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CRESE
30, avenue de l'Observatoire
25009 Besançon
France
http://crese.univ-fcomte.fr/

Nullified equal loss property and equal division values

Sylvain Ferrières[a,b]

[a] *CRESE EA3190, Univ. Bourgogne Franche-Comté, F-25000 Besançon, France*
[b] *Chair of Economics and Information Systems, HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany*

Abstract

We provide characterizations of the equal division values and their convex mixtures, using a new axiom on a fixed player set based on player nullification which requires that if a player becomes null, then any two other players are equally affected. Two economic applications are also introduced concerning bargaining under risk and common-pool resource appropriation.

Keywords: Player nullification, equal division, equal surplus division, bargaining under risk, common-pool resource.

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1. Introduction

Reconciling individual and social interests is a common theme in many economics fields. For instance, solutions for bankruptcy problems often possess an egalitarian flavor (see Thomson, 2015, for a recent survey). Similarly, egalitarian considerations are also central in fair division problems as pointed out by Thomson (2011).

Cooperative games with transferable utility (TU-games henceforth) are often used to model analogous allocation situations. A solution for a class of TU-games is called a value and assigns to each TU-game in the class and to each player a payoff for her participation. This article deals with egalitarian solutions by introducing a new axiom for TU-games called the nullified equal loss property. This axiom rests on the nullification operation studied in Béal et al. (2014) and Béal et al. (2016). A player is nullified if the worth of any coalition to which she belongs becomes equal to the worth of the same coalition without the player, i.e. the player is null in the resulting new game. The nullified equal loss property requires that if a player is nullified, then all other players experience the same payoff variation. Our results detailed in the next paragraph suggest that this axiom captures an essential feature of egalitarian values such as the equal division and equal surplus division values, as opposed to marginalistic values such as the Shapley value (Shapley, 1953). These results are in line with a recent and growing literature on the axiomatic characterizations of classes of egalitarian values (van den Brink and Funaki, 2009; van den Brink et al., 2016), their axiomatic comparisons with the Shapley value (van den Brink, 2007; Béal et al., 2015), and axiomatic characterizations of combination of both types of values (Ju et al., 2007; Casajus and Hüttner, 2014).

Email address: `sylvain.ferrieres@univ-fcomte.fr` (Sylvain Ferrières)
URL: `http://crese.univ-fcomte.fr/sylvain_ferrieres.html` (Sylvain Ferrières)

The main results are as follows. Firstly, if two values satisfy the nullified equal loss property and efficiency, and furthermore coincide on the class of additive TU-games, then they are equal for all TU-games (proposition 1). This result provides the principle of a unique extension from additive TU-games to all TU-games. Secondly, proposition 2 extends this principle for values that are linear, symmetric and efficient, and proves that the extended value must be a linear combination of the equal division value and the equal surplus division value. As a corollary, the latter class of values is characterized by linearity, symmetry, efficiency and the nullified equal loss property. Thirdly, the more natural class of convex combinations of the equal division value and equal surplus division value is singled out by invoking efficiency, additivity, the nullified equal loss property together with desirability and superadditive monotonicity (theorem 1). Desirability (Maschler and Peleg, 1966) requires that if a first player contributes not less than a second player to coalitions, then the first player should not obtain a smaller payoff than the second player. Superadditive monotonicity is newly introduced and imposes that the players' payoff are nonnegative in a TU-game that is both superadditive and monotone. The axiom is implied by both monotonicity (Weber, 1988), which does not require the superadditivity of the monotone TU-game, and the axiom of nonnegativity in van den Brink et al. (2016) which imposes nonnegative payoff for nonnegative TU-games in which the grand coalition achieves a worth not less than the sum of the singletons' worth. This class emerges naturally in auction games as a mean for the player who obtains the indivisible good to compensate the other players (see van den Brink, 2007). Interestingly all axioms in theorem 1 except the nullified equal loss property are also satisfied by the Shapley value. This enables comparisons: replacing the nullified equal loss property by the classical null player axiom yields a characterization of the Shapley value, and replacing the nullified equal loss property by the null player in a productive environment (Casajus and Hüttner, 2013) characterizes the egalitarian Shapley values, even if some axioms may be redundant. Fourthly, thanks to proposition 1, an elegant characterization of the equal surplus division value is obtained by adding the well-known inessential game property to efficiency and the nullified equal loss property.

Although there are very few applications of egalitarian solutions for TU-games to economic models, the last part of this article presents two such applications. The first one considers the nullification of a player as a random event in a context of bargaining under risk. It shows that the nullified equal loss property is compatible with non-linear values that incorporate the risk aversion of the players. The second one endogeneizes a choice of a value in a non-cooperative model of common-pool management. It is shown that the unique value which maximizes the social welfare at equilibrium is a specific convex combination of equal division value and equal surplus division value.

The closest axiom to the nullified equal loss property is perhaps nullified solidarity (Béal et al., 2014). Both axioms describe the consequences of a player's nullification with two notable exception: our axiom (a) does not specify the payoff variation for the nullified player, and (b) imposes equal payoff variation for all other players while nullified solidarity requires that all payoffs vary in the same direction. Other characterizations of the convex combinations of equal division value and equal surplus division value are due to van den Brink et al. (2016), while characterizations of the equal surplus division can be found in Chun and Park (2012) and Béal et al. (2015). Our results are given for fixed player sets while player sets can vary in Chun and Park (2012) and van den Brink et al. (2016). The approach by axioms of invariance in Béal et al. (2015) is very different from ours.

The rest of the article is organized as follows: section 2 presents notation and definitions. Section

3 contains the results. Section 4 presents the two applications. Section 5 provides concluding remarks.

2. Basic definitions and notations

2.1. Cooperative games with transferable utility

The cardinality of any set S is denoted by s. Let N be a finite and fixed set of players such that $n \geq 3$. A **TU-game** v on N is a map $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. Define $\mathbb{V}$ as the class of all TU-games on this fixed player set N. $\mathbb{V}$ is endowed with the natural vector space structure. A non-empty subset $S \subseteq N$ is a coalition, and $v(S)$ is the worth of this coalition. For simplicity, we write the singleton $\{i\}$ as i.

The **null game** is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. A TU-game $v \in \mathbb{V}$ is **additive** if for all $S \subseteq N$, $v(S) = \sum_{i \in S} v(i)$. We will denote the class of additive TU-games by $\mathbb{V}_A$. For any TU-game $v \in \mathbb{V}$, let define the 0**-normalized** TU-game v^0 by $v^0(S) = v(S) - \sum_{i \in S} v(i)$ for any $S \subseteq N$ so that any additive TU-game v is characterized by $v^0 = \mathbf{0}$. A TU-game $v \in \mathbb{V}$ is **superadditive** if for all $S, T \subseteq N$ such that $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$. A TU-game $v \in \mathbb{V}$ is **monotone** if for all $S, T \subseteq N$ such that $S \subseteq T$, $v(S) \leq v(T)$. For any nonempty $S \in 2^N$, the **unanimity** TU-game induced by S is denoted by u_S and such that $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. It is well-known that any TU-game $v \in \mathbb{V}$ admits a unique decomposition in the unanimity games basis:

$$v = \sum_{S \in 2^N, S \neq \emptyset} \Delta_S(v) u_S,$$

where $\Delta_S(v)$ is called the Harsanyi dividend of S.

Player $i \in N$ is **null** in $v \in \mathbb{V}$ if $v(S) = v(S \backslash i)$ for all $S \subseteq N$ such that $S \ni i$. Following Béal et al. (2014), for $v \in \mathbb{V}$ and $i \in N$, we denote by v^i the TU-game in which player i is **nullified**: $v^i(S) = v(S \backslash i)$ for all $S \subseteq N$. Note that $(v^i)^j = (v^j)^i$ so that v^S is well-defined by nullifying all players of $S \subseteq N$, in any order. Moreover, if $S, T \subseteq N$, then $(v^S)^T) = v^{S \cup T}$. For any given $v \in \mathbb{V}$, define

$$G(v) = \{v^S, S \subseteq N\}$$

the lattice generated by v using the nullification operation. Note that $v^{\emptyset} = v$. Moreover, $v^N = \mathbf{0}$ and $v^{N \backslash i} = v(i) \cdot u_i$ for any $i \in N$ and these TU-games are additive. At last, note that the nullification operation is compatible with the vector space structure, i.e. for all $v, w \in \mathbb{V}$, $S \subseteq N$ and $\lambda \in \mathbb{R}$, $(v + \lambda w)^S = v^S + \lambda w^S$.

2.2. Values

A **value** on $\mathbb{V}$ is a function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathbb{V}$. For any player $i \in N$, $\varphi_i(v)$ represents her payoff for participating in $v \in \mathbb{V}$. We consider the following values.

The **Equal division value** is the value ED given by:

$$\mathrm{ED}_i(v) = \frac{v(N)}{n} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **Equal surplus division value** is the value ESD given by:

$$\mathrm{ESD}_i(v) = v(i) + \frac{1}{n}\left(v(N) - \sum_{j \in N} v(j)\right) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **Shapley value** (Shapley, 1953) is the value Sh given by:

$$\mathrm{Sh}_i(v) = \sum_{S \ni i} \frac{\Delta_S(v)}{s} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

2.3. Punctual and relational Axioms

In this article, we divide axioms in two categories: punctual axioms if they impose restrictions on the payoff vector of a fixed TU-game, and relational axioms if they impose a particular relation between the payoff vectors of two different but interrelated TU-games. Two new axioms are introduced (one punctual and one relational). Let us recall first classical punctual axioms.

Efficiency, E. For all $v \in \mathbb{V}$, $\sum_{i \in N} \varphi_i(v) = v(N)$.
Symmetry, S. For all $v \in \mathbb{V}$, all $i, j \in N$ such that $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$, we have $\varphi_i(v) = \varphi_j(v)$.
Desirability, D. (Maschler and Peleg, 1966) For all $v \in \mathbb{V}$, all $i, j \in N$ such that, for all $S \subseteq N \backslash \{i, j\}$, $v(S \cup i) \geq v(S \cup j)$, then $\varphi_i(v) \geq \varphi_j(v)$.
Inessential game property, IGP. For all additive TU-games $v \in \mathbb{V}_A$, for all $i \in N$, $\varphi_i(v) = v(i)$.

The following new axiom imposes that a player's payoff is non negative in a superadditive and monotone TU-game.
Superadditive monotonicity, SM. For any superadditive and monotone TU-game $v \in \mathbb{V}$, all $i \in N$, $\varphi_i(v) \geq 0$.

This axiom echoes monotonicity (Weber, 1988) in which a player's payoff is required to be non negative for monotonic TU-games only. While the latter is satisfied by ED, Sh but not ESD, these three values satisfy the weaker axiom **SM**.

Below is a list of relational axioms containing our main axiom, called Nullified equal loss property. It links an arbitrary TU-game v to the TU-game v^h in which a player h is nullified, by imposing that the payoff variation should affect all the other players equally, thus preserving payoff differences among them.

Nullified equal loss property, NEL. For all $v \in \mathbb{V}$, all $h \in N$, all $i, j \in N \backslash h$,

$$\varphi_i(v) - \varphi_i(v^h) = \varphi_j(v) - \varphi_j(v^h).$$

Linearity, L. φ is a linear map $\mathbb{V} \longrightarrow \mathbb{R}^N$.
Additivity, A. For all $v, w \in \mathbb{V}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

3. Axiomatic study of the Nullified equal loss property

3.1. General formula for efficient values satisfying the Nullified equal loss property

We begin the axiomatic study by showing that the combination of Nullified equal loss property and Efficiency implies that the values only depend on $v(S)$ for $s \in \{1, n\}$, i.e. they are determined by $n+1$ parameters out of the $2^n - 1$ given by an arbitrary $v \in \mathbb{V}$. The following lemma is central in this approach as it allows to apprehend how these two axioms work together to restrict the value and, as corollaries, two general formulas are obtained.

Lemma 1. *Given a TU-game $v \in \mathbb{V}$, consider two values φ and φ' on $G(v)$ satisfying Efficiency (**E**) and Nullified equal loss property (**NEL**). If they coincide on v^S for all $S \subseteq N$ such that $s \geq n-1$, they are equal on $G(v)$.*

Proof. Remind that $n \geq 3$ throughout the article. The proof that $\varphi(v^S) = \varphi'(v^S)$ is done by (descending) induction on the cardinal s of S.
Initialization. If $s \geq n-1$, $\varphi = \varphi'$ by hypothesis.
Induction hypothesis. Assume that $\varphi(v^S) = \varphi'(v^S)$ for all $S \subseteq N$ such that $s \geq k$ for a given $k \leq n-1$.
Induction step. Choose any $S \subseteq N$ such that $s = k-1$. Because $s < n-1$, there exists at least two distinct players $h, h' \in N \backslash S$. For all $i, j \neq h$, **NEL** and the induction hypothesis imply:

$$\varphi_i(v^S) - \varphi_j(v^S) \overset{\mathbf{NEL}}{=} \varphi_i(v^{S\cup h}) - \varphi_j(v^{S\cup h}) = \varphi'_i(v^{S\cup h}) - \varphi'_j(v^{S\cup h}) \overset{\mathbf{NEL}}{=} \varphi'_i(v^S) - \varphi'_j(v^S) \qquad (1)$$

Let us show that (1) holds for all $i, j \in N$ without making horses the same color. Indeed, (1) similarly holds for $i, j \neq h'$. Thanks to $n \geq 3$, with the help of an existing $l \neq h, h'$, we have $\varphi_h(v^S) - \varphi_l(v^S) = \varphi'_h(v^S) - \varphi'_l(v^S)$ and $\varphi_l(v^S) - \varphi_{h'}(v^S) = \varphi'_l(v^S) - \varphi'_{h'}(v^S)$. Summing these last two equalities brings $\varphi_h(v^S) - \varphi_{h'}(v^S) = \varphi'_h(v^S) - \varphi'_{h'}(v^S)$, and so (1) holds for all $i, j \in N$. Now by summing this last equality over $j \in N$ and using **E**, one gets:

$$n \cdot \varphi_i(v^S) - v^S(N) = n \cdot \varphi'_i(v^S) - v^S(N)$$

This immediatly leads to $\varphi_i(v^S) = \varphi'_i(v^S)$ for every $i \in N$. Conclude that $\varphi = \varphi'$ on $G(v)$. ■

Remark 1. *Note that if a value φ satisfies **NEL** on $\mathbb{V}$, then for all $h \in N$, the quantity $\varphi_i(v) - \varphi_i(v^h)$ is independent of $i \neq h$ and so is equal to its average when i runs through $N \backslash h$. If φ is also efficient, this leads to:*

$$\varphi_i(v) - \varphi_i(v^h) = \frac{1}{n-1}\left[v(N) - \varphi_h(v) - \left(v^h(N) - \varphi_h(v^h)\right)\right]. \qquad (2)$$

We are now ready to characterize an efficient value φ satisfying **NEL** by means of a general formula.

Corollary 1. *A value φ on $\mathbb{V}$ satisfies the Nullified equal loss property **NEL** and Efficiency **E** if and only if:*

$$\varphi_i(v) = \varphi_i(v^{N\backslash i}) - \frac{1}{n-1}\sum_{j\in N\backslash i}\left[\varphi_j(v^{N\backslash j}) - \varphi_j(v^N) - \frac{v(j)}{n}\right] + \frac{v(N) - v(i)}{n} \qquad (3)$$

Proof. On the one hand, the right hand side of (3) defines a value ψ on $\mathbb{V}$ satisfying **NEL**: for any given $v \in \mathbb{V}$, $h \in N$ and $i \in N \backslash h$, we have:

$$\begin{aligned}
\psi_i(v) - \psi_i(v^h) = \;& \varphi_i(v^{N\backslash i}) - \varphi_i((v^h)^{N\backslash i}) \\
& - \frac{1}{n-1}\left(\sum_{j\in N\backslash i}\left[\varphi_j(v^{N\backslash j}) - \varphi_j((v^h)^{N\backslash j}) - \varphi_j(v^N) + \varphi_j((v^h)^N) - \frac{v(j) - v^h(j)}{n}\right]\right) \\
& + \frac{v(N) - v^h(N) - v(i) + v^h(i)}{n} \\
= \;& \frac{1}{n-1}\left(\varphi_h(v^{N\backslash h}) - \varphi_h(v^N) - \frac{v(h)}{n}\right) + \frac{v(N) - v(N\backslash h)}{n}
\end{aligned}$$

which is independent of $i \in N\backslash h$. And ψ also satisfies **E**:

$$\begin{aligned}
\sum_{k\in N}\psi_k(v) = & \sum_{k\in N}\varphi_k(v^{N\backslash k}) - \sum_{k\in N}\left(\frac{1}{n-1}\sum_{j\in N\backslash k}\left[\varphi_j(v^{N\backslash j}) - \varphi_j(v^N) - \frac{v(j)}{n}\right]\right) + v(N) - \sum_{k\in N}\frac{v(k)}{n} \\
= & \sum_{k\in N}\varphi_k(v^{N\backslash k}) - \frac{1}{n-1}\left(\sum_{j\in N}\sum_{k\in N\backslash j}\left[\varphi_j(v^{N\backslash j}) - \varphi_j(v^N) - \frac{v(j)}{n}\right]\right) + v(N) - \sum_{k\in N}\frac{v(k)}{n} \\
= & \sum_{j\in N}\varphi_j(v^N) + v(N) \ = \ v(N).
\end{aligned}$$

On the other hand, for any given $v \in \mathbb{V}$ and $i \in N$, ψ_i and φ_i coincide obviously on $\mathbf{0} = v^N$. Moreover:

$$\begin{aligned}
\psi_i(v^{N\backslash i}) = & \ \varphi_i((v^{N\backslash i})^{N\backslash i}) \\
& -\frac{1}{n-1}\left(\sum_{j\in N\backslash i}\left[\varphi_j((v^{N\backslash i})^{N\backslash j}) - \varphi_j((v^{N\backslash i})^N) - \frac{v^{N\backslash i}(j)}{n}\right]\right) + \frac{v^{N\backslash i}(N) - v^{N\backslash i}(i)}{n} \\
= & \ \varphi_i(v^{N\backslash i}).
\end{aligned}$$

Lastly, let $h \in N\backslash i$, (2) leads to:

$$\begin{aligned}
\varphi_i(v^{N\backslash h}) - \varphi_i(v^N) &= \frac{1}{n-1}\left[v^{N\backslash h}(N) - \varphi_h(v^{N\backslash h}) + \varphi_h(v^N)\right] \\
&= -\frac{1}{n-1}\left[\varphi_h(v^{N\backslash h}) - \varphi_h(v^N) - \frac{v(h)}{n}\right] + \frac{v(h)}{n} \\
&= \psi_i(v^{N\backslash h}) - \psi_i(v^N)
\end{aligned}$$

where the second equality results from:

$$\frac{1}{n-1} = \frac{1}{n(n-1)} + \frac{1}{n}.$$

Therefore ψ_i and φ_i coincide on $v^{N\backslash h}$ too. By lemma 1, $\psi = \varphi$ on $G(v)$, and so $\psi(v) = \varphi(v)$ for any $v \in \mathbb{V}$. ■

Formula (3) may be written in the following simpler form:

Corollary 2. *A value φ on $\mathbb{V}$ satisfies the Nullified equal loss property* **NEL** *and Efficiency* **E** *if and only if it exists n functions $(F_i)_{i\in N}$ and n numbers $(a_i)_{i\in N}$ such that $\sum_{i\in N} a_i = 0$, $F_i(0) = 0$ for all $i \in N$ and :*

$$\varphi_i(v) = a_i + F_i(v(i)) - \frac{1}{n-1}\sum_{j\in N\backslash i} F_j(v(j)) + \frac{v(N)}{n} \tag{4}$$

Proof. Clear: formula (4) is only a recoding of formula (3) by setting $F_i(x) = \varphi_i(x\cdot u_i) - \varphi_i(\mathbf{0}) - x/n$ for any $x \in \mathbb{R}$ and $a_i = \varphi_i(\mathbf{0})$. ■

Remark 2. *Formula (4), applied to the 0-normalized TU-game v^0, simplifies to an affine equal division value: for any TU-game $v \in \mathbb{V}$, $\varphi_i(v^0) - \varphi_i(\mathbf{0}) = v^0(N)/n = \mathrm{ED}_i(v^0)$. As a consequence, there is no value on $\mathbb{V}$ that satisfies* **NEL**, **E** *and the well-known* **Null player axiom, N.** *defined by: for all $v \in \mathbb{V}$, all null players $i \in N$ in v, $\varphi_i(v) = 0$.*

The following proposition is weaker than lemma 1 to characterize values satisfying **NEL** and **E** but is more convenient for the forthcoming applications.

Proposition 1. *Consider two values φ and φ' on $\mathbb{V}$ satisfying Efficiency (**E**) and Nullified equal loss property (**NEL**). If they coincide on the class of additive TU-games $\mathbb{V}_A$, they are equal on $\mathbb{V}$.*

Proof. The proof is immediate. For any $v \in \mathbb{V}$, $v^N \in \mathbb{V}_A$ and $v^{N\setminus i} \in \mathbb{V}_A$, for all $i \in N$. Then lemma 1 applies. ■

3.2. Linear symmetric and efficient values satisfying the Nullified equal loss property

Our next result extends linear symmetric and efficient values defined on the class $\mathbb{V}_A$ of additive TU-games to an efficient value on $\mathbb{V}$ satisfying **NEL** in a unique way. Moreover, the class of linear symmetric efficient values satisfying **NEL** on $\mathbb{V}$ correponds to the class of (efficient) linear combinations of ED and ESD which, by the way, is characterized.

Proposition 2. *If ψ is a linear symmetric and efficient value only defined on $\mathbb{V}_A$ (i.e. satisfying **L**, **S** and **E** on $\mathbb{V}_A$), there exists a unique value φ satisfying Efficiency **E** and the Nullified equal loss property **NEL** on $\mathbb{V}$ such that $\varphi = \psi$ on $\mathbb{V}_A$. Moreover, φ is also linear and symmetric on $\mathbb{V}$ and there is $\lambda \in \mathbb{R}$ such that $\varphi = \lambda\text{ED} + (1-\lambda)\text{ESD}$.*

Proof. The proof relies on the fact that, on $\mathbb{V}_A$, **S** and **L** imply **NEL**. Indeed, let ψ be a linear symmetric value defined on $\mathbb{V}_A$ and let $v \in \mathbb{V}_A$. Thus $v = \sum_{i\in N} v(i)\cdot u_i$ and $v^h \in \mathbb{V}_A$ for any $h \in N$. More precisely, for any $h \in N$, one has $v - v^h = v(h)\cdot u_h$ so that, for any $i, j \neq h$ and $S \subseteq N\setminus\{i,j\}$, $(v-v^h)(S\cup i) = (v-v^h)(S\cup j)$. By **S**, this implies that $\psi_i(v-v^h) = \psi_j(v-v^h)$ and **L** allows to conclude that ψ satisfies **NEL**.
Suppose that ψ is also efficient and let us show that $\psi_i(u_i) = \psi_j(u_j)$ for two different players $i, j \in N$: firstly, **S** and **E** imply that:

$$1 - \psi_j(u_j) \overset{\mathbf{E}}{=} \sum_{k\in N\setminus j} \psi_k(u_j) \overset{\mathbf{S}}{=} (n-1)\psi_i(u_j). \tag{5}$$

Then consider $u_i + u_j \in \mathbb{V}_A$. Players i and j are symmetric, so **S** implies:

$$\begin{aligned}
&\psi_i(u_i+u_j) = \psi_j(u_i+u_j)\\
\overset{\mathbf{L},(5)}{\iff}\ &\psi_i(u_i) + \frac{1-\psi_j(u_j)}{n-1} = \psi_j(u_j) + \frac{1-\psi_i(u_i)}{n-1}\\
\iff\ &\psi_i(u_i) = \psi_j(u_j)
\end{aligned} \tag{6}$$

Now let us construct a value φ on $\mathbb{V}$ which extends ψ from $\mathbb{V}_A$. For $k \in N$, by analogy with formula (4), let $a_k = \psi_k(\mathbf{0}) = 0$ and $F_k(x) = \psi_k(x\cdot u_k) - \psi_k(\mathbf{0}) - x/n = x\cdot(\psi_k(u_k) - 1/n)$. Consider now the following value φ on $\mathbb{V}$:

$$\begin{aligned}
\varphi_i(v) &= a_i + F_i(v(i)) - \frac{1}{n-1}\sum_{j\in N\setminus i} F_j(v(j)) + \frac{v(N)}{n}\\
&= \left(\psi_i(u_i) - \frac{1}{n}\right)v(i) - \frac{1}{n-1}\sum_{j\in N\setminus i}\left[\left(\psi_j(u_j) - \frac{1}{n}\right)v(j)\right] + \frac{v(N)}{n}
\end{aligned}$$

Then φ satisfies **E**, **NEL** and **L** on $\mathbb{V}$. Moreover, if $v(i) = v(j)$ for $i, j \in N$, then $\varphi_i(v) - \varphi_j(v) = (1 + 1/(n-1))(\psi_i(u_i) - \psi_j(u_j))v(i) = 0$ hence φ satisfies **S**. Finally, for all $i \in N$, $\varphi_i(u_i) = \psi_i(u_i)$ and, for all $j \in N \backslash i$:

$$\varphi_i(u_j) = -\frac{\psi_j(u_j) - 1/n}{n-1} + \frac{1}{n} = \frac{1 - \psi_j(u_j)}{n-1} = \psi_i(u_j)$$

so, by linearity, $\varphi = \psi$ on $\mathbb{V}_A$. The uniqueness of ψ's extension from $\mathbb{V}_A$ to $\mathbb{V}$ is a direct consequence of proposition 1.

Finally, define $\lambda = n(1 - \varphi_i(u_i))/(n-1)$, independent of a chosen $i \in N$ by formula (6). Then $\varphi_i(u_i) = \lambda/n + (1-\lambda)$ for all $i \in N$ and, by formula (5), $\varphi_j(u_i) = \lambda/n$ for any $j \in N \backslash i$. Hence, for all $i, j \in N$, $\varphi_i(u_j) = \lambda \mathrm{ED}_i(u_j) + (1-\lambda)\mathrm{ESD}_i(u_j)$. By linearity, $\varphi = \lambda \mathrm{ED} + (1-\lambda)\mathrm{ESD}$ holds on $\mathbb{V}_A$ and finally on $\mathbb{V}$ by proposition 1. ∎

Remark 3. *As it appears in proposition 2's proof, the general formula (4) for efficient values satisfying* **NEL** *can be particularized for linear values (satisfying* **L***) by assuming linearity on $\mathbb{V}_A$. The corresponding functions F_i are then linear and $a_i = 0$ for all $i \in N$. Likewise, symmetric values (satisfying* **S***) can be generated by imposing a symmetric treatment of players on $\mathbb{V}_A$ only. The corresponding functions F_i are then equal and $a_i = 0$ for all $i \in N$. Clearly, these assumptions are logically independent of* **E** *and* **NEL** *on $\mathbb{V}$ and leads to simpler formulas.*

Not all symmetric efficient values defined on additive TU-games satisfies **NEL** as shown in the following example.

Example 1. *Take $n = 3$ and ψ defined on $\mathbb{V}_A = \{x_1u_1 + x_2u_2 + x_3u_3, (x_1, x_2, x_3) \in \mathbb{R}^3\}$ by:*

$$\psi_i(v) = x_i(x_{i+1} - x_{i-1})^2 + \frac{x_1 + x_2 + x_3}{3} - \sum_{i \in \{1,2,3\}} x_i(x_{i+1} - x_{i-1})^2$$

where $x_4 = x_1$ and $x_0 = x_3$. Then ψ is symmetric and efficient but:

$$\psi_1(v) - \psi_2(v) = (x_3^2 - x_1x_2)(x_1 - x_2) \neq -x_1x_2(x_1 - x_2) = \psi_1(v^3) - \psi_2(v^3)$$

so that it does not satisfy **NEL***.*

Similarly, not all linear efficient values defined on additive TU-games satisfies **NEL** as shown in the following example.

Example 2. *Take $n = 3$ and ψ defined on $\mathbb{V}_A = \{x_1u_1 + x_2u_2 + x_3u_3, (x_1, x_2, x_3) \in \mathbb{R}^3\}$ by:*

$$\psi_i(v) = \frac{1}{6}(3x_i + 2x_{i+1} + x_{i-1})$$

where $x_4 = x_1$ and $x_0 = x_3$. Then ψ is linear and efficient but:

$$\psi_1(v) - \psi_2(v) = \frac{1}{6}(2x_1 - x_2 - x_3) \neq \frac{1}{6}(2x_1 - x_2) = \psi_1(v^3) - \psi_2(v^3)$$

so that it does not satisfy **NEL***.*

As a direct consequence of proposition 2, we have the following characterization.

Corollary 3. *A value φ on $\mathbb{V}$ satisfies Efficiency (***E***), Nullified equal loss property (***NEL***), Linearity (***L***) and Symmetry (***S***) if and only if there is $\lambda \in \mathbb{R}$ such that $\varphi = \lambda \mathrm{ED} + (1-\lambda)\mathrm{ESD}$.*

3.3. *Characterization of the class of convex combinations of* ED *and* ESD

By relying on the previous result, we characterize the more natural class of convex combinations of ED and ESD.

Theorem 1. *A value φ on $\mathbb{V}$ satisfies Efficiency (**E**), Nullified equal loss property (**NEL**), Additivity (**A**), Desirability (**D**) and Superadditive Monotonicity (**SM**) if and only if there is $\lambda \in [0,1]$ such that:*

$$\varphi = \lambda \text{ED} + (1-\lambda)\text{ESD}.$$

Proof. For any superadditive and monotone TU-game $w \in V$, note that for all $i \in N$, $0 \leq w(i) \leq w(N)$ and $w(N) \geq \sum_{i\in N} w(i)$ so that $\text{ED}_i(w) \geq 0$ and $\text{ESD}_i(w) \geq 0$. Thus any convex combination φ of ED and ESD satisfies **SM**. Moreover, for any $i, j \in N$,

$$\varphi_i(v) - \varphi_j(v) = (1-\lambda)(v(i) - v(j)) \tag{7}$$

so that φ satisfies **D** and, by corollary 3, φ satisfies all the other involved axioms.
Reciprocally, let φ be a value satisfying the five aforementioned axioms. By Casajus and Hüttner (2013, Lemma 5), **A**, **E** and **D** imply **L**. Moreover, **D** implies **S**. By corollary 3, there is $\lambda \in \mathbb{R}$ such that $\varphi = \lambda\text{ED} + (1-\lambda)\text{ESD}$. From formula (7) and **D**, we get $\lambda \leq 1$. Finally, **SM** applied to the superadditive and monotone TU-game u_i for a fixed $i \in N$ brings $\varphi_j(u_i) = \lambda/n \geq 0$ for all $j \in N\backslash i$. ■

The axioms invoked in Theorem 1 (as well as in corollary 3) are logically independent:

- The value $\varphi = 2\text{ED}$ satisfies all axioms except **E**.
- The value $\varphi = \text{Sh}$ satisfies all axioms except **NEL**, as a consequence of remark 2.
- Single out a player $i_0 \in N$ and define φ for all $v \in \mathbb{V}$ by:

$$\varphi_i(v) = \begin{cases} -\frac{1}{n-1} \cdot \frac{2}{3} \cdot v(i_0) + \frac{v(N)}{n} & \text{if } i \in N\backslash i_0 \\ \\ \frac{2}{3} \cdot v(i_0) + \frac{v(N)}{n} & \text{if } i = i_0 \end{cases}$$

  Then φ satisfies **E**, **A** and **NEL**. Moreover if w is a superadditive and monotone TU-game, $v(N) \geq w(i_0) \geq 0$. Because $n/(n-1) \geq 2/3$ for $n \geq 3$, $w(N)/n \geq 2w(i_0)/3(n-1)$ so that $\varphi_i(w) \geq 0$ for all $i \in N$. Hence, **SM** is also satisfied by φ. However, **S** is clearly not satisfied so that **D** is violated.

- The value $\varphi = 2\text{ESD} - \text{ED}$ satisfies all axioms except **SM**. Indeed, consider for instance the unanimity game u_i, for a given player $i \in N$, it is a superadditive and monotone TU-game and for any $j \in N\backslash i$, $\varphi_j(u_i) = -1/n < 0$.
- The value φ defined by $\varphi_i(v) = \max(0, v(i)) + \big(v(N) - \sum_{j\in N} \max(0, v(j))\big)/n$ for all $i \in N$ satisfies all axioms except **A**.

Remark 4. *As mentioned in the introduction, theorem 1 can be used to compare the class of convex combinations of* ED *and* ESD *with the Shapley value and the class of egalitarian Shapley values. The latter class consists of all convex combinations of* Sh *and* ED*. It is easy to check that all the aforementioned values satisfy* **SM***. Replacing* **NEL** *by* **N** *(resp.* **Null player in a productive environment, NPE.**[1] *introduced in Casajus and Hüttner, 2013) yields a (redundant) characterization of* Sh *(resp. the class of egalitarian Shapley values).*

3.4. Punctual characterization of equal division values

This section provides characterizations of ESD and ED that only differ with respect to the requirements on additive TU-games, attesting to the centrality of **NEL** axiom in the context of equal division values.

Proposition 3. *A value φ on $\mathbb{V}$ satisfies Efficiency (**E**), Nullified equal loss property (**NEL**) and the Inessential game property (**IGP**) if and only if it is the equal surplus division value $\varphi = \mathrm{ESD}$.*

Proof. The result is a straight consequence of proposition 1: **IGP** characterizes a unique value on $\mathbb{V}_A$ and ESD satisfies **NEL**, **IGP** and **E**. By proposition 2, the logical independence is obvious. ■

Remark 5. *The equal division value can be characterized with a similar set of three axioms. For this purpose, we introduce an ad hoc axiom* **Equal division for inessential games, EIG***: for all additive TU-games $v \in \mathbb{V}_A$, for all $i \in N$, $\varphi_i(v) = (\sum_{j\in N} v(j))/n$. One easily gets that* **E**, **NEL** *and* **EIG** *characterizes* ED*. More generally, for a fixed $\lambda \in [0;1]$, the convex combination $\lambda\mathrm{ED} + (1-\lambda)\mathrm{ESD}$ is characterized by* **E**, **NEL** *and* **λ-IGP** *where the latter axiom is defined by: for all additive TU-games $v \in \mathbb{V}_A$, for all $i \in N$, $\varphi_i(v) = \lambda v(i) + (1-\lambda)(\sum_{j\in N} v(j))/n$. By lemma 1, these characterizations can be weakened by only requiring* **λ-IGP** *for all multiple of unanimity games $x \cdot u_i$ where $i \in N$, $x \in \mathbb{R}$.*

4. Applications

This section presents two applications of the values involved in the preceding sections. Our aim is not to characterize them in other axiomatic contexts and this aspect is left for future work. The first one rests on formula (4). This expression does not specify the shape of functions F_i and so allows to grasp situations in which non-linearity and individual specificities are important features. More specifically, we consider a situation of bargaining under risk, dealing with risk aversion, which cannot be handled with symmetric or linear values only.

4.1. Bargaining under risk

Most economic models of bargaining assume certainty of outcomes or risk-neutrality of negotiators although many real life situations involve pay-off uncertainty which may arise from various random events. A typical n-bargaining situation à la Nash is usually described by a pair $(\mathcal{C}, d)$ composed of a convex, comprehensive subset $\mathcal{C} \subseteq \mathbb{R}^n$ of feasible outcomes and a disagreement point $d \in \mathcal{C}$. If all players agree on a point $x \in \mathcal{C}$, they get x. Otherwise, they obtain d. A solution is a function associating with every $(\mathcal{C}, d)$ a feasible outcome $F(\mathcal{C}, d) \in \mathcal{C}$ representing the compromise

[1] For all $v \in V$ with $v(N) \geq 0$, all null players $i \in N$ in v, we have $\varphi_i(v) \geq 0$.

unanimously reached by the players. Here we consider a fixed set N of players in a risky bargaining situation where every player may be independently affected by an entire loss of productivity as modeled by a nullification, involving both the disagreement point and the set of feasible outcomes. Hence players may face one of the 2^n different n-bargaining situations resulting from all possible nullifications. If $S \subsetneq N$ is the set of nullified players, we limit the corresponding set of feasible outcomes to $\mathcal{C}_S = \{(x_i)_{i\in N} \in \mathbb{R}^n, \sum_{i\in N} x_i \leq W_S\}$ where $W_S > 0$ is a worth to be shared. This generates a positive TU-game $v \in \mathbb{V}$ by setting $v(N\backslash S) = v^S(N) = W_S$ and we assume that v is also superadditive. The objective is to fairly distribute the worth finally achievable by the society N among its members. Besides, players are not allowed to await the realizations of the potential nullifications before deciding upon a joint sharing scheme, i.e. they have to design a value φ on $G(v)$. At last, each player is characterized by an individual risk aversion, which alters her bargaining power accordingly and incents her to hedge her stand-alone risk by prior monetary transfers. Formally, this situation is essentially described by three elements:

- an individual and independent probability $p_i \in]0,1[$ that measures the risk of being nullified faced by player $i \in N$, in the sense that i is fully productive with probability $1 - p_i$ and loses her productivity with probability p_i. Note $p = (p_i)_{i\in N} \in]0,1[^N$ the corresponding vector;
- a positive and monotone TU-game v on N so that $v^S(N) = v(N\backslash S)$ evaluates, for any $S \subseteq N$, the worth to be shared in the bargaining situation where S is the set of nullified players;
- an individual utility function w_i for each player $i \in N$ which takes player i's risk aversion into account. We require that these utility functions should be defined on $\mathbb{R}$, strictly increasing and strictly concave such that $w_i(0) = 0$. A well-known example is the CARA utility function (see for instance Pratt, 1964) of the form $w_i(x) = (1 - \mathrm{e}^{-\alpha_\mathrm{i} \mathrm{x}})/\alpha_\mathrm{i}$ for $x \in \mathbb{R}$ where $\alpha_i > 0$ is the individual constant absolute risk aversion parameter. Note $w = (w_i)_{i\in N}$ so that the disagreement point d_S is $\big(w_i(v^S(i))\big)_{i\in N}$ in the bargaining situation where S is the set of nullified players.

For a situation (p, v, w) on N, define the average bargaining situation by:

$$v_p = \sum_{S\subseteq N} \prod_{j\in S} p_j \prod_{i\notin S} (1 - p_i) v^S.$$

This expression is similar to Owen's multilinear extension of TU-games (see Owen, 1972). For each coalition $S \subseteq N$, $v_p(S)$ can be considered as the average worth to be shared given that (i.e. conditionally to) $N\backslash S$ is nullified for sure.

Given an efficient value φ, a solution to the bargaining situation $(\mathcal{C}_S, d_S)$ may be denoted by $\varphi(v^S)$. For each player $k \in N$, the average allocation is denoted by:

$$\varphi^p_k(v) = \sum_{S\subseteq N} \prod_{j\in S} p_j \prod_{i\notin S} (1 - p_i) \varphi_k(v^S).$$

Requiring efficiency for φ in this context can be seen as the risk-neutrality of the grand coalition:

$$\sum_{i\in N} \varphi_i(v_p) = v_p(N) = \sum_{i\in N} \varphi^p_i(v). \tag{8}$$

The following axiom is a collective variant of our axiom **NEL** and is defined by:

Group-Nullified equal loss property, GNEL. For all $v \in \mathbb{V}$, all $S \subseteq N$, all $i, j \in N \backslash S$,

$$\varphi_i(v) - \varphi_i(v^S) = \varphi_j(v) - \varphi_j(v^S).$$

The **GNEL** axiom is interpreted similarly to **NEL**: bargaining players in $N \backslash S$ incur the same difference in payoff when coalition S becomes nullified. This axiom is a natural requirement for φ in this context of risk hedging and turns out to be equivalent to **NEL** (by successive application). Thus values on $G(v)$ compatible with **GNEL** and **E** are given by formula (4). An example of such a value can be obtained by setting for instance $a_i = 0$ and $F_i(x) = (1 - 1/n) \cdot w_i(x)$. This particular value can be naturally extended from $G(v)$ to $\mathbb{V}$ so that $\varphi(v_p)$ may be computed. After simplifications, this brings:

$$\varphi_i(v) = w_i(v(i)) + \frac{v(N) - \sum_{j \in N} w_j(v(j))}{n}.$$

The allocation $\varphi(v^S)$ in our context corresponds to the egalitarian solution (Kalai, 1977) applied on the bargaining situation $(\mathcal{C}_S, d_S)$. Remark also that, in the particular case where the utility functions are CARA, $w_i(v(i))$ tends to $v(i)$ when α_i tends to 0 so that, when all players are risk-neutral, the disagreement point is $d = (v(i))_{i \in N}$. In this case, we recover ESD as the egalitarian solution when utility is transferable. Let us emphasize the following fact: when players become risk-adverse, with possibly different individual risk-aversion parameters, only the disagreement points d_S are impacted. This is consistent with the *individual* aspect of these outcomes and we do not exclude that d_S may be outside $\mathcal{C}_S$. However, if an agreement is found, the worth to be shared is transferable, independent of individual utilities and only depends, in our context, on the set of nullified players. Bearing this idea in mind, the players have the possibility of making transfers prior to the actual realization of the potential nullifications. We obtain the following result:

Result 1. *There exists a unique budget-balanced transfer scheme π of risk premia between the players so that any player's average allocation equals her allocation in the non-random (or certainty equivalent) bargaining situation*[2] $v_p - \pi$*:*

$$\varphi_i(v_p - \pi) = \varphi_i^p(v) \text{ for all } i \in N.$$

Proof. Firstly, φ is efficient. Equation (8) implies that $\sum_{i \in N} \pi_i = 0$, so that the aforesaid transfer scheme π is budget-balanced. Secondly, it is easy to show that:

$$\varphi_i^p(v) = (1 - p_i) w_i(v(i)) + \frac{v_p(N) - \sum_{j \in N} (1 - p_j) w_j(v(j))}{n}$$

$$\varphi_i(v_p - \pi) = w_i\big((1 - p_i)v(i) - \pi_i\big) + \frac{v_p(N) - \sum_{j \in N} w_j\big((1 - p_j)v(j) - \pi_j\big)}{n}.$$

The equation $\varphi_i(v_p - \pi) = \varphi_i^p(v)$ then becomes:

$$T_i(\pi_i) = \frac{1}{n} \sum_{j \in N} T_j(\pi_j) \tag{9}$$

[2] Abusing notation, π also denotes the induced additive TU-game.

where $T_i(x) = w_i((1-p_i)v(i) - x) - (1-p_i)w_i(v(i))$, for all $i \in N$. The function T_i is strictly decreasing. Denote by $q_i = \lim_{-\infty} T_i > 0$ which exists and may be infinite. Remark that $\lim_{+\infty} T_i = -\infty$. Hence T_i is a continuous strictly decreasing bijection between $\mathbb{R}$ and $]-\infty, q_i[$. Note also that $T_i(0) > 0$ whenever $v(i)(1-p_i) \neq 0$ by strict concavity.

The system (9) of n linear equations in $t_i = T_i(\pi_i)$ is underdetermined of rank $n-1$. Indeed, the solutions are parametrized by $t \in \mathbb{R}$ such that $t_i = t$ for all $i \in N$. Define $q = \min_{i \in N} q_i > 0$ (which may be infinite) and consider the continuous strictly decreasing function $Q(x) = \sum_{i \in N} T_i^{-1}(x)$ defined on $]-\infty, q[$. One has $\lim_{-\infty} Q = +\infty$ and $\lim_q Q = -\infty$ so there exists a unique $t^\star \in]-\infty, q[$ such that $Q(t^\star) = 0$. Finally the transfer scheme defined by $\pi_i = T_i^{-1}(t^\star)$ satisfies all desired conditions and depends on p, w and the stand-alone capacities $(v(i))_{i \in N}$ only. ■

In our context, transfers only result from the non-linearity of φ, through the non-linearity of w. Indeed, for linear values, we have $\varphi_i(v_p) = \varphi_i^p(v)$ unconditionally. Likewise, if $T_i = T$ for all players $i \in N$ (for instance when all individual variables $v(i)$, p_i and w_i are equal), i.e. in a symmetric framework, then $\pi_i = 0$ and no transfer is needed. Moreover, when there is no random effect, i.e. if $p_i = 0$ (resp. $p_i = 1$) for all $i \in N$, one may also show that $t = 0$ and $\pi_i = 0$ for all $i \in N$.

4.2. Softening the tragedy of the Commons

The second application illustrates the interest of convex combinations of equal division values in a well-known economic context. Consider a perfectly divisible common-pool resource (CPR) for which no storage is feasible and operated by a fixed community N of potential consumers, facing pure appropriation externalities (see Ostrom et al., 1994, for a wide overview). In this context, uncoordinated individual consumption leads the aggregated society to deviates from an optimal social welfare. Suppose that the socially optimal overall consumption is independent of how this consumption is divided among the players. If any player chooses not to consume the CPR, this will not affect the community's consumption optimum but, in a symmetric framework, other players will have to equally compensate this gap so that the community's consumption remains optimal. This last comment allows an analogy with the **NEL** principle.

Let us now present a model close to Funaki and Yamato (1999). Suppose that a constant and common marginal labor cost $q > 0$ is needed to exploit the CPR and denote by $x = (x_i)_{i \in N}$ the vector of individual work efforts so that $x_i \in \mathbb{R}^+$ for player i. Furthermore, let f be the technology function, which assigns to each total effort $x_N = \sum_{i \in N} x_i$ the production per unit $f(x_N)$. Thus $x_N f(x_N)$ is the total production. The function f is supposed to be positive, strictly decreasing and concave on an interval $[0, \bar{x}]$, and null thereafter so that $f(0) > q$ and $f(\bar{x}) = 0$. This reflects that the more the CPR is exploited, the less it is productive.

Unlike Funaki and Yamato (1999), we assume that the players would like to agree upon a distribution method of the total production, prior to choosing their efforts. For this purpose, define the additive TU-game v_x so that $v_x(S) = f(x_N) \sum_{i \in S} x_i$ represents S's total production when the overall effort in the society is given by x_N. The aforementioned distribution will be implemented by an efficient value φ. Therefore the income of player i is defined by $\theta_i(x) = \varphi_i(v_x) - qx_i$. Thus, each value φ induces a non-cooperative game $(N, (\mathbb{R}^+, \theta_i)_{i \in N})$.

Let $\widehat{x_N}$ be the the total effort that achieves the social optimum, i.e. the greatest total of incomes. Indeed $\sum_{i \in N} \theta_i(x) = x_N(f(x_N) - q)$ is maximum when the following equation, independent of φ

and n, is satisfied:

$$\psi(\widehat{x_N}) = q \tag{10}$$

where $\psi(t) = f(t) + tf'(t)$ for $t \in \mathbb{R}^+$ is a strictly decreasing function on $[0, \bar{x}]$. We also have $\widehat{x_N} \in]0, \bar{x}[$.

In this illustration, we aim at implementing the social optimum by a Nash equilibrium through the choice of a value φ. For any value φ, denote by x^φ any pure-strategy Nash equilibrium of $(N, (\mathbb{R}^+, \theta_i)_{i\in N})$ if there exists one. Let us start by two particular cases. For $\varphi = \text{ESD}$, one has $\varphi_i(v_x) = x_i f(x_N)$ so that $\theta_i(x) = x_i(f(x_N) - q)$ and the first order condition is $x_i^{\text{ESD}} f'(x_N^{\text{ESD}}) + f(x_N^{\text{ESD}}) = q$. Averaging these conditions gives:

$$\frac{x_N^{\text{ESD}} f'(x_N^{\text{ESD}})}{n} + f(x_N^{\text{ESD}}) = q$$

so that x^{ESD} exists, is unique and symmetric. Moreover $\psi(\widehat{x_N}) = q > \psi(x_N^{\text{ESD}})$. Hence:

$$\widehat{x_N} < x_N^{\text{ESD}} \tag{11}$$

and we find, as in Hardin (1968), that the CPR is overused when each player enjoys a share of the production in proportion to her effort. Note that $x_N^{\text{ESD}} < \bar{x}$.

For $\varphi = \text{ED}$, one has $\varphi_i(v_x) = x_N f(x_N)/n$ so that $\theta_i(x) = x_N f(x_N)/n - qx_i$ and the first order condition is:

$$x_N^{\text{ED}} f'(x_N^{\text{ED}}) + f(x_N^{\text{ED}}) = nq$$

so that any x such that $x_N = x_N^{\text{ED}}$ is a Nash equilibrium. Note that if $nq > f(0)$, $x_N^{\text{ED}} = 0$. Moreover $\psi(\widehat{x_N}) = q < \psi(x_N^{\text{ED}})$. Hence:

$$\widehat{x_N} > x_N^{\text{ED}} \tag{12}$$

and now the CPR is underused as the equal division rule gives the players no incentive to exploit the resource.

At this point, it is quite intuitive that some convex combination of ESD and ED will allow to implement the social optimum $\widehat{x_N}$ by a Nash equilibrium. This particular class of values has otherwise a special interest in this context: it corresponds to levy a proportional tax on individual performances, which is afterward distributed equally within the society (see Casajus, 2015, for an axiomatic foundation of this approach).

Thus, let us consider $\varphi = (1-\lambda)\text{ESD}+\lambda\text{ED}$, the first order condition becomes $(1-\lambda)(x_i^\varphi f'(x_N^\varphi) + f(x_N^\varphi)) + \lambda(x_N^\varphi f'(x_N^\varphi) + f(x_N^\varphi))/n = q$ for player i. Summing all these conditions brings the following equation:

$$nq = (n(1-\lambda)+\lambda)f(x_N^\varphi) + f'(x_N^\varphi)\underbrace{((1-\lambda)x_N^\varphi + \lambda x_N^\varphi)}_{x_N^\varphi}. \tag{13}$$

The two inequalities (11,12) for $\lambda = 0$ and $\lambda = 1$ respectively, and the implicit function theorem applied on equation (13) allow to prove existence and uniqueness of a $\lambda^\star$ such that $x_N^\varphi = \widehat{x_N}$. To see this, note that the partial derivative of the right member of (13) with respect to x_N^φ is $(n(1-\lambda)+1+\lambda)f'(x_N^\varphi) + f''(x_N^\varphi)x_N^\varphi < 0$ so that $x_N^\varphi(\lambda) \in \mathcal{C}^1([0,1])$. Note that the partial derivative of the right member of (13) with respect to λ is $(1-n)f(x_N^\varphi) < 0$ so that $\mathrm{d}x_N^\varphi/\mathrm{d}\lambda < 0$. Substitute $nq = (n-1)q + \psi(\widehat{x_N})$ and $x_N^\varphi = \widehat{x_N}$ in equation (13) finally brings the following result.

Result 2. *There exists a unique internal tax $\lambda^\star$ which allows to implement the social optimum of a CPR consumption by a unique Nash equilibrium x^φ through a redistribution $\varphi = \lambda^\star \mathrm{ED} + (1-\lambda^\star)\mathrm{ESD}$ of the total production. One has $x_i^\varphi = \widehat{x_N}/n$ for all player $i \in N$. Moreover,*

$$\lambda^\star = 1 - \frac{q}{f(\widehat{x_N})} \tag{14}$$

does not depend on the population's size n. Finally, the Nash equilibrium x^φ is strong.

Proof. It remains to prove that the Nash equilibrium $x^\varphi = (\widehat{x_N}/n)_{i\in N}$ is strong for the non-cooperative game $(N, (\mathbb{R}^+, \theta_i)_{i\in N})$ defined by the value $\varphi = \lambda^\star\mathrm{ED} + (1-\lambda^\star)\mathrm{ESD}$. For any coalition $S \subseteq N$, define the vector $x_{-S} = (\widehat{x_N}/n)_{i\in N\setminus S}$, the real number $\widehat{x_{-S}} = (n-s)\widehat{x_N}/n$ and, for all vector $x = (x_i)_{i\in S} \in R^S$, the sum of utility functions of players in S:

$$\Theta_S(x_S) = f(x_S + \widehat{x_{-S}})\big(s\lambda^\star \frac{x_S + \widehat{x_{-S}}}{n} + (1-\lambda^\star)x_S\big) - qx_S$$

where $x_S = \sum_{i\in S} x_i \in \mathbb{R}$. Let us show that $\Theta_S(x_S)$ reaches its maximum when $x_S = s\cdot\widehat{x_N}/n$:

$$\begin{aligned}
\Theta_S'(\frac{s}{n}\widehat{x_N}) &= -q + f'(\widehat{x_N})\cdot\frac{s\widehat{x_N}}{n} + f(\widehat{x_N})\left(1 - \lambda^\star + \frac{s\lambda^\star}{n}\right)\\
&\overset{(10)}{=} f'(\widehat{x_N})\widehat{x_N}\cdot\left(\frac{s}{n} - 1\right) + \lambda^\star f(\widehat{x_N})\left(\frac{s}{n} - 1\right)\\
&\overset{(10)}{=} \left(\frac{s}{n} - 1\right)\cdot\left(q - f(\widehat{x_N}) + \lambda^\star f(\widehat{x_N})\right)\\
&\overset{(14)}{=} 0
\end{aligned}$$

Moreover, one may show that $\Theta_S''(x_S) < 0$ for $x_S \in [0, \bar{x}]$ so that Θ_S is a strictly concave function and has at most one maximum. ■

Lastly, let us show that $\mathrm{d}\lambda^\star/\mathrm{d}q < 0$. Starting by differentiating $\widehat{x_N}(q)$ accordingly to the implicit equation (10), we have:

$$\widehat{x_N}'(q) = \frac{1}{2f'(\widehat{x_N}) + \widehat{x_N}f''(\widehat{x_N})} < 0.$$

A straight computation gives:

$$\begin{aligned}
\frac{\mathrm{d}\lambda^\star}{\mathrm{d}q} &= -\frac{f(\widehat{x_N}) - qf'(\widehat{x_N})\widehat{x_N}'(q)}{f(\widehat{x_N})^2}\\
&= \frac{-\widehat{x_N}'(q)}{f(\widehat{x_N})^2}\Big(f(\widehat{x_N})\big(2f'(\widehat{x_N}) + \widehat{x_N}f''(\widehat{x_N})\big) - qf'\widehat{x_N}\Big)\\
&\overset{(10)}{=} \underbrace{\frac{-\widehat{x_N}'(q)}{f(\widehat{x_N})^2}}_{>0}\Big(\underbrace{f(\widehat{x_N})f'(\widehat{x_N})}_{<0} + \widehat{x_N}\big(\underbrace{f(\widehat{x_N})f''(\widehat{x_N})}_{<0} - f'(\widehat{x_N})^2\big)\Big) < 0
\end{aligned}$$

To conclude, this organization can be interpreted as a cooperative company whose owners / workers are remunerated partly by their individual efforts and partly by an equal pension levied through an internal tax. We have also shown that the harder the CPR to exploit, the lesser should be the internal tax, in order to encourage players to reach the social optimum. A large literature tackles this crucial tragedy of common-pool resources overuse. Let us emphasize that our approach internalizes the Nash implementation locally without market or social planner.

5. Concluding remarks

An ultimate argument in favor of **NEL** is that **NEL** implies the following axiom:
Balanced cycle contributions under nullification, BCyCN.(Béal et al., 2016) For all $v \in \mathbb{V}$, all ordering $(i_1, \ldots, i_n)$ of N,

$$\sum_{p=1}^{n} \Big(\varphi_{i_p}(v) - \varphi_{i_p}(v^{i_{p+1}}) \Big) = \sum_{p=1}^{n} \Big(\varphi_{i_p}(v) - \varphi_{i_p}(v^{i_{p-1}}) \Big)$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

Indeed, the term $\varphi_{i_{p-1}}(v) - \varphi_{i_{p-1}}(v^{i_p})$ in the left hand side of the preceding equation corresponds to the payoff variation of player i_{p-1} when player i_p is nullified, whereas the term $\varphi_{i_{p+1}}(v) - \varphi_{i_{p+1}}(v^{i_p})$ in the right hand side corresponds to the payoff variation of player i_{p+1} when player i_p is nullified. These terms are equal if **NEL** is invoked. **BCyCN** is an interesting and very weak axiom as, following (Béal et al., 2016), any linear symmetric value satisfies it.

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