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A Study of Conservation Laws of Dynamical Systems by Means of the Differential Variational Principles of Jourdain and Gauss*

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Summary

In this report we consider the possibility of using the differential variational principles of Jourdain and Gauss as a starting point for the study of conservation laws of holonomic conservative and nonconservative dynamical systems with a finite number of degrees of freedom. We demonstrate that this approach has the same status as the method based on the D'Alembert's differential variational principle developed in a previous paper.

1. Introduction

It is generally accepted that the conservation laws (or first integrals) of the classical dynamical systems are always of mathematical importance and at the same time, they are regarded as the manifestation of some profound physical principle.

In general, there are a variety of approaches to take in finding conservation laws. For example, in the early times of the development of analytical mechanics the attention was paid mainly to the conservation laws of the total energy, momentum and moment of momentum which can be easily derived directly from the differential equations of motion. It is also interesting to note that some conservation laws were discovered much before the appearance of the differential calculus and the Newton's laws of motion [1]. For example Jordanus de Nemore (about 1300 A.D.) was acquainted with a primitive form of the law of conservation of energy and the Kepler's second and third laws are also the classical conservation laws for the two body problem.

Probably the best known and mostly used modern method for finding conservation laws is based on the study of the invariant (or gauge-invariant) prop-

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erties of the Hamilton's action with respect to the infinitesimal transformations of generalized coordinates and time. Namely as shown by E. Noether [2], for every infinitesimal transformation which leaves the Hamilton's action absolute or gauge invariant, there exists a conservation law of the dynamical system.

However, the Noether's theory can be successfully applied to so called monogenic or Lagrangian dynamical systems i.e. systems which can be completely described by means of the Lagrangian function. Naturally, for purely non-conservative dynamical systems which are subject to nonconservative (non-potential or polygenic) forces, the Noether's theory is not applicable since the action does not exists for this class of systems.

In order to study the conservation laws of nonconservative dynamical systems several attempts have been made by employing a version of the Hamilton's variational principle suitable for the study of nonconservative systems and also the D'Alembert's differential variational principle [3], [4]. Recently, an integrating multipliers method for the study of conservation laws of nonconservative systems have been established in [5].

In this work, we consider the possibility of using the differential variational principles of Jourdain and Gauss as a starting point for the study of conservation laws of conservative and nonconservative dynamical systems. We demonstrate that this approach has the same status as methods described in [3] and [4]. In addition, for the case of purely conservative systems, the general feature of our theory is the same as the Noether's theory.

2. A Short Outline of the Differential Variational Principles

1. Let us consider a holonomic dynamical system consisting of M particles. We denote coordinates of the particles, referred to fixed orthogonal axes, by $y_1, y_2, \dots, y_N$ where $N = 3M$. The x, y and z coordinates of a particle enumerated by the integer β are $y_{3\beta-2}, y_{3\beta-1}, y_{3\beta}$. The mass of this particle is denoted indifferently by $m_{3\beta-1}, m_{3\beta-2}, m_{3\beta}$. The system is subject to the applied forces Y_α and constraint forces R_α , ($\alpha = 1, 2, \dots, N$). Note that this classification is performed in such a way that R_α are so called ideal reaction foces i.e. their virtual work is zero

$$\sum_1^N R_\alpha \delta y_\alpha = 0, \quad (2.1)$$

where δy_α , ($\alpha = 1, 2, \dots, n$) stands for the virtual displacement vector or simultaneous (Lagrangian) variation. Note also, that the nonideal components of the reaction forces are usually classified as the nonconservative applied forces for which, as a rule, we have to know in advance the character of the physical interaction between particles and constraints, as for example: dry friction forces, laminar friction forces depending on the velocities of particles etc.

Denoting by $\ddot{y}_\alpha$ the acceleration vector, we define the D'Alembert's principle of the virtual work in the following invariant form

$$\sum_1^N (Y_\alpha - m_\alpha \ddot{y}_\alpha) \delta y_\alpha = 0. \quad (2.2)$$

It is of interest to underline, that the virtual displacement vector δy_α refers to an arbitrary and infinitesimal change of the configuration of the system consistent with the imposed constraints at the given instant of time t . In other words, the virtual displacement is of purely kinematic character and the operator δ does not affects the time $\delta t = 0$. This fact can be used for the proof that the symbols d/dt and δ are commutative i.e. $d\delta - \delta d = 0$, which is one of the basic supposition of the classical variational calculus.

Let us introduce the independent generalized coordinates $x^i(t)$, ($i = 1, \dots, n$) whose values at time t determine the configuration of the system. We suppose that all Cartesian coordinates can be uniquely represented in terms of x^i and t

$$y_\alpha = y_\alpha(t, x^1, \dots, x^n), \quad (\alpha = 1, \dots, N). \quad (2.3)$$

From this relation we have

$$\delta y_\alpha = \frac{\partial y_\alpha}{\partial x^i} \delta x^i, \quad (i = 1, \dots, n, \quad \alpha = 1, \dots, N) \quad (2.4)$$

where the summation convention with respect to the Latin repeated indeces is assumed. Note, that this convention will be used throughout the subsequent text.

The time derivative of (2.4) is

$$\delta \dot{y}_\alpha = \left(\frac{\partial^2 y_\alpha}{\partial x^i \partial x^j} \dot{x}^j + \frac{\partial^2 y_\alpha}{\partial x^i \partial t} \right) \delta x^i + \frac{\partial y_\alpha}{\partial x^i} \delta \dot{x}^i \quad (2.5)$$

where an overdot denotes the time derivative. Naturally, the same relation can be derived by varying the velocity vector

$$v_\alpha = \frac{\partial y_\alpha}{\partial x^i} \dot{x}^i + \frac{\partial y_\alpha}{\partial t}. \quad (2.6)$$

An expression similar to (2.5) can be obtained for the Lagrangian variation of the acceleration vector which is given by the relation

$$\ddot{y}_\alpha = \frac{\partial y_\alpha}{\partial x^i} \ddot{x}^i + \frac{\partial^2 y_\alpha}{\partial x^i \partial x^j} \dot{x}^i \dot{x}^j + 2 \frac{\partial^2 y_\alpha}{\partial x^i \partial t} \dot{x}^i + \frac{\partial^2 y_\alpha}{\partial t^2} \quad (2.7)$$

($\alpha = 1, \dots, N, \quad i, j = 1, \dots, n$)

2. Next, we introduce the Jordain's variational principle by the scalar product [5]

$$\sum_1^N (Y_\alpha - m_\alpha \ddot{y}_\alpha) \delta_1 \dot{y}_\alpha = 0 \quad (2.8)$$

where $\delta_1 \dot{y}_\alpha$ denotes the Jourdain's variation which affects only the infinitesimal arbitrary changes of the velocity vector but without time and space deformations:

$$\delta_1 \dot{y}_\alpha = (\delta_1 y_\alpha)' \neq 0, \quad \delta_1 \ddot{y}_\alpha = (\delta_1 \ddot{x}_\alpha)'' \neq 0, \dots, \quad \delta_1 y_\alpha = 0, \quad \delta_1 t = 0. \quad (2.9)$$

Since the Jourdainian type of variations are introduced in full accordance with the holonomic constraints, we have by varying (in the sense of Jourdain) the Eq. (2.6), or by putting $\delta x^i = 0$ in (2.5):

$$\delta_1 v_\alpha = \frac{\partial y_\alpha}{\partial x^i} \delta_1 \dot{x}^i. \quad (2.10)$$

The Jourdainian variation of the acceleration vector (2.7) is found to be

$$\delta_1 \ddot{y}_\alpha = \frac{\partial y_\alpha}{\partial x^i} \delta_1 \ddot{x}^i + 2 \left(\frac{\partial^2 y_\alpha}{\partial x^i \partial x^j} \dot{x}^j + \frac{\partial^2 y_\alpha}{\partial x^i \partial t} \right) \delta_1 \dot{x}^i. \quad (2.11)$$

3. Let us introduce the Gauss' differential variational principle in the form

$$\sum_1^N (Y_\alpha - m_\alpha \ddot{y}_\alpha) \delta_2 \ddot{y}_\alpha = 0, \quad (2.12)$$

where $\delta_2 \ddot{y}_\alpha$ denotes the Gauss' infinitesimal variation of the acceleration vector by supposing that the configuration y_α , velocities $\dot{y}_\alpha$ and the time t are given i.e.

$$\begin{aligned} \delta_2 \ddot{y}_\alpha &= (\delta_2 y_\alpha)'' \neq 0, \quad \delta_2 \ddot{y}_\alpha = (\delta_2 y_\alpha)''' \neq 0, \dots \\ \delta_2 y_\alpha &= 0, \quad \delta_2 \dot{y}_\alpha = 0, \quad \delta_2 t = 0. \end{aligned} \quad (2.13)$$

By varying the expression (2.7) in the sense of Gauss, we find

$$\delta_2 \ddot{y}_\alpha = \frac{\partial y_\alpha}{\partial x^i} \delta_2 \ddot{x}^i \quad (2.14)$$

It is obvious, that the last expression can be obtained from (2.11) by changing the symbol δ_1 into the symbol δ_2 and applying the rule $\delta_2 \dot{y}_\alpha = 0$.

Note that in contrast to the D'Alembert's and Jourdain's variational principles, the Gauss' variational principle can be formulated as a minimum problem with respect to the true components of the acceleration vector (see, for example [6]). Namely, let us consider the constraint in the sense of Gauss:

$$Z = \sum_1^N \frac{1}{m_\alpha} (Y_\alpha - m_\alpha \ddot{y}_\alpha)^2. \quad (2.15)$$

Suppose that $\ddot{y}_\alpha$ is the actual acceleration and let $\ddot{\bar{y}}_\alpha$ denotes any other possible acceleration which is compatible with the constraints acting on the system and which differs infinitesimally from $\ddot{y}_\alpha$ i.e.

$$\ddot{\bar{y}}_\alpha = \ddot{y}_\alpha + \delta_2 \ddot{y}_\alpha = \ddot{y}_\alpha + \varepsilon K_\alpha \quad (2.16)$$

where ε is a small positive dimensionless parameter and K_α is a continuous and differentiable function which can depend on time, configuration and velocities of the system. The change in the value of the Gauss' constraint Z subject to the small changes of $\ddot{y}_\alpha$ is found to be

$$\bar{Z} - Z = -2 \sum_1^N (Y_\alpha - m_\alpha \ddot{y}_\alpha) \delta_2 \ddot{y}_\alpha + \sum_1^N m_\alpha (\delta_2 \ddot{y}_\alpha)^2. \quad (2.17)$$

Since by the supposition (2.12) the first term on the right-hand side vanishes for the actual acceleration, we have $\bar{Z} - Z > 0$.

However, it is of importance to note, that if we interpret the Eq. (2.16) as a characteristic *infinitesimal transformation of the acceleration vector*, then the expression (2.12) can be contemplated as the *condition of invariance* (up to order ε) of the Gauss constraint Z . This fact will be used in the subsequent text.

3. Lagrangian, Jourdainian and Gaussian Generalized (Nonsimultaneous) Variations

In order to enlarge the class of conservation laws of dynamical systems, it is necessary to introduce the deformation (variation) of the time into the Lagrangian, Jourdainian and Gaussian variations.

1. For the sake of clarity, consider first the classical nonsimultaneous variations which are usually arising in the variable end-points problems of the classical variational calculus.

As defined in the previous section, the simultaneous variation δx^i of a generalized coordinate x^i is an infinitesimally small increment of this coordinate without the time change

$$\begin{aligned} \bar{x}^i(\bar{t}) &= x^i(t) + \delta x^i \\ \bar{t} &= t \end{aligned} \quad (3.1)$$

where an overbar denotes the varied path. Obviously, we can look at (3.1) as an infinitesimal transformation correlating the point $(x^1, \dots, x^n)$ on the actual path to a contemporaneous point $(x^1 + \delta x^1, \dots, x^n + \delta x^n)$ on the varied trajectory.

Next, let us introduce the varied path $\bar{x}^i(t + \Delta t)$ which is infinitesimally close to the actual path $x^i(t)$ but suffers a small continuous variation of time Δt . Developing and retaining the first order terms, we have

$$\bar{x}^i(t + \Delta t) = \bar{x}^i(t) + \dot{\bar{x}}^i \Delta t. \quad (3.2)$$

Inserting (3.1) into this relation and defining so called generalized or nonsimultaneous (noncontemporaneous) variation Δx^i as

$$\Delta x^i = \bar{x}^i(t + \Delta t) - x^i(t) \quad (3.3)$$

we have

$$\Delta x^i = \delta x^i + \dot{x}^i \Delta t. \quad (3.4)$$

As in the case of the simultaneous variations, we can consider the generalized variations as an infinitesimal transformation of the generalized coordinates and time correlating the point $x^1, \dots, x^n$ on the actual trajectory at the time t with a point $x^1 + \Delta x^1, \dots, x^n + \Delta x^n$ on the varied path at the time $t + \Delta t$, namely

$$\begin{aligned}\bar{x}^i(\bar{t}) &= x^i(t) + \Delta x^i, \\ \bar{t} &= t + \Delta t.\end{aligned}\tag{3.5}$$

The relation (3.4) which relates the generalized variations Δ and classical variations δ can serve as a pattern for the variation of any scalar, vector or tensor quantity. In other words, we can establish the relation

$$\Delta(*) = \delta(*) + (*)' \Delta t.\tag{3.6}$$

For example, applying (3.6) to the generalized velocity vector one finds

$$\Delta \dot{x}^i = \delta \dot{x}^i + \ddot{x}^i \Delta t.\tag{3.7}$$

Differentiating (3.4) we have

$$(\Delta x^i)' = \delta \ddot{x}^i + \ddot{x}^i \Delta' + \dot{x}^i (\Delta t)'.\tag{3.8}$$

Combining (3.7) and (3.8) we find

$$\Delta \ddot{x}^i = (\Delta \dot{x}^i)' - \dot{x}^i (\Delta t)'.\tag{3.9}$$

therefore, the symbols Δ and d/dt are not commutative.

Repeating the same process we find the following relations for the generalized variations of the acceleration vector

$$\Delta \ddot{x}^i = \delta \ddot{x}^i + \ddot{\ddot{x}}^i \Delta t\tag{3.10}$$

$$(\Delta \ddot{x}^i)'' = \delta \ddot{\ddot{x}}^i + \ddot{\ddot{x}}^i (\Delta t)'' + \ddot{\ddot{x}}^i \Delta t + 2\ddot{x}^i (\Delta t)'. \tag{3.11}$$

Differentiating (3.9) we also have

$$(\Delta \ddot{x}^i)' = (\Delta \dot{x}^i)'' - \ddot{x}^i (\Delta t)' - \dot{x}^i (\Delta t)''.\tag{3.12}$$

Finally, from (3.10) and (3.11) one finds

$$\Delta \ddot{\ddot{x}}^i = (\Delta \ddot{x}^i)'' - \ddot{x}^i (\Delta t)'' - 2\ddot{x}^i (\Delta t)'. \tag{3.13}$$

At this point it should be noted that the structure of the generalized variations Δx^i and Δt introduced by (3.5) is of a great importance in the study of conservation laws. We will suppose that this structure is of the form (see also [4])

$$\begin{aligned}\Delta x^i &= \varepsilon F^i(t, x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n), \\ \Delta t &= \varepsilon f(t, x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n).\end{aligned}\tag{3.14}$$

We usually call F^i and f the space and time generators of the infinitesimal transformation (3.5).

2. Let us define the Jourdain's generalized (nonsimultaneous) variations $\Delta_1 \dot{x}^i$, $(\Delta_1 \dot{x}^i)^\cdot$ and $(\Delta_1 t)^\cdot$ together with the requirement:

$$\Delta_1 \dot{x}^i = 0, \quad \Delta_1 t = 0. \quad (3.15)$$

By changing the symbols Δ and δ into Δ_1 and δ_1 and applying (3.15), we obtain from (3.7)—(3.9) respectively:

$$\Delta_1 \dot{x}^i = \delta_1 \dot{x}^i, \quad (3.16)$$

$$(\Delta_1 \dot{x}^i)^\cdot = \delta_1 \dot{x}^i + \dot{x}^i (\Delta_1 t)^\cdot, \quad (3.17)$$

$$\Delta_1 \dot{x}^i = (\Delta_1 \dot{x}^i)^\cdot - \dot{x}^i (\Delta_1 t)^\cdot. \quad (3.18)$$

The following infinitesimal transformations of the generalized coordinates, time and velocity can be of use for better understanding the nature of the Jourdainian variations:

$$\begin{aligned} \bar{x}^i &= x^i, & \bar{t} &= t, \\ \frac{d\bar{x}^i}{d\bar{t}} - \frac{dx^i}{dt} &= \delta_1 \dot{x}^i = \Delta_1 \dot{x}^i \\ \frac{d\bar{x}^i}{d\bar{t}} - \frac{dx^i}{dt} &= (\Delta_1 \dot{x}^i)^\cdot. \end{aligned} \quad (3.19)$$

We will consider the infinitesimal quantities $(\Delta_1 \dot{x}^i)^\cdot$ and $(\Delta_1 t)^\cdot$ as the constitutive elements or primitives of the Jourdain's infinitesimal transformation, and we introduce the following space and time Jourdainian generators of the transformation

$$\begin{aligned} (\Delta_1 \dot{x}^i)^\cdot &= \varepsilon F_1^i(t, x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n) \\ (\Delta_1 t)^\cdot &= \varepsilon f_1(t, x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n) \end{aligned} \quad (3.20)$$

From the relation (3.16)—(3.18) we express the simultaneous Jourdainian variation in terms of these generators

$$\delta_1 \dot{x}^i = \varepsilon [F_1^i(t, x, \dot{x}) - \dot{x}^i f_1(t, x, \dot{x})], \quad (3.21)$$

where $x = \{x^1, \dots, x^n\}$ and $\dot{x} = \{\dot{x}^1, \dots, \dot{x}^n\}$.

3. Finally, let us introduce the Gauss' nonsimultaneous variations: $\Delta_2 \dot{x}^i$, $(\Delta_2 \dot{x}^i)^\cdot$, $(\Delta_2 \dot{x}^i)^\cdot$ and $(\Delta_2 t)^\cdot$ together with the following requirements

$$\Delta_2 \dot{x}^i = 0, \quad \Delta_2 t = 0, \quad \Delta_2 \dot{x}^i = 0, \quad (\Delta_2 \dot{x}^i)^\cdot = 0, \quad (\Delta_2 t)^\cdot = 0 \quad (3.22)$$

and also

$$\delta_2 \dot{x}^i = 0, \quad \delta_2 t = 0, \quad \delta_2 \dot{x}^i = 0. \quad (3.23)$$

By changing the symbol Δ and δ into Δ_2 and δ_2 and applying (3.22), we obtain from (3.10)–(3.13) the following relations

$$(\Delta_2 x^i)^{\cdot\cdot} = \delta_2 \ddot{x}^i + \dot{x}^i (\Delta_2 t)^{\cdot\cdot}, \quad (3.24)$$

$$\Delta_2 \ddot{x}^i = \delta_2 \ddot{x}^i, \quad (3.25)$$

$$(\Delta_2 \dot{x}^i)^{\cdot} = (\Delta_2 x^i)^{\cdot\cdot} - \dot{x}^i (\Delta_2 t)^{\cdot\cdot}, \quad (3.26)$$

$$\Delta_2 \ddot{x}^i = (\Delta_2 x^i)^{\cdot\cdot} - \dot{x}^i (\Delta_2 t)^{\cdot\cdot}. \quad (3.27)$$

A simple comparison of these equations shows that we also have

$$(\Delta_2 \dot{x}^i)^{\cdot} = \Delta_2 \ddot{x}^i = \delta_2 \ddot{x}^i. \quad (3.28)$$

We can now establish the following infinitesimal transformations of the acceleration vector in the sense of Gauss' which include the variations of the time elements:

$$\begin{aligned} \bar{x}^i &= x^i, & \bar{t} &= t, \\ \frac{d\bar{x}^i}{dt} - \frac{dx^i}{dt} &= \frac{d\bar{x}^i}{d\bar{t}} - \frac{dx^i}{dt} = 0, \\ \frac{d^2\bar{x}^i}{dt^2} - \frac{d^2x^i}{dt^2} &= \delta_2 \ddot{x}^i = (\delta_2 x^i)^{\cdot\cdot} = (\Delta_2 \dot{x}^i)^{\cdot} = \Delta_2 \ddot{x}^i, \\ \frac{d^2\bar{x}^i}{d\bar{t}^2} - \frac{d^2x^i}{dt^2} &= (\Delta_2 x^i)^{\cdot\cdot}. \end{aligned} \quad (3.29)$$

Let us introduce the space and time Gauss' generators F_2^i and f_2 of the infinitesimal transformation of the generalized acceleration vector

$$\begin{aligned} (\Delta_2 x^i)^{\cdot\cdot} &= \varepsilon F_2^i(t, x, \dot{x}), \\ (\Delta_2 t)^{\cdot\cdot} &= \varepsilon f_2(t, x, \dot{x}). \end{aligned} \quad (3.30)$$

By using the relation (3.24), we express the Gauss' simultaneous variation in terms of the generators as

$$\delta_2 \ddot{x}^i = \varepsilon [F_2^i(t, x, \dot{x}) - \dot{x}^i f_2(t, x, \dot{x})]. \quad (3.31)$$

4. Condition of Invariance of the Gauss Constraint

In this section we study the condition of the invariance of Gauss' constraint, given by (2.15). Namely we make the following transformation of the Gauss' principle (2.12) into an expression involving the Gauss' nonsimultaneous variations of the generalized acceleration vector and time elements.

Substituting (3.31) into (2.14) and entering with it into (2.12), we have

$$\varepsilon \sum_1^N \left(Y_\alpha \frac{\partial y_\alpha}{\partial x^i} - m_\alpha \ddot{y}_\alpha \frac{\partial y_\alpha}{\partial x^i} \right) (F_2^i - \dot{x}^i f_2) = 0 \quad (i = 1, \dots, n) \quad (4.1)$$

where (as we already mentioned) the summation convention with respect to the Latin repeated indices is assumed.

Let us suppose that each particle of the dynamical system is subject to the conservative force $-\partial\Pi/\partial x^i$ and a purely nonconservative force $Q_i = Q_i(t, x, \dot{x})$, where $\Pi = \Pi(x, t)$ is the potential function. Thus we have

$$\sum_{\alpha=1}^N Y_\alpha \frac{\partial y_\alpha}{\partial x^i} = -\frac{\partial\Pi}{\partial x^i} + Q_i(t, x, \dot{x}). \quad (4.2)$$

Performing a simple transformation of the second group of terms in (4.1) we write this equation as

$$\begin{aligned} \varepsilon \left\{ \left(Q_i - \frac{\partial\Pi}{\partial x^i} \right) (F_2^i - \dot{x}^i f_2) - \frac{d}{dt} \left[\sum_{\alpha=1}^N m_\alpha \dot{y}_\alpha \frac{\partial y_\alpha}{\partial x^i} (F_2^i - \dot{x}^i f_2) \right] \right. \\ \left. + \sum_{\alpha=1}^N m_\alpha \dot{y}_\alpha \left(\frac{\partial y_\alpha}{\partial x^i} \right)^{\cdot} (F_2^i - \dot{x}^i f_2) + \sum_{\alpha=1}^N m_\alpha \dot{y}_\alpha \frac{\partial y_\alpha}{\partial x^i} (F_2^i - \dot{x}^i f_2)^{\cdot} \right\} = 0. \end{aligned} \quad (4.3)$$

Using the well known identities

$$\frac{\partial y_\alpha}{\partial x^i} = \frac{\partial \dot{y}_\alpha}{\partial \dot{x}^i}, \quad \left(\frac{\partial y_\alpha}{\partial x^i} \right)^{\cdot} = \frac{\partial \dot{y}_\alpha}{\partial x^i} \quad (4.4)$$

and introducing the kinetic energy of the system $T = (1/2) \sum m_\alpha \dot{y}_\alpha^2$, the Eq. (4.3) becomes

$$\begin{aligned} \varepsilon \left\{ \left(Q_i - \frac{\partial\Pi}{\partial x^i} \right) (F_2^i - \dot{x}^i f_2) - \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}^i} (F_2^i - \dot{x}^i f_2) \right] + \frac{\partial T}{\partial x^i} (F_2^i - \dot{x}^i f_2) \right. \\ \left. + \frac{\partial T}{\partial x^i} (F_2^i - \dot{x}^i f_2) + \frac{\partial T}{\partial \dot{x}^i} (\dot{F}_2^i - \ddot{x}^i f_2 - \dot{x}^i \dot{f}_2) \right\} = 0. \end{aligned} \quad (4.5)$$

Introducing the Lagrangian function $L = T - \Pi$ and noting that $\partial T/\partial \dot{x}^i = \partial L/\partial \dot{x}^i$, the last equation, after adding and subtracting the term $(\partial L/\partial t) f_2$, reads

$$\begin{aligned} \varepsilon \left\{ Q_i (F_2^i - \dot{x}^i f_2) + \frac{\partial L}{\partial \dot{x}^i} F_2^i + \frac{\partial L}{\partial \dot{x}^i} \dot{F}_2^i - \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x^i} \dot{x}^i + \frac{\partial L}{\partial \dot{x}^i} \ddot{x}^i \right) f_2 \right. \\ \left. + \frac{\partial L}{\partial t} f_2 - \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \dot{f}_2 - \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{x}^i} (F_2^i - \dot{x}^i f_2) \right] \right\} = 0. \end{aligned} \quad (4.6)$$

Since $\dot{L} = \partial L / \partial t + (\partial L / \partial x^i) \dot{x}^i + (\partial L / \partial \dot{x}^i) \ddot{x}^i$, we have by adding and subtracting the time derivative of a gauge function $\varepsilon \dot{P}(t, x, \dot{x})$, the following relation

$$\varepsilon \left\{ Q_i (F_2^i - \dot{x}^i f_2) + \frac{\partial L}{\partial x^i} F_2^i + \frac{\partial L}{\partial \dot{x}^i} \dot{F}_2^i + \left(L - \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \right) \dot{f}_2 + \frac{\partial L}{\partial t} f_2 - \dot{P}(t, x, \dot{x}) - \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{x}^i} F_2^i + \left(L - \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \right) f_2 - P(t, x, \dot{x}) \right] \right\} = 0 \quad (4.7)$$

which is the transformation of the condition of the invariance of the Gauss' principle we have been seeking.

However, from this equation follows immediately, that if the relation

$$\frac{\partial L}{\partial x^i} F_2^i + \frac{\partial L}{\partial \dot{x}^i} \dot{F}_2^i + \left(L - \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \right) \dot{f}_2 + \frac{\partial L}{\partial t} f_2 + Q_i (F_2^i - \dot{x}^i f_2) - P(t, x, \dot{x}) = 0 \quad (4.8)$$

is satisfied, the dynamical system admits a conservation law of the form

$$\frac{\partial L}{\partial x^i} F_2^i + \left(L - \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \right) \dot{f}_2 - P(t, x, \dot{x}) = C = \text{const.} \quad (4.9)$$

Note that the condition for the existence of the conserved quantity (4.8) and corresponding conservation law are identical with those obtained by the study of the invariant properties of a special form of the Hamilton's principle [3] and also from the D'Alembert's differential variational principle [4]. For the case of a conservative dynamical system, i.e. $Q_i = 0$, the Eqs. (4.8) and (4.9) constitute the classical form of the Noether's theorem (see [7] and [8]), except, the meaning of the generators F_2^i and f_2 introduced here, is completely different than used in the cited references.

It is interesting to note that the condition of the invariance of the Gauss' variational principle (4.7) is producing simultaneously the condition for the existence of a conservation law (4.8) and the conservation law itself (4.9). The situation in the classical Noetherian approach is different. The condition of the invariance of the Hamilton's action (with respect to the infinitesimal transformations of the generalized coordinates and time) is generating first the single relation (4.8) and after an additional transformation of this equation, we arrive to the conservation law (4.9) (see [7] and [8]). Note that this fact is not surprising, since the Gauss differential variational principle is at the higher differential level than the Hamilton's integral variational principle whose structure does not contain the acceleration in itself. However, despite these differences it is clear that the Gauss' principle can be used for finding conservation laws of conservative and nonconservative dynamical systems, whose behaviour is described by the monogenic function i.e. the Gauss' constraint Z .

It is also of interest to note, that the derivation of the condition of the invariance (4.7) can commence from the Z which is preliminary expressed in terms of the independent generalized coordinates x^i as (see [9])

$$Z = \frac{1}{2} g_{ij} \frac{D\dot{x}^i}{Dt} \frac{D\dot{x}^j}{Dt} - Q_i^* \frac{D\dot{x}^i}{Dt} + \frac{1}{2} g^{ij} Q_i^* Q_j^*, \quad (i, j = 1, \dots, n). \quad (4.10)$$

As in ordinary tensor dynamics, the following notation in the equation (4.10) is introduced: g_{ij} and g^{ij} are covariant and contravariant forms of the fundamental metric tensors respectively and the symbol $D\dot{x}^i/Dt$ denotes the intrinsic derivative of $\dot{x}^i$ with respect to time, namely

$$\frac{D\dot{x}^i}{Dt} = \dot{x}^i + \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} \dot{x}^j \dot{x}^k \quad (4.11)$$

where $\left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\}$ stands for the Christoffel symbol of the second kind, formed with respect to the metric tensor. Finally, Q_i^* denotes the generalized force which includes conservative and nonconservative parts.

Applying the symbol of the Gauss' variation δ_2 , we have from (4.11): $\delta_2(D\dot{x}^i/Dt) = \delta_2\ddot{x}^i$. Therefore from (4.10) we find that the condition of invariance of (4.10) is

$$\left(g_{ij} \frac{D\dot{x}^j}{Dt} - Q_i^* \right) \delta_2\ddot{x}^i = 0. \quad (4.12)$$

Noting that $L = (1/2) g_{ij} \dot{x}^i \dot{x}^j - \Pi(x, t)$ and $Q_i^* = -(\partial\Pi/\partial\dot{x}^i) + Q_i(t, x, \dot{x})$, we can easily show that (4.12) is equivalent to (4.7).

We conclude this section with the following theorem: For every set of Gauss' generators $F_2^i(t, x, \dot{x})$, $f_2(t, x, \dot{x})$ and the gauge function $P = P(t, x, \dot{x})$ which satisfy the functional relation (4.8), there exists a conserved quantity of the dynamical system in the form of (4.9).

5. An Equivalent Transformation of the Jourdain's Principle

In this section we will demonstrate that the Jourdain's variational principle (2.8) can be transformed into a relation equivalent to (4.7). Thus, this variational principle can be also used as a basis for obtaining of conservation laws.

The proof is simple. Substituting (3.21) into (2.10) and inserting this into (2.8), we find

$$\sum_{\alpha=1}^N \left(Y_\alpha \frac{\partial y_\alpha}{\partial x^i} - m_\alpha \ddot{y}_\alpha \frac{\partial y_\alpha}{\partial x^i} \right) (F_1^i - \dot{x}^i f_1) = 0, \quad (i = 1, \dots, n) \quad (5.1)$$

which is structurally identical to the Eq. (4.1). Therefore, repeating the same procedure as in the previous section, we arrive at the relation equivalent to (4.7), which confirms our statement.

6. Brief Comments on the Relation (4.8). Generation of Conservation Laws

Here we shortly discuss the question: Can the theorem established in the section 4, be used as a working tool for obtaining conservation laws? Actually, the problem of generating constant quantities of a dynamical system depends on the possibility of finding a set of $n + 2$ functions

$$F_2^i = F_2^i(t, x, \dot{x}), \quad f_2 = f_2(t, x, \dot{x}), \quad P = P(t, x, \dot{x}) \quad (6.1)$$

from the single relation (4.8), which we usually call the basic Noether's identity. Generally it is a difficult problem and it is not easy to trace any universal approach for the systematic selection of these functions. Sometimes, with experience as a guide, the set (6.1) is readily evident directly from (4.8). But generally, for non-conservative systems with more than one degree of freedom it is not an easy task.

As proposed in the ref. [7] and accepted by many authors, the relation (4.8) can be frequently decomposed into a system of partial differential equations of the first order with respect to F_2^i , f_2 and P . We call this system the generalized Killing's equations. Naturally, this decomposition is based on the premise that it is easier to find a solution of the generalized Killing's equations than to analyze the basic Noether's identity (4.8) directly. As demonstrated in numerous practical situations solutions of these equations are frequently obtainable. A detailed study of the generalized Killing's equations is out of scope of this paper. For more details see [3], [4], [7], [8], [10] and [11].

It is of interest to note that the structure of the generators F_2^i , f_2 and the gauge function P , especially their velocity dependence, as indicated by (6.1), has been the subject of a controversial discussion from the theoretical point of view. A rather exhaustive account concerning this question is given in the ref. [8]. Note, that the velocity and even acceleration dependent infinitesimal transformations were introduced by E. Noether in her pioneering work [2]. It seems, apart from the various subtle theoretical analysis, that the structure of infinitesimal transformations is mainly dictated by the form of a conservation law we wish to obtain. Namely, it is easy to demonstrate that many important conservation laws are not derivable by means of the Noether-type theory without the supposition of the velocity dependent infinitesimal transformations. To show this, we turn to a simple concrete example.

Let us consider the differential equation which arise in gas dynamics

$$\ddot{x} = ax^{-1/2}, \quad (a = \text{const.}). \quad (6.2)$$

The Lagrangian function is

$$L = (1/2) \dot{x}^2 + 2ax^{1/2}. \quad (6.3)$$

Therefore, the basic Noether's identity (4.8) becomes

$$ax^{-1/2}F_2 + \dot{x}\dot{F}_2 - \dot{f}_2\left(\frac{\dot{x}^2}{2} - 2ax^{1/2}\right) - \dot{P}(t, x, \dot{x}) = 0. \quad (6.4)$$

Let us try to find a conservation law of (6.2) by supposing that the structure of the space and time generators is

$$F_2 = F(x), \quad f_2 = f(\dot{x}). \quad (6.5)$$

Substituting this into (6.4) and employing (6.2), we find

$$\dot{P} = \dot{x}^2 \left[\frac{dF}{dx} - \frac{a}{2} x^{-1/2} \frac{df(\dot{x})}{d\dot{x}} \right] + ax^{-1/2}F(x) + 2a^2 \frac{df(\dot{x})}{d\dot{x}}. \quad (6.6)$$

By taking $df(\dot{x})/d\dot{x} = \text{const.} = 1$, i.e.

$$f_2(\dot{x}) = \dot{x} \quad (6.7)$$

the expression in brackets is independent of $\dot{x}$. Equating this to zero, one obtains $dF/dx = (a/2)x^{-1/2}$, namely

$$F = ax^{1/2}. \quad (6.8)$$

The Eq. (6.6) becomes $\dot{P} = 3a^2$, namely

$$P = 3a^2t. \quad (6.9)$$

Therefore, with (6.7), (6.8) and (6.9) the expression (4.9) generates the following cubic conservation law of the Eq. (6.2) which was found by Parsons [12] using a different approach

$$\dot{x}^3 - 6ax^{1/2}\dot{x} + 6a^2t = \text{const.} \quad (6.10)$$

It is clear that this conservation law cannot be obtained by using the Noether's approach without the supposition (6.7).

7. An Example. Energy Like Conservation Laws of a Linear Nonconservative Dynamical System

As an illustration of the foregoing theory, we consider the problem of finding quadratic conservation laws of the linear dynamical system whose differential equations of motion are

$$\ddot{x}_i + 2k\dot{x}_i + a_{ij}x_j = 0, \quad (i, j = 1, \dots, n) \quad (7.1)$$

where k and a_{ij} are given constant parameters. It should be noted, that the position forces $X_i = -a_{ij}x_j$ are nonconservative, namely $\partial X_i/\partial x_j \neq \partial X_j/\partial x_i$.

To analyze the system (7.1) we introduce the Lagrangian function

$$L = \frac{1}{2} (\Sigma \dot{x}_i^2) e^{2kt} \quad (7.2)$$

and nonconservative generalized forces

$$Q_{x_i} = -(a_{ij}x_j) e^{2kt}. \quad (7.3)$$

It is easy to verify that the Lagrangian equations of the second kind: $(\partial L / \partial \dot{x}^i) - (\partial L / \partial x^i) = Q_{x_i}$, are producing the differential equations of motion (7.1).

Denoting the space and time generators by F_i , ($i = 1, \dots, n$) and f , we write the basic Noether's identity (4.8) as

$$[\dot{x}_i \dot{F}_i + (\Sigma \dot{x}_i^2) (kf - \dot{f}/2) - \langle F_i - \dot{x}_i f \rangle a_{ij}] e^{2kt} - \dot{P} = 0. \quad (7.4)$$

The following considerations will be simplified if we suppose that the infinitesimal transformation does not contain the time transformations i.e. $f = 0$. We select the space generators in the form

$$F_1 = F, \quad F_\alpha = \lambda_\alpha F, \quad (\alpha = 2, 3, \dots, n) \quad (7.5)$$

where $\lambda_2, \lambda_3, \dots, \lambda_n$ are constant factors to be determined and F is an unknown function of time, position and velocity.

Equation (7.5) becomes

$$\begin{aligned} & \{ \dot{F}(\dot{x}_1 + \lambda_\alpha \dot{x}_\alpha) - F[(a_{11} + \lambda_\alpha a_{\alpha 1}) x_1 + (a_{12} + \lambda_\alpha a_{\alpha 2}) x_2 + \dots \\ & + (a_{1n} + \lambda_\alpha a_{\alpha n}) x_n] \} e^{2kt} - \dot{P} = 0 \end{aligned} \quad (7.6)$$

or

$$\begin{aligned} & \left\{ \dot{F}(\dot{x}_1 + \lambda_\alpha \dot{x}_\alpha) - F(a_{11} + \lambda_\alpha a_{\alpha 1}) \left[x_1 + \frac{a_{12} + \lambda_\alpha a_{\alpha 2}}{a_{11} + \lambda_\alpha a_{\alpha 1}} x_2 + \dots \right. \right. \\ & \left. \left. + \frac{a_{1n} + \lambda_\alpha a_{\alpha n}}{a_{11} + \lambda_\alpha a_{\alpha 1}} x_n \right] \right\} e^{2kt} - \dot{P} = 0. \end{aligned} \quad (7.7)$$

Denoting by

$$\Omega^2 = a_{11} + \lambda_\alpha a_{\alpha 1} \quad (7.8)$$

and selecting $\lambda_2, \lambda_3, \dots, \lambda_n$ as

$$\frac{a_{12} + \lambda_\alpha a_{\alpha 2}}{a_{11} + \lambda_\alpha a_{\alpha 1}} = \lambda_2, \quad \frac{a_{13} + \lambda_\alpha a_{\alpha 3}}{a_{11} + \lambda_\alpha a_{\alpha 1}} = \lambda_3, \dots, \quad \frac{a_{1n} + \lambda_\alpha a_{\alpha n}}{a_{11} + \lambda_\alpha a_{\alpha 1}} = \lambda_n \quad (7.9)$$

we write (7.7) in the form

$$(\dot{F}\dot{\theta} - \Omega^2 F\theta) e^{2kt} - \dot{P} = 0 \quad (7.10)$$

where

$$\theta = x_1 + \lambda_\alpha x_\alpha. \quad (7.11)$$

Let us select the space generator in the form

$$F = A\dot{\theta} + B\theta \quad (7.12)$$

where A and B are constants. Substituting (7.12) into (7.10), we have

$$[A(\dot{\theta}\ddot{\theta} - \Omega^2\theta\dot{\theta}) + B(\dot{\theta}^2 - \Omega^2\theta^2)]e^{2kt} - \dot{P} = 0. \quad (7.13)$$

For $A = 1$ and $B = k$, (7.13) can be integrated. Therefore, the gauge function is

$$P = \frac{1}{2} (\dot{\theta}^2 - \Omega^2 \theta^2) e^{2kt}. \quad (7.14)$$

Inserting $f = 0$, (7.5), (7.12), (7.14) and (7.2) into (7.9), we have by employing (7.11)

$$\left(\frac{\dot{\theta}^2}{2} + \frac{1}{2} \Omega^2 \theta^2 + k\theta\dot{\theta}\right) e^{2kt} = D = \text{const.} \quad (7.15)$$

Or, in original notation

$$\left[\frac{1}{2} (\dot{x}_1 + \lambda_\alpha \dot{x}_\alpha)^2 + \frac{1}{2} \Omega^2 (x_1 + \lambda_\alpha x_\alpha)^2 \right. \\ \left. + k(x_1 + \lambda_\alpha x_\alpha) (\dot{x}_1 + \lambda_\alpha \dot{x}_\alpha) \right] e^{2kt} = D = \text{const.} \quad (7.16)$$

It should be noted that we have obtained n independent quadratic conservation laws of the dynamical system (7.1) without any restrictions on the coefficients a_{ij} .

To show this, we turn to the algebraic problem of finding λ_α ($\alpha = 2, 3, \dots, n$) and Ω^2 from n relations (7.8) and (7.9). Actually, using (7.8), we write (7.9) in the form

[illegible]

Solving this system of linear algebraic equations with respect to λ_α , ($\alpha = 2, 3, \dots, n$) and substituting the solution into (7.8), we conclude after simple calculation, that Ω^2 can be obtained as a root of the determinantal equation

$$\begin{pmatrix} a_{11} - \Omega^2, a_{12}, \dots, a_{1n} \\ a_{21}, a_{22} - \Omega^2, \dots, a_{2n} \\ \vdots \\ a_{n1}, a_{n2}, \dots, a_{nn} - \Omega^2 \end{pmatrix} = 0 \quad (7.18)$$

which is an equation of the n -th order with respect to Ω^2 . Solving (7.18) we find $\Omega_1^2, \Omega_2^2, \dots, \Omega_n^2$. Entering with this into (7.17) we find also $\lambda_{\alpha(1)}, \dots, \lambda_{\alpha(n)}$, which completes our proof. Note, that the case of the multiple roots of the Eq. (7.18) will be not considered here.

1. Since the dissipative parameter k is not involved in the algebraic problem (7.18)–(7.17) our analysis is valid for the special case $k = 0$. Thus, we have the following result: Linear dynamical nonconservative system $\ddot{x}_i = -a_{ij}x_j$, ($i, j = 1, \dots, n$) has n independent quadratic conservation laws of the form $\theta_{(i)}^2/2 + \Omega_{(i)}^2 \theta_{(i)}^2/2 = D_{(i)} = \text{const.}$ (the summation convention with respect to (i) is not applied), where $\theta_{(i)} = x_1 + \lambda_{2(i)}x_2 + \dots + \lambda_{n(i)}x_n$. The coefficients $\Omega_{(i)}^2$, $\lambda_{\alpha(i)}$, ($\alpha = 2, 3, \dots, n$, $(i) = 1, \dots, n$) can be determined from the algebraic problem (7.18)–(7.17).

2. The analysis given in the previous point is valid also for the case of conservative dynamical systems i.e. $k = 0$, $a_{ij} = a_{ji}$. Naturally, the conservation law of the total mechanical energy can be represented as linear combination of the conservation laws $D_{(i)}$ mentioned above.

3. Note that for conservative and oscillatory dynamical systems the Eq. (7.11) is identical with the principal coordinates of the vibrating system. The conservation laws $D_{(i)}$ thus obtained can be interpreted as the energy associated to each normal mode of vibration. As in the previous point, the total mechanical energy of the vibrating system can be expressed as a linear combination of the conservation law $D_{(i)}$, ($i = 1, \dots, n$).

As a simple concrete example, consider the nonconservative dynamical system analyzed in [3] and [13]:

$$\ddot{x}_1 + 2k\dot{x}_1 + ax_1 - bx_2 = 0, \quad \ddot{x}_2 + 2k\dot{x}_2 + bx_1 + ax_2 = 0. \quad (7.19)$$

From (7.18) and (7.17) we find $\Omega_{(1)}^2 = a + bi$, $\Omega_{(2)}^2 = a - bi$, $\lambda_{2(1)} = i$, $\lambda_{2(2)} = -i$, where $i = (-1)^{1/2}$. Therefore, using (7.16) we have following two conservation laws

$$\left[\frac{1}{2} (\dot{x}_1 + i\dot{x}_2)^2 + \frac{1}{2} (a + bi) (x_1 + ix_2)^2 + k(x_1 + ix_2) (\dot{x}_1 + i\dot{x}_2) \right] e^{2kt} = D_{(1)}, \quad (7.20)$$

$$\left[\frac{1}{2} (\dot{x}_1 - i\dot{x}_2)^2 + \frac{1}{2} (a - bi) (x_1 - ix_2)^2 + k(x_1 - ix_2) (\dot{x}_1 - i\dot{x}_2) \right] e^{2kt} = D_{(2)}.$$

Separating the real and imaginary parts, one has

$$\left[\frac{1}{2} (\dot{x}_1^2 - \dot{x}_2^2) + \frac{1}{2} a(x_1^2 - x_2^2) - bx_1x_2 + k(x_1\dot{x}_1 - x_2\dot{x}_2) \right] e^{2kt} = C_1, \quad (7.21)$$

$$\left[\dot{x}_1\dot{x}_2 + ax_1x_2 + \frac{1}{2} b(x_1^2 - x_2^2) + k(x_1\dot{x}_2 - \dot{x}_1x_2) \right] e^{2kt} = C_2.$$

Next, consider the conservative linear system

$$\ddot{x}_1 + ax_1 + bx_2 = 0, \quad \ddot{x}_2 + bx_1 + ax_2 = 0. \quad (7.22)$$

Repeating the same procedure as in the previous case we arrive to following two independent conservation laws

$$(\dot{x}_1 + \dot{x}_2)^2 + (a + b)(x_1 + x_2)^2 = 2E_1, \quad (\dot{x}_1 - \dot{x}_2)^2 + (a - b)(x_1 - x_2)^2 = 2E_2 \quad (7.23)$$

The total energy integral of (7.22) is $2E = E_1 + E_2$ i.e. the linear combination of the conservation laws E_1 and E_2 .

Note, that the conservation laws (7.21) were obtained in [13] by means of a different approach.

Conclusions

The main purpose of this report is to show that the differential variational principles of Jourdain and Gauss can be employed as a starting point for the study of conservation laws of conservative and nonconservative systems with a finite number of degrees of freedom. It is demonstrated that the introduction of the generalized (nonsimultaneous) variations in the sense of Jourdain and Gauss, a Noether's-type theory for finding conservation laws can be established. For the case of purely conservative dynamical systems, the outcome is identical with the classical Noether's theory obtained by means of the Hamilton's principle. As a characteristic feature of the theory, the condition of the invariance of the Gauss' constraint with respect to the infinitesimal transformations of the acceleration vector (which includes the time deformations), produces simultaneously the condition for the existence of a conserved quantity (the basic Noether's identity) and corresponding conservation law.

In recent years a lot of publications have been devoted to the quadratic conservation laws of linear dynamical systems. A great deal of them are not based on the Noetherian approach. Contrary to the opinion expressed in [13] that the considerations builded on the basic Noether's identity (4.8) "restrict the range of possibilities" we demonstrate by means of a general example that the linear conservative and nonconservative systems can be advantageously studied in the light of generalized Noether's theory described in this paper. The author believes that the form of the conservation laws given by (7.16) (including the special cases discussed therein) did not appear previously.

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