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September 2016

Working paper No. 2016–9

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The optimal contract under adverse selection in a moral-hazard model with a risk-averse agent

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Abstract

This paper studies the optimal contract offered by a risk-neutral principal to a risk-averse agent when the agent's hidden efficiency and action both improve the probability of the project being successful. We show that if the agent is sufficiently prudent and efficient, the principal induces a higher probability of success than under moral hazard, despite the costly informational rent given up. Moreover, the conditions to avoid pooling are difficult to satisfy because of the different kinds of incentives to be managed and the overall trade-off between rent extraction, insurance, and efficiency involved.

Keywords Adverse selection, moral hazard, risk aversion, prudence.

JEL Classification Numbers D82.

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1 Introduction

The theory of incentives has made considerable advances in the last forty years. The implications of pure adverse-selection or pure moral-hazard models are now well known.[1] However, there are many examples of contracts designed to solve adverse-selection and moral-hazard problems simultaneously. Chief-executive-officer (CEO) and financial contracts are particularly archetypal. In the former, a CEO has private information on how good a manager he is and how tirelessly he works. In the latter, a borrower has private information on how risky his project is and how hazardously he carries it out. Paradoxically, despite the plethora of mixed models in the incentives literature, such problems where the principal's payoff is closely bound up with the agent's private information and unobservable action have come in for little study.

In this paper, a stochastic production taking two values, low (i.e. failure) or high (i.e. success), is considered in order to study the optimal contract offered by a risk-neutral principal to a risk-averse agent when the agent's hidden efficiency and hidden action both improve the probability of success.

We begin by recalling the well-known full-insurance property of the complete information contract: to generate a positive probability of success the agent receives a (full information) fixed payment whether the production fails or succeeds. Furthermore, he gets no rent (i.e. he only receives his reservation utility). Then, we consider that the agent's action becomes non-observable and we recall the main characteristics of the moral-hazard contract. The agreement offers high-powered incentives: the (moral-hazard) fixed payment is increased by a positive bonus in the event of success. Moreover, because the agent bears some risk, the contract pays him a risk-premium without giving up a rent for him. These elements constitute the moral-hazard cost incurred by the principal to induce a positive probability of success despite the non-observability of the action. It is optimal to distort the efficient probability of success in order to take this cost into account. Such distortions reflect the usual trade-off between insurance and efficiency.

Next, we consider that efficiency is no longer observable. Thus adverse selection comple-

[1] See Laffont and Martimort (2002), Bolton and Dewatripont (2005).

ments moral hazard to form a mixed model. When the principal faces asymmetric information about the agent's type, she needs to elicit truthful information. To do this, she must give up an informational rent to the agent. Compared with moral hazard, such a rent introduces two modifications into the marginal cost to induce a positive probability of success. First, this rent constitutes a cost that raises the moral-hazard cost. So the moral-hazard marginal cost to induce a positive probability of success is increased, except for the most efficient agent, by the marginal cost of the informational rent. This is the standard adverse-selection cost for the principal.

Second, the rent affects the two components of the moral-hazard cost. The consequence for the bonus is inevitably costly. Because the bonus incentivizes the risk-averse agent to bear a risk, the higher the fixed payment is, the more costly high-powered incentives are. That is precisely what the informational rent does since the agent gets no rent under moral-hazard alone.

By contrast, the consequence of the informational rent for the risk premium is ambiguous. It is important to notice that a rent is equivalent to transferring the risk borne by the agent toward higher utility levels than the reservation utility. This forces the principal to pay a risk premium that guarantees a higher utility in the mixed model than in the moral-hazard model. On the one hand, it is costly at the margin because of risk aversion again. On the other hand, in accordance with decision theory, this transfer is not welcomed by an imprudent agent, but the contrary is true for a prudent one.[2] In the former case, this prompts an increase in the risk premium (at the margin). In the latter, the contrary arises.

Ultimately, the presence of adverse selection can reinforce or mitigate the moral-hazard trade-off. More specifically, the informational rent given up implies that[3]

- risk aversion and prudence both contribute to increasing the moral-hazard marginal cost if the agent is imprudent,
- prudence alleviates the increase in the moral-hazard marginal cost due to risk aversion if the agent is weakly or moderately prudent,

[2]See Eeckhoudt et al. (1995), Crainich and Eeckhoudt (2005).

[3]The notions of weakly, moderately, and highly prudent agent will be stated rigorously in the core of the paper.

- the moral-hazard marginal cost is reduced because prudence offsets risk aversion if the agent is highly prudent.

A somewhat surprising effect arises if we consider both a highly prudent and sufficiently efficient agent.[4] Indeed, as shown above, great prudence implies a reduction in the moral-hazard marginal cost. In parallel, in accordance with pure adverse-selection models, a sufficiently efficient agent implies an adverse-selection marginal cost relatively close to zero (even null at the top). So in this case, the mixed-model marginal cost to induce a positive probability of success is lower than under moral hazard. The mixed-model contract entails a higher probability of success, despite the costly informational rent given up. Otherwise, there is a reduction in the probability of success when the agent is imprudent or weakly/moderately prudent since the informational rent also increases the moral-hazard marginal cost. It follows there is inevitably a distortion for all the agent's efficiency compared to moral hazard alone, even at the top, which contradicts the well-known result under adverse selection alone. Such distortions reflect the overall trade-off between rent extraction, insurance, and efficiency.

Nevertheless, the nature of these distortions varies with agent's efficiency. We show that moving from the lowest to the highest efficiency leads the overall trade-off to substitute distortions due to moral hazard for distortions due to adverse selection. Indeed, again according to adverse-selection models alone, rather inefficient agents get a low rent and are associated with a high marginal cost of the informational rent. The contrary is true for rather efficient agents. So the distortion associated with the inefficient types is due rather to the rent extraction efficiency trade-off. The principal is more concerned about the adverse-selection cost. The probability of success is distorted to limit the informational rent. The reverse is true for the distortion associated with the efficient types. It occurs from the insurance-efficiency trade-off, because the principal is more interested in the moral-hazard cost. The distortion comes from the desire to limit the cost to induce a positive probability of success, i.e. the bonus and the risk premium.

This result raises the question whether a fully separating contract can be implemented. Pooling can occur because of the common-value nature of the model and the possible lack of monotonicity of the marginal cost of the informational rent due to risk aversion. So

[4]The notion of sufficiently efficient agent will be stated rigorously in the core of the paper.

pooling in the mixed contract does not have different causes from those already known in pure adverse-selection models. However, pooling is most likely to emerge since the conditions for avoiding pooling are more difficult to satisfy because of the different kinds of incentives to be managed and the overall trade-off between rent extraction, insurance, and efficiency involved.

The paper is organized as follows. Section 2 presents the related literature. The model is stated in section 3. Sections 4 and 5 respectively analyze complete information and moral hazard. The mixed case is studied in section 6. An example is given in section 7. We briefly conclude in an eighth section. Section 9 is devoted to appendices.

2 Related literature

The literature on screening under true moral-hazard can be split into two distinct parts. The first part concerns closed models. Faynzilberg and Kumar (2000) analyze a similar model, but with a continuum of output. They study more specifically the validity of the well-known first-order approach. However, this degree of generality implies that the optimal contract cannot be fully characterized. On the one hand, the two-output model constitutes a limit. But, on the other hand, it allows us to use the first-order approach and to fully determine the properties of the optimal contract and the overall trade-off between rent extraction, insurance, and efficiency.

Ollier (2007) and Ollier and Thomas (2013) also study a two output model, but with a risk-neutral agent protected by limited liability or ex-post participation constraints. They show that a fully pooling contract is optimal. Such a contract allows the principal to reduce the overall (informational plus limited liability or ex-post participation) rent left to the agent. In our model, pooling can also arise. But this is due to standard reasons: the common-value nature of the model or the lack of monotonicity of the marginal cost of the informational rent. So pooling is not a cause for the principal to limit rent but a consequence of the difficulty in managing different kinds of incentives in the presence of risk aversion.

The second part of the literature studies insurance models. In Jullien et al. (2007), a two-type/two-output model is analyzed. The major difference with the present paper is that the agent's private knowledge affects the level of risk-aversion instead of the technology. One result can be underlined: the properties of the optimal contract crucially depend on the high power incentives existing in the outside option. In our paper, we do not investigate an endogenous outside-option utility. By contrast, we assume that it is constant.

3 The Model

The basic data of the model follow Ollier and Thomas (2013). A principal contracts with an agent to produce a pecuniary output with random value $x, x \in \{\underline{x}, \overline{x}\}$. High (resp. low) production $\overline{x}$ (resp. $\underline{x}$) is associated with success (resp. failure). The realization of high production requires the agent to perform an action generating a probability of success $\rho = \Pr(x = \overline{x}) \in (0, 1)$. But action is costly for the agent. He incurs an indirect disutility $\psi(\rho, \theta)$, with θ his efficiency. The principal does not observe the action or the efficiency. But she knows that efficiency is drawn from a density $f > 0$ on $[\underline{\theta}, \overline{\theta}]$ with cumulative F. The principal offers a contract $\langle a, b \rangle$ with a, a fixed payment, and b, a bonus in the event of success. In other words, a is the non-contingent component of the contract and b the contingent component. Let U_0 be the agent's reservation utility.

The principal. The principal is risk-neutral. For a given probability of success and a given contract, her objective function is

$$
\begin{aligned}
V &= \big(1 - \rho\big)\big(\underline{x} - a\big) + \rho\big(\overline{x} - (a + b)\big) \\
&= \underline{x} + \rho \Delta x - a - \rho b,
\end{aligned} \tag{1}
$$

where $\Delta x = \overline{x} - \underline{x} > 0$, is the increase in output value due to success.

The agent. The agent is risk-averse. The utility of the payment in the event of success (resp. failure) is $u(a + b)$ (resp. $u(a)$). The agent's expected utility is

$$
\begin{aligned}
U &= \big(1 - \rho\big)u(a) + \rho u(a + b) - \psi(\rho, \theta) \\
&= u(a) + \rho\big(u(a + b) - u(a)\big) - \psi(\rho, \theta).
\end{aligned} \tag{2}
$$

The properties of the functions u and ψ are the following.

Property 1. *The functions u and ψ verify*

$$u(0) = 0, u'(.) > 0, u''(.) < 0, u'''(.) \text{ has a constant sign,}$$

and [5]

$$\psi(0,\theta) = 0, \psi_1(\rho,\theta) > 0, \psi_{11}(\rho,\theta) > 0, \psi_2(\rho,\theta) < 0, \psi_{22}(\rho,\theta) > 0, \psi_{12}(\rho,\theta) < 0.$$

The conditions on the function u imply that the agent has no utility if he receives no payment, is risk-averse since the marginal utility of the payment is positive and decreasing, and is either imprudent when $u''' < 0$ or prudent when $u''' > 0$. The conditions on the function ψ reflect standard assumptions about the probability of success and the disutility of effort.[6] For all types, the disutility incurred to implement no probability of success is null. The marginal indirect disutility is positive and increasing. In more productive states, the agent's disutility diminishes but at an increasing rate. Higher values of θ correspond to states in which a higher probability of success is less costly to generate.

Let $r_A = -\frac{u''}{u'}$ (resp. $r_P = -\frac{u'''}{u''}$) be the absolute risk aversion (resp. prudence) coefficient. For the analysis, it is useful to introduce different levels of prudence.

Definition 1. *The agent is said to be,* $\forall(\rho,\theta) \in (0,1) \times [\underline{\theta},\overline{\theta}]$,

- *weakly prudent if*

$$0 < r_P \leq 3r_A,$$

- *moderately prudent if*

$$3r_A < r_P \leq 3r_A + \frac{u'}{\rho(1-\rho)\psi_{11}(\rho,\theta)},$$

- *highly prudent if*

$$3r_A + \frac{u'}{\rho(1-\rho)\psi_{11}(\rho,\theta)} < r_P.$$

[5] Subscript i denotes the partial derivative with the i^{th} argument.

[6] Indeed, an equivalent way to analyze this problem is to consider that the agent exerts an effort that, like his efficiency, increases the probability of success $\rho(e,\theta)$. But he incurs a disutility $\varphi(e)$. See Ollier and Thomas (2013) for details.

The problem. The principal's problem is to maximize the expectation of (1) with respect to $\langle a, b \rangle$, subject to the participation and the incentive compatibility constraints. The former implies that the agent voluntarily agrees to the contract. The latter requires the agent to be honest and obedient.

For the sake of clarity in resolution whatever the informational framework, it is useful to change the variables as follows. Let $\underline{u}$ be the utility in the event of failure, $\overline{u}$ the utility in the event of success, and Δu the spread of utility (hereafter, the power of incentives). We obtain

$$\begin{cases} \underline{u} = u(a), \\ \overline{u} = u(a+b), \\ \Delta u = \overline{u} - \underline{u}. \end{cases} \tag{3}$$

Denoting $u^{-1} = w$, we get a first lemma.

Property 2. *The function w is such that*

$$\begin{aligned} w' &= \frac{1}{u'} > 0, \\ w'' &= -\frac{u''}{u'^3} > 0, \\ w''' &= -\frac{u'''u' - 3u''^2}{u'^5} \lesseqgtr 0 \Leftrightarrow r_P \gtreqless 3r_A. \end{aligned}$$

Proof. Straightforward. □

The signs of w' and w'' directly follow from Property 1. Using Definition **1**, the sign of w''' depends on the level of the agent's prudence. If the agent is at least (resp. most) moderately (resp. weakly) prudent, w''' is negative (resp. positive).

Considering payments, we get the following lemma.

Lemma 1. *The payments are such that*

$$\begin{aligned} a &= w(\underline{u}), \\ b &= w(\overline{u}) - w(\underline{u}). \end{aligned}$$

Proof. Straightforward. □

Using this lemma, the objective function (1) is

$$V = \underline{x} + \rho\Delta x - w(\underline{u}) - \rho\big(w(\overline{u}) - w(\underline{u})\big). \tag{4}$$

From (3), the agent's utility (2) becomes

$$U = \underline{u} + \rho\Delta u - \psi(\rho, \theta). \tag{5}$$

Moreover, manipulating (5) and since $\overline{u} = \Delta u + \underline{u}$, we get

$$\begin{cases} \underline{u} = U + \psi(\rho, \theta) - \rho\Delta u, \\ \bar{u} = U + \psi(\rho, \theta) + \big(1 - \rho\big)\Delta u. \end{cases} \tag{6}$$

Thus offering the contract $\langle U, \Delta u\rangle$ specifying the agent's expected utility, U, and the power of incentives, Δu, is equivalent to offering the initial contract $\langle a, b\rangle$.

To get a well-behaved model, we assume that the indirect disutility ψ verifies Inada's conditions and has a convex marginal indirect disutility.

Assumption 1. *ψ is such that* $\lim_{\rho\to 0} \psi_1(\rho, \theta) = 0, \lim_{\rho\to 1} \psi_1(\rho, \theta) = \infty$*, and* $\psi_{111}(\rho, \theta) \geq 0$.

The rest of the paper studies the optimal contract according to the following informational frameworks: complete information, pure moral hazard, and mixed model (i.e. adverse selection plus moral hazard).

4 Complete information

When information is complete, the principal observes the agent's efficiency and action. Only the participation constraint needs to be satisfied. Given (6), the problem is $\max_{(\rho,\Delta u,U)}(4)$ subject to the participation constraint

$$U \geq U_0. \tag{PC}$$

This constraint ensures that the agent is not forced to accept the contract.

We get the following lemma.

Lemma 2. *The first-best contract entails* , $\forall\theta \in [\underline{\theta}, \overline{\theta}]$

- $U^{FB}(\theta) = U_0$,

- $\Delta u^{FB}(\theta) = 0$,

so $\underline{u}^{FB}(\theta) = U^{FB}(\theta) + \psi(\rho^{FB}(\theta), \theta)$, *with* $\rho^{FB}(\theta)$ *given by*

$$\Delta x = w'(\underline{u}^{FB}(\theta))\psi_1(\rho^{FB}(\theta), \theta). \tag{FB}$$

Proof. See appendix 9.1. □

The interpretation is the following. To benefit from the increase in output value, Δx in (FB), the principal must implement a positive probability of success. This implies that the agent incurs the indirect disutility. According to (4) and (5), the principal must make the expected payment, $w(\underline{u}) + \rho\big(w(\Delta u + \underline{u}) - w(\underline{u})\big)$, in order to ensure the participation constraint. But it is increasing in $\underline{u}$ and Δu. It follows that U is costly (make use of (6)) as is Δu. It is optimal to set $U = U_0$ and $\Delta u = 0$. In other words, it is optimal not to give up a rent to the agent, $U = U_0$, nor a contingent contract, $\Delta u = 0$ and the power of incentives is null in the complete information contract. Instead, the agent receives a full insurance contract since $\Delta u = 0 \Leftrightarrow \overline{u} = \underline{u}$. The first-best payment is thus $w(\underline{u})$, with $\underline{u} = U_0 + \psi$ and the first-best marginal cost to induce a positive probability of success corresponds to the marginal payment $w'(\underline{u})\psi_1$ in (FB).

5 Moral hazard

In this framework, the agent's efficiency is still observable, but the action is not. The principal faces a moral-hazard problem. The incentive question requires the agent to be obedient. Using Assumption **1**, the agent faced with an incentive contract $\langle U, \Delta u\rangle$ chooses to generate the probability of success

$$\begin{aligned} p(\Delta u, \theta) \quad &= \quad \arg\max_{\rho} \{\underline{u} + \rho\Delta u - \psi(\rho, \theta)\} \\ &\Rightarrow \Delta u = \psi_1(p(\Delta u, \theta), \theta) > 0. \end{aligned} \tag{7}$$

Equation (7) represents the moral-hazard incentive constraint. Now, the power of incentives must be strictly positive to ensure a positive action from the agent.

Thus (6) becomes

$$\begin{cases} \underline{u} = U + \psi(p(\Delta u, \theta), \theta) - p(\Delta u, \theta)\Delta u, \\ \bar{u} = U + \psi(p(\Delta u, \theta), \theta) + \big(1 - p(\Delta u, \theta)\big)\Delta u. \end{cases} \tag{8}$$

Given (8), the problem is $\max_{(\Delta u, U)}(4)$ subject to (PC).

We can state the following lemma.

Lemma 3. *The moral-hazard contract entails*

- $U^{MH}(\theta) = U_0,$ $\qquad (MH1)$
- $\Delta u^{MH}(\theta)$ *such that*

$$\begin{aligned} \Delta x = &\big(w(\Delta u^{MH}(\theta) + \underline{u}^{MH}(\theta)) - w(\underline{u}^{MH}(\theta))\big) \\ &+ p(\Delta u^{MH}(\theta), \theta)\big(1 - p(\Delta u^{MH}(\theta), \theta)\big) \times \\ &\quad \big(w'(\Delta u^{MH}(\theta) + \underline{u}^{MH}(\theta)) - w'(\underline{u}^{MH}(\theta))\big)\psi_{11}(p(\Delta u^{MH}(\theta), \theta), \theta), \end{aligned} \tag{MH2}$$

with

$$\underline{u}^{MH}(\theta) = U^{MH}(\theta) + \psi(p(\Delta u^{MH}(\theta), \theta), \theta) - p(\Delta u^{MH}(\theta), \theta)\Delta u^{MH}(\theta). \tag{MH3}$$

Proof. See appendix 9.2. □

This Lemma deserves some comments because the marginal payment on the right-hand side of $(MH2)$ differs from complete information. It is composed of two terms. To induce a positive action despite its non-observability, the power of incentives can no longer be set to 0, i.e. $\Delta u > 0$ (see (7)). This forces the principal to offer a contingent contract or high-powered incentives to the agent. Thus, the principal must give up the bonus to the agent, or the contingent component of the contract. This is the first term on the right-hand side of $(MH2)$ (see Lemma 1). It represents the high-powered incentives marginal cost (HPIMC).

In parallel, as with complete information, the agent does not get a positive rent, $U = U_0$ in $(MH1)$, because it is costly for the principal. However, because the contract is contingent, the agent bears some risk. It follows that the moral-hazard payment is higher than the first-best payment. Indeed, for a given ρ, we have

$$\begin{aligned}
w(\underline{u}) + \rho\big(w(\overline{u}) - w(\underline{u})\big) &= \big(1-\rho\big)w(\underline{u}) + \rho w(\overline{u}) \\
&> w((1-\rho)\underline{u} + \rho\overline{u}), \text{ by Jensen's inequality} \\
&= w(\underline{u} + \rho\Delta u) \\
&= w(U_0 + \psi), \text{ because } U = U_0 \text{ and } \Delta u = 0 \text{ with complete information.}
\end{aligned}$$

This higher payment arises because the principal has to pay a risk-premium to the agent to ensure that the participation constraint is binding despite the risk borne. The second term on the right-hand side of ($MH2$) is the risk-premium marginal cost (RPMC), given that $\Delta u = \psi_1$ from (7).

All in all, the right-hand side of ($MH2$) is the moral-hazard marginal cost to induce a positive probability of success. It differs from the first-best. So the efficient action is distorted to take into account the fact that a contingent contract implies a moral-hazard cost. This is the standard insurance-efficiency trade-off in moral-hazard problems with a risk-averse agent.

6 Mixed model

In this framework, the principal observes neither the agent's efficiency nor the agent's action. Following Myerson (1982), there is no loss of generality in focusing on direct revelation mechanisms. The contract offered by the principal is then $\langle U(\hat{\theta}), \Delta u(\hat{\theta})\rangle$, where $\hat{\theta}$ is the agent's report on his efficiency.

From the moral-hazard section, we know that the agent chooses to generate a probability of success such that

$$\begin{aligned}
p(\Delta u(\hat{\theta}), \theta) \quad &= \quad \arg\max_{\rho}\{\underline{u}(\hat{\theta}) + \rho\Delta u(\hat{\theta}) - \psi(\rho, \theta)\} \\
&\Rightarrow \Delta u(\hat{\theta}) = \psi_1(p(\Delta u(\hat{\theta}), \theta), \theta) > 0.
\end{aligned} \tag{9}$$

Let us denote by $\upsilon(\hat{\theta}, \theta)$ the indirect expected utility of an agent with efficiency θ who reports $\hat{\theta}$. We have

$$\upsilon(\hat{\theta}, \theta) = \underline{u}(\hat{\theta}) + p(\Delta u(\hat{\theta}), \theta)\Delta u(\hat{\theta}) - \psi(p(\Delta u(\hat{\theta}), \theta), \theta).$$

Hence, the incentive constraint is $\forall\hat{\theta},\theta\in\Theta$

$$U(\theta)=\upsilon(\theta,\theta)\geq\upsilon(\hat{\theta},\theta). \tag{IC}$$

That is, the agent is better off reporting the truth about his efficiency.
The participation constraint is $\forall\theta\in\Theta$

$$U(\theta)\geq U_0. \tag{PC'}$$

Moreover (8) becomes

$$\begin{cases}\underline{u}(\theta)=U(\theta)+\psi(p(\Delta u(\theta),\theta),\theta)-p(\Delta u(\theta),\theta)\Delta u(\theta),\\ \bar{u}(\theta)=U(\theta)+\psi(p(\Delta u(\theta),\theta),\theta)+\big(1-p(\Delta u(\theta),\theta)\big)\Delta u(\theta).\end{cases} \tag{10}$$

Given (10) and using (4), the principal's problem is

$$\max_{\Delta u(.),U(.)}\int_{\underline{\theta}}^{\overline{\theta}}\big\{\underline{x}+p(\Delta u(\theta),\theta)\Delta x-w(\underline{u}(\theta))-p(\Delta u(\theta),\theta)\big(w(\overline{u}(\theta))-w(\underline{u}(\theta))\big)\big\}f(\theta)d\theta, \tag{11}$$

subject to (IC) and (PC').

Let us begin to resolve this problem by a reformulation. The following lemma characterizes necessary and sufficient conditions for (IC).

Lemma 4. *(Ollier and Thomas (2013)). The allocation $\langle U(\theta),\Delta u(\theta)\rangle$ is incentive compatible if and only if, $\forall\theta\in\Theta$*

$$U'(\theta)=-\psi_2(p(\Delta u(\theta),\theta),\theta), \tag{$IC1$}$$

$$\Delta u'(\theta)\geq 0. \tag{$IC2$}$$

To ensure revelation, the constraint $(IC1)$ tells us that the agent's expected utility, $U(\theta)$, must follow the path $U'(\theta)=-\psi_2(p(\Delta u(\theta),\theta),\theta)$. Thus, using Property 1, it is increasing with efficiency. Moreover, following $(IC2)$, the principal must ensure that the power of incentives, $\Delta u(\theta)$, increases with the type. This is the implementability condition.

Next, we already know that U is costly for the principal. Since it is increasing with the agent's efficiency, it is optimal for the principal not to give up an informational rent to the least efficient agent. That is

$$U(\underline{\theta})=U_0. \tag{$PC1$}$$

Finally, the principal's problem is $\max_{(\Delta u(.),U(.))}(11)$ s.t. $(IC1)$, $(IC2)$, and $(PC1)$. This is an optimal control problem, where $U(\theta)$ and $\Delta u(\theta)$ are state variables.

For ease of analysis, we begin by assuming a fully separating contract, i.e. $\Delta u'(\theta) > 0, \forall\theta \in \Theta$. This allows us to identify the first-round effects of adding adverse selection to moral hazard. We study the possibility of pooling in a subsequent step.

6.1 Fully separating contract

We can present our first result.

Proposition 1. *Assume a fully separating contract. The mixed contract entails*

- $$U^*(\theta) = U_0 - \int_{\underline{\theta}}^{\theta} \psi_2(p(\Delta u^*(\tau),\tau),\tau)d\tau, \tag{MM1}$$
- $\Delta u^*(\theta)$ *such that*

$$\begin{aligned}\Delta x = {} & \big(w(\Delta u^*(\theta) + \underline{u}^*(\theta)) - w(\underline{u}^*(\theta))\big) \\ & + p(\Delta u^*(\theta),\theta)\big(1 - p(\Delta u^*(\theta),\theta)\big)\times \\ & \qquad \big(w'(\Delta u^*(\theta) + \underline{u}^*(\theta)) - w'(\underline{u}^*(\theta))\big)\psi_{11}(p(\Delta u^*(\theta),\theta),\theta) \\ & \qquad - \frac{\int_{\theta}^{\overline{\theta}} \big\{w'(\underline{u}^*(\tau)) + p(\Delta u^*(\tau),\tau)\big(w'(\Delta u^*(\tau) + \underline{u}^*(\tau)) - w'(\underline{u}^*(\tau))\big)\big\} f(\tau)d\tau}{f(\theta)}\times \\ & \qquad \psi_{12}(p(\Delta u^*(\theta),\theta),\theta), \end{aligned} \tag{MM2}$$

with

$$\underline{u}^*(\theta) = U^*(\theta) + \psi(p(\Delta u^*(\theta),\theta),\theta) - p(\Delta u^*(\theta),\theta)\Delta u^*(\theta). \tag{MM3}$$

Proof. See appendix 9.3. ☐

Several comments can be made.

Marginal cost to induce a positive probability of success. When the principal does not observe the agent's type, she needs to elicit truthful information. To do this, she must give up an informational rent to the agent, $U > U_0$, except at $\underline{\theta}$ (see $(MM1)$). Such a rent implies that the marginal cost to induce a positive probability of success is modified compared to pure moral hazard (i.e. the right-hand side of $(MH2)$).

The first change is the presence of the last term in $(MM2)$, which reflects the marginal cost of the informational rent and is usual in adverse-selection problems. It is composed of two factors. Because U is costly and must satisfy $(IC1)$, i.e. $U' = -\psi_2$, it is optimal to moderate its slope, through the probability of success, to reduce the informational rent. This corresponds to ψ_{12}. The second factor is the shadow cost of U weighted by $\frac{1}{f}$. Indeed, to benefit from the participation of the type θ, the principal must increase the informational rent, and so the expected payment, $w(\underline{u}) + \rho\big(w(\overline{u}) - w(\underline{u})\big)$ (make use of (10)), of all agents with higher efficiency. Thus, the shadow cost is $-\int_\theta^{\overline{\theta}} \{w'(\underline{u}) + p(w'(\Delta u + \underline{u}) - w'(\underline{u}))\} f d\tau$.[7]

The second modification arises because U is higher than U_0. Therefore, using $(MH2)$, $(MM2)$, and $(MM3)$, adverse selection also influences the moral-hazard marginal cost. Combining the right-hand sides of $(MH2)$ and $(MM2)$ with $(MH3)$ and $(MM3)$, the difference in marginal costs is given by

$$\int_{U_0}^{U} \int_0^{\Delta u} \{w''(\gamma + \epsilon + \psi - p\Delta u) + p\big(1-p\big) w'''(\gamma + \epsilon + \psi - p\Delta u)\psi_{11}\} d\gamma d\epsilon. \qquad (12)$$

Thus, the informational rent influences the HPIMC through w'' and the RPMC through $p(1-p)w'''\psi_{11}$. Consider first the HPIMC. The effect depends on w'', or equivalently on $-u''$ using Property 2 since $u' > 0$. This is positive. Indeed, due to risk aversion, the bonus incentivizing the agent to bear a risk is even more costly when the agent obtains a higher utility in the event of failure. But, using $(MH3)$ and $(MM3)$, the informational rent increases such utility compared with pure moral hazard. Thus, adverse selection implies that the HPIMC is increased.

Then, consider the RPMC. Since $\psi_{11} > 0$ from Property 1, the effect of the informational rent depends on the sign of w''', or equivalently on $3r_A - r_P$ using Property 2. So the agent's risk aversion and prudence have a key role. In fact, a rent is equivalent to transferring the risk borne by the agent toward higher utility levels than the reservation utility. This constrains the principal to pay a risk premium that guarantees a higher utility in the mixed model than in the moral-hazard model. At the margin, this is costly for the principal because of risk aversion. The term $3r_A$ reflects this phenomenon. The role of prudence is more

[7]Since the agent is not risk-neutral, the shadow cost does not reduce to $-\int_\theta^{\overline{\theta}} f d\tau = -(1-F)$, as with risk-neutrality in the canonical adverse-selection model.

complex. According to decision theory, the transfer of the risk toward higher levels of utility is not welcomed by an imprudent agent, but the contrary is true for a prudent one. So, in the former case, i.e. $-r_P > 0$, imprudence makes the transfer of risk costly, as does risk aversion. The RPMC increases in the presence of adverse selection. In the latter case, i.e. $-r_P < 0$, risk aversion and prudence work in opposite directions. Thus prudence reduces the RPMC contrarily to risk aversion. The ultimate effect depends on the level of prudence. Using Definition **1**, when the agent is weakly prudent, risk aversion dominates prudence and the informational rent still increases the RPMC because $3r_A - r_P$ remains positive. But when the agent is moderately or highly prudent, adverse selection decreases the RPMC since $3r_A - r_P < 0$.

Finally, the overall effect of adverse selection on the moral-hazard marginal cost depends on the sign of the integrand in (12), or equivalently using Property 2, of (since $u' > 0$)

$$-(u''u'^2 + p(1-p)(u'''u' - 3u''^2)\psi_{11})$$

or

$$\frac{u'}{p(1-p)\psi_{11}} + 3r_A - r_P.$$

by factoring $-u''u'$ and making use of Property 1, Definition **1**, and $u''u' < 0$.

From the above analysis, we can conclude that in presence of adverse selection

a. if the agent is imprudent, (12) is positive. Risk aversion and prudence both contribute to increasing the moral-hazard marginal cost,

b. if the agent is weakly or moderately prudent, (12) is still positive. But prudence alleviates the increase in moral-hazard marginal cost due to risk aversion,

c. if the agent is highly prudent, (12) is negative. So, a reduction in the moral-hazard marginal cost occurs because prudence offsets risk aversion.

Altogether, the right-hand side of ($MM2$) is the mixed-model marginal cost to induce a positive probability of success.

Distortions with respect to moral hazard. These two modifications in the marginal cost to induce a positive probability of success imply two distortions in the probability of success, compared to its moral-hazard level. To see this, notice that $(MM2)$ is equivalent to

$$\Delta x + \frac{\int_{\theta}^{\overline{\theta}} \{w'(\underline{u}) + p\big(w'(\overline{u}) - w'(\underline{u})\big)\} f d\tau}{f(\theta)} \psi_{12}(p,\theta)$$
$$= \big(w(\overline{u}) - w(\underline{u})\big) + p\big(1-p\big)\big(w'(\overline{u}) - w'(\underline{u})\big)\psi_{11}(p,\theta). \qquad (MM2')$$

We know that $\Delta x + \frac{\int_{\theta}^{\overline{\theta}} \{w'(\underline{u}) + p(w'(\Delta u + \underline{u}) - w'(\underline{u}))\} f d\tau}{f} \psi_{12} \leq \Delta x$, with equality holding at $\overline{\theta}$. So comparing $(MH2)$ and $(MM2')$, the first modification forces a reduction in p because the moral-hazard marginal cost is increasing in p.[8] This reduction is standard in adverse-selection problems and reflects the usual rent extraction efficiency trade-off: this tends to distort the probability downward from its reference level (here its moral-hazard level) in order to reduce the informational rent of more efficient agents.

Moreover, the second modification is due to the increase in U. As shown above, this can increase (cases a. and b.) or reduce (case c.) the moral-hazard marginal cost. In cases a. and b., adverse selection contributes to a second reduction in p because the transfer of risk toward higher levels of utility involved makes the insurance-efficiency trade-off worse. In case c., adverse selection leads to an opposite effect and tends to increase p. In this case, the informational rent combined with the high level of prudence softens the trade-off due to moral hazard.

These distortions reflect the overall trade-off between rent extraction, insurance, and efficiency. More specifically, we observe that the adverse-selection cost acts through two channels. The first distortion implies that it is added to the moral-hazard cost. The second distortion shows that it can reinforce or mitigate the moral-hazard trade-off.

Using the above analysis, we can conclude in the following corollary that adverse-selection can have a (somewhat) surprising increasing effect on the probability of success, despite the informational rent cost involved.

Corollary 1. *Consider a highly prudent agent. The mixed probability of success $p(\Delta u^*(\theta), \theta)$ is higher than the moral-hazard probability $p(\Delta u^{MH}(\theta), \theta)$ if the agent is sufficiently prudent, that is, an agent for whom the decrease in the moral-hazard marginal cost is higher than the*

[8] By concavity of the moral-hazard problem, the moral-hazard marginal cost increases in p. See equation (**??**) in appendix 9.2.

marginal cost of the informational rent.

Otherwise, it is lower.

Moral-hazard cost versus adverse-selection cost. From the preceding discussion, it follows that the nature of distortions is not equivalent among $[\underline{\theta}, \overline{\theta}]$. Indeed, when θ is close to $\underline{\theta}$, we have $U \simeq U_0$ and $-\frac{\int_{\theta}^{\overline{\theta}}\{w'(\underline{u})+p(w'(\Delta u+\underline{u})-w'(\underline{u}))\}fd\tau}{f}\psi_{12} \gg 0$. So the distortion associated with inefficient types is due rather to the rent extraction-efficiency trade-off. The principal is more concerned about the adverse-selection cost. The probability of success is distorted to limit the informational rent. The reverse is true for θ close to $\overline{\theta}$ since $U \gg U_0$ and $-\frac{\int_{\theta}^{\overline{\theta}}\{w'(\underline{u})+p(w'(\Delta u+\underline{u})-w'(\underline{u}))\}fd\tau}{f}\psi_{12} \simeq 0$. The distortion associated with efficient types occurs rather from the insurance-efficiency trade-off, because the principal is more interested in the moral-hazard cost. The distortion arises from the desire to limit the cost to induce a positive probability of success, i.e. the high-powered incentives and the risk-premium. It follows that in presence of adverse selection, each moral-hazard probability of success is distorted. In particular, even if the marginal cost of the informational rent is null for the highest type, because $\frac{\int_{\theta}^{\overline{\theta}}\{w'(\underline{u})+p(w'(\Delta u+\underline{u})-w'(\underline{u}))\}fd\tau}{f} = 0$ at $\theta = \overline{\theta}$, there is a distortion at the top. This contradicts the well-known result of pure adverse selection.

The following corollary summarizes this phenomenon.

Corollary 2. *Moving from the lowest to the highest agent's efficiency, the mixed contract replaces distortions due to adverse-selection cost by distortions due to moral-hazard cost.*

6.2 Partially separating contract

The difference in the nature of distortions raises the question whether the assumption of full separation in Proposition 1 is relevant. Let $\eta(\theta)$ represent the shadow cost of U, $\eta(\theta) = -\int_{\theta}^{\overline{\theta}}\{w'(\underline{u}) + p(w'(\Delta u + \underline{u}) - w'(\underline{u}))\}fd\tau$. The following proposition answers this question.

Proposition 2. *The optimal adverse-selection moral-hazard contract is separating if,* $\forall\theta \in$

$[\underline{\theta}, \overline{\theta}]$

$$\begin{aligned} & p_{12}(\Delta u^*(\theta), \theta)\big(\Delta x - \big(w(\overline{u}^*(\theta)) - w(\underline{u}^*(\theta))\big)\big) \\ & \quad - p_2(\Delta u^*(\theta), \theta)\big(1 - 2p(\Delta u^*(\theta), \theta)\big)\big(w'(\overline{u}^*(\theta)) - w'(\underline{u}^*(\theta))\big) \\ & \quad + p_2(\Delta u^*(\theta), \theta)\left(\frac{\eta^*(\theta)}{f}\right)' + p_{22}(\Delta u^*(\theta), \theta)\frac{\eta^*(\theta)}{f(\theta)} > 0. \end{aligned} \tag{13}$$

Proof. See appendix 9.4. □

Two comments can be made. First, this proposition highlights that pooling in this mixed framework can have two sources. The first source is the common-value nature of the model. So non-responsiveness can occur. That is, $\Delta u^{MH}(\theta)$ does not satisfy the incentive constraint $(IC2)$. In this case, the principal would like to implement a power of incentives decreasing in θ for reasons of moral hazard. Thus, a conflict appears with the implementability condition. The first two terms in (13) reflect this phenomenon. But, because $\underline{u}$ depends on $U > U_0$ when there is adverse selection, the conflict between the principal's preference and the monotonicity condition $(IC2)$ is somewhat modified compared to moral-hazard alone. The second source is due to the possible lack of monotonicity of the marginal cost of the informational rent. In such a situation, this cost is not ranked exactly as the agent's efficiency and the principal would like to implement a decreasing power of incentives over some interval. Again, this conflicts with the implementability condition. This concerns the final two terms in (13). This effect is well-known in pure adverse-selection models. But, because the agent is risk-averse, the standard monotone hazard rate property, i.e. $\frac{1-F}{f}$ non-increasing in θ, is no longer sufficient to avoid non-monotonicity.

Second, examining (13), we observe that since $p_2 = -\frac{\psi_{12}}{\psi_{11}} > 0$ and $w'' > 0$ (see Property 2), a probability less than $\frac{1}{2}$ contributes to the non-responsiveness of the model. The reverse is true if $p \geq \frac{1}{2}$. Moreover, since $\Delta x - \big(w(\overline{u}) - w(\underline{u})\big) > 0$, $\eta' > 0$ and $\eta < 0$, (13) is more easily satisfied when

$$p_{12} \geq 0 \text{ and } p_{22} \leq 0. \tag{14}$$

These conditions structure the function ψ even more than Property 1 and Assumption **1** do.

Indeed, we get

$$p_{12}(\Delta u,\theta)=\frac{\psi_{111}(p(\Delta u,\theta),\theta)\psi_{12}(p(\Delta u,\theta),\theta)-\psi_{112}(p(\Delta u,\theta),\theta)\psi_{11}(p(\Delta u,\theta),\theta)}{\psi_{11}^3(p(\Delta u,\theta),\theta)},$$

$$\begin{aligned}p_{22}(\Delta u,\theta)=&\frac{\psi_{112}(p(\Delta u,\theta),\theta)\psi_{12}(p(\Delta u,\theta),\theta)-\psi_{212}(p(\Delta u,\theta),\theta)\psi_{11}(p(\Delta u,\theta),\theta)}{\psi_{11}^2(p(\Delta u,\theta),\theta)}\\&-\frac{\psi_{12}(p(\Delta u,\theta),\theta)}{\psi_{11}(p(\Delta u,\theta),\theta)}\frac{\psi_{111}(p(\Delta u,\theta),\theta)\psi_{12}(p(\Delta u,\theta),\theta)-\psi_{112}(p(\Delta u,\theta),\theta)\psi_{11}(p(\Delta u,\theta),\theta)}{\psi_{11}^2(p(\Delta u,\theta),\theta)}.\end{aligned}$$

So a separating contract is subject to a combination of many second and third partial derivatives of the indirect disutility ψ. Thus, except maybe for very simple functions ψ, one can reasonably have doubts about the existence of a contract that strictly satisfies $(IC2)$ for all types. In this case, the monotonicity constraint is binding on some intervals.

We state the shape of the optimal mixed contract in the following proposition.

Proposition 3. *Consider a single interior interval $[\theta_0,\theta_1]$ where there is pooling. The mixed contract entails*

- $U^{**}(\theta)=U_0-\displaystyle\int_{\underline{\theta}}^{\theta}\psi_2(p(\Delta u^{**}(\tau),\tau),\tau)d\tau,$
- $\Delta u^{**}(\theta)$ *equal to*
  - $\Delta u^*(\theta)$ *if* $\theta\in[\underline{\theta},\theta_0]\cup[\theta_1,\overline{\theta}]$,
  - Δu^k *if* $\theta\in[\theta_0,\theta_1]$, *with*

$$\begin{aligned}\Delta x-\int_{\theta_0}^{\theta_1}\Big\{&\big(w(\Delta u^k+\underline{u}^{**}(\theta))-w(\underline{u}^{**}(\theta))\big)\\&-p(\Delta u^k,\theta)\big(1-p(\Delta u^k,\theta)\big)\times\\&\quad\big(w'(\Delta u^k+\underline{u}^{**}(\theta))-w'(\underline{u}^k(\theta))\big)\psi_{11}(p(\Delta u^k,\theta),\theta)\\&+\frac{\int_\theta^{\overline{\theta}}\left\{w'(\underline{u}^{**}(\tau))+p(\Delta u^k,\theta)\big(w'(\Delta u^k+\underline{u}^{**}(\tau))-w'(\underline{u}^{**}(\tau))\big)\right\}f(\tau)d\tau}{f(\theta)}\times\\&\quad\psi_{12}(p(\Delta u^k,\theta),\theta)\Big\}f(\theta)d\theta=0\end{aligned}$$

and $\qquad \underline{u}^{**}(\theta)=U^{**}(\theta)+\psi(p(\Delta u^{**}(\theta),\theta),\theta)-p(\Delta u^{**}(\theta),\theta)\Delta u^{**}(\theta).$

This proposition shows that when pooling arises, the optimal contract consists in verifying $(MM2)$ on average, and no longer pointwise. This result is well-known in pure adverse-selection models.

In the light of Propositions 2 and 3, we observe that pooling in the mixed contract has no different causes and consequences from those already known in adverse-selection models alone. However, pooling is most likely to emerge since the conditions to avoid pooling are more difficult to satisfy because of the different kinds of incentives the principal has to manage and the overall trade-off between rent extraction, insurance, and efficiency involved.

7 An example

It is important to notice that the optimal power of incentives Δu^* defined in Proposition 1 is the solution of a non-linear integral state equation. Thus, the goal of this example is not to find an explicit solution but to give a simple setting in order (1) to have a concrete overview of what distortions can be, (2) to see whether the conditions ensuring the existence of a highly prudent agent or a separating contract can arise.

To do this, let $x = u(y) = (\alpha y)^{\frac{1}{\alpha}}$, with $1 < \alpha \leq 2$.[9] We get

$$u' = \alpha^{\frac{1-\alpha}{\alpha}} y^{\frac{1-\alpha}{\alpha}}; u'' = \frac{1-\alpha}{\alpha} \alpha^{\frac{1-\alpha}{\alpha}} y^{\frac{1-2\alpha}{\alpha}}; u''' = \frac{1-2\alpha}{\alpha} \frac{1-\alpha}{\alpha} \alpha^{\frac{1-\alpha}{\alpha}} y^{\frac{1-3\alpha}{\alpha}},$$

so

$$r_A = -\frac{1-\alpha}{\alpha} y^{-1}, r_P = -\frac{1-2\alpha}{\alpha} y^{-1},$$

and

$$r_P - 3r_A = \frac{2-\alpha}{\alpha} y^{-1}. \tag{15}$$

Moreover, let $\psi(\rho,\theta) = \frac{\rho^2}{\theta}$. It follows immediately that $\psi_1 = \frac{2\rho}{\theta}, \psi_2 = -\frac{\rho^2}{\theta^2}, \psi_{11} = \frac{2}{\theta}$, and $\psi_{12} = -\frac{2\rho}{\theta^2}$.

7.1 Overview of distortions

In this subsection, we let $\alpha = 2$ and denote $p(\theta) = p(\Delta u(\theta), \theta)$ to simplify. So, we obtain $y = w(x) = \frac{x^2}{2}$, $w'(x) = x$ and $w''(x) = 1$.

[9] Using this utility function requires a positive payment in the event of failure. Since $\rho\psi_1 - \psi = \frac{\rho^2}{\theta} > 0$, $\Delta u = \psi_1$ from (7), and $\underline{u} = U + \psi(\rho,\theta) - \rho\Delta u$ from (6), we assume that $U_0 > \frac{\rho^2}{\theta}$ to ensure that $\underline{u}$, and so a, are positive.

Complete information. Since $U = U_0$, $\Delta u = 0$, and $\underline{u} = \psi + U_0$, (FB) becomes

$$\Delta x = \left(U_0 + \frac{\rho^{FB}(\theta)^2}{\theta} \right) \frac{2\rho^{FB}(\theta)}{\theta}. \tag{16}$$

Moral hazard. We have

$$\begin{aligned} w(\Delta u + \underline{u}) - w(\underline{u}) &= (U_0 + \psi)\psi_1 + \frac{1-2\rho}{2}\psi_1^2 \\ &= \left(U_0 + \frac{\rho^2}{\theta} \right) \frac{2\rho}{\theta} + \frac{1-2\rho}{2} \left(\frac{2\rho}{\theta} \right)^2, \\ w'(\Delta u + \underline{u}) - w'(\underline{u}) &= \psi_1 = \frac{2\rho}{\theta}. \end{aligned}$$

So $(MH2)$ becomes

$$\begin{aligned} \Delta x = &\left(U_0 + \frac{p^{MH}(\theta)^2}{\theta} \right) \frac{2p^{MH}(\theta)}{\theta} \\ &+ \frac{1-2p^{MH}(\theta)}{2} \left(\frac{2p^{MH}(\theta)}{\theta} \right)^2 + p^{MH}(\theta)(1-p^{MH}(\theta)) \left(\frac{2p^{MH}(\theta)}{\theta} \right) \left(\frac{2}{\theta} \right). \end{aligned} \tag{17}$$

Comparing (16) and (17), the distortion due to moral hazard is

$$\frac{1-2p}{2} \left(\frac{2p}{\theta} \right)^2 + p(1-p) \left(\frac{2p}{\theta} \right) \left(\frac{2}{\theta} \right).$$

Mixed model. Since $-\int_{\underline{\theta}}^{\theta} \psi_2 d\tau = \int_{\underline{\theta}}^{\theta} \frac{\rho^2}{\tau^2} d\tau$, we have

$$\begin{aligned} w(\Delta u + \underline{u}) - w(\underline{u}) &= (U + \psi)\psi_1 + \frac{1-2\rho}{2}\psi_1^2 \\ &= \left(U_0 + \int_{\underline{\theta}}^{\theta} \frac{\rho^2}{\tau^2} d\tau + \frac{\rho^2}{\theta} \right) \frac{2\rho}{\theta} + \frac{1-2\rho}{2} \left(\frac{2\rho}{\theta} \right)^2, \\ w'(\Delta u + \underline{u}) - w'(\underline{u}) &= \psi_1 = \frac{2\rho}{\theta}, \\ w'(\underline{u}) + \rho(w'(\Delta u + \underline{u}) - w'(\underline{u})) &= U + \psi \\ &= U_0 + \int_{\underline{\theta}}^{\theta} \frac{\rho^2}{\tau^2} d\tau + \frac{\rho^2}{\theta}. \end{aligned}$$

So, using Proposition 1, the optimal probability of success is such that

$$\begin{aligned}\Delta x = &\left(U_0 + \frac{p^*(\theta)^2}{\theta}\right)\frac{2p^*(\theta)}{\theta} \\ &+ \frac{1-2p^*(\theta)}{2}\left(\frac{2p^*(\theta)}{\theta}\right)^2 + p^*(\theta)(1-p^*(\theta))\left(\frac{2p^*(\theta)}{\theta}\right)\left(\frac{2}{\theta}\right) \\ &+ \int_{\underline{\theta}}^{\theta}\frac{p^*(\tau)^2}{\tau^2}d\tau\frac{2p^*(\theta)}{\theta} \\ &+ \frac{2p^*(\theta)}{\theta}\frac{\int_{\theta}^{\overline{\theta}}\left(U_0 + \int_{\underline{\theta}}^{\epsilon}\frac{p^*(\epsilon)^2}{\epsilon^2}d\epsilon + \frac{p^*(\theta)^2}{\theta}\right)f(\tau)d\tau}{f(\theta)}. \end{aligned} \tag{18}$$

If we compare (18) to (17), the two last lines of the preceding equation reflect the distortions due to adverse selection. The first term is the increase in the moral hazard marginal cost because the agent is weakly prudent (i.e. $r_P - 3r_A = 0$ when $\alpha = 2$ from (15)). The last term is the marginal cost of the informational rent. So the mixed model probability of success is distorted twice downward from its moral-hazard level.

7.2 Highly prudent agent

According to Definition **1** and (15), a highly prudent agent satisfies

$$\begin{aligned} y^{-1}\left(\frac{2-\alpha}{\alpha}\right) &> \frac{\alpha^{\frac{1-\alpha}{\alpha}}y^{\frac{1-\alpha}{\alpha}}}{p(1-p)\frac{2}{\theta}} \\ \Leftrightarrow \frac{2-\alpha}{\alpha} &> \frac{\alpha^{\frac{1-\alpha}{\alpha}}y^{\frac{1}{\alpha}}}{p(1-p)\frac{2}{\theta}}. \end{aligned} \tag{19}$$

Taking the limit of (19) when $\alpha \to 1$, we get

$$1 > \frac{y}{p(1-p)\frac{2}{\theta}}. \tag{20}$$

Moreover, from (12), we obtain $\gamma \in [0, \Delta u] \Rightarrow u \in [\underline{u}, \overline{u}] \Rightarrow y \in [a, a+b]$. So if there exists a contract $\langle a, a+b\rangle$, implying a probability of success p and a payment y between a and $a+b$ such that (20) is satisfied, then the agent is highly prudent. The presence of adverse selection will lead to a decrease in the moral-hazard marginal cost. Sufficiently efficient agents are thus induced to generate a higher probability of success in the mixed model than in the pure moral hazard setting.

7.3 Separating contract

According to (9), the agent chooses to generate a probability of success such that

$$p(\Delta u(\hat{\theta}), \theta) = \frac{\Delta u(\hat{\theta})\theta}{2}. \tag{21}$$

It follows immediately that $p_{12} = \frac{1}{2}$ and $p_{22} = 0$. Thus the conditions in (14) are satisfied. They contribute to obtaining a separating contract.

8 Conclusion

In this paper, we have studied the contract between a risk-neutral principal and a risk-averse agent who has private information about his efficiency and action that both improve the probability of success. In a two-output model, we have shown that if the agent is highly prudent and sufficiently efficient, the principal induces a higher probability of success than under moral-hazard, despite the costly informational rent given up. Moreover the conditions for avoiding pooling are difficult to satisfy because of the different kinds of incentives to be managed and the overall trade-off between rent extraction, insurance, and efficiency involved.

Two natural extensions would be interesting. The first one would be to consider more than two outputs, even a continuum. The complexity arises from finding a tractable form of the moral-hazard marginal cost to be able to analyze the influence of the informational rent on the moral-hazard trade-off. Secondly, this two-output model can be used to investigate an insurance relationship. However, such a contract must consider an outside option corresponding to the expected utility obtained by the agent when he does not purchase insurance. The difficulty is thus to measure the influence of the outside option on the contract offered by the principal.

9 Appendix

For the sake of simplicity in the appendix, we focus on interior solutions.

9.1 Proof of Lemma 2

In this informational setting, an interior solution means that $\Delta u \in \mathbb{R}$ and $\rho \in (0,1)$.

We denote μ the Kuhn and Tucker multiplier associated to (PC). The Lagrangian is

$$\begin{aligned} L = \underline{x} + \rho\Delta x - w(U + \psi - \rho\Delta u) \\ - \rho(w(U + \psi + (1-\rho)\Delta u) - w(U + \psi - \rho\Delta u))) + \mu(U - U_0). \end{aligned} \tag{22}$$

Necessary conditions are

$$\frac{\partial L}{\partial \rho} = \Delta x - w'(\underline{u})(\psi_1 - \Delta u) - (w(\Delta u + \underline{u}) - w(\underline{u}))$$
$$- \rho(w'(\Delta u + \underline{u})(\psi_1 - \Delta u) - w'(\underline{u})(\psi_1 - \Delta u)) = 0 \tag{23}$$
$$\frac{\partial L}{\partial U} = -w'(\underline{u}) - \rho(w'(\Delta u + \underline{u}) - w'(\underline{u})) + \mu = 0 \tag{24}$$
$$\frac{\partial L}{\partial \Delta u} = w'(\underline{u})\rho - \rho(w'(\Delta u + \underline{u})(1-\rho) + w'(\underline{u})\rho) = 0 \tag{25}$$
$$\mu \geq 0, \mu(U - U_0) = 0. \tag{26}$$

From (25), we have

$$w'(\Delta u + \underline{u}) = w'(\underline{u}) \Rightarrow \Delta u = 0.$$

Then, plugging this result into (24), we have $\mu = w'(\underline{u}) > 0$ using Property 2. So, from (26), we get $U = U_0$. After simplifications in (23), the probability $\rho^{FB}(\theta)$ is given by (FB).

9.2 Proof of Lemma 3

Looking for $\rho \in (0,1)$, the Lagrangian is similar to (22),

$$\begin{aligned} L &= \underline{x} + p\Delta x - w(U + \psi - p\Delta u) \\ &\quad -p(w(U + \psi + (1-p)\Delta u) - w(U + \psi - p\Delta u)) + \mu(U - U_0). \end{aligned}$$

Necessary conditions. Given (7), the necessary conditions are

$$\frac{\partial L}{\partial U} = -w'(\underline{u}) - p(w'(\Delta u + \underline{u}) - w'(\underline{u})) + \mu = 0 \tag{27}$$
$$\frac{\partial L}{\partial \Delta u} = p_1\Delta x + w'(\underline{u})p - p_1(w(\Delta u + \underline{u}) - w(\underline{u}))$$
$$- p(w'(\Delta u + \underline{u})(1-p) + w'(\underline{u})p) = 0 \tag{28}$$
$$\mu \geq 0, \mu(U - U_0) = 0. \tag{29}$$

From (27), we have $\mu > 0$, since $w'' > 0$ by Property 2 and $\Delta u > 0$ by (7). Then $U = U_0$ using (29). Moreover, collecting terms in (28), we get $(MH2)$, since $p_1 = \frac{1}{\psi_{11}}$ from (7).

Sufficient conditions. Since the constraint is linear in U, necessary conditions are sufficient if V in (4) is concave in $(\Delta u, U)$. We need to verify

$$\frac{\partial^2 V}{\partial U^2} = -pw''(\Delta u + \underline{u}) - (1-p)w''(\underline{u}) < 0 \tag{30}$$

$$\begin{aligned}\frac{\partial^2 V}{\partial \Delta u^2} = p_{11}\big(\Delta x - \big(w(\Delta u + \underline{u}) - w(\underline{u})\big)\big) \\ - p_1\big(w'(\Delta u + \underline{u})(1-p) + w'(\underline{u})p\big) - p_1(1-2p)\big(w'(\Delta u + \underline{u}) - w'(\underline{u})\big) \\ - p(1-p)\big(w''(\Delta u + \underline{u})(1-p) + w''(\underline{u})p\big) < 0\end{aligned} \tag{31}$$

$$\frac{\partial^2 V}{\partial \Delta u^2}\frac{\partial^2 V}{\partial U^2} - \left(\frac{\partial^2 V}{\partial \Delta u \partial U}\right)^2 \geq 0, \tag{32}$$

with $\frac{\partial^2 V}{\partial \Delta u \partial U} = -p_1(w'(\Delta u + \underline{u}) - w'(\underline{u})) - p(1-p)(w''(\Delta u + \underline{u}) - w''(\underline{u}))$.

Using Property 2, (30) is indeed negative. By contrast, the sign of (31) is not warranted. Recall that, $w' > 0, w'' > 0$ from Property 2, and $p_1 = \frac{1}{\psi_{11}} > 0$ from Property 1, so the second and the fourth terms are negative. But the first and the third are undetermined. Finally, it is difficult to compute the sign of (32). Thus, unfortunately, we cannot be sure that necessary conditions ensure a global maximum.

However, notice that $p_{11} = -\frac{\psi_{111}}{\psi_{11}^3} \leq 0$ from Property 1 and Assumption **1**, and $\Delta x - \big(w(\Delta u + \underline{u}) - w(\underline{u})\big) > 0$ from $(MH2)$. So the first term in (31) is negative at the solution $(MH2)$. Moreover, the third term in (31) is non positive if $p \leq \frac{1}{2}$. It is positive otherwise. So (31) is verified, in particular, as long as p is not too much higher than $\frac{1}{2}$, when $1-2p < 0$. So a local maximum is likely to emerge. But the difficulty to ensure the sign of (32) remains.

9.3 Proofs of Propositions 1 and 3

We omit arguments for the sake of clarity. First, using $(IC1)$ and $(PC1)$, we get

$$U = U_0 - \int_{\underline{\theta}}^{\theta} \psi_2 d\tau.$$

Second, consider the optimal control problem. Let $\Delta u' = y \geq 0$ where y is a control. We associate the adjoint variable η (resp. μ) with U (resp. Δu). The Hamiltonian is

$$H = \big\{\underline{x} + p\Delta x - w(U+\psi - p\Delta u) - p\big(w(U+\psi+(1-p)\Delta u) - w(U+\psi-p\Delta u)\big)\big\}f$$
$$- \eta\psi_2 + \mu y.$$

Necessary and transversality conditions. Using (9) and (10), the maximum principle yields[10]

- as necessary conditions

$$\begin{aligned} \frac{\partial H}{\partial y} &= \mu \le 0; y\frac{\partial H}{\partial y} = y\mu = 0 && (33)\\ \eta' &= -\frac{\partial H}{\partial U} = \big\{w'(\underline{u}) + p\big(w'(\Delta u + \underline{u}) - w'(\underline{u})\big)\big\}f > 0 && (34)\\ \mu' &= -\frac{\partial H}{\partial \Delta u} = -\big\{p_1\Delta x + w'(\underline{u})p - p_1\big(w(\Delta u + \underline{u}) - w(\underline{u})\big) \\ &\qquad -p\big(w'(\Delta u+\underline{u})(1-p) + w'(\underline{u})p\big)\big\}f + \eta\psi_{12}p_1, && (35)\end{aligned}$$

- as transversality conditions

$$\begin{aligned} &\eta(\underline{\theta}) \text{ no condition,}\\ &\eta(\overline{\theta}) = 0. && (36)\end{aligned}$$

From (34) and (36), we have

$$\eta = -\int_\theta^{\overline{\theta}} \big\{w'(\underline{u}) + p\big(w'(\Delta u + \underline{u}) - w'(\underline{u})\big)\big\}f(\tau)d\tau \le 0. \tag{37}$$

Proof of Proposition 1. A fully separating contract, i.e. $\Delta u' > 0$, implies $y > 0$. Using (33), we get $\mu = 0$ on Θ. So μ' is equal to 0 on Θ. Using (35) and (37), Δu is equal to Δu^* since $p_1 = \frac{1}{\psi_{11}}$ from (9). Using (33), if it is increasing, i.e. $y^* > 0$, this is the solution.

Proof of Proposition 3. If $y^* < 0$, y must be set equal to 0, and Δu is constant. Consider this occurs on a single interior interval $[\theta_0, \theta_1]$. By continuity, $\mu(\theta_0)$ and $\mu(\theta_1)$ are both equal to 0. Thus, the constant solution, denoted Δu^k is obtained by integration of (35) between θ_0 and θ_1, knowing that $\Delta u^*(\theta_0) = \Delta u^*(\theta_1) = \Delta u^k$.

[10]See Seierstadt and Sydsaeter (1987).

Sufficient conditions. Sufficient conditions require H to be concave in $(\Delta u, U)$. First, it is straightforward to check that $\frac{\partial^2 H}{\partial U^2} = \left(\frac{\partial^2 V}{\partial U^2}\right) f$ and $\frac{\partial^2 H}{\partial \Delta u \partial U} = \left(\frac{\partial^2 V}{\partial \Delta u \partial U}\right) f$.

Second, let us compute $\frac{\partial^2 H}{\partial \Delta u^2}$. Since $p_1 = \frac{1}{\psi_{11}}$ and $p_2 = -\frac{\psi_{12}}{\psi_{11}}$ from (9), we get using (35)

$$\frac{\partial H}{\partial \Delta u} = (p_1(\Delta x - (w(\Delta u + \underline{u}) - w(\underline{u}))) - p(1-p)(w'(\Delta u + \underline{u}) - w'(\underline{u})))f + \eta p_2. \tag{38}$$

So we have

$$\frac{\partial^2 H}{\partial \Delta u^2} = \left(\frac{\partial^2 V}{\partial \Delta u^2}\right) f + \eta p_{12}.$$

By analogy with the proof of Lemma 3, we cannot warranty the concavity of H. But it is important to notice that since $f > 0$, V concave implies that H is concave for sure if $\eta p_{12} < 0$. Since $\eta \leq 0$, a sufficient condition is $p_{12} > 0$ (see the discussion below Proposition 2). So if a global maximum with moral hazard is ensured, a global maximum with mixed model is.

9.4 Proof of Proposition 2

Using the concavity of H in Δu, differentiating (38) and using $(IC1)$ and (9), we get (13).

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