded subspaces $V_m \subset Sol_2^1(D) \cap C^\infty(D)$, $\dim V_m = m$, and construct a solution $(\rho^m, \mathbf{u}^m, \mathbf{n}^m)$ to the finite-dimensional problem such that

$\mathbf{u}^m \in L_2((0,t); V_m)$ satisfies (2.10) for almost all $t < T$ and any test function $\varphi \in V_m$;

$\mathbf{n}^m$ satisfies almost everywhere an equation similar to the third equation in (2.5) with $\mathbf{u}^m$ instead of $\mathbf{u}$; moreover, $\mathbf{n}^m = \mathfrak{n}_0$ at the initial time and on the boundary,

ρ^m is constant on the trajectories of the vector field $\mathbf{u}^m$ and satisfies the initial condition $\rho^m(x,0) = \mathfrak{p}_0(x)$.

The solution is looked for by the method of successive approximations and the fixed point theorem, which guarantees the existence of a solution on a small time interval $(0,T)$.

Second, we deduce a priori estimates for the solutions, based on the energy inequality (see similar technique in [9, Section 2]), which is obtained by differentiating the integral

$$\int_D \left(\frac{\rho^m}{2} |\mathbf{u}^m|^2 + \frac{1}{2} |\nabla \mathbf{n}^m|^2 + \rho^m \mathcal{F}(\mathbf{n}^m) \right) \mathrm{d}x$$

with respect to t and further rearrangement of terms. We have

$$\int_D \left(\frac{\rho^m}{2} |\mathbf{u}^m|^2 + \frac{1}{2} |\nabla \mathbf{n}^m|^2 + \rho^m \mathcal{F}(\mathbf{n}^m) \right) \mathrm{d}x\Big|_0^t \leqslant -\int_{Q_T} \left(\mu_1 \left(A^m_{kp}\mathbf{n}^m_k \mathbf{n}^m_p \right)^2 \right.$$
$$\left. + \mu_4 \left(A^m_{ij} \right)^2 \right) \mathrm{d}x\mathrm{d}t - \int_{Q_T} (\mu_5 + \mu_6) \left| A^m_{ij}\mathbf{n}^m_i \right|^2 \mathrm{d}x\mathrm{d}t + \int_{Q_T} \lambda_1 \left| N^m \right|^2 \mathrm{d}x\mathrm{d}t$$
$$+ \int_{Q_T} 2\lambda_2 N^m_j A^m_{ij} \mathbf{n}^m_i \mathrm{d}x\mathrm{d}t + \mathrm{Const}(\mathfrak{p}_0, \mathfrak{u}_0, \mathfrak{n}_0). \tag{2.11}$$

Now estimating the coefficient of λ_2 on the right-hand side of (2.11). By the Cauchy inequality,

$$\lambda_2 \int_{Q_T} 2N_j^m A_{ij}^m \mathbf{n}_i^m \mathrm{d}x\mathrm{d}t \leqslant \lambda_2 \left(\int_{Q_T} \left| A_{ij}^m \mathbf{n}_i^m \right|^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{Q_T} \left| N^m \right|^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}},$$

keeping in mind the condition (2.9), we conclude that the right-hand side of this inequality does not exceed

$$(1-\delta)(\mu_5+\mu_6) \int_{Q_T} \left| A_{ij}^m \mathbf{n}_i^m \right|^2 \mathrm{d}x\mathrm{d}t + (1-\delta)\left|\lambda_1\right| \int_{Q_T} \left| N^m \right|^2 \mathrm{d}x\mathrm{d}t.$$

Finally, we derive

$$\begin{aligned} &\int_D \left(\frac{\rho^m}{2} \left|\mathbf{u}^m\right|^2 + \frac{1}{2}\left|\nabla \mathbf{n}^m\right|^2 + \rho^m \mathcal{F}(\mathbf{n}^m) \right) \mathrm{d}x \Big|_0^t \\ &\quad + \int_{Q_T} \left(\mu_1 \left(A_{kp}^m \mathbf{n}_k^m \mathbf{n}_p^m \right)^2 + \mu_4 \left(A_{ij}^m \right)^2 \right) \mathrm{d}x \\ &\quad + \int_{Q_T} \delta(\mu_5+\mu_6) \left| A_{ij}^m \mathbf{n}_i^m \right|^2 \mathrm{d}x\mathrm{d}t - \int_{Q_T} \delta\lambda_1 \left| N^m \right|^2 \mathrm{d}x\mathrm{d}t \leqslant \mathrm{Const}(\mathfrak{p}_0, \mathfrak{u}_0, \mathfrak{n}_0). \end{aligned} \tag{2.12}$$

Then, from the equation

$$\Delta \mathbf{n}_i - \rho \frac{\partial \mathcal{F}}{\partial \mathbf{n}_i} = -\left(\lambda_1 \dot{\mathbf{n}}_i + \lambda_1 \mathbf{u}^m \mathbf{n}_{i\,x_k} + \lambda_1 \omega_{ki} \mathbf{n}_k + \lambda_2 A_{ij} \mathbf{n}_j \right)$$

in view of known results [13, Section 6] and the inequality (2.12), we deduce

$$||\mathbf{n}^m||_{W_2^{0,2}(Q_T)} \leqslant \mathrm{Const}\left(\mathfrak{p}_0, \mathfrak{u}_0, \mathfrak{n}_0, T\right). \tag{2.13}$$

The inequality (2.12) implies the time-global existence of solutions to finite-dimensional problems and the $*$-weak precompactness in the corresponding spaces.

Passing to the limit and taking into account the Sobolev embedding theorem, we establish the existence of a weak solution to the initial problem. We note that this solution satisfies the energy inequality of type (2.12) and (2.13); moreover,

$$\mathrm{ess\,inf}\ \mathfrak{p}_0 \leqslant \rho \leqslant \mathrm{ess\,sup}\ \mathfrak{p}_0.$$

□

3. Randomness

Assume that $(\Omega, \mathcal{A}, \mu)$ is a probability space, i.e. the set Ω with σ-algebra $\mathcal{A}$ of its subsets and σ-additive nonnegative measure μ on $\mathcal{A}$ such that $\mu(\Omega) = 1$.

Definition 3.1 A family of measurable maps $T_y : \Omega \to \Omega$, $y \in \mathbb{R}^3$ we call a *dynamical system*, if the following properties hold true:

(1) *group property*: $T_{y_1+y_2} = T_{y_1} T_{y_2}\ \forall y_1, y_2 \in \mathbb{R}^3$; $T_0 = Id$ (Id is the identical mapping);

(2) *isometry property* (the mapping T_y preserves the measure μ on Ω): $T_y A \in \mathcal{A}$, $\mu(T_y A) = \mu(A)\ \forall y \in \mathbb{R}^3,\ A \in \mathcal{A}$;
(3) *measurability*: for any measurable functions $\psi(\omega)$ on Ω the function $\psi(T_y\omega)$ is measurable on $\Omega \times \mathbb{R}^3$ and continuous in y.

Let $L_q(\Omega, \mu)$ $(q \geqslant 1)$ be the space of measurable functions integrable in the power q with respect to the measure μ. If $U_y : \Omega \to \Omega$ is a dynamical system, then in the space $L_2(\Omega, \mu)$ we define a parametric group of operators $\{U_y\}$, $y \in \mathbb{R}^3$ (we keep the same notation) by the formula $(U_y\psi)(\omega) := \psi(U_y\omega)$, $\psi \in L_2(\Omega, \mu)$.

From the condition (3) of the definition it follows that the group U_y is strongly continuous, i.e. we have $\lim_{y\to 0}\|U_y\psi - \psi\|_{L_2(\Omega,\mu)} = 0$ for any $\psi \in L_2(\Omega, \mu)$.

Definition 3.2 Suppose that $\psi(\omega)$ is a measurable function on Ω. The function $\psi(T_y\omega)$ of $y \in \mathbb{R}^3$ for fixed $\omega \in \Omega$ is called the *realization of the function* ψ.

The following assertion is proved, for instance, in [5,7].

PROPOSITION 3.1 *Assume that $\psi \in L_q(\Omega, \mu)$, then almost all realizations $\psi(T_y\omega)$ belong to $L_q^{loc}(\mathbb{R}^3)$.*

If the sequence $\psi_k \in L_q(\Omega, \mu)$ converges in $L_q(\Omega, \mu)$ to the function ψ, then there exists a subsequence k' such that almost all realizations $\psi_{k'}(T_y\omega)$ converges in $L_q^{loc}(\mathbb{R}^3)$ to the realization $\psi(T_y\omega)$.

Definition 3.3 A measurable function $\psi(\omega)$ on Ω is called *invariant*, if $\psi(T_y\omega) = \psi(\omega)$ for any $y \in \mathbb{R}^3$ and almost all $\omega \in \Omega$.

Definition 3.4 The dynamical system T_y is called *ergodic*, if any invariant function almost everywhere coincides with a constant.

We denote by $\mathcal{B}$ the natural Borel σ-algebra of subsets of the space $\mathbb{R}^3$. Suppose that $\digamma(y) \in L_1^{loc}(\mathbb{R}^3)$.

Definition 3.5 We say that the function $\digamma(y)$ has a spatial average, if the limit

$$M(\digamma) = \lim_{\varepsilon\to 0} \frac{1}{|B|} \int_B \digamma\left(\frac{y}{\varepsilon}\right) \mathrm{d}y$$

does exist for any bounded Borel sets $B \in \mathcal{B}$ and does not depend on the choice of B, and $M(\digamma)$ is called the *spatial average value* of the function $\digamma$.

In equivalent form

$$M(\digamma) = \lim_{t\to +\infty} \frac{1}{|B_t|} \int_{B_t} \digamma(y)\,\mathrm{d}y,$$

where $B_t = \{y \in \mathbb{R}^3 \mid \frac{y}{t} \in B\}$.

The following statement could be found, for instance, in [7].

PROPOSITION 3.2 *Let the function* $\mathcal{F}(y)$ *have a spatial meanvalue in* $\mathbb{R}^3$*, and suppose that the family* $\{\mathcal{F}\left(\frac{y}{\varepsilon}\right),\ 0 < \varepsilon \leqslant 1\}$ *is bounded in* $L_q(\mathcal{K})$ *for some* $q \geqslant 1$*, where* $\mathcal{K}$ *is compact in* $\mathbb{R}^3$*. Then* $\mathcal{F}\left(\frac{y}{\varepsilon}\right) \rightharpoonup M(\mathcal{F})$ *weakly in* $L_q^{loc}(\mathbb{R}^3)$ *as* $\varepsilon \to 0$.

In further analysis, we use the Birkhoff theorem in the following form (see, for instance, [5,7]):

THEOREM 3.1 (Birkhoff ergodic theorem) *Let* T_y *satisfy the Definition 3.1 and assume that* $\psi \in L_q(\Omega, \mu), q \geqslant 1$*. Then for almost all* $\omega \in \Omega$*, the realization* $\psi(T_y\omega)$ *has the spatial meanvalue* $M(\psi(T_y\omega))$*. Moreover, the spatial meanvalue* $M(\psi(T_y\omega))$ *is a conditional mathematical expectation of the function* $\psi(\omega)$ *with respect to the* σ*-algebra of invariant subsets. Hence,* $M(\psi(T_y\omega))$ *is an invariant function and*

$$\mathbb{E}(\psi) \equiv \int_\Omega \psi(\omega)\, \mathrm{d}\mu = \int_\Omega M(\psi(T_y\omega))\, \mathrm{d}\mu.$$

In particular, if the dynamical system T_y *is ergodic, then for almost all* $\omega \in \Omega$ *the following formula*

$$\mathbb{E}(\psi) = M(\psi(T_y\omega))$$

holds true.

Definition 3.6 A random function $\psi(y, \omega)$ $(y \in \mathbb{R}^3, \omega \in \Omega)$ is called *statistically homogeneous*, if the following representation $\psi(y, \omega) = \Psi(T_y\omega)$ holds for some measurable function $\Psi : \Omega \to \mathbb{R}$, where T_y is a dynamical system in Ω.

4. Homogenization

4.1. *Statement of the problem*

We consider functions $\mathfrak{p}_\varepsilon(x, \omega) := \mathfrak{p}_0(\frac{x}{\varepsilon}, \omega)$ and $\mathfrak{u}_\varepsilon(x, \omega) := \mathfrak{u}_0(\frac{x}{\varepsilon}, \omega)$ depending on a small parameter ε and such that

(1) $\mathfrak{p}_0$ and $\mathfrak{p}_0\mathfrak{u}_0$ are statistically homogeneous, i.e. $\mathfrak{p}_0(y, \omega) = \mathfrak{P}_0(T_y\omega)$, $(\mathfrak{p}_0\mathfrak{u}_0)(y, \omega) = \mathfrak{V}_0(T_y\omega)$, where $\mathfrak{P}_0 : \Omega \to \mathbb{R}$, $\mathfrak{V}_0 : \Omega \to \mathbb{R}^3$ are measurable,
(2) T_y is ergodic dynamical system,
(3) $\mathfrak{p}_0$ and $\mathfrak{p}_0\mathfrak{u}_0$ are almost surely uniformly bounded, i.e. $\|\mathfrak{P}_0\|_{L_\infty(\Omega,\mu)} < K_1 < \infty$, $\|\mathfrak{V}_0\|_{L_\infty(\Omega,\mu)} < K_2 < \infty$,
(4) almost surely $\mathfrak{P}_0(\omega) \geqslant M_1 > 0$

Moreover, for almost all $\omega \in \Omega$ we have the convergence $\mathfrak{p}_\varepsilon \rightharpoonup \mathfrak{p}_0^{hom}$ and $\mathfrak{p}_\varepsilon\mathfrak{u}_\varepsilon \rightharpoonup \mathfrak{v}_0^{hom}$ weakly in $L_\infty(D)$, where $\mathfrak{p}_0^{hom} = \mathbb{E}\left(\mathfrak{p}_0(y, \omega)\right) = \mathbb{E}\left(\mathfrak{P}_0(\omega)\right)$, as $\varepsilon \to 0$ and $\mathfrak{v}_0^{hom} = \mathbb{E}\left(\mathfrak{v}_0(y, \omega)\right) = \mathbb{E}\left(\mathfrak{V}_0(\omega)\right)$, as $\varepsilon \to 0$, respectively.

Remark 4.1 Since the dynamical system is ergodic, the invariant functions are ω-almost everywhere constants and keeping in mind the Birkhoff theorem we conclude that $\mathfrak{p}_0^{hom}$ and $\mathfrak{v}_0^{hom}$ are constants independent of y.

To each initial distribution of densities $\mathfrak{p}_\varepsilon$ and momenta $\mathfrak{p}_\varepsilon \mathfrak{u}_\varepsilon$ we associate a triple of random functions $(\rho^\varepsilon(x,t,\omega), \mathbf{u}^\varepsilon(x,t,\omega), \mathbf{n}^\varepsilon(x,t,\omega))$ that is a solution to the system (2.5) with the initial conditions

$$\rho^\varepsilon(x,0,\omega) = \mathfrak{p}_\varepsilon(x,\omega), \quad \mathbf{n}^\varepsilon(x,0,\omega) = \mathfrak{n}_0(x), \quad \rho^\varepsilon \mathbf{u}^\varepsilon(x,0,\omega) = \mathfrak{p}_\varepsilon \mathfrak{u}_\varepsilon(x,\omega). \tag{4.1}$$

We study the behavior of the solution to this problem as the small parameter tends to zero. In this section, we prove the following assertion.

THEOREM 4.1 *Suppose that $\mathfrak{p}_\varepsilon$ for $\varepsilon \geqslant 0$, $\mathfrak{u}_\varepsilon$, and $\mathfrak{n}_0$ satisfy the assumptions of Theorem 2.1, the families $\mathfrak{p}_\varepsilon$ and $\mathfrak{u}_\varepsilon$ possesses properties (1)—(4), and the constants μ_i satisfy the condition (2.8). In addition, assume that the limit problem has a unique weak solution. Then the family of weak solutions $(\rho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon)$ to the problem (2.5) and (4.1) converges to the solution $(\rho^{hom}, \mathbf{u}^{hom}, \mathbf{n}^{hom})$ of the problem (2.5) and (2.6) with initial conditions*

$$\mathbf{u}^{hom}(x,0) = \frac{\mathfrak{v}_0^{hom}(x)}{\mathfrak{p}_0^{hom}(x)}, \quad \mathbf{n}^{hom}(x,0) = \mathfrak{n}_0(x), \quad \rho^{hom}(x,0) = \mathfrak{p}_0^{hom}(x). \tag{4.2}$$

in the following sense: for almost all $\omega \in \Omega$, we have

$$\begin{aligned}
&\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}^{hom} \quad \textit{weakly in } L_2((0,T); H^1(D)),\\
&\mathbf{n}^\varepsilon \rightharpoonup \mathbf{n}^{hom} \quad \textit{weakly in } L_2(0,T; H^2(D)),\\
&\rho^\varepsilon \overset{*}{\rightharpoonup} \rho^{hom} \quad *\textit{-weakly in } L_\infty(Q_T),\\
&\mathbf{u}^\varepsilon \to \mathbf{u}^{hom} \quad \textit{strongly in } L_3(Q_T),\\
&\mathbf{n}^\varepsilon \to \mathbf{n}^{hom} \quad \textit{strongly in } L_{8-\delta}(Q_T), \quad \delta > 0.
\end{aligned}$$

4.2. *Proof of the main theorem*

By the definition of a weak solution, $(\rho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon)$ satisfies a.s. the integral identities

$$\int_{Q_T} \left(\rho \mathbf{u}_{i,t}\varphi^i + \rho \mathbf{u}_j \mathbf{u}_i \varphi_{i\,x_j}\right) \mathrm{d}x\mathrm{d}t + \int_D \rho u_i \varphi_i \Big|_T \mathrm{d}x = \int_{Q_T} \sigma_{ij}\varphi_{i\,x_j} \,\mathrm{d}x\mathrm{d}t + \int_D \mathfrak{p}_0 \mathfrak{u}_{0i}\varphi_i \,\mathrm{d}x, \tag{4.3}$$

$$\int_{Q_T} \left(g_i^\varepsilon + \pi^\varepsilon_{ji\,x_j}\right) \psi \,\mathrm{d}x\mathrm{d}t = 0, \tag{4.4}$$

$$\int_{Q_T} \rho^\varepsilon \dot{w} \,\mathrm{d}x\mathrm{d}t = \int_D \rho^\varepsilon w \mathrm{d}x \Big|_0^T, \tag{4.5}$$

where φ is an arbitrary vector in $Sol(Q_T)$, $\psi \in L_2(Q_T)$, $w \in H^1(Q_T)$.

The main goal is to prove the precompactness of the family of solutions in some function spaces. For this purpose, we turn to the inequalities (2.12) and (2.13).

First of all, we note that ρ^ε are a.s. essentially uniformly bounded and are a.s. uniformly separated from zero in Q_T. Hence $\mathbf{u}^\varepsilon$, $\mathbf{n}^\varepsilon$ are a.s. uniformly bounded in the $L_2(Q_T)$– and $H^1(Q_T)$–norm.

To estimate the gradient of the velocity vector, we consider the first term on the right-hand side. The functional

$$f(\mathbf{u}) = \left(\int |A_{ij}|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

defines the norm in $C_0^\infty(D)$. It is easy to verify that $f(\mathbf{u})$ is equivalent to the H^1-norm. Indeed, the inequality $f(\mathbf{u}) \leqslant C||\mathbf{u}||_{H^1(Q)}$ is obvious. The inverse inequality follows from the estimate

$$||\mathbf{u}_{1x}||_2^2 + ||\mathbf{u}_{2y}||_2^2 + ||\mathbf{u}_{3z}||_2^2 \leqslant f^2(\mathbf{u})$$

and the formula

$$\int \mathbf{u}_{2x}\mathbf{u}_{1y}\ \mathrm{d}x = \int \mathbf{u}_{2y}\mathbf{u}_{1x}\ \mathrm{d}x,$$

which is valid for any function $\mathbf{u} \in C_0^\infty$.

Thus, the family of vector-valued functions $\mathbf{u}^\varepsilon$ is a.s. bounded in the H^1-norm. Therefore, we can select a sequence of solutions in such a way that for almost all $\omega \in \Omega$ we have

$$\begin{aligned}
&\mathbf{u}^{\varepsilon_n} \rightharpoonup \mathbf{u}^{hom} \text{ in } L_2(0, T; H^1), \\
&\mathbf{n}^{\varepsilon_n} \overset{*}{\rightharpoonup} \mathbf{n}^{hom} \text{ in } L_2(0, T; H^2), \\
&\mathbf{n}_t^{\varepsilon_n} \rightharpoonup \mathbf{n}_t^{hom} \text{ in } L_2(Q), \\
&\rho^{\varepsilon_n} \overset{*}{\rightharpoonup} \rho^{hom}.
\end{aligned} \tag{4.6}$$

Since the assumptions of Lemma 5.1 in [12] are satisfied a.s., the uniform boundedness of $\mathbf{u}^{\varepsilon_n}$ in $L_2(0, T; Sol_2^1) \cap L_\infty(0, T; Sol_2)$ implies the strong convergence $\mathbf{u}^{\varepsilon_n} \to \mathbf{u}^{hom}$ in $L_3(Q)$ for almost all $\omega \in \Omega$.

We prove the weak convergence of $\sigma_{ij}^\varepsilon, \pi_{ij}^\varepsilon, g_i^\varepsilon$. By the embedding theorem for Sobolev spaces, we have

$$\mathbf{n}^\varepsilon \in L_2(0, T; C(D)) \cap L_\infty(0, T; L_6) \cap L_4(Q);$$

moreover,

$$\mathbf{n}^{\varepsilon_n} \to \mathbf{n}^{hom} \quad \text{in } L_4(Q).$$

By the energy inequality (2.12), $A_{kp}^{\varepsilon_n}\mathbf{n}_k^{\varepsilon_n}\mathbf{n}_p^{\varepsilon_n}$ are a.s. uniformly bounded in L_2, and, by the strong convergence of $\mathbf{n}^{\varepsilon_n}$ a.s. in L_4, we can assume that a.s. $A_{kp}^{\varepsilon_n}\mathbf{n}_k^{\varepsilon_n}\mathbf{n}_p^{\varepsilon_n} \rightharpoonup A_{kp}^{hom}\mathbf{n}_k^{hom}\mathbf{n}_p^{hom}$. This, in turn, guarantees the weak convergence of $\mathbf{n}_k^{\varepsilon_n}\mathbf{n}_p^{\varepsilon_n}A_{kp}^{\varepsilon_n}\mathbf{n}_i^{\varepsilon_n}\mathbf{n}_j^{\varepsilon_n}$ to $\mathbf{n}_k^0\mathbf{n}_p^0A_{kp}^0\mathbf{n}_i^0\mathbf{n}_j^0$ in L_1 for almost all $\omega \in \Omega$. The weak convergence of the remaining terms is proved in a similar way.

Thus, we can conclude that a.s.

$$\begin{aligned}
&\sigma_{ij}^{\varepsilon_n} \rightharpoonup \sigma_{ij}^{hom} \text{ weakly in } L_1, \\
&\pi_{ij}^{\varepsilon_n} \rightharpoonup \pi_{ij}^{hom} \text{ weakly in } L_2, \\
&g_i^{\varepsilon_n} \rightharpoonup g_i^{hom} \text{ weakly in } L_2.
\end{aligned}$$

Passing to the limit in the identities (4.3)–(4.5), we find that the triple $\left(\rho^{hom}, \mathbf{u}^{hom}, \mathbf{n}^{hom}\right)$ is a weak solution to the problem (2.5), (2.6), and (4.2). The theorem is proved.

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