Nonequivalence of Controllability Properties for Piecewise Linear Markov Switch Processes

Dan Goreac

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Abstract

In this paper we study the exact null-controllability property for a class of controlled PDMP of switch type with switch-dependent, piecewise linear dynamics and multiplicative jumps. First, we show that exact null-controllability induces a controllability metric. This metric is linked to a class of backward stochastic Riccati equations. Using arguments similar to the euclidian-valued BSDE in [4], the equation is shown to be equivalent to an iterative family of deterministic Riccati equations that are solvable. Second, we give an example showing that, for switch-dependent coefficients, exact null-controllability is strictly stronger than approximate null-controllability. Finally, we show by convenient examples that no hierarchy holds between approximate (full) controllability and exact null-controllability. The paper is intended as a complement to [15] and [14].

Résumé

Nous étudions la propriété de zéro-contrôlabilité exacte pour une classe de processus de type Markovien à switch ayant des dynamiques linéaires par morceaux à coefficients switchés et bruit multiplicatif. Premièrement, nous montrons que la zéro-contrôlabilité exacte induit une métrique de contrôlabilité. Celle-ci est liée à une classe d'équations stochastiques rétrogrades de type Riccati. En employant des arguments similaires aus EDSR classiques ([4]), l'équation de Riccati se réduit à une famille d'équations itératives déterministes de type Riccati qui admettent une solution unique. Deuxièmement, nous présentons un exemple montrant que, pour des systèmes à coefficients switchés, la propriété de zéro-contrôlabilité exacte est strictement plus forte que celle de zéro-contrôlabilité approchée. Finalement, nous montrons, à l'aide d'exemples particuliers, l'impossibilité à établir une hiérarchie entre les propriétés de contrôlabilité approchée (vers une cible arbitraire) et celle de zéro-contrôlabilité exacte. Ce travail doit être regardé en complément des études menées dans [15] et [14].

1 Introduction

We study the exact null-controllability property for a class of piecewise deterministic Markov processes of switch type. More precisely, our model belongs to Markovian systems consisting of a couple mode/ trajectory (Γ, X). The mode Γ is a pure jump uncontrolled Markov process corresponding to spikes inducing regime switching. The second component X obeys a controlled linear stochastic differential equation (SDE) with respect to the compensated random measure associated to Γ. The linear coefficients governing the dynamics depend on the current mode.

The exact null-controllability problem concerns criteria allowing one to drive the X_T component to zero. This property is of particular importance in the study of regulatory networks (e.g. [5], [16], [22], [6], etc.) to distinguish, for example, lytic pathways (e.g. [16]).

An extensive literature on controllability is available in different frameworks: finite-dimensional deterministic setting (Kalman's condition, Hautus test [17]), infinite dimensional settings (via invariance criteria in [26], [7], [25], [20], [19], etc.), Brownian-driven control systems (exact terminal-controllability in [23], approximate controllability in [3], [10], mean-field Brownian-driven systems in [13], infinite-dimensional setting in [9], [27], [1], [11], etc.), jump systems ([12], [14], etc.). We refer to [14] for more details on the literature as well as applications one can address using switch models.

The recent papers [14] and [15] consider different characterizations of approximate and approximate null-controllability properties for the same class of systems. However, they do not address the question of exact

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null-controllability (which, in non-stochastic time-homogeneous framework, is identified with approximate null-controllability).

The aim of the present paper is to offer a complement to the research topics in [14] and [15]. In [14], a Riccati-type argument is used to characterize approximate controllability for systems with constant coefficients. Our first aim (in Section 3.1) is to give an answer to the problem left open in [14, Remark 4] where a family of backward stochastic Riccati equations are presented and absence of results on the solvability is mentioned. As a by-product, the result provides a metric-type characterization of exact null-controllability in Section 3.2. In Section 3.3, we give an example of approximate null-controllable system which fails to be exactly null-controllable. Finally, in Section 3.4, we show by convenient examples that no hierarchy can be established between approximate (full) controllability and exact null-controllability. In other words, we present an example of approximate controllable yet non exactly null-controllable system and an example of null-controllable system which fails to be approximately controllable in certain directions.

We begin with presenting the model and the standing assumptions in Section 2.1. The technical constructions allowing to prove the theoretical results are gathered in Section 2.2. The controllability notions (exact null, approximate null, approximate) are given in Section 2.3. We gather in the same section some useful results on approximate and approximate null-controllability given in [14] and [15]. The main results on Riccati BSDEs are given in Section 3.1. The method allowing to deal with this stochastic system is based on the recent ideas in [4]. As a by-product, the result on Riccati BSDE provides a metric-type characterization of exact null-controllability in Section 3.2. Hierachy (or absence of) between exact null-controllability and approximate null-controllability (resp. approximate full controllability) make the object of Section 3.3 (resp. Section 3.4).

2 Model and Preliminaries

2.1 The Model

We briefly recall the construction of a particular class of pure jump, non explosive processes on a space Ω and taking their values in a metric space $(E, \mathcal{B}(E))$. Here, $\mathcal{B}(E)$ denotes the Borel σ-field of E. The elements of the space E are referred to as modes. These elements can be found in [8] in the particular case of piecewise deterministic Markov processes (see also [2]). To simplify the arguments, we assume that E is finite and we let $p \geq 1$ be its cardinal. The process is completely described by a couple (λ, Q), where $\lambda : E \longrightarrow \mathbb{R}_+$ and the measure $Q : E \longrightarrow \mathcal{P}(E)$, where $\mathcal{P}(E)$ stands for the set of probability measures on $(E, \mathcal{B}(E))$ such that $Q(\gamma, \{\gamma\}) = 0$. Given an initial mode $\gamma_0 \in E$, the first jump time satisfies $\mathbb{P}^{0,\gamma_0}(T_1 \geq t) = \exp(-t\lambda(\gamma_0))$. The process $\Gamma_t^{\gamma_0} := \gamma_0$, on $t < T_1$. The post-jump location γ^1 has $Q(\gamma_0, \cdot)$ as conditional distribution. Next, we select the inter-jump time $T_2 - T_1$ such that $\mathbb{P}^{0,\gamma_0}\left(T_2 - T_1 \geq t \;/\; T_1, \gamma^1\right) = \exp\left(-t\lambda\left(\gamma^1\right)\right)$ and set $\Gamma_t^{\gamma_0} := \gamma^1$, if $t \in [T_1, T_2)$. The post-jump location γ^2 satisfies $\mathbb{P}^{0,\gamma_0}\left(\gamma^2 \in A \;/\; T_2, T_1, \gamma^1\right) = Q\left(\gamma^1, A\right)$, for all Borel set $A \subset E$. And so on. To simplify arguments on the equivalent ordinary differential system, following [4, Assumption (2.17)], we will assume that the system stops after a non-random, fixed number $M > 0$ of jumps i.e. $\mathbb{P}^{0,\gamma_0}(T_{M+1} = \infty) = 1$.

We look at the process Γ^{γ_0} under $\mathbb{P}^{0,\gamma_0}$ and denote by $\mathbb{F}^0$ the filtration $\left(\mathcal{F}_{[0,t]} := \sigma\{\Gamma_r^{\gamma_0} : r \in [0,t]\}\right)_{t \geq 0}$. The predictable σ-algebra will be denoted by $\mathcal{P}^0$ and the progressive σ-algebra by $Prog^0$. As usual, we introduce the random measure q on $\Omega \times (0, \infty) \times E$ by setting $q(\omega, A) = \sum_{k \geq 1} 1_{\left(T_k(\omega), \Gamma^{\gamma_0}_{T_k(\omega)}(\omega)\right) \in A}$, for all $\omega \in \Omega$, $A \in \mathcal{B}(0, \infty) \times \mathcal{B}(E)$. The compensated martingale measure is denoted by $\widetilde{q}$. (Further details on the compensator are given in Section 2.2.)

We consider a switch system given by a process $(X(t), \Gamma^{\gamma_0}(t))$ on the state space $\mathbb{R}^N \times E$, for some $N \geq 1$ and the family of modes E. The control state space is assumed to be some Euclidian space $\mathbb{R}^d$, $d \geq 1$. The component $X(t)$ follows a controlled differential system depending on the hidden variable γ. We will deal with the following model.

$$\left\{ \begin{array}{l} dX_s^{x,u} = \left[A\left(\Gamma_s^{\gamma_0}\right) X_s^{x,u} + B\left(\Gamma_s^{\gamma_0}\right) u_s\right] ds + \int_E C\left(\Gamma_{s-}^{\gamma_0}, \theta\right) X_{s-}^{x,u} \widetilde{q}(ds, d\theta), \; s \geq 0, \\ X_0^{x,u} = x. \end{array} \right. \tag{1}$$

The operators $A(\gamma) \in \mathbb{R}^{N \times N}$, $B(\gamma) \in \mathbb{R}^{N \times d}$ and $C(\gamma, \theta) \in \mathbb{R}^{N \times N}$, for all $\gamma, \theta \in E$. For linear operators, we denote by ker their kernel and by Im the image (or range) spaces. Moreover, the control process $u : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ is an $\mathbb{R}^d$-valued, $\mathbb{F}^0-$ progressively measurable, locally square integrable process. The space of all such processes will be denoted by $\mathcal{U}_{ad}$ and referred to as the family of admissible control processes. The explicit structure of such processes can be found in [21, Proposition 4.2.1], for instance. Since the control process does not (directly) intervene in the noise term, the solution of the above system can be explicitly computed with $\mathcal{U}_{ad}$ processes instead of the (more usual) predictable processes.

2.2 Technical Preliminaries

Before giving the reduction of our backward Riccati stochastic equation to a system of ordinary Riccati differential equations, we need to introduce some notations making clear the stochastic structure of several concepts : final data, predictable and càdlàg adapted processes and compensator of the initial random measure. The notations in this subsection follow the ordinary differential approach from [4]. Since we are only interested in what happens on $[0,T]$, we introduce a cemetery state $(\infty,\overline{\gamma})$ which will incorporate all the information after $T\wedge T_M$. It is clear that the conditional law of T_{n+1} given $\left(T_n,\Gamma_{T_n}^{\gamma_0}\right)$ is now composed by an exponential part on $[T_n\wedge T,T]$ and an atom at ∞. Similarly, the conditional law of $\Gamma_{T_{n+1}}^{\gamma_0}$ given $\left(T_{n+1},T_n,\Gamma_{T_n}^{\gamma_0}\right)$ is the Dirac mass at $\overline{\gamma}$ if $T_{n+1}=\infty$ and given by Q otherwise. Finally, under the assumption $\mathbb{P}^{0,\gamma_0}\left(T_{M+1}=\infty\right)=1$, after T_M, the marked point process is concentrated at the cemetery state.

We set $\overline{E}_T:=([0,T]\times E)\cup\{(\infty,\overline{\gamma})\}$. For every $n\geq 1$, we let $\overline{E}_{T,n}\subset\left(\overline{E}_T\right)^{n+1}$ be the set of all marks of type $e=((t_0,\gamma_0),...,(t_n,\gamma_n))$, where

(2) $t_0=0,\ (t_i)_{0\leq i\leq n}$ are non-decreasing; $t_i<t_{i+1},$ if $t_i\leq T$; $(t_i,\gamma_i)=(\infty,\overline{\gamma})$, if $t_i\leq T,\ \forall 0\leq i\leq n-1,$

and endow it with the family of all Borel sets $\mathcal{B}_n$. For these sequences, the maximal time is denoted by $|e|:=t_n$. Moreover, by abuse of notation, we set $\gamma_{|e|}:=\gamma_n$. Whenever $T\geq t>|e|$, we set

$$e\oplus(t,\gamma):=((t_0,\gamma_0),...,(t_n,\gamma_n),(t,\gamma))\in\overline{E}_{T,n+1}. \tag{3}$$

By defining

$$e_n:=\left((0,\gamma_0),\left(T_1,\Gamma_{T_1}^{\gamma_0}\right),...,\left(T_n,\Gamma_{T_n}^{\gamma_0}\right)\right), \tag{4}$$

we get an $\overline{E}_{T,n}-$valued random variable, corresponding to our mode trajectories.

A càdlàg process Y **continuous except, maybe, at switching times** T_n and taking its values in a topological vector space $\mathcal{S}$ is given by the existence of a family of $\mathcal{B}_n\otimes\mathcal{B}([0,T])/\mathcal{B}(\mathcal{S})$-measurable functions y^n such that, for all $e\in\overline{E}_{T,n}$, $y^n(e,\cdot)$ is continuous on $[0,T]$ and constant $[0,T\wedge|e|]$ and

(5) If $|e|=\infty,$ then $y^n(e,\cdot)=0$. Otherwise, on $T_n(\omega)\leq t<T_{n+1}(\omega),\ y_t(\omega)=y^n(e_n(\omega),t),\ t\leq T.$

Similar, an $\mathcal{S}-$valued $\mathbb{F}$**-predictable process** Z defined on $\Omega\times[0,T]\times E$ is given by the existence of a family of $\mathcal{B}_n\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(E)/\mathcal{B}(\mathcal{S})$ $-$measurable functions z^n satisfying

(6)

If $|e|=\infty,$ then $z^n(e,\cdot,\cdot)=0$. On $T_n(\omega)<t\leq T_{n+1}(\omega),\ z_t(\omega,\gamma)=z^n(e_n(\omega),t,\gamma),$ for $t\leq T,\ \gamma\in E.$

To deduce the form of **the compensator**, one simply writes

$$\widehat{q}(\omega,dt,d\gamma):=\sum_{n\geq 0}\widehat{q}_{e_n(\omega)}^n(dt,d\gamma)\,1_{T_n(\omega)<t\leq T_{n+1}(\omega)\wedge T}$$

such that

(7)

If $n\geq M,$ then $\widehat{q}_e^n(dt,d\gamma)=\delta_{\overline{\gamma}}(d\gamma)\,\delta_\infty(dt)$.

If $n\leq M-1,\widehat{q}_e^n(dt,d\gamma)=\lambda(\gamma_{|e|})Q(\gamma_{|e|},d\gamma)1_{|e|<\infty,t\in[|e|,T]}Leb(dt)+\delta_{\overline{\gamma}}(d\gamma)\,\delta_\infty(dt)\,1_{(|e|<\infty,t>T)\cup|e|=\infty}.$

The **coefficient** function $A\left(\Gamma_t^{\gamma_0}\right)$ is adapted and can be seen as follows: if $|e|=\infty$, then $A=0$; otherwise, one works with $A\left(\gamma_{|e|}\right)$. Similar constructions hold true for C. In fact, the results of the present paper can be generalized to more general path-dependence of the coefficients.

2.3 Approximate Controllability, Exact and Approximate Null-Controllability

We will be dealing with the following notions of controllability.

Definition 1 *(i) Given the finite time horizon $T>0$, the system (1) is said to be approximately controllable (with initial mode $\gamma_0 \in E$) if, for every final data $\xi \in \mathbb{L}^2\left(\Omega, \mathcal{F}_{[0,T]}, \mathbb{P}^{0,\gamma_0}; \mathbb{R}^N\right)$ (i.e. $\mathcal{F}_{[0,T]}$-measurable, square integrable), every initial condition $x \in \mathbb{R}^N$ and every $\varepsilon > 0$, there exists some admissible control process $u \in \mathcal{U}_{ad}$ such that $\mathbb{E}^{0,\gamma_0}\left[\left|X_T^{x,u} - \xi\right|^2\right] \leq \varepsilon$.*
(ii) The system (1) is said to be approximately null-controllable if the previous condition holds for $\xi = 0$.
(iii) The system (1) is said to be (exactly) null-controllable (with initial mode $\gamma_0 \in E$) if, for every initial condition $x \in \mathbb{R}^N$ there exists some admissible control process $u \in \mathcal{U}_{ad}$ such that $X_T^{x,u} = 0$, $\mathbb{P}^{0,\gamma_0}$-a.s..

The approach of [14, Theorem 1] relies on the duality between the concepts of controllability and observability. For these reasons, one introduces the backward stochastic differential equation.

$$(8) \qquad \begin{cases} dY_t^{T,\xi} = \int_E Z_t^{T,\xi}(\theta)\, \widetilde{q}(dt, d\theta) - A^*\left(\Gamma_t^{\gamma_0}\right) Y_t^{T,\xi} dt - \int_E C^*\left(\Gamma_t^{\gamma_0}, \theta\right) Z_t^{T,\xi}(\theta)\, \widehat{q}(dt, d\theta), \\ Y_T^{T,\xi} = \xi \in \mathbb{L}^2\left(\Omega, \mathcal{F}_{[0,T]}, \mathbb{P}^{0,\gamma_0}; \mathbb{R}^N\right). \end{cases}$$

The following characterization follows from standard considerations on the controllability linear operator(s) (cf. [14, Theorem 1]).

Theorem 2 ([14, Theorem 1]) *The necessary and sufficient condition for approximate null-controllability (resp. approximate controllability) of (1) with initial mode $\gamma_0 \in E$ is that any solution $\left(Y_t^{T,\xi}, Z_t^{T,\xi}(\cdot)\right)$ of the dual system (8) for which $Y_t^{T,\xi} \in \ker B^*\left(\Gamma_t^{\gamma_0}\right)$, $\mathbb{P}^{0,\gamma_0} \otimes Leb$ almost everywhere on $\Omega \times [0,T]$ should equally satisfy $Y_0^{T,\xi} = 0$, $\mathbb{P}^{0,\gamma_0}-$almost surely (resp. $Y_t^{T,\xi} = 0$, $\mathbb{P}^{0,\gamma_0} \otimes Leb - a.s.$).*

Equivalent assertions are easily obtained by interpreting the system (8) as a controlled, forward one :

$$(9) \qquad dY_t^{y,v} = \int_E v_t(\theta)\, \widetilde{q}(dt, d\theta) - A^*\left(\Gamma_t^{\gamma_0}\right) Y_t^{y,v} dt - \int_E C^*\left(\Gamma_t^{\gamma_0}, \theta\right) v_t(\theta)\, \widehat{q}(dt, d\theta), \; Y_0^{y,v} = y \in \mathbb{R}^N.$$

The family of admissible control processes is given by $v \in \mathcal{L}^2\left(q; \mathbb{R}^N\right)$ i.e. the space of all $\mathcal{P}^0 \otimes \mathcal{B}(E)$ - measurable, $\mathbb{R}^N-$valued functions $v_s(\omega, \theta)$ on $\Omega \times \mathbb{R}_+ \times E$ such that

$$\mathbb{E}^{0,\gamma_0}\left[\int_0^T \int_E |v_s(\theta)|^2\, \widehat{q}(ds, d\theta)\right] < \infty,$$

for all $T < \infty$.

Similar duality arguments yield the following characterization of (exact) null-controllability.

Proposition 3 *The necessary and sufficient condition for exact null-controllability at time $T > 0$ of (1) with initial mode $\gamma_0 \in E$ is the existence of a positive constant $C_T > 0$ such that for every initial data $y \in \mathbb{R}^N$ and every $v \in \mathcal{L}^2\left(q; \mathbb{R}^N\right)$, one has $|y|^2 \leq C_T \mathbb{E}^{0,\gamma_0}\left[\int_0^T \left|B^*\left(\Gamma_t^{\gamma_0}\right) Y_t^{y,v}\right|^2 dt\right]$.*

The proof is quasi-identical to the duality arguments in [14, Theorem 1] by invoking [24, Appendix B, Proposition B.1].

In the remaining of the section, unless stated otherwise, we assume the control matrix B to be mode-independent (constant). Using the explicit construction of BSDE with respect to marked-point processes, an invariance (algebraic) necessary and sufficient criterion for approximate null-controllability has been given in [15, Theorem 6]. We recall the following invariance concepts (cf. [7], [26]).

Definition 4 *Given a linear operator $\mathcal{A} \in \mathbb{R}^{N \times N}$ and a family $\mathcal{C} = (\mathcal{C}_i)_{1 \leq i \leq k} \subset \mathbb{R}^{N \times N}$, a set $V \subset \mathbb{R}^N$ is said to be $(\mathcal{A}; \mathcal{C})$- invariant if $\mathcal{A}V \subset V + \sum_{i=1}^{k} \operatorname{Im} \mathcal{C}_i$.*

We construct a mode-indexed family of linear subspaces of $\mathbb{R}^N$ denoted by $\left(V_\gamma^{M,n}\right)_{0 \leq n \leq M,\ \gamma \in E}$ by setting

$$(10) \qquad \mathcal{A}^*(\gamma) := A^*(\gamma) - \int_E \left(C^*(\gamma, \theta) + I\right) \lambda(\gamma) Q(\gamma, d\theta) \text{ and } V_\gamma^{M,M} = \ker B^*,$$

for all $\gamma \in E$, and computing, for every $0 \leq n \leq M-1$,

(11)

$$V_\gamma^{M,n} \text{ the largest } \left(\mathcal{A}^*(\gamma); \left[\left(C^*(\gamma, \theta) + I\right) \Pi_{V_\theta^{M,n+1}} : \theta \in E,\ Q(\gamma, \theta) > 0\right]\right) - \text{invariant subspace of } \ker B^*.$$

Here, Π_V denotes the orthogonal projection operator onto the linear space $V \subset \mathbb{R}^N$. The explicit criterion is the following

Theorem 5 ([15, Theorem 6]) *The switch system (1) is approximately null-controllable (in time $T > 0$) with γ_0 as initial mode, if and only if the generated set $V_{\gamma_0}^{M,0}$ reduces to $\{0\}$.*

In the same paper [15], the property of approximate null-controllability for general systems is shown (using convenient examples) to be strictly weaker than approximate controllability. The following sufficient criterion is proven to guarantee the approximate controllability.

Proposition 6 ([15, Condition 10]) *Let us assume that the largest $(\mathcal{A}^*(\gamma);[(C^*(\gamma,\theta)+I)\Pi_{\ker B^*}:Q(\gamma,\theta)>0])$-invariant subspace of $\ker B^*$ is reduced to $\{0\}$, for every $\gamma \in E$. Then, for every $T > 0$ and every $\gamma_0 \in E$, the system (1) is approximately controllable in time $T > 0$.*

3 A Backward Stochastic Riccati Equation Approach to Exact Null-Controllability

3.1 A Riccati Equation

A simple look at [14, Remark 4] shows that a key argument in the analysis of controllability properties resides in a family of backward stochastic Riccati equations. The authors of [14, Remark 4] argue that their analysis is limited by solvability of the general BSDE of the form

(12)
$$\begin{cases} dK_t^{\varepsilon,\mathcal{B}} = \left(K_t^{\varepsilon,\mathcal{B}} A^*\left(\Gamma_t^{\gamma_0}\right) + A\left(\Gamma_t^{\gamma_0}\right) K_t^{\varepsilon,\mathcal{B}} - \mathcal{B}\left(\Gamma_t^{\gamma_0}\right)\right) dt + \int_E H_t^{\varepsilon,\mathcal{B}}(\theta)\, q(dt,d\theta) \\ \qquad + \int_E \left[\left(f_t^{\varepsilon,\mathcal{B}}(\theta)\right)^* g_t^{\varepsilon,\mathcal{B}}(\theta)\, f_t^{\varepsilon,\mathcal{B}}(\theta) - H_t^{\varepsilon,\mathcal{B}}(\theta)\right] \widehat{q}(dt,d\theta), \\ \text{where } f_t^{\varepsilon,\mathcal{B}}(\theta) := \left(C\left(\Gamma_t^{\gamma_0},\theta\right) K_t^{\varepsilon,\mathcal{B}} - H_t^{\varepsilon,\mathcal{B}}(\theta)\right) \text{ and } g_t^{\varepsilon,\mathcal{B}}(\theta) := \left(\varepsilon I + K_t^{\varepsilon,\mathcal{B}} + H_t^{\varepsilon,\mathcal{B}}(\theta)\right)^{-1},\ t \in [0,T]. \\ K_t^{\varepsilon,\mathcal{B}} = 0,\ \varepsilon I + K_t^{\varepsilon,\mathcal{B}} + H_t^{\varepsilon,\mathcal{B}}(\theta) > 0, \text{ for almost all } t \in [0,T]. \end{cases}$$

Here, $\mathcal{B}\left(\Gamma_t^{\gamma_0}\right)$ are positive semi-definite matrix. However, by using the structure of the jumps and inspired by [4], existence of the solution of the previous BSDE will be reduced to a family of itterated (classical) Riccati equations.

The first result gives existence and uniqueness for the solution of the previous equation. Before stating and proving this result, let us concentrate on the specific form of the jump contribution H. We consider a càdlàg process $K^{\varepsilon,\mathcal{B}}$ continuous except, maybe, at switching times T_n. Then, as explained before, this can be identified with a family $\left(k^{n,\varepsilon,\mathcal{B}}\right)$. We construct, for every $n \geq 0$,

$$\widehat{k}^{n+1,\varepsilon,\mathcal{B}}(e,t,\gamma) := k^{n+1,\varepsilon,\mathcal{B}}(e \oplus (t,\gamma), t)\, 1_{|e|<t} \tag{13}$$

and $K_t^{\varepsilon,\mathcal{B}}$ can be obtained by simple integration of the previous quantity with respect to the conditional law of $\left(T_{n+1}, \Gamma_{T_{n+1}}^{\gamma_0}\right)$ knowing $\mathcal{F}_{T_n}$. Then, H is simply given by $h^{n,\varepsilon,\mathcal{B}}(e,t,\gamma) := \widehat{k}^{n+1,\varepsilon,\mathcal{B}}(e,t,\gamma) - k^{n,\varepsilon,\mathcal{B}}(e,t)$.

The main theoretical contribution of the subsection is the following.

Theorem 7 *We assume that $\mathbb{P}^{0,\gamma_0}(T_{M+1} = \infty) = 1$, for some $M \geq 1$. For every $\varepsilon > 0$ and every $T > 0$, the Riccati BSDE (12) admits a unique solution $\left(K_\cdot^{\varepsilon,\mathcal{B}}, H_\cdot^{\varepsilon,\mathcal{B}}(\cdot)\right)$ consisting of an $\mathcal{S}_+^N$-valued (i.e. positive semi-definite) càdlàg process $K_\cdot^{\varepsilon,\mathcal{B}}$ continuous everywhere except, maybe, at jump times and an $\mathcal{S}^N$-valued (i.e. symmetric matrix-valued) $\mathbb{F}$-predictable process H^ε.*

Proof. For notation purposes, we will consider $\mathcal{B}$ to be fixed and drop the dependency on $\mathcal{B}$. The proof consists of two steps.

Step 1. Using the previous structure of the candidate to the solution of the Riccati equation, one gets an equivalent system of ordinary Riccati-type differential equations

$$\begin{cases} k^{M,\varepsilon}(\cdot) = 0, k^{n,\varepsilon}(e_n,T) = 0, \text{ for every } 0 \leq n \leq M-1, \\ dk^{n,\varepsilon}(e_n,t) = \left[k^{n,\varepsilon}(e_n,t) A^*\left(\gamma_{|e_n|}\right) + A\left(\gamma_{|e_n|}\right) k^{n,\varepsilon}(e_n,t) - \mathcal{B}\left(\gamma_{|e_n|}\right)\right] dt \\ + \int_E \left[(f^n(e_n,t,\theta))^* g^n(e_n,t,\theta)\, f^n(e_n,t,\theta)\right] \widehat{q}(dt,d\theta) - \int_E \left[\widehat{k}^{n+1,\varepsilon}(e_n,t,\theta) - k^{n,\varepsilon}(e_n,t)\right] \widehat{q}(dt,d\theta), \\ \text{for } t \in [0,T], \text{ where } f^n(e_n,t,\theta) = \left(C\left(\gamma_{|e_n|},\theta\right) + I\right) k^{n,\varepsilon}(e_n,t) - \widehat{k}^{n+1,\varepsilon}(e_n,t,\theta) \text{ and} \\ g^n(e_n,t,\theta) := \left(\varepsilon I + k^{n+1,\varepsilon}(e_n \oplus (t,\gamma), t)\right)^{-1} 1_{|e|<t}, \\ \text{Under the condition that } k^{n,\varepsilon}(e_n,t) \geq 0, \text{ for almost all } t \in [0,T]. \end{cases}$$

The fact that the two systems are indeed equivalent follow from the same arguments as those in [4, Theorem 2]. Step 2. Thus, solvability of the Riccati backward stochastic equation reduces to the solvability of the previous system or, again, to the solvability (in $\mathcal{S}_+^N$) of the following equation

$$\dot{p}(t) = p(t)\,a + a^*p(t) - \Pi + p(t)\int_E \left[b(\theta)\,r_t^{-1}(\theta)\,b^*(\theta)\right]\nu(d\theta)\,p(t)\,,\ \text{for } t\in[t_0,T]\,,\ p(T)=\varepsilon I.$$

by setting, for a fixed e_n (and $t > t_0 := |e_n|$)

$$\begin{aligned} a &:= A^*\left(\gamma_{|e_n|}\right) - \lambda\left(\gamma_{|e_n|}\right)\left[\begin{array}{c} \frac{1}{2}I + \int_E C^*\left(\gamma_{|e_n|},\theta\right)Q\left(\gamma_{|e_n|},d\theta\right) \\ -\varepsilon\int_E\left(C^*\left(\gamma_{|e_n|},\theta\right)+I\right)\left(\varepsilon I + k^{n+1,\varepsilon}\left(e_n\oplus(t,\theta),t\right)\right)^{-1}Q\left(\gamma_{|e_n|},d\theta\right)\end{array}\right], \\ \Pi &:= \mathcal{B}\left(\gamma_{|e_n|}\right) + \varepsilon\int_E\left(\varepsilon I + k^{n+1,\varepsilon}\left(e_n\oplus(t,\theta),t\right)\right)^{-1}k^{n+1,\varepsilon}\left(e_n\oplus(t,\theta),t\right)Q\left(\gamma_{|e_n|},d\theta\right) \\ b(\theta) &:= C^*\left(\gamma_{|e_n|},\theta\right)+I,\ r_t(\theta) := \left(\varepsilon I + k^{n+1,\varepsilon}\left(e_n\oplus(t,\theta),t\right)\right) \text{ and } \nu(d\theta) = \lambda\left(\gamma_{|e_n|}\right)Q\left(\gamma_{|e_n|},d\theta\right)\end{aligned}$$

Existence and uniqueness for this equation is standard. Indeed, one notes that $\Pi \geq 0$ and $r \gg 0$ (provided that $k^{n+1,\varepsilon} \geq 0$). If E reduces to a singletone, then this is the classical equation for deterministic control problems (see [28, Chapter 6, Equation 2.34]). The existence and uniqueness is guaranteed by [28, Chapter 6, Corollary 2.10]. For the general case, one assumes that E is given by the standard basis of $\mathbb{R}^p$ and works with

$$b = \left(\sqrt{\nu(e^1)}b\left(e^1\right),...,\sqrt{\nu(e^p)}b\left(e^p\right)\right) \text{ and}$$
$$r_t := \begin{pmatrix} r_t\left(e^1\right) & 0 & ... & 0 \\ 0 & r_t\left(e^2\right) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & r_t\left(e^p\right)\end{pmatrix} \geq \varepsilon I \gg 0.$$

The proof is complete by descending recurrence over $n \leq M$. ■

3.2 First Application: Null-Controllability Metric(s)

Proposition 8 *A necessary and sufficient condition for exact null-controllability of (1) with initial mode $\gamma_0 \in E$ at time $T > 0$ is that the pseudonorm*

$$\text{(14)}\qquad \mathbb{R}^N \ni y \longmapsto p(y)\,,\ \textit{where } p^2(y) := \inf_{v\in\mathcal{L}^2(q;\mathbb{R}^N)}\mathbb{E}^{0,\gamma_0}\left[\int_0^T\left|\Pi_{\left(\ker B^*\left(\Gamma_t^{\gamma_0}\right)\right)^\perp}\left(Y_t^{y,v}\right)\right|^2dt\right]$$

be a norm on $\mathbb{R}^N$.

Proof. It is clear that the application p has non-negative values. Homogeneity is a consequence of the equality $Y_\cdot^{ay,av} = aY_\cdot^{y,v}$, for all $y \in \mathbb{R}^N$, all $a \in \mathbb{R}$ and all $v \in \mathcal{L}^2\left(q;\mathbb{R}^N\right)$ (due to the linearity of (9)). To prove the subadditivity, one simply notes $Y_\cdot^{y_1+y_2,v^1+v^2} = Y_\cdot^{y_1,v^1} + Y_\cdot^{y_2,v^2}$, for all $y^1, y^2 \in \mathbb{R}^N$ and all $v^1, v^2 \in \mathcal{L}^2\left(q;\mathbb{R}^N\right)$. It follows that

$$\begin{aligned} p(y_1+y_2) &\leq \left(\mathbb{E}^{0,\gamma_0}\left[\int_0^T\left|\Pi_{\left(\ker B^*\left(\Gamma_t^{\gamma_0}\right)\right)^\perp}\left(Y_t^{y_1+y_2,v^1+v^2}\right)\right|^2dt\right]\right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E}^{0,\gamma_0}\left[\int_0^T\left|\Pi_{\left(\ker B^*\left(\Gamma_t^{\gamma_0}\right)\right)^\perp}\left(Y_t^{y_1,v^1}\right)\right|^2dt\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}^{0,\gamma_0}\left[\int_0^T\left|\Pi_{\left(\ker B^*\left(\Gamma_t^{\gamma_0}\right)\right)^\perp}\left(Y_t^{y_2,v^2}\right)\right|^2dt\right]\right)^{\frac{1}{2}},\end{aligned}$$

for all $v^1, v^2 \in \mathcal{L}^2\left(q;\mathbb{R}^N\right)$. The conclusion follows by taking infimum over such control processes. It follows that p is a pseudonorm (independently of the fact that the system is approximately null-controllable). Necessity follows from Proposition 3 and sufficiency from the equivalence of norms on $\mathbb{R}^N$ by applying Proposition 3. ■

Using the form of the Riccati BSDE (12), one infers the following explicit condition.

Corollary 9 *A necessary and sufficient condition for exact null-controllability of (1) with initial mode $\gamma_0 \in E$ at time $T > 0$ is that the positive-semidefinite matrix $k_0 := \inf_{\varepsilon>0} K_0^\varepsilon$, where, for every $\varepsilon > 0$, K^ε is the unique solution of the Riccati equation (12) for $\mathcal{B} := BB^*$ be positive definite. In this case, the metric p given in (14) is induced by k_0 i.e.*

$$p(y) = \sqrt{\langle k_0 y, y\rangle}, \text{ for all } y \in \mathbb{R}^N.$$

Proof. This result is quite classical (see, e.g. [27] for the Brownian-noise case). For our readers' sake, we sketch the proof. Let us fix, for the time being $\varepsilon > 0$. Then, according to Theorem 7, the Riccati equation (12) admits a unique solution. A simple application of Itô's formula (cf. [18, Chapter II, Section 5, Theorem 5.1]) to $\langle K_t^\varepsilon Y_t^{y,v}, Y_t^{y,v}\rangle$ on $[0,T]$ yields

$$\begin{aligned}&\mathbb{E}^{0,\gamma_0}\left[\int_0^T \left|\Pi_{\left[Ker\left(B^*\left(\Gamma_t^{\gamma_0}\right)\right)\right]^\perp} Y_t^{y,v}\right|^2 dt\right]\\ &= \langle K_0^\varepsilon y, y\rangle - \varepsilon \mathbb{E}^{0,\gamma_0}\left[\int_0^T |u_t|^2 dt\right] + \mathbb{E}^{0,\gamma_0}\left[\int_0^T \left|\begin{array}{c}(\varepsilon I + K_t^\varepsilon + H_t^\varepsilon(\theta))^{-\frac{1}{2}} f_t(\theta) Y_{t-}^{y,v}\\ -(\varepsilon I + K_t^\varepsilon + H_t^\varepsilon(\theta))^{\frac{1}{2}} v_t(\theta)\end{array}\right|^2 dt\right].\end{aligned}$$

One easily notes that $\inf_{v\in\mathcal{L}^2(q;\mathbb{R}^N)} \mathbb{E}^{0,\gamma_0}\left[\int_0^T \left|\Pi_{\left[Ker\left(B^*\left(\Gamma_t^{\gamma_0}\right)\right)\right]^\perp} Y_t^{y,v}\right|^2 dt\right] = \liminf_{\varepsilon\to 0} \langle K_0^\varepsilon y, y\rangle$ and the conclusion follows. ■

3.3 Non-equivalence Between Exact and Approximate Null-Controllability

The following example presents a switching system which is approximately null-controllable without being exactly null-controllable.

Example 10 *We consider a two-dimensional state space and a one-dimensional control space. Moreover, we consider the mode to switch randomly between three states (for simplicity, $E = \{e^1, e^2, e^3\}$ is taken to be the standard basis of $\mathbb{R}^3$). The transition measure is given by $Q := \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$. The coefficients are given by*

$$A(e^1) = A := \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix},\ A(\gamma) := 0_{2\times 2},\ \text{if } \gamma \neq e^1,\ B := \begin{pmatrix} 1\\ 0\end{pmatrix},\ C(\gamma) := 0_{2\times 2}.$$

a) Approximate null-controllability

With the definition (11), one easily establishes $V_\gamma^{M,n} = \begin{cases} \ker B^*, \text{ if } \gamma \neq e^1,\\ \{0\}, \text{ if } \gamma = e^1.\end{cases}$, for every $M \geq 1$ and every $0 \leq n \leq M$. It follows that the system is null-controllable if and only if $\gamma_0 = e^1$.

b) Limit of the Riccati equations in the approximate null-controllable case *(initial mode $\gamma_0 = e^1$). Quid est for the controllability metric ? In this case, we recall that the limit of the solutions of the Riccati equations is given by*

$$p^2(y) := \inf_{v\in\mathcal{L}^2(q;\mathbb{R}^N)} \mathbb{E}^{0,\gamma_0}\left[\int_0^T |B^* Y_t^{y,v}|^2 dt\right],\ \text{where } dY_t^{y,v} = \int_E v_t(\theta)\,\widetilde{q}(dt,d\theta) - A^*(\Gamma_t^{\gamma_0}) Y_t^{y,v} dt,\ Y_0^{y,v} = y \in \mathbb{R}^2.$$

Starting from $y := \begin{pmatrix} 0\\ 1\end{pmatrix}$, with the feedback control process $v_t^\varepsilon := \left(-\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} Y_t + \begin{pmatrix} 0\\ \frac{e^{2\varepsilon}-1}{-e^\varepsilon+1+\varepsilon} 1_{t\in[\varepsilon,2\varepsilon]}\end{pmatrix}\right) 1_{t\leq T_1}$, one gets

$$\left\langle Y_t^{y,v}, \begin{pmatrix} 1\\ 0\end{pmatrix}\right\rangle = \left[\left(1-e^t\right) 1_{t\leq\varepsilon} + \left[1 - e^t + \left(t - \varepsilon + 1 - e^{t-\varepsilon}\right)\frac{e^{2\varepsilon}-1}{-e^\varepsilon+1+\varepsilon}\right] 1_{\varepsilon\leq t\leq 2\varepsilon}\right] 1_{t\leq T_1}.$$

One easily notes that $0 \geq \left\langle Y_t^{y,v}, \begin{pmatrix} 1\\ 0\end{pmatrix}\right\rangle \geq c_\varepsilon := -2\left(e^{2\varepsilon}-1\right)$. Then, by taking infimum over $\varepsilon > 0$, it follows that $p^2\begin{pmatrix} 0\\ 1\end{pmatrix} = 0$ and it cannot induce a norm. As consequence, by invoking Corollary 9, the system fails to be (exactly) null-controllable.

Remark 11 *Of course, a direct proof of null-controllability can also be given based on the eigenvector* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. *Absence of null-controllability is obvious for* $\gamma_0 \neq e^1$ *(the system is not even approximately null-controllable). In the case* $\gamma_0 = e^1$, *we reason by contradiction. Let us assume that, for some admissible control process* u, *the system is exactly controllable at time* T *starting from* $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. *Then, prior to the first jump time,*

$$X_t^{x_0,u} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & 0 \\ t-s & 1 \end{pmatrix} \begin{pmatrix} u_s \\ 0 \end{pmatrix} ds = \begin{pmatrix} \int_0^t u_s ds \\ 1 + \int_0^t (t-s) u_s ds \end{pmatrix}.$$

Since, on $[0, T_1)$, u *is deterministic and square integrable, there exists* $T > t_0 > 0$ *such that* $1+\int_0^t (t-s) u_s ds > \frac{1}{2}$, *for every* $t \leq t_0$ *(consequence of the absolute continuity). Hence, on* $T_1 \leq t_0$, *one gets* $\left\langle X_{T_1-}^{x_0,u}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle > \frac{1}{2}$. *Second, one notes that* $X_{T_1-}^{x_0,u} = X_{T_1}^{x_0,u}$ *and* $d\left\langle X_t, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 0$ *for* $t \geq T_1$ *to conclude that* $\left\langle X_T^{x_0,u}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle > \frac{1}{2}$, *on* $T_1 \leq t_0$. *Since* $\mathbb{P}(T_1 \leq t_0) = 1 - e^{-t_0} > 0$, *it follows that* u *cannot lead to* 0 *with full probability. Therefore, although it is approximately null-controllable for some initial modes, the switch system is never exactly null-controllable.*

3.4 Exact Null-Controllability vs. Approximate (Full) Controllability

It has been shown in [15] that, in general, approximate controllability is strictly stronger than approximate null-controllability. In the light of the previous example, it is then natural to ask oneself whether the condition on p (given by (14)) being a metric implies approximate controllability of the initial system. The answer is negative. We begin with an example of a system governed by an off/on mode which is approximately null-controllable iff the initial mode is set off and is never approximately controllable. We show that, for this system, the Riccati equations give a controllability metric (iff the initial mode is set off).

Example 12 *We consider a two-dimensional state space and a one-dimensional control space. Moreover, we consider the mode to switch randomly between inactive* 0 *and active* 1 *(i.e.* $E = \{0, 1\}$*). The coefficients are given by*

$$A(\gamma) = A := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C(\gamma, 1-\gamma) := \begin{pmatrix} -1 & 0 \\ \gamma & -1 \end{pmatrix}.$$

a) Approximate null-controllability
One easily establishes $V_\gamma^{M,n} = span\{\gamma e_2\}$, *for every* $M \geq 1$ *and every* $0 \leq n \leq M$. *It follows that the system is null-controllable if and only if* $\gamma_0 = 0$.
b) Approximate controllability *(initial mode* $\gamma_0 = 0$*)*
One easily checks that $Y_t := \begin{pmatrix} 0 \\ \Gamma_t^0 \end{pmatrix}$, $Z_t(\cdot) = \begin{pmatrix} 0 \\ (-1)^{\Gamma_{t-}^0} \end{pmatrix}$, *for* $0 \leq t \leq T$ *satisfies the BSDE (8) with final data* $Y_T = \begin{pmatrix} 0 \\ \Gamma_T^0 \end{pmatrix}$. *Since this solution stays in* $\ker B^*$ *and it is not trivially zero, it follows that the system is (never) approximately controllable.*
c) Riccati equations in the approximate null-controllable case *(initial mode* $\gamma_0 = 0$*)*
One easily notes that the Riccati equations lead to

$$\begin{aligned} dk^{0,\varepsilon}(0,t) &= \left[k^{0,\varepsilon}(0,t) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} k^{0,\varepsilon}(0,t) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k^{0,\varepsilon}(0,t) \right] dt \\ &+ \left[\widehat{k}^{1,\varepsilon}(0,0,t,1) \left(\varepsilon I + \widehat{k}^{1,\varepsilon}(0,0,t,1) \right)^{-1} \widehat{k}^{1,\varepsilon}(0,0,t,1) - \widehat{k}^{1,\varepsilon}(0,0,t,1) \right] dt, t \in [0,T]. \end{aligned}$$

Since $\widehat{k}^{n+1,\varepsilon}(0,0,t,1) \geq 0$, *this solution is at least equal to the one given by* $\overline{K}_t := \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}$, *where* $a(t) := 1 - e^{t-T}$, $b(t) := (T+1-t) e^{t-T} - 1$ *and* $c(t) := 2 - \left(1 + (T+1-t)^2\right) e^{t-T}$. *Hence,* $\overline{K}_0$ *is positive definite (for every* $T > 0$*) and so is* $\liminf_{\varepsilon \to 0+} K_0^{0,\varepsilon}$ *(to prove this, one simply studies the sign of the function* $T \mapsto \left(1 - e^{-T}\right) \left[2 - \left(2 + T^2 + 2T\right) e^{-T}\right] - \left[(T+1) e^{-T} - 1\right]^2$ *on* $\mathbb{R}_+^*$*).*

Remark 13 *The reader may want to note that in the case presented in the previous example, the system is approximately null-controllable if and only if it is (exactly) null-controllable. To give an explicit control leading from x_0 to 0 for the initial setting $\gamma_0 = 0$, one proceeds as follows. Prior to the first jump, the system is a deterministic one and given by*

$$d\Phi_t^{x_0,u^1} = \left((A - C(0,1)) \Phi_t^{x_0,u^1} + Bu_t^1 \right) dt, \ \Phi_0^{x_0,u^1} = x_0 \in \mathbb{R}^2.$$

Since Kalman's condition is satisfied for this dterministic system, it is exactly null-controllable at time $T > 0$. An explicit control is obtained by considering the (deterministic) controllability Gramian G_T, respectively the induced (open-loop) control process u^1 given by

$$G_T := \int_0^T e^{(A-C(0,1))(T-s)} BB^* e^{(A-C(0,1))^*(T-s)} ds, \ u^1(t) := -B^* e^{(A-C(0,1))^*(T-t)} G_T^{-1} e^{(A-C(0,1))^* T} x_0 1_{t \le T}.$$

We obtain a stochastic control by setting $u^p(t) = 0$, i.e. we take null-control after the first jumping time. Then, it is obvious that, on $T_1 \ge T$, $X_T^{x_0,u} = \Phi_T^{x_0,u^1} = 0$. On $T_1 < T$, $X_{T_1}^{x_0,u} = 0$ and the conclusion follows.

Finally, one is entitled to ask whether approximate null-controllability implies exact null-controllability. The answer is, again, negative proving that approximate controllability and exact null-controllability are, in general, completely different properties. To illustrate this, let us take, once again, a glance at the first example.

Example 14 *We consider a two-dimensional state space and a one-dimensional control space. Moreover, we consider the mode to switch randomly between three states (for simplicity, $E = \{e_1, e_2, e_3\}$ is taken to be the standard basis of $\mathbb{R}^3$). The transition measure is given by $Q := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. The coefficients are given by*

$$A(e_1) = A := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ A(\gamma) := 0_{2\times 2}, \ \textit{if } \gamma \ne e_1, \ B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ C(\gamma) := 0_{2\times 2}$$

As we have seen before (in Example 10), this system is never exactly null-controllable. We assume the system to only jump once. We consider a solution of

$$dY_t^{y,v} = \int_E v_t(\theta) \widetilde{q}(dt, d\theta) - A^* (\Gamma_t^{\gamma_0}) Y_t^{y,v} dt, Y_0^{y,v} = y \in \mathbb{R}^N$$

that belongs to $\ker B^*$, $\mathbb{P}^{0,e_1} - a.s.$ Due to the approximate null-controllability, it follows that $y = 0$. Prior to the first jump, v is given by a deterministic function $v^1 = \begin{pmatrix} v^{1,1} \\ v^{1,2} \end{pmatrix}$ and $Y_t^{y,v}$ coincides with the deterministic solution of $d\Phi_t^{v^1} = \left(-v_t^1 - A^* \Phi_t^{v^1} \right) dt$, $\Phi_0^{v^1} = 0$. Since $\Phi_t^{v^1} \in \ker B^* = span \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ (for all $t \in [0,T]$) one has, for Leb$-$almost all $t \in [0,T]$, $-v_t^{1,1} = \left\langle \Phi_t^{v^1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$. Hence, at the first jumping time, one has

$$0 = \left\langle Y_{T_1}^{y,v}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle \Phi_{T_1}^{v}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle - \left\langle \Phi_{T_1}^{v^1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = -\left\langle \Phi_{T_1}^{v^1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle.$$

One deduces that $\left\langle \Phi_t^{v^1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 0$ and $v_t^{1,1} = 0$, Leb-almost surely on $[0,T]$. Then the derivative of $\left\langle \Phi_t^{v^1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$ is null i.e. $v_t^{1,2} = 0$, Leb-almost surely on $[0,T]$. As a consequence, for all (due to right continuity) $t \in [0, T_1]$, $Y_t^{y,v} = 0$, $\mathbb{P}^{0,e_1}$-almost surely. Since the process is no longer allowed to jump after T_1, it follows that the equality actually holds on $[0,T]$ and the initial system (1) is approximately controllable (cf. Theorem 2).

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