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DISPLACEMENT POTENTIALS IN NON-SMOOTH DYNAMICS

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Abstract. The paper treats the evaluation of the accelerations in rigid multibody systems which are subjected to displacement dependent set-valued force interactions. The interaction laws are represented by non-smooth displacement potentials and derived through generalized differentiation. The resulting multifunctions contain the cases of smooth force characteristics, bilateral constraints, as well as combinations of them like unilateral constraints or prestressed springs with play. Impacts are excluded. A generalization of the classical principles of d'Alembert, Jourdain, and Gauss in terms of hemi-variational inequalities is given. A strictly convex minimization problem depending on the unknown accelerations is stated, known in classical mechanics as the Principle of Least Constraints. The theory is applied to unilaterally constrained systems.

1. Introduction

In classical mechanics the concept of virtual work is used in order to classify displacement-dependent forces. Forces are called "active" or "applied forces" if they produce virtual work, otherwise they are called "passive" or "constraint forces". Both types are treated completely different with respect to formulation and evaluation: Applied forces are continuous functions of the displacements and can therefore be directly evaluated when the displacements are known. Passive forces originate from bilateral constraints and can not be represented by such functions. Usually certain displacements are prescribed, but the forces are arbitrary. The evaluation of the passive forces is done by either inverting the corresponding force characteristics

which yield additional equations depending on the displacements known as constraints, or by choosing appropriate reduced sets of new coordinates, called “minimal coordinates”, such that the constraints are automatically satisfied and thus eliminated from the equations of motion. This approach makes sense as long as the above mentioned classification applies to all forces within a multibody system. However, there are forces of intermediate type acting in some regions as applied forces, in other regions as constraints. Usually one becomes aware of them when dealing with dry friction, unilateral constraints, or impact effects. This type of forces leads to hemivariational inequalities which are variational expressions for differential inclusions obtained by generalized differentiation of certain (non-smooth) superpotentials. In classical mechanics only small attention is paid to this case. Even in modern literature only a few works have appeared concerning inequality constraints and nondifferentiable energy functions. We refer especially to the papers [1], [4], [6] and to the few references given there. The aim of this paper is to introduce general interaction laws including both, active type forces and constraint type forces, and to give a new interpretation of d’Alembert’s Principle of Virtual Work which leads to a strictly convex optimization problem in terms of the unknown accelerations of the system. As an example the theory will be applied to systems with arbitrary unilateral constraints.

2. The Principles of d’Alembert, Jourdain, and Gauss

In this section we define the Principles of d’Alembert, Jourdain, and Gauss for holonomic scleronomous multibody systems, and we show that the Principle of Gauss is equivalent to a strictly convex optimization problem which corresponds to the classical Principle of Least Constraints.

We consider a dynamical system consisting of a certain number of rigid bodies. The generalized displacements of the system may be described by a coordinate vector $q \in \mathbb{R}^f$ which depends on time t . The velocities are assumed to be functions of bounded variations [4], [5], i.e. their left and right limits $\dot{q}^-$ and $\dot{q}^+$ exist at every point t , the set of points at which $\dot{q}$ is discontinuous is at most countable, the displacements are therefore absolutely continuous, and the differential measure of $\dot{q}$ consists of three parts: The Lebesgue measurable part which is absolutely continuous, the atomic part which is concentrated on the countable set of discontinuities of $\dot{q}$, and a part with support on sets with Hausdorff dimensions between 0 and 1. Neglecting the latter and splitting off the atomic part one may state the equilibrium of the linear and angular momenta and forces which yields

$$M(q) \ddot{q}^+ - h(q, \dot{q}^+) = f^+ \quad (1)$$

with a symmetric positive definite mass matrix M and a vector h consisting of the gyroscopical accelerations. In order to obtain a complete description of the dynamics of the system we must introduce some force laws which connect the forces f^+ and the displacements q . We assume these interaction laws to be derived from a *displacement potential* $V(q)$ through generalized differentiation [6], and we will equip it with the following properties:

Let $V : \mathfrak{R}^f \rightarrow \mathfrak{R} \cup \{+\infty\}$ be a (non-smooth, non-convex) lower semicontinuous (l.s.c.) function with effective domain C such that its epigraph $\text{epi } V$ consists of regular points only, i.e. the tangent cone $T_{\text{epi } V}$ and the contingent cone $K_{\text{epi } V}$ to the set $\text{epi } V$ coincide for every point $(q, V(q)) \in \text{epi } V$. With that assumption we have excluded the so-called “reentrant corners” [3], the physical meaning of which being not yet well understood in multi-body dynamics. The potentials $V(q)$ considered here thus “share the properties of nonsmooth convex and smooth nonconvex functions” rather than being general which, however, seems to be sufficient at the present time with respect to application problems. We will now assume force laws of the form

$$-f^+ \in \bar{\partial}V(q) \quad (2)$$

where $\bar{\partial}V(q)$ denotes the generalized gradient of V at q . The generalized gradient $\bar{\partial}V(q)$ is a convex set [8]. Roughly speaking, it consists of the convex hull of the gradients $\lim_{i \rightarrow \infty} \nabla V(q_i)$ of all sequences $\{q_i\}$ approaching q for $i \rightarrow \infty$. If we are interested in the accelerations from the past ($\ddot{q}^-$) we have to consider, instead of (1), the equation $M(q)\ddot{q}^- - h(q, \dot{q}^-) = f^-$ together with the force law $-f^- \in \bar{\partial}V(q)$. Note that both, $-f^-$ and $-f^+$ belong to the same set $\bar{\partial}V(q)$ but might be different. They coincide, for example, if $\bar{\partial}V(q)$ consists of one element only, i.e. for example in the smooth case where we have $-f^+ = -f^- = \nabla V(q)$.

By using the force law (2) we are able to take into account “smooth” interaction forces in the classical sense, for example springs with nonlinear characteristics, as well as unilateral and additional (this also means all) smooth bilateral constraints, and certain set-valued interactions like prestressed springs with plays. Every kind of viscous and dry friction is excluded as well as nonholonomic constraints. They demand a representation via velocity potentials which are not considered here. Combining (1) and (2) we get the differential inclusions of the system [1]

$$-(M\ddot{q}^+ - h) \in \bar{\partial}V(q). \quad (3)$$

It is known from non-smooth analysis that inclusion (2) can also be expressed by means of a hemivariational inequality, i.e.

$$-f^+ \in \bar{\partial}V(q) \Leftrightarrow V^\uparrow(q, q^* - q) \geq \langle -f^+, q^* - q \rangle, \quad q \in C, \quad \forall q^*, \quad (4)$$

where $V^\uparrow(q, q^\star - q)$ denotes the generalized directional derivative of V at q in the direction $(q^\star - q)$, and C is the set of the admissible values of q , see [8]. Together with (1) we obtain the expression

$$-\langle M\ddot{q}^+ - h, q^\star - q \rangle \leq V^\uparrow(q, q^\star - q), \quad q \in C, \forall q^\star \quad (5)$$

which holds for every fixed time t and which we will call the *Principle of d'Alembert* [1]. Note that the variations $(q^\star - q)$ are not restricted to any subset of $\mathbb{R}^f$, thus values $V^\uparrow(q, q^\star - q) = +\infty$ which occur when q is at the boundary of C are allowed. The physical dimension of eq. (5) is that of “work” [Nm]. Indeed, we will see in Section 3 when dealing with unilateral constraints that eq. (5) leads to Fourier’s classical principle that the virtual work produced by constraint forces is always greater than or equal to zero.

Although providing a complete description of the dynamics of the system, equations (3) and (5) are too general for further investigations. Particularly, there is no direct way to obtain the accelerations $\ddot{q}^+$ if the displacements q and velocities $\dot{q}^+$ are given. However, one can show that d'Alembert’s principle (5) is equivalent to a variational expression in terms of velocities which yields an expression with the physical dimension of a power [Nm/s]. This approach is also based on potential functions. We define the function

$$\Phi : p \rightarrow V^\uparrow(q, p) \quad (6)$$

and call it the *velocity potential* of the multibody system. Note that the velocity potential Φ is nothing more than the generalized directional derivative of V at a point q considered as a function of the direction p . The velocity potential is a convex, l.s.c., positively homogeneous, and subadditive function, and obviously $\text{dom } \Phi = T_C(q)$ for every fixed q , where $T_C(q)$ denotes the tangent cone to C at q . We define the *Principle of Jourdain* for every fixed time t to be

$$-\langle M\dot{q}^+ - h, \dot{q}^\diamond - \dot{q}^+ \rangle \leq \Phi^\uparrow(\dot{q}^+, \dot{q}^\diamond - \dot{q}^+), \quad q \in C, \dot{q}^+ \in T_C(q), \forall \dot{q}^\diamond. \quad (7)$$

The magnitude $\dot{q}^\diamond$ denotes just any arbitrary velocity in order to express that the inequality in eq. (7) has to be valid for every direction $(\dot{q}^\diamond - \dot{q}^+)$. Like in classical mechanics $\dot{q}^\diamond$ has to be understood as the (right) time derivative of any arbitrary trajectory starting at time t in point q , i.e. there are sequences $\{q^\diamond(t_n)\} \subset \{q^\star(t_n)\}$ approaching q asymptotic to the half-line emanating from q in the direction $\dot{q}^\diamond$ when $t_n \rightarrow t$ for $n \rightarrow \infty$, [1], [9]. Analogously to (3) a differential inclusion being equivalent to Jourdain’s Principle (7) can be stated

$$-(M\dot{q}^+ - h) \in \partial\Phi(\dot{q}^+), \quad (8)$$

where due to the convexity of Φ one may use the subdifferential from Convex Analysis ∂ instead of the generalized gradient $\bar{\partial}$, [9].

In order to define a variational principle on acceleration level we proceed in the same manner as above. We introduce the *acceleration potential* to be the generalized directional derivative of the velocity potential as a function of its direction

$$\Psi : p \rightarrow \Phi^\uparrow(\dot{q}^+, p) \quad (9)$$

and notice that $\Psi(p)$ is also convex, l.s.c., positively homogeneous, and subadditive, and its effective domain is given by $\text{dom } \Psi = T_{T_C(q)}(\dot{q}^+)$ for every fixed $q, \dot{q}^+$. We define the *Principle of Gauss* for every fixed time t to be

$$\begin{aligned} -\langle M\ddot{q}^+ - h, \ddot{q}^\times - \ddot{q}^+ \rangle &\leq \Psi^\uparrow(\ddot{q}^+, \ddot{q}^\times - \ddot{q}^+), \\ q \in C, \dot{q}^+ \in T_C(q), \ddot{q}^+ \in T_{T_C(q)}(\dot{q}^+), \forall \ddot{q}^\times \end{aligned} \quad (10)$$

which has the physical dimension of a power per time $[\text{Nm/s}^2]$, and where the term $\ddot{q}^\times$ has to be understood in the same sense as $\dot{q}^\circ$ above. As in (8) we obtain the differential inclusion

$$-(M\ddot{q}^+ - h) \in \partial\Psi(\ddot{q}^+) \quad (11)$$

which is an equivalent representation of (10).

Equation (11) constitutes the necessary and sufficient optimality conditions of a strictly convex optimization problem [9] which is obvious by rewriting it in the form

$$0 \in M\ddot{q}^+ - h + \partial\Psi(\ddot{q}^+). \quad (12)$$

The accelerations $\ddot{q}^+$ are hence the optimal solutions of the program

$$\ddot{q}^+ = \arg \min \{f(\ddot{q}^+)\}; \quad f(\ddot{q}^+) = \frac{1}{2}\langle M\ddot{q}^+, \ddot{q}^+ \rangle - \langle h, \ddot{q}^+ \rangle + \Psi(\ddot{q}^+) \quad (13)$$

which is called in classical mechanics the *Principle of Least Constraints*. The cost function f is strictly convex because M is a symmetric and positive definite matrix and the acceleration potential Ψ is convex. Moreover, since f is strictly convex, the optimal solutions $\ddot{q}^+$ are unique.

In [1] it has been proved that the principles of d'Alembert, Jourdain, and Gauss are equivalent. One important property of such systems may be expressed via the inequalities

$$-\langle M\ddot{q}^+ - h, \ddot{\eta}^\times \rangle \leq \Psi^\uparrow(\ddot{q}^+, \ddot{\eta}^\times) \leq \Phi^\uparrow(\dot{q}^+, \ddot{\eta}^\times) \leq V^\uparrow(q, \ddot{\eta}^\times), \quad \forall \ddot{\eta}^\times, \quad (14)$$

or, by rewriting (14) in terms of the corresponding differential inclusions (3), (8), (11), via

$$-(M\ddot{q}^+ - h) \in \partial\Psi(\ddot{q}^+) \subseteq \partial\Phi(\dot{q}^+) \subseteq \bar{\partial}V(q). \quad (15)$$

The advantages of introducing acceleration potentials and solving Gauss' principle instead of d'Alembert's principle are multiple: First of all we were able to proceed from a non-convex displacement potential to a convex and hence much better behaved acceleration potential which is positively homogeneous in addition. Furthermore the number of inequalities describing the sets (15) becomes smaller, because $\partial\Psi(\ddot{q}^+)$ is contained in $\bar{\partial}V(q)$ (15). Sometimes it may happen that $\partial\Psi(\ddot{q}^+)$ consists of one element only whereas $\bar{\partial}V(q)$ is really "big." For systems allowing the formulation of the generalized gradients via linear complementarity conditions the dimension of the matrix of the corresponding Linear Complementarity Problem is directly related to the dimensions of the sets in (15). Finally, by the Principle of Gauss we achieved a representation of the dynamics equations which enables a direct access to the right accelerations of the system. Moreover, these accelerations are uniquely determined by the corresponding strictly convex optimization problem (13).

3. Unilateral Constraints

It is well known by now that unilateral constraints may be taken into account by indicator functions. We consider a multibody system, the displacements of which being subjected to geometrical restrictions $q \in \text{dom } V = C$. The indicator function I_C of a set C is defined by

$$I_C(q) = \begin{cases} 0 & \text{if } q \in C \\ +\infty & \text{if } q \notin C \end{cases} \quad (16)$$

and has some important connections to the normal and the tangential cone of C . In [8] it is shown that

$$\bar{\partial}I_C(q) = N_C(q) \quad \text{and} \quad I_C^\dagger(q, p) = I_{T_C(q)}(p), \quad (17)$$

where $N_C(q)$ is the normal cone to C at q defined by $N_C(q) = \{x \mid \langle x, y \rangle \leq 0, \forall y \in T_C(q)\}$. We assume that only forces resulting from unilateral constraints act on the multibody system. In this case the displacement potential $V(q)$ is

$$V(q) = I_C(q). \quad (18)$$

In order to derive the velocity potential (6) and the acceleration potential (9) we just have to apply equation (17). We obtain

$$\Phi(p) = I_C^\dagger(q, p) = I_{T_C(q)}(p), \quad \Psi(p) = I_{T_C(q)}^\dagger(\dot{q}^+, p) = I_{T_{T_C(q)}(\dot{q}^+)}(p). \quad (19)$$

With the help of (17) we get immediately from (18) and (19) the three sets $\bar{\partial}V(q)$, $\partial\Phi(\dot{q}^+)$, and $\partial\Psi(\ddot{q}^+)$. They are

$$\bar{\partial}V(q) = N_C(q), \quad \partial\Phi(\dot{q}^+) = N_{T_C(q)}(\dot{q}^+), \quad \partial\Psi(\ddot{q}^+) = N_{T_{T_C(q)}(\dot{q}^+)}(\ddot{q}^+), \quad (20)$$

and the differential inclusion (15) becomes

$$-(M\ddot{q}^+ - h) \in N_{T_{T_C(q)}(\dot{q}^+)}(\ddot{q}^+) \subseteq N_{T_C(q)}(\dot{q}^+) \subseteq N_C(q). \quad (21)$$

At that point, note the polarity between the normal and the tangential cone. Due to (21) we obviously have

$$T_C(q) \subseteq T_{T_C(q)}(\dot{q}^+) \subseteq T_{T_{T_C(q)}(\dot{q}^+)}(\ddot{q}^+). \quad (22)$$

The tangential cones increase from the displacement level to the acceleration level in the same manner as the corresponding normal cones decrease. One may say that the accelerations are in some sense less constrained than the displacements, a fact completely different from bilaterally constrained motion. Using the potential functions in (18), (19) the three principles (5), (7), and (10) become

$$\begin{aligned} -\langle M\ddot{q}^+ - h, q^* - q \rangle &\leq I_C^\dagger(q, q^* - q), \quad \forall q^*, q \in C \\ -\langle M\ddot{q}^+ - h, \dot{q}^\circ - \dot{q}^+ \rangle &\leq I_{T_C(q)}^\dagger(\dot{q}^+, \dot{q}^\circ - \dot{q}^+), \quad \forall \dot{q}^\circ, q \in C, \dot{q}^+ \in T_C(q) \\ -\langle M\ddot{q}^+ - h, \ddot{q}^\times - \ddot{q}^+ \rangle &\leq I_{T_{T_C(q)}(\dot{q}^+)}^\dagger(\ddot{q}^+, \ddot{q}^\times - \ddot{q}^+), \\ &\quad \forall \ddot{q}^\times, q \in C, \dot{q}^+ \in T_C(q), \ddot{q}^+ \in T_{T_C(q)}(\dot{q}^+). \end{aligned} \quad (23)$$

We apply again equation (17) to the right-hand sides of the inequalities in (23) and choose the variations of the displacements, velocities, and accelerations such that they belong to the sets specified by the resulting indicator functions. The values of the indicators are then equal to zero and (23) becomes

$$\begin{aligned} -\langle M\ddot{q}^+ - h, q^* - q \rangle &\leq 0, \quad \forall (q^* - q) \in T_C(q), q \in C \\ -\langle M\ddot{q}^+ - h, \dot{q}^\circ - \dot{q}^+ \rangle &\leq 0, \quad \forall (\dot{q}^\circ - \dot{q}^+) \in T_{T_C(q)}(\dot{q}^+), q \in C, \dot{q}^+ \in T_C(q) \\ -\langle M\ddot{q}^+ - h, \ddot{q}^\times - \ddot{q}^+ \rangle &\leq 0, \quad \forall (\ddot{q}^\times - \ddot{q}^+) \in T_{T_{T_C(q)}(\dot{q}^+)}(\ddot{q}^+), \\ &\quad q \in C, \dot{q}^+ \in T_C(q), \ddot{q}^+ \in T_{T_C(q)}(\dot{q}^+). \end{aligned} \quad (24)$$

Setting $M\ddot{q}^+ - h = f^+$ (cf. eq. (1)) we see that all scalar virtual expressions such as virtual work or virtual power produced by forces from unilateral constraints are always greater than or equal to zero. The Principle of d'Alembert for unilateral constraints was stated in 1821 by Fourier and can also be found in [2], [3]. Moreau's work is completely based on Jourdain's principle in the second line of (24) but contains additionally impacts and Coulomb friction [4], [5]. The Principle of Gauss as stated in the third equation of (24) can be found in [2] and is often heuristically used in multi-body dynamics, cf. for example [7]. Taking the acceleration potential (19) we may state the Principle of Least Constraints (13),

$$\ddot{q}^+ = \arg \min \{f(\ddot{q}^+)\}: \quad f(\ddot{q}^+) = \frac{1}{2} \langle M\ddot{q}^+, \ddot{q}^+ \rangle - \langle h, \ddot{q}^+ \rangle + I_{T_{T_C(q)}(\dot{q}^+)}(\ddot{q}^+) \quad (25)$$

which is a strictly convex program with inequality constraints, i.e. find $\ddot{q}^+$ such that $\frac{1}{2}\langle M\ddot{q}^+, \ddot{q}^+ \rangle - \langle h, \ddot{q}^+ \rangle$ becomes minimal under the restriction that $\ddot{q}^+$ belongs to the closed convex cone $T_{T_C(q)}(\dot{q}^+)$.

4. Conclusion

The topic of the presented paper may be described as follows: When considering the velocities of a multibody system as functions of bounded variation the accelerations of the system exist almost everywhere. In this case there must be a way to determine their values directly from the equations of motion. For that purpose we considered the equations of motion of a system under the influence of forces which were derived by generalized differentiation from a displacement potential. The corresponding differential inclusion was reformulated in terms of a hemivariational inequality; this is the Principle of d'Alembert. Following classical mechanics we also introduced a variational principle on the velocity and on the acceleration level, i.e. the Principles of Jourdain and Gauss, respectively. We also showed that unilateral constraints were covered by the chosen approach. The presented theory can be extended to non-integrable potential functions on velocity level in order to take into account additional forces resulting from viscous damping, dry friction, and nonholonomic constraints. At a first view these forces seem to be of completely different nature, only sharing the property of non-integrability of some related functions. It is a remarkable fact that all of them can be treated by the unified concept of using velocity potentials.

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