A new nonconvex variational approach for sensory neurons receptive field estimation

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Abstract. Determining the receptive field of a visual sensory neuron is crucial to characterize the region of the visual field and the stimuli this neuron is sensitive to. We propose a new method to estimate receptive fields by a nonconvex variational approach, thus relaxing the simplifying and unrealistic assumption of convexity made by standard approaches. The method consists of solving a relaxed discrete energy minimization problem using a proximal alternating algorithm. We compare our approach with the classical spike-triggered-average technique on simulated data, considering a typical retinal ganglion cell as ground truth. Results show a high improvement in terms of accuracy and convergence with respect to the duration of the experiment.

1. Introduction

This paper deals with the estimation of receptive fields of individual visual neurons. Knowing the receptive field of a particular neuron is a crucial information to understand which region of the visual field and which stimulus the neuron is sensitive to. Thus, biologists always devote efforts in their experimental protocol to characterize the neurons they are recording.

The goal is to characterize the relation between the stimulus and the neuron response which is a set of action potentials (also called spikes). To do so, it is classical to assume that response follows a linear-nonlinear Poissonian (LNP) model [5, 9, 12] (Fig. 1): given a visual stimulus $s(x,t) : \Omega \subset \mathbb{R}^2 \times [0,T] \to \mathbb{R}$, where Ω is the spatial domain and T is the duration of the experiment, the neuron generates a sequence of $n(T)$ spikes times $\{t_i\}_{1\leq i\leq n(T)}$ such that

$$\{t_i\}_{1\leq i\leq n(T)} \text{ is generated by a Poisson process of rate } r(t) = \mathcal{S}\left((s \times u)(t)\right), \qquad (\mathcal{P})$$

where $\mathcal{S}(.)$ is a nonlinear function and $u : \Omega_d = \Omega \times [-d,0] \longrightarrow \mathbb{R}$ is the so-called receptive field which corresponds to the linear part of the processing with $d > 0$ the length of its temporal support.

The operator $\times$ is defined by $(s \times u)(t) = \int_{-\infty}^{0} \langle s(.,t+\tau), u(.,\tau)\rangle_\Omega d\tau$, where $< .,. >_\Omega$ denotes the inner product on Ω. Given hypothesis $(\mathcal{P})$ the problem of estimating the receptive field $u(x,t)$ is an inverse problem: given a stimulus $s(x,t)$ of duration T and the $n(T)$ spikes $\{t_i\}_{1\leq i\leq n(T)}$ from the sensory neuron responding to $s(x,t)$, recover the unknown receptive field $u(x,t)$.

This inverse problem can be formulated using a Bayesian approach. Using the Bayes rule, a classical method is to search for the receptive field $u(x,t)$ as minimum of the anti-log of the *a posteriori* probability (see, e.g., [12, 14, 13]):

$$\inf_u \left\{ -log\left(\rho(u|\{t_i\}_{1\leq i\leq n(T)})\right) \right\} = \inf_u \left\{ -log(\rho(\{t_i\}_{1\leq i\leq n(T)}|u)) - log(\rho(u)) \right\}, \qquad (1)$$

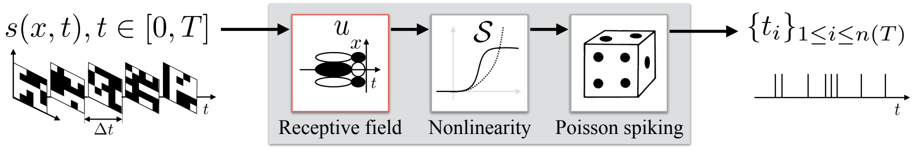

Figure 1. Illustration of the LNP model ($\mathcal{P}$). Two kinds of $\mathcal{S}(.)$ are illustrated: ramp- or sigmoid-like nonlinearities. Here the stimulus is a white noise sequence which is the classical stimulus used for the spike-triggered averaged approach (STA, see [5]).

where $\rho(u)$ denotes the probability density of the random variable u. Then, assuming that $\rho(u)$ is a distribution of the form $\rho(u) \propto e^{-J(u)}$, one can show that (1) can be rewritten as [6]:

$$\inf_u \ \mathcal{E}(u) = \psi(s \times u) + J(u), \tag{2}$$

where $\psi(s \times u) = -log(\rho(\{t_i\}_{1\le i\le n(T)}|u))$ is the data fidelity term defined by

$$\psi(z) = \int_0^T \mathcal{S}(z(\tau))d\tau - \sum_{i=1}^{n(T)} log(\mathcal{S}(z(t_i))), \tag{3}$$

with $z(t) = (s \times u](t)$, and $J(u)$ is the prior term to infer qualitative properties to the solution, according to what is known about the general shape of a receptive field. To study this problem, it is classical to assume that the nonlinearity $\mathcal{S}(.)$ is a ramp function or an exponential. The computation of (2) is then simple since it is a convex problem to solve [12, 14, 9].

On the opposite, in this paper, we consider the case when the nonlinearity $\mathcal{S}(.)$ is a sigmoid, which is more realistic from a physiological point of view; in particular it models the fact that the neuron has a bounded firing rate. However, since $\mathcal{S}(.)$ is nonconvex, the data fidelity term $\psi(s \times u)$ also becomes nonconvex. This is known to be harder to minimize numerically.

Concerning $J(.)$, we choose it to impose two properties on the solution. The first is that u should be localized in space and time since neurons are sensitive to a particular region of the visual field. This can be imposed by a sparsity constraint term. Here we choose to use a convex relaxation of the sparsity and we penalise the L^1–norm of u. The second is that u should be smooth. We propose that u should belong to the space BV_2 that contains piecewise linear functions. This kind of regularity constraint has also been previously used in the context of image restoration [3]. This allows to recover functions with fast smooth variations more precisely than using a simple $\|\nabla u\|_{L^2}^2$–regularity constraint. So the prior term will be

$$J(u) = \lambda \|u\|_{L^1(\Omega_d)} + \mu |u|_{BV_2(\Omega_d)}, \tag{4}$$

where $\lambda, \mu > 0$ are weights associated respectively to the sparsity and to the regularity of u.

So this paper is about the study of the problem (2) with (3)–(4) In Sect. 2, we study the discrete version of the problem (2). We introduce a relaxed problem so that the solution can be computed more easily. We justify it theoretically and introduce an alternating minimizing algorithm converging toward a critical point of the relaxed energy. In Sect. 3, we test the approach on simulated data to provide a quantitative evaluation with comparisons to the classical spike triggered averaged technique (STA, see [5]). In Sect. 4, we describe future work.

2. Study of the discrete problem: Well-posedness and algorithm

2.1. Problem definition

In this section, we study a discrete version of (2). The stimulus is a sequence of N_t images each presented during a period of Δt so that the duration of the experiment is $T = N_t\Delta t$. Each image is of size $N_x \times N_y$ pixels. Receptive field is of size $N_x \times N_y \times D$ where D is a fixed depth. We introduce the real vector spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ so that:

$$u \in \mathcal{X} = \mathbb{R}^{N_x \times N_y \times D}, \quad s \in \mathcal{Y} = \mathbb{R}^{N_x \times N_y \times N_t}, \quad z \in \mathcal{Z} = \mathbb{R}^{N_t}, \tag{5}$$

endowed with the scalar product $\langle .,. \rangle$ and the associated norm $\|.\|_2 = \sqrt{\langle .,. \rangle}$. We denote by $\|.\|_1$ the l_1–norm. We do the following hypothesis on the sigmoid function $\mathcal{S}(.)$.

Hypothesis 1. *There exist $c > 0$ and $\theta_1 < \theta_2$ such that :*

(i) *$\mathcal{S} = 0$ on $]-\infty, \theta_1]$, $\mathcal{S}$ increasing in $[\theta_1, \theta_2]$ and $\mathcal{S} = c$ on $[\theta_2, +\infty[$;*

(ii) *$\mathcal{S} \in C^1(\mathbb{R})$ and analytic on $[\theta_1, \theta_2]$;*

(iii) *$z \mapsto -log(\mathcal{S}(z))$ is convex on $]\theta_1, +\infty[$.*

Given these notations, if we denote $\xi = (\xi_i)_{1 \le i \le N_t}$ the number of spikes per time step and if we assume that $\Delta t = 1$ without loss of generality, then the data fidelity term (3) can be written as

$$\psi_\xi(z) = \sum_{i=1}^{N_t} \psi_{\xi_i}(z_i), \quad \text{with} \quad \psi_{\xi_i}(z_i) = \mathcal{S}(z_i) - \xi_i log(\mathcal{S}(z_i)), \tag{6}$$

where $\psi_{\xi_i}(z)$ has a limited domain of definition denoted by $\mathcal{Z}^+_{\xi_i}$ equal to $\mathbb{R}$ if $\xi_i = 0$, and $]\theta_1, +\infty[$ if $\xi_i > 0$. One can easily verify that the data fidelity term is nonconvex when $\xi_i \le c$.

The prior term is approximated by

$$J_\varepsilon(u) = \lambda \|u\|_1 + \mu I_\varepsilon(u) \quad \text{with} \quad I_\varepsilon(u) = \sum_{(i,j,k)} \sqrt{\varepsilon^2 + \|(Hu)_{i,j,k}\|_2^2}, \tag{7}$$

where the introduced parameter $\varepsilon > 0$, is small enough to make the prior term differentiable and $H : \mathcal{X} \to \mathcal{X}^9$ is a matrix equal to the discrete Hessian. It is clear that by introducing a dual variable, by following for example [7], we could have avoid to introduce this regularization. We did not use this approach for computational time reason: we have to solve a primal dual problem at each iteration and compute a projection on the set $\{H^\star p,\ p \in \mathcal{X}^9,\ |p_{i,j,k}| \le 1\}$ by a fixed point algorithm based on [4]. Denoting by $\times_d$ the discretization of $\times$ we get the discrete problems associated to (2):

$$\inf_{u \in \mathcal{X}} \mathcal{E}(u) = \psi_\xi(s \times_d u) + i_{\mathcal{Z}^+_\xi}(s \times_d u) + \lambda \|u\|_1 + \mu I_\varepsilon(u), \tag{8}$$

where $i_{\mathcal{Z}^+_\xi}$ denotes the characteristic function associated to $\mathcal{Z}^+_\xi = \prod_{i=1}^{N_t} \mathcal{Z}^+_{\xi_i}$.

In (8), the data fidelity term depends only on $s \times_d u$ and when minimizing w.r.t. u it is difficult to guarantee that $s \times_d u$ remains in the domain of definition of ψ_ξ. To solve this problem, we propose to introduce an auxiliary variable z and the following relaxed formulation:

$$\inf_{z \in \mathcal{Z}, u \in \mathcal{X}} \mathcal{E}_\alpha(z, u) = \psi_\xi(z) + i_{\mathcal{Z}^+_\xi}(z) + \frac{\alpha}{2} \|s \times_d u - z\|_2^2 + \lambda \|u\|_1 + \mu I_\varepsilon(u), \tag{9}$$

where the term in α penalizes the difference between z and $s \times_d u$. Another interest of this relaxed problem is that now the problem in z containing the nonconvexity is separable (it leads to N_t one-dimensional independent problems) and we will show that the energy is separately convex w.r.t. to z and u under some assumptions.

2.2. Well-posedness

We start with the following two propositions which result from standard arguments:

Proposition 2.1 (Existence)**.** *Let* $\alpha > 0$ *and* $\lambda > 0$ *(resp.* $\lambda > 0$*), the energy* $\mathcal{E}_\alpha(z,u)$ (9) *(resp.* $\mathcal{E}(u)$ (8)*) is coercive and admits at least a minimizer.*

Proposition 2.2. *If* $\alpha > -\inf_{\mathbb{R}} \text{ess } \mathcal{S}''$ *then problem* $\mathcal{E}_\alpha(z,u)$ (9) *is convex w.r.t to* z *and* u *separately (but not convex w.r.t.* (z,u)*).*

The following proposition establishes the link between solutions of the relaxed problem and solutions of the initial one.

Proposition 2.3. *Let* α *be a positive integer, then the sequence* $\{(z_\alpha, u_\alpha)\}_{\alpha \geq 1}$ *solution of the minimization problem* (9) *associated to* $\mathcal{E}_\alpha(z,u)$ *is bounded and*

(i) *All its cluster points are couples* $(s \times \bar{u}, \bar{u})$ *such that* $\bar{u}$ *is a solution of* (8).

(ii) *The infimum converges:* $\inf_{(z,u)} \mathcal{E}_\alpha(z,u) = \mathcal{E}_\alpha(z_\alpha, u_\alpha) \underset{\alpha \to +\infty}{\longrightarrow} \inf_u \mathcal{E}(u)$.

(iii) *If* $\mathcal{E}(u)$ *admits a unique minimizer* $\bar{u}$*, then* $(z_\alpha, u_\alpha) \underset{\alpha \to +\infty}{\longrightarrow} (s \times \bar{u}, \bar{u})$.

Proof. The proof of this proposition uses the epi-convergence of $\mathcal{E}_\alpha(z,u)$ toward $\mathcal{E}(u)\ i_{\{z = s \times_d u\}}$ (see [15]). □

2.3. Algorithm

To compute a solution of (9), we propose the proximal alternating minimization introduced in [1] in a general context. Given an initial condition $u_0 \in \mathcal{X}$ (e.g., $u_0 = 0$), the algorithm consists of the following two steps:

$$z^{(k+1)} \in \arg\min_z \ \mathcal{E}_\alpha(z, u^{(k)}) + \frac{1}{2\beta^{(k)}} \|z - z^{(k)}\|_2^2, \tag{10a}$$

$$u^{(k+1)} \in \arg\min_u \ \mathcal{E}_\alpha(z^{(k+1)}, u) + \frac{1}{2\gamma^{(k)}} \|u - u^{(k)}\|_2^2, \tag{10b}$$

where $\beta^{(k)}, \gamma^{(k)}$ are sequences of parameters belonging to $[r^-, r^+]$ with $0 < r^- < r^+$ for all $k \geq 0$ (note that this is the only condition on these parameters to obtain convergence). Note that quadratic terms in (10) are necessary to show the convergence and that the definition of (10) leads to a decrease of the energy verifying:

$$\mathcal{E}_\alpha(z^{(k+1)}, u^{(k+1)}) + \frac{1}{2\beta^{(k)}} \|z^{(k+1)} - z^{(k)}\|_2^2 + \frac{1}{2\gamma^{(k)}} \|u^{(k+1)} - u^{(k)}\|_2^2 \leq \mathcal{E}_\alpha(z^{(k)}, u^{(k)}).$$

From a numerical point of view, this algorithm becomes easy to tackle:

- If we choose $\mathcal{S}(.)$ as a piecewise cubic function verifying (1), when $\xi_i = 0$, then problem (10a) can be solved analytically thanks to its separability: it is equivalent to compute the proximal operator [10] of $\mathcal{S}(.)$ up to a multiplicative constant depending on α and $\beta^{(k)}$. When $\alpha + \frac{1}{\beta^{(k)}} > -\inf \text{ess } \mathcal{S}''(.)$ the problem is strictly convex so that the proximal operator is uni-valued. Otherwise, in the general case, we use a Newton algorithm.
- Problem (10b) is convex and we solve it by a standard forward-backward proximal algorithm of type Nesterov [2].

Theorem 2.1. (Convergence) *Algorithm* (10a)–(10b) *generates a sequence* $(z^{(k)}, u^{(k)})_{k \in \mathbb{N}}$ *which converges to a critical point* $(\overline{z}, \overline{u})$ *of* $\mathcal{E}_\alpha(z,u)$ *(i.e.,* $0 \in \partial \mathcal{E}_\alpha(\overline{z}, \overline{u})$*) and the sequence* $(z^{(k)}, u^{(k)})$ *is* $l_1(\mathbb{N})$.

Proof. Energy $\mathcal{E}_\alpha(z,u)$ verifies the Kurdyka-Lojasiewicz property (see [8] and Definition 7 in [1]) and is coercive from Proposition 2.1. By applying Theorem 9 of [1] we get the result. □

3. Numerical results

In this section we show quantitative results obtained using simulated data. We assume that we look for the receptive field of a neuron which can be approximated by a LNP model and that the nonlinearity is a known sigmoid (a piecewise cubic approximation verifying Hypothesis 1). Here we choose as the ground truth a receptive field shape that corresponds to a class of retinal ganglion cells: the function u is separable in space and time, it is a difference of Gaussian in space and a difference of exponential in time. It is discretized as a spatio-temporal volume of size $20 \times 20 \times 30$ pixels. This is illustrated in Fig. 2 (first row, sample temporal slices).

For the stimulus, we choose a sequence of uniform random binary images each presented during a time step $\Delta t = 1$. Here we use images of size 20×20 pixels, with block size 4×4 pixels (as illustrated in Fig. 1). Using this stimulus allows to apply the STA technique [5] to make a comparison.

In Fig. 2 we show a comparison between STA [5] and our approach ($\beta^{(k)} = \gamma^{(k)} = 10\ \forall k$, $\lambda = 10$, $\epsilon = 10^{-8}$, $\mu = 100$, $c = 33$, $\theta_1 = -7$, $\theta_2 = 107$). This result is obtained using a stimulus of $N_t = 1000$ images (giving around $n(T) = 500$ spikes). Results show that the spatial resolution of STA is constrained by the size of the block of the stimulus, by definition of the STA (PSNR=22.1dB), while we obtain a much better estimation (PSNR=33.1dB).

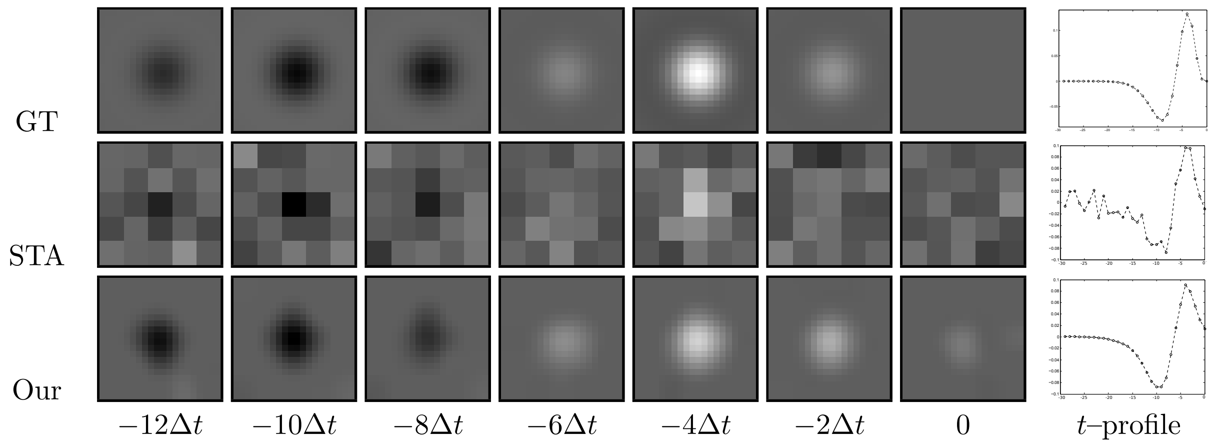

Figure 2. Example of result for simulated data. Each column shows a sample of temporal slices of (first line) the ground truth (GT), i.e., the receptive field to recover, (second line) the result obtained using the classical STA approach, and (third line) the result obtained using our approach. Last column shows temporal profile corresponding to a central position.

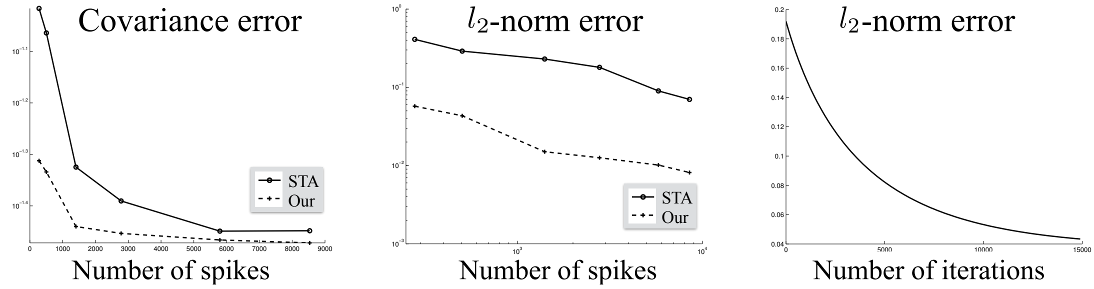

Figure 3. Convergence of the approach: (left) covariance error $(1 - \frac{cov(u,u_0)}{\sigma_u \sigma_{u_0}})$ as a function of the number of spikes in semi-log scale, (middle) l_2-norm error ($\|u - u_0\|_2$) as a function of the number of spikes in log-log scale, and (right) l_2-norm error as a function of the number of iterations for $N_t = 1000$.

Concerning the convergence, different aspects are shown in Fig. 3. In Fig. 3(left), we show that our approach converges faster than STA as the number of spikes increases, which is directly related to the duration of the experiment. In other words, for a fixed duration of experiment, our approach provides a better estimate of the receptive field than STA. In Fig. 3(middle), the main observation is that error decays linearily (in log scale) with a slope of the convergence equal for both methods which can be explained by the central limit Theorem. Finally, Fig. 3(right) illustrates the speed of convergence of our algorithm.

4. Conclusion

Overall, this study presents for the first time an approach to deal with the non-convex case allowing efficient extraction of receptive fields with great accuracy, as exemplified on synthetic data. Another advantage of this approach is that we are no longer constrained to use white noise as input as needed in STA, so that one can investigate what could be an optimal stimulus (e.g., [11]) to further improve the results. Future work will focus on validating our approach on real cell recordings.

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