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Level set method for optimization of contact problems

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This paper deals with the numerical solution of topology and shape optimization problems of an elastic body in unilateral contact with a rigid foundation. The contact problem with the prescribed friction is described by an elliptic variational inequality of the second order governing a displacement field. The structural optimization problem consists in finding such shape of the boundary of the domain occupied by the body that the normal contact stress along the contact boundary of the body is minimized. The shape of this boundary and its evolution is described using the level set approach. Level set methods are numerically efficient and robust procedures for the tracking of interfaces. They allow domain boundary shape changes in the course of iteration. The evolution of the domain boundary and the corresponding level set function is governed by the Hamilton–Jacobi equation. The speed vector field driving the propagation of the level set function is given by the Eulerian derivative of an appropriately defined functional with respect to the free boundary.

In this paper the necessary optimality condition is formulated. The level set method, based on the classical shape gradient, is coupled with the bubble or topological derivative method, which is precisely designed for introducing new holes in the optimization process. The holes are supposed to be filled by weak phase mimicking voids. Since both methods capture a shape on a fixed Eulerian mesh and rely on a notion of gradient computed through an adjoint analysis, the coupling of these two method yields an efficient algorithm. Moreover the finite element method is used as the discretization method. Numerical examples are provided and discussed.

1. Introduction

The paper is concerned with the numerical solution of a structural optimization problem for an elastic body in unilateral contact with a rigid foundation. The contact with a given friction, described by Coulomb law, is assumed to occur at a portion of the boundary of the body. The displacement field of the body in unilateral contact is governed by an elliptic variational inequality of the second order. The results concerning the existence, regularity and finite-dimensional approximation of solutions to contact problems are given, among others, in [1]. The structural optimization problem for the elastic body in contact consists in finding such shape of the boundary of the domain occupied by the body that the normal contact stress along the boundary of the body is minimized. It is assumed, in a case of shape optimization problem, that the volume of the body is constant.

Shape optimization of contact problems is considered, among others, in [1,2], where necessary optimality conditions, results concerning convergence of finite-dimensional approximation and numerical results are provided. The material derivative method is employed in monograph [2] to calculate the sensitivity of solutions to contact problems as well as the derivatives of domain depending functionals with respect to variations of the boundary of the domain occupied by the body. Shape optimization of a dynamic contact problem with a given Coulomb friction and heat flow is considered in [3]. In this paper the material derivative method is employed to formulate a necessary optimality condition. The finite element method for the spatial derivatives and the finite difference method for the time derivatives are employed to discretize the optimization problem. The level set based method is applied to find numerically the optimal solution.

Topology optimization deals with the optimal material distribution within the body resulting in its optimal shape [4]. The topological derivative is employed to account variations of the solutions to state equations or shape functionals with respect to emerging of small holes in the interior of the domain occupied by the body. The notion of topological derivative and results concerning its application in optimization of elastic structures are reported in the series of papers [4–9]. Among others, paper [9] deals with the calculation of topological derivatives of solutions to Signorini and elastic contact problems. Asymptotic expansion method combined with transformation of energy functional are employed to calculate these derivatives. Simultaneous shape and topology optimization of Signorini and elastic frictionless contact problems are analyzed in papers [10,11]. In these papers the level set method is incorporated in numerical algorithms.

In structural optimization the level set method [12,13] is employed in numerical algorithms for tracking the evolution of the domain boundary on a fixed mesh and finding an optimal domain. This method is based on an implicit representation of the boundaries of the optimized structure. A level set model describes the boundary of the body as an isocontour of a scalar function of a higher dimensionality. While the shape of the structure may undergo major changes the level set function remains to be simple in its topology. Level set methods are numerically efficient and robust procedures for the tracking of interfaces, which allow domain boundary shape changes in the course of iteration. Applications of the level set methods in structural optimization can be found, among others, in [3,7,14–17]. The speed vector field driving the propagation of the level set function is given by the Eulerian derivative of an appropriately defined cost functional with respect to the variations of the free boundary. Recently, in the series of papers [18–20] different numerical improvements of the level set method employed for the numerical solution of the structural optimization problems are proposed and numerically tested.

This paper deals with topology and shape optimization of an elastic contact problems. The structural optimization problem for elastic contact problem is formulated. Shape as well as topological derivatives formulae of the cost functional are provided using the material derivative [2] and the asymptotic expansion [4] methods, respectively. These derivatives are employed to formulate necessary optimality condition for simultaneous shape and topology optimization. Level set based numerical algorithm for the solution of the shape or topology optimization problem is proposed. The finite element method is used as the discretization method. Numerical examples are provided and discussed. This paper extends results of [11] to contact problems with the prescribed friction.

2. Problem formulation

Consider deformations of an elastic body occupying a two-dimensional domain Ω with Lipschitz continuous boundary Γ, i.e., the function describing the boundary Γ of the domain Ω is continuous and the increment of its value is bounded. Assume $\Omega \subset D$ where D is a given bounded hold-all subset of R^2 with piecewise smooth boundary containing domain Ω as well as all perturbations of domain Ω. The body is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. We assume that the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i,j = 0,1,2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i,j = 1,2$, the stress field in the body. Consider elastic bodies obeying Hooke's law [1]:

$$\sigma_{ij}(u(x)) = a_{ijkl}(x) e_{kl}(u(x)), \quad i,j,k,l = 1,2, \ x \in \Omega. \tag{1}$$

Functions $a_{ijkl}(x)$ are components of elasticity tensor satisfying usual symmetry, boundedness and ellipticity conditions [1]. We use here and throughout the paper the summation convention over repeated indices [1]. The strain $e_{kl}(u(x))$, $k,l = 1,2$, is defined by

$$e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \quad u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}. \tag{2}$$

The stress field $\sigma(u(x))$ satisfies the system of equations [1–3]

$$-\sigma_{ij}(u(x))_{,j} = f_i(x), \quad x \in \Omega, \ i,j = 1,2, \tag{3}$$

$\sigma_{ij}(u(x))_{,j} = \partial \sigma_{ij}(u(x))/\partial x_j$, $i,j = 1,2$. The following boundary conditions are imposed:

$$u_i(x) = 0, \quad i = 1,2 \text{ on } \Gamma_0, \tag{4}$$

$$\sigma_{ij}(x) n_j = p_i, \quad i,j = 1,2 \text{ on } \Gamma_1, \tag{5}$$

$$u_N \leqslant 0, \quad \sigma_N \leqslant 0, \quad u_N \sigma_N = 0 \text{ on } \Gamma_2, \tag{6}$$

$$|\sigma_T| \leqslant 1, \quad u_T \sigma_T + |u_T| = 0 \text{ on } \Gamma_2, \tag{7}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i,j = 1,2$, represent the normal components of displacement u and stress σ, respectively. The tangential components of displacement u and stress σ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i,j = 1,2$, respectively. $|u_T|$ denotes the Euclidean norm in R^2 of the tangent vector u_T.

2.1. *Variational formulation of contact problem*

Let us formulate contact problem (3)–(7) in variational form. Denote by V_{sp} and K the space and set of kinematically admissible displacements:

$$V_{sp} = \{z \in [H^1(\Omega)]^2 : z_i = 0 \text{ on } \Gamma_0, \ i = 1,2\}, \tag{8}$$
$$K = \{z \in V_{sp} : z_N \leqslant 0 \text{ on } \Gamma_2\}.$$

$H^1(\Omega)$ denotes Sobolev space of square integrable functions and their first derivatives. $[H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega)$. Denote also by Λ the set

$$\Lambda = \{\zeta \in L^2(\Gamma_2) : |\zeta| \leqslant 1\}.$$

Variational formulation of problem (3)–(7) has the form: *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying*

$$\int_\Omega a_{ijkl} e_{ij}(u) e_{kl}(\varphi - u)\, dx - \int_\Omega f_i(\varphi_i - u_i)\, dx - \int_{\Gamma_1} p_i(\varphi_i - u_i)\, ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T)\, ds \geqslant 0, \quad \forall \varphi \in K, \tag{9}$$

$$\int_{\Gamma_2} (\zeta - \lambda) u_T\, ds \leqslant 0, \quad \forall \zeta \in \Lambda, \tag{10}$$

$i,j,k,l = 1,2$. Function λ is interpreted as a Lagrange multiplier corresponding to term $|u_T|$ in equality constraint in (7) [1,2]. This function is equal to tangent stress along the boundary Γ_2, i.e., $\lambda = \sigma_{T|\Gamma_2}$. Function λ belongs to the space $H^{-1/2}(\Gamma_2)$, i.e., the space of traces on the boundary Γ_2 of functions from the space $H^1(\Omega)$. Here following [1] function λ is assumed to be more regular, i.e., $\lambda \in L^2(\Gamma_2)$. The results concerning the existence of solutions to system (9)–(10) can be found, among others, in [1].

2.2. *Optimization problem*

Before formulating a structural optimization problem for (9)–(10) let us introduce the set U_{ad} of admissible domains. Denote by $Vol(\Omega)$ the volume of the domain Ω equal to

$$Vol(\Omega) = \int_\Omega dx. \tag{11}$$

Domain Ω is assumed to satisfy the volume constraint of the form

$$Vol(\Omega) - Vol^{giv} \leqslant 0, \tag{12}$$

where the constant $Vol^{giv} = const_0 > 0$ is given. In a case of shape optimization of problem (9)–(10) the optimized domain Ω is assumed to satisfy equality volume condition, i.e., (12) is assumed to be satisfied as equality. In a case of topology optimization Vol^{giv} is assumed to be the initial domain volume and (12) is satisfied in the form $Vol(\Omega) = r_{fr} Vol^{giv}$ with $r_{fr} \in (0,1)$ [20]. The set U_{ad} has the following form:

$$U_{ad} = \{\Omega : E \subset \Omega \subset D \subset R^2 : \Omega \text{ is Lipschitz continuous}, \\ \Omega \text{ satisfies condition (12)}, \ P_D(\Omega) \leqslant const_1\}, \tag{13}$$

where $E \subset R^2$ is a given domain such that Ω as well as all perturbations of it satisfy $E \subset \Omega$. $P_D(\Omega) = \int_\Gamma \mathrm{d}x$ is a perimeter of a domain Ω in D [2, p. 126, 21]. The perimeter constraint is added in (13) to ensure the compactness of the set U_{ad} in the square integrable topology of characteristic functions as well as the existence of optimal domains. The constant $const_1 > 0$ is assumed to exist. The set U_{ad} is assumed to be nonempty. In order to define a cost functional we shall also need the following set M^{st} of auxiliary functions

$$M^{st} = \{\phi = (\phi_1, \phi_2) \in [H^1(D)]^2 : \phi_i \leqslant 0 \text{ on } D, \ i = 1, 2, \\ \|\phi\|_{[H^1(D)]^2} \leqslant 1\}, \tag{14}$$

where the norm $\|\phi\|_{[H^1(D)]^2} = (\sum_{i=1}^2 \|\phi_i\|^2_{H^1(D)})^{1/2}$.

In order to formulate an optimization problem for system (9)–(10) we have to define the cost functional. Measurements and engineering practice indicate that when two surfaces are in contact a large stress along the contact boundary occurs. The goal of structural engineers is to reduce this maximal value of the stress as much as possible. Thus the cost functional $S(\Gamma_2) = \max_{x \in \Gamma_2} |\sigma_N(x)|$ is natural criterion of optimization directly reflecting the design objectives. Unfortunately, the optimization problem with the cost functional $S(\Gamma_2)$ is nonsmooth and difficult for analysis and numerical solution [1]. This is the reason, that the criterion of maximal contact stress is approximated by integral, differentiable functionals. Recall from [3] the cost functional approximating the normal contact stress on the contact boundary

$$J_\phi(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\phi_N(x)\,\mathrm{d}s, \tag{15}$$

depending on the auxiliary given bounded function $\phi(x) \in M^{st}$. σ_N and ϕ_N are the normal components of the stress field σ corresponding to a solution u satisfying system (9)–(10) and the function ϕ, respectively.

Consider the following structural optimization problem: *for a given function $\phi \in M^{st}$, find a domain $\Omega^\star \in U_{ad}$ such that*

$$J_\phi(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\phi(u(\Omega)). \tag{16}$$

The existence of an optimal domain $\Omega^\star \in U_{ad}$ follows by standard arguments (see [2,21]).

3. Optimality conditions

Let us recall the optimality conditions for structural optimization problem (16). We consider this problem either as the shape optimization problem or the topological optimization problem.

3.1. Shape derivative

Consider variations of domain $\Omega \subset D$ with respect to the boundary Γ only. Assume that in (13) volume condition is satisfied as equality, i.e., constant volume condition holds. Let τ be a given parameter such that $0 \leqslant \tau < \tau_0$, τ_0 is prescribed, and $V = V(x, \tau)$, $x \in \Omega$, be a given admissible velocity field. The set of admissible velocity fields V consists from vector fields regular enough (C^k class, $k \geqslant 1$, for details see [2]) with respect to x and τ and such that on the boundary ∂D of D either $V = 0$ at singular points of this boundary or normal component $V \cdot n$ of V equals to $V \cdot n = 0$ at points of this boundary where the outward unit normal field n exists. Therefore the perturbations of domain Ω are governed by the transformation $T(\tau, V) : \bar{D} \to \bar{D}$, i.e., $\Omega_\tau = T(\tau, V)(\Omega)$ [2]. Since only small perturbations of Ω are considered this transformation can have the form of perturbation of the identity operator I in R^2. An example of such transformation is $T(\tau, \tilde{V}) = I + \tau\tilde{V}(x)$, where $\tilde{V}$ denotes a smooth vector field defined on R^2 [2]. The Euler derivative of the domain functional $J_\phi(\Omega)$ is defined as

$$\mathrm{d}J_\phi(\Omega, V) = \lim_{\tau \to 0^+} \frac{J_\phi(\Omega_\tau) - J_\phi(\Omega)}{\tau}. \tag{17}$$

In [3], using the material derivative approach [2], the Euler derivative of the cost functional (15) has been calculated and a necessary optimality condition for the shape optimization problem (16) has been formulated. This Euler derivative has the form

$$\begin{aligned} &\mathrm{d}J_\phi(u(\Omega); V) \\ &= \int_\Gamma (\sigma_{ij} e_{kl}(\phi + p^{adt}) - f \cdot \phi) V(0) \cdot n\,\mathrm{d}s \\ &\quad - \int_{\Gamma_1} \left[\frac{\partial(p \cdot (p^{adt} + \phi))}{\partial n} + \kappa p \cdot (p^{adt} + \phi) \right] V(0) \cdot n\,\mathrm{d}s \\ &\quad + \int_{\Gamma_2} [\lambda(p_T^{adt} + \phi_T) + q^{adt} u_T] \kappa V(0) \cdot n\,\mathrm{d}s, \end{aligned} \tag{18}$$

where $i, j, k, l = 1, 2$, $V(0) = V(x, 0)$, the displacement $u \in V_{sp}$ and the stress $\lambda \in \Lambda$ satisfy state system (9)–(10). κ denotes the mean curvature of the boundary Γ. The adjoint functions $p^{adt} \in K_1$ and $q^{adt} \in \Lambda_1$ satisfy for $i, j, k, l = 1, 2$, the following system:

$$\int_\Omega a_{ijkl} e_{ij}(\phi + p^{adt}) e_{kl}(\varphi)\,\mathrm{d}x + \int_{\Gamma_2} q^{adt} \varphi_T\,\mathrm{d}s = 0, \quad \forall \varphi \in K_1 \tag{19}$$

and

$$\int_{\Gamma_2} \zeta(p_T^{adt} + \phi_T)\,\mathrm{d}s = 0, \quad \forall \zeta \in \Lambda_1, \tag{20}$$

where the cones K_1 and Λ_1 are given by [3]

$$K_1 = \{\xi \in V_{sp} : \xi_N = 0 \text{ on } A^{st}\},$$
$$\Lambda_1 = \{\zeta \in L^2(\Gamma_2) : \zeta(x) = 0 \text{ on } B_1 \cup B_2 \cup B_1^+ \cup B_2^+\},$$

while the coincidence set $A^{st} = \{x \in \Gamma_2 : u_N = 0\}$. Moreover $B_1 = \{x \in \Gamma_2 : \lambda(x) = -1\}$, $B_2 = \{x \in \Gamma_2 : \lambda(x) = +1\}$, $\tilde{B}_i = \{x \in B_i : u_N(x) = 0\}$, $i = 1, 2$, $B_i^+ = B_i \backslash \tilde{B}_i$, $i = 1, 2$. The necessary optimality condition is formulated in [22].

Lemma 3.1. *Let $\Omega^\star \in U_{ad}$ be an optimal solution to the problem* (16). *Then there exist Lagrange multipliers $\mu_1 \in R$ associated with the constant volume constraint and $\mu_2 \in R$, $\mu_2 \geqslant 0$, associated with the finite perimeter constraint such that for all admissible vector fields V and such that all perturbations $\delta\Omega \in U_{ad}$ of domain $\Omega \in U_{ad}$ satisfy $E \subset \Omega \cup \delta\Omega \subset D$, at any optimal solution $\Omega^\star \in U_{ad}$ to the shape optimization problem* (16) *the following conditions are satisfied*:

$$\mathrm{d}J_\phi(u(\Omega^\star); V) + \mu_1 \int_{\Gamma^\star} V(0) \cdot n\,\mathrm{d}s + \mu_2\,\mathrm{d}P_D(\Omega^\star; V) \geqslant 0, \tag{21}$$

$$\mu_1 \left(\int_{\Omega^\star} \mathrm{d}x - const_0 \right) = 0, \tag{22}$$

$$(\tilde{\mu_2} - \mu_2)(P_D(\Omega^\star) - const_1) \leqslant 0, \quad \forall \tilde{\mu_2} \in R, \ \tilde{\mu_2} \geqslant 0, \tag{23}$$

where $u(\Omega^\star)$ denotes the solution to (9)–(10) *in the domain $\Omega^\star$, $\Gamma^\star = \partial\Omega^\star$, the Euler derivative $\mathrm{d}J_\phi(u(\Omega^\star); V)$ is given by* (18) *and $\mathrm{d}P_D(\Omega; V)$ denotes the Euler derivative of the finite perimeter functional $P_D(\Omega)$ (see* [2, p. 126]*). The given constant $const_0 > 0$ and constant $const_1 > 0$ are the same as in* (13).

3.2. Topological derivative

Classical shape optimization is based on the perturbation of the boundary of the initial shape domain. The initial and final shape domains have the same topology. The aim of the topological optimization is to find an optimal shape without any a priori assumption about the structure's topology.

The value of the goal functional (15) can be minimized by the topology variation of the domain Ω. The topology variations of geometrical domains are defined as a function of a small parameter ρ such that $0<\rho<R$, $R>0$ given. They are based on the creation of a small hole $B(x,\rho)=\{z\in R^2 : |x-z|<\rho\}$ of radius ρ at a point $x\in\Omega$ in the interior of the domain Ω. The Neumann boundary conditions are prescribed on the boundary ∂B of the hole. Denote by $\Omega_\rho=\Omega\backslash\overline{B(x,\rho)}$ the perturbed domain. The topological derivative $TJ_\phi(\Omega,x)$ of the domain functional $J_\phi(\Omega)$ at $\Omega\subset R^2$ is a function depending on a center x of the small hole and is defined by [4,6,14]

$$TJ_\phi(\Omega,x)=\lim_{\rho\to0^+}[J_\phi(\Omega\backslash\overline{B(x,\rho)})-J_\phi(\Omega)]/\pi\rho^2. \tag{24}$$

This derivative is calculated by the asymptotic expansion method [4]. To minimize the cost functional $J_\phi(\Omega)$ the holes have to be created at the points of domain Ω where TJ_ϕ is negative.

The formulae for topological derivatives of cost functionals for plane elasticity systems or contact problems are provided, among others, in papers [5,8,9]. Using the methodology from [4] as well as the results of differentiability of solutions to variational inequalities [2], we can calculate the formulae of the topological derivative $TJ_\phi(\Omega;x_0)$ of the cost functional (15) at a point $x_0\in\Omega$. This derivative is equal to

$$\begin{aligned}TJ_\phi(\Omega,x_0)&\\ =-&\left[f(\phi+w^{adt})+\frac{1}{E}(a_u a_{w^{adt}+\phi}+2b_u b_{w^{adt}+\phi}\cos2\delta)\right]_{|x=x_0}\\ &-\int_{\Gamma_2}(s^{adt}u_T+\lambda(w_T^{adt}+\phi_T))\kappa\,\mathrm{d}s,\end{aligned} \tag{25}$$

where $a_{\tilde\beta}=\sigma_I(\tilde\beta)+\sigma_{II}(\tilde\beta)$, $b_{\tilde\beta}=\sigma_I(\tilde\beta)-\sigma_{II}(\tilde\beta)$, and either $\tilde\beta=$ "u" or $\tilde\beta=$ "$w^{adt}+\phi$", $\sigma_I(u)$ and $\sigma_{II}(u)$ denote principal stresses for displacement u, δ is the angle between principal stresses directions. E denotes Young's modulus. The dependance of tangent displacement and stress functions on ρ along Γ_2 is assumed. The adjoint state $(w_\rho^{adt},s_\rho^{adt})\in K_1\times\Lambda_1$ satisfies system (19)–(20) in domain Ω_ρ rather than Ω, i.e.,

$$\int_{\Omega_\rho}a_{ijkl}e_{ij}(\phi+w_\rho^{adt})e_{kl}(\varphi)\,\mathrm{d}x+\int_{\Gamma_2}s_\rho^{adt}\varphi_T\,\mathrm{d}s=0,\quad\forall\varphi\in K_1 \tag{26}$$

and

$$\int_{\Gamma_2}\zeta(w_{\rho T}^{adt}+\phi_T)\,\mathrm{d}s=0,\quad\forall\zeta\in\Lambda_1, \tag{27}$$

where $w^{adt}_{\rho\ |\rho=0}=w^{adt}(x_0)$. By standard arguments [2,8,21] it can be shown that if $\Omega^\star\in U_{ad}$ is an optimal domain to the problem (16) it satisfies for all $x_0\in\Omega^\star$ the necessary optimality condition of the form (21)–(23) with topological derivative (25) rather than Euler derivative (18) in (21) and inequality in (22) rather than equality as well as with Lagrange multiplier $\mu_1\geqslant0$.

3.3. *Domain differential*

Finally, consider the variation of the functional (15) resulting both from the nucleation of the internal small hole as well as from the boundary variations. In order to take into account these perturbations, in [8] the notion of the domain differential of the domain functional has been introduced. The domain differential $DJ_\phi(\Omega;V,x_0)$ of the shape functional (15) at $\Omega\subset R^2$ in direction V and at point $x_0\in\Omega$ is defined as

$$DJ_\phi(\Omega;V,x_0)(\tau,\rho)=\tau\,\mathrm{d}J_\phi(\Omega,V)+\pi\rho^2TJ_\phi(\Omega,x_0). \tag{28}$$

This differential completely characterizes the variation of the cost functional $J_\phi(\Omega)$ with respect to the simultaneous shape and topology perturbations provided that the constant volume condition holds (for details see [8]). The shape derivative $\mathrm{d}J_\phi(\Omega,V)$ and the topological derivative $TJ_\phi(\Omega,x_0)$ are provided by (18) and (25), respectively. Using standard arguments [8] we can show that if $\Omega^\star\in U_{ad}$ is an optimal domain to problem (16) it satisfies for all admissible velocity fields V, for all admissible pairs (ρ,τ) of parameters and for all $x_0\in\Omega^\star$ the necessary optimality condition of the form (21)–(23) with the domain differential (28) rather than Euler derivative (18).

4. Shape representation by level set method

In order to solve structural optimization problem (16) in numerical algorithm we employ the level set method [13] to describe the position of the boundary $\partial\Omega$ of the design domain $\Omega\subset D\subset R^2$ as well as its evolution. It is well established [13,14,17,23] that the level set formulations of moving interface problems possess several advantages including flexibility with respect to topology changes, the possibility to use fixed grids, low computational cost and robustness.

The level set method is based on implicit description of the boundary of the domain. An implicit representation of the boundary of the domain is based on defining it as the isocontour of some function Φ defined on the hold-all domain D. Consider the evolution of a domain Ω under a velocity field V. Let $t\in[0,t_0)$, $t_0>0$ given, denote the (artificial) time variable. Under the mapping $T(t,V)$ we have $\Omega_t=T(t,V)(\Omega)$. By Ω_t^- and Ω_t^+ we denote the interior and the outside of the domain Ω_t, respectively. The domain Ω_t and its boundary $\partial\Omega_t$ are defined by a function $\Phi=\Phi(x,t):R^2\times[0,t_0)\to R$ satisfying

$$\begin{cases}\Phi(x,t)=0 & \text{if } x\in\partial\Omega_t,\\ \Phi(x,t)<0 & \text{if } x\in\Omega_t^-,\\ \Phi(x,t)>0 & \text{if } x\in\Omega_t^+,\end{cases} \tag{29}$$

i.e., the boundary $\partial\Omega_t$ is the level curve of the function Φ. Recall [13], the gradient of the implicit function is defined as $\nabla\Phi=(\partial\Phi/\partial x_1,\partial\Phi/\partial x_2)$, the local unit outward normal n to the boundary is equal to $n=\nabla\Phi/|\nabla\Phi|$, the mean curvature $\kappa=\nabla\cdot n$. In the level set approach Heaviside function $H(\Phi)$ and Dirac function $\delta(\Phi)$ are used to transform integrals from domain Ω into domain D. These functions are defined as

$$H(\Phi)=1\ \text{if}\ \Phi\geqslant0,\quad H(\Phi)=0\ \text{if}\ \Phi<0, \tag{30}$$

$$\delta(\Phi)=H'(\Phi),\quad \delta(x)=\delta(\Phi(x))|\nabla\Phi(x)|,\quad x\in D. \tag{31}$$

The implicit function Φ is used both to represent and to evolve the domain boundary. Recall [13] differentiating with respect to t the interface equation

$$\Phi(x(t),t)=0,$$

and using $\mathrm{d}x/\mathrm{d}t=V(x,t)$ for all x with $\Phi(x,t)=0$ as well as using the formula for the local unit outward normal n to the boundary leads to Hamilton–Jacobi equation governing the evolution of the domain boundary

$$\Phi_t(x,t)+V(x,t)\cdot n|\nabla\Phi(x,t)|=0\ \text{in}\ D\times[0,t_0), \tag{32}$$

where Φ_t denotes a partial derivative of Φ with respect to the time variable t and $V\cdot n$ is the normal component of velocity field V on the boundary of the domain. The initial condition is $\Phi(x,0)=\Phi_0(x)$ with $\Phi_0(x)=\pm\mathrm{dist}(x,\partial\Omega_0)$, i.e., it is chosen as the signed distance function to the initial boundary $\partial\Omega_0$ with the minus sign if the point x is inside the initial domain Ω_0. The homogeneous Neumann boundary condition is imposed on the whole boundary ∂D.

4.1. Structural optimization problem in domain D

Using the notion of the level set function (29) as well as functions (30) and (31) structural optimization problem (16) may be reformulated in the following way: *for a given function* $\phi \in M^{st}$, *find function* Φ *such that*

$$J_\phi(u(\Phi^\star)) = \min_{\Phi \in U_{ad}^\Phi} J_\phi(u(\Phi)), \tag{33}$$

where

$$J_\phi(u(\Phi)) = \int_D \sigma_N(u)\phi_N(x)\delta(\Phi)|\nabla\Phi|\, \mathrm{d}s, \tag{34}$$

$$\begin{aligned} U_{ad}^\Phi = \{&\Phi : \Phi \text{ satisfies (29)},\ Vol(\Phi) \leqslant Vol^{giv}, \\ &P_D(\Phi) \leqslant const_1\}, \end{aligned} \tag{35}$$

$$Vol(\Phi) = \int_D H(\Phi)\, \mathrm{d}x, \quad P_D(\Phi) = \int_D \delta(\Phi)|\nabla\Phi|\, \mathrm{d}x.$$

Moreover, a pair $(u, \lambda) \in K \times \Lambda$ satisfies system

$$\begin{aligned} &\int_D a_{ijkl} e_{ij}(u) e_{kl}(\varphi - u) H(\Phi)\, \mathrm{d}x - \int_D f_i(\varphi_i - u_i) H(\Phi)\, \mathrm{d}x \\ &\quad - \int_D p_i(\varphi_i - u_i)\delta(\Phi)|\nabla\Phi|\, \mathrm{d}x \\ &\quad + \int_D \lambda(\varphi_T - u_T)\delta(\Phi)|\nabla\Phi|\, \mathrm{d}x \geqslant 0, \quad \forall\varphi \in K, \end{aligned} \tag{36}$$

$$\int_D (\zeta - \lambda) u_T \delta(\Phi)|\nabla\Phi|\, \mathrm{d}x \leqslant 0, \quad \forall\zeta \in \Lambda, \tag{37}$$

while V_{sp} and K are defined by (8) and (9), respectively, on domain D rather than Ω and $i, j, k, l = 1, 2$.

5. Level set based numerical algorithm

The topological derivative can provide better prediction of the structure topology with different levels of material volume than the method based on updating the shape of initial structure containing many regularly distributed holes [4,14]. Our approach is based on the application of the topological derivative to predict the structure topology and substitute material according to the material volume constraint and next to optimize the structure topology to merge the unreasonable material interfaces and to change the shape of material boundary. For the sake of simplicity in the description of the algorithm we omit the bounded perimeter constraint in (13). Therefore, the level set method combined with the shape or topological derivatives results in the following conceptual algorithm (A1) to solve structural optimization problem (16):

Step 1: Choose: a computational domain D such that $\Omega \subset D$, an initial level set function $\Phi^0 = \Phi_0$ representing $\Omega^0 = \Omega$, function $\phi \in M^{st}$, parameters r^0, $\varepsilon_1, \varepsilon_2, q, r_{fr} \in (0, 1)$. Set $m_0 = Vol(\Omega^0)$, $\tilde{\mu}_1^0 = \mu_1^0 = 0$, $k = n = 0$.

Step 2: Calculate the solution (u^n, λ^n) to the state system (36)–(37).

Step 3: Calculate the solution $((w^{adt})^n, (s^{adt})^n)$ to the adjoint system (26)–(27) as well as the topological derivative $TJ_\phi(\Omega^n, x)$ of the cost functional (15) given by (25).

Step 4: For given $\tilde{\mu}_1^n$ set $\Omega^{n+1} = \{x \in \Omega^n : TJ_\phi(\Omega, x) \geqslant \chi_{n+1}\}$ where χ_{n+1} is chosen in such a way that $Vol(\Omega^{n+1}) = m_{n+1}$, $m_{n+1} = qm_n$. Fill the void part $D \backslash \Omega^{n+1}$ with a very weak material with Young's modulus $E^w = 10^{-5}E$. Update $\tilde{\mu}_1^{n+1} = \tilde{\mu}_1^n + r^n(Vol_1^{giv})$, $r^n > 0$, $Vol_1^{giv} = Vol(\Omega^{n+1}) - r_{fr}Vol^{giv}$. If $|\tilde{\mu_1}^{n+1} - \tilde{\mu_1}^n| \leqslant \varepsilon_1$ then set $\Omega^k = \Omega^{n+1}$ and go to Step 5. Otherwise set $n = n + 1$, goto Step 2.

Step 5: Calculate the solution $((p^{adt})^k, (q^{adt})^k)$ to the adjoint system (19)–(20). Calculate the shape derivative $\mathrm{d}J_\phi(u(\Omega^k))$ of the cost functional (15) given by (18).

Step 6: For given μ_1^k solve the level set equation (32) to calculate the level set function Φ^{k+1}.

Step 7: Set Ω^{k+1} equal to the zero level set of function Φ^{k+1}. Calculate $\mu_1^{k+1} = \mu_1^k + r^k(Vol(\Omega^{k+1}) - Vol_1^{giv})$, $r^k > 0$. If $|\mu_1^{k+1} - \mu_1^k| \leqslant \varepsilon_2$ then Stop. Otherwise set $k = k + 1$, $\Omega^n = \Omega^{k+1}$ and go to Step 2.

Let us describe some details of this algorithm and indicate its modifications to solve shape and/or topology optimization problems.

5.1. Extended normal velocity

In order to solve the level set equation (32) in Step 6 of the aforementioned algorithm (A1) the normal velocity $V(0) \cdot n$ has to be determined on the whole domain D. Since the normal velocity is determined on the boundary Γ_2 only it has to be extended to the domain D. Following [13] the extension $V_{ext}(x, t) \cdot n$ of $V(x, t) \cdot n$ is calculated as a solution q to the following auxiliary equation up to the stationary state:

$$q_\tau + S(\Phi)\frac{\nabla\Phi}{|\nabla\Phi|}\nabla q = 0 \quad \text{in } D \times (0, t_0), \tag{38}$$

$$q(x, 0) = p(x, t), \quad x \in D, \tag{39}$$

where $p(x, t) = V(x, t) \cdot n$ on Γ_2 and 0 elsewhere. The function $S(\Phi)$ approximating the sign distance function is given by

$$S(\Phi) = \frac{\Phi}{\sqrt{\Phi^2 + |\nabla\Phi|^2\gamma_{min}}},$$

where $\gamma_{min} = \min(\Delta x_1, \Delta x_2)$ and Δx_i, $i = 1, 2$, denote discretizations steps in directions x_i, respectively. Let us remark that the extension of velocity V to the whole domain D allows to enforce the solution Φ to the level set equation to remain close to a distance function. For the discussion of the other extension methods of normal velocity see [19,20].

5.2. Shape optimization problem

First we solve numerically the structural optimization problem (16) as the shape optimization problem only. We can employ algorithm (A1) omitting the Steps 3 and 4 dealing with the topology optimization. In this paper, following [23,24], we slightly modify this procedure for solving shape optimization problem (16). For the sake of simplicity we assume that the measure of the boundary Γ_2 is not equal to 0. In the modified algorithm first we neglect some inequality type boundary conditions on the boundary Γ_2 and solve the systems (9)–(10) and (19)–(20) with equality type boundary conditions only. Next the violation of inequality type boundary conditions is taken into account. This violation defines a distance of the actual computed configuration Γ_2 to the optimal one $\Gamma_2^\star$. The violation of the unilateral boundary conditions (6)–(7) is measured by an appropriate penalty type cost functional $K(\Gamma_2)$ depending on the boundary Γ_2. Let us define this cost functional as equal to

$$K(\Gamma_2) = \int_{\Gamma_2} \{c_1 \max{}^2(0, u_N) + c_2 \max{}^2(0, |\lambda| - 1)\}\, \mathrm{d}\Gamma, \tag{40}$$

where $(u, \lambda) \in V_{sp} \times L^2(\Gamma_2)$ satisfies the equality state system (9)–(10) and c_1, c_2 are given positive constants. Using the formulae from [2] Euler derivative of the cost functional (40) at Γ_2 in direction of velocity field V can be characterized as

$$\begin{aligned} \mathrm{d}K(\Gamma_2, V) = 2\int_{\Gamma_2} \Big\{ &c_1 \max(0, u_N)\left(\frac{\partial u_N}{\partial n} + \kappa u_N\right) p_N^{adt} \\ &+ c_2 \max(0, |\lambda| - 1) sgn(\lambda)\left(\frac{\partial\lambda}{\partial n} + \kappa\lambda\right) q_T^{adt} \Big\} \end{aligned}$$

$$V(0) \cdot n\, d\Gamma, \tag{41}$$

where the adjoint pair $(p^{adt}, q^{adt}) \in V_{sp} \times L^2(\Gamma_2)$ satisfies system (19)–(20) in a whole space $V_{sp} \times L^2(\Gamma_2)$ rather than in the cone $K_1 \times \Lambda_1$. $sgn(\cdot)$ denotes signum function. We solve the following optimization problem (P_E): *for given $\phi \in M^{st}$, find $\Omega \in U_{ad}$ minimizing*

$$J^{\sim}_{\phi}(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\phi_N(x)\, ds + K(\Gamma_2) + \mu_1(Vol(\Omega) - Vol^{giv}), \tag{42}$$

μ_1 is Lagrange multiplier associated with constant volume condition. Euler derivative of this functional is equal to

$$\begin{aligned} dJ^{\sim}_{\phi}(u(\Omega); V) &= \int_{\Gamma_2} GV(0) \cdot n\, d\Gamma \\ &= dJ_{\phi}(u(\Omega); V) + dK(\Gamma_2; V) \\ &\quad + \mu_1 \int_{\Gamma_2} V(0) \cdot n\, d\Gamma, \end{aligned} \tag{43}$$

where the Euler derivatives $dJ_{\phi}(u(\Omega), V)$ and $dK(\Gamma_2, V)$ are given by (18) and (41), respectively. The shape gradient G of the cost functional (42) with respect to the variation of the boundary Γ_2 is used as the velocity field in Eq. (32) to define a family of propagating interfaces $\Gamma_2(t) = \{x(t) : x_0 \in \Gamma_2(0)\}$ in time $t \in [0, t_0)$. Therefore, the conceptual level set based algorithm (A2) for solving the shape optimization problem (16) can be described as follows:

Step 1: Choose an initial domain Ω^0, $\mu_1^0 = 0$, r^0, $\varepsilon_1 \in (0, 1)$. Set $n = 0$.

Step 2: Evaluate the cost functional (42) and compute its Euler derivative (43) with respect to perturbation of Γ_2^n.

Step 3: For given μ_1^n, calculate an extension of the shape gradient G^n of cost functional (42) and use it as the speed function in the level set equation (32) for updating the level set function Φ^{n+1}.

Step 4: Set Ω^{n+1} equal to the zero level set of the calculated level set function Φ^{n+1}. Set: $\mu_1^{n+1} = \mu_1^n + r^n(Vol(\Omega^{n+1}) - Vol^{giv})$, $r^n > 0$. If $|\mu_1^{n+1} - \mu_1^n| \leqslant \varepsilon_1$ then Stop. Otherwise set $n = n + 1$, and go to Step 2.

5.3. *Topology optimization problem*

For the sake of comparison next the optimization problem (16) is solved as a topology optimization problem only. Algorithm (A1) without level set method is employed to solve numerically this problem, i.e., in this case Steps 5–7 of algorithm (A1) dealing with shape optimization are omitted. The topological derivative is calculated at each grid point of design domain [5]. These points are sorted with respect to the calculated sensitivity and the points with the lowest sensitivity are removed, i.e., the circular small holes are inserted. The number of points removed at each step is given by a ratio equal to volume of elements removed divided by volume of elements of the previous structure. This ratio is usually taken between 5% and 15%. Void parts of the computational domain are filled with a very weak material having Young's modulus E^w much smaller than Young's modulus E of solid initial material. The computational process stops when a given volume constraint equal to the prescribed fraction of the initial volume of the structure is reached. It means that the Lagrange multiplier $\tilde{\mu}_1$ calculated in Step 4 associated to this volume constraint differs from the Lagrange multiplier calculated in the previous iteration less than prescribed tolerance and the optimality condition (21)–(22) is satisfied. In this algorithm $(m)_{n \geqslant 0}$ is assumed to be a decreasing sequence of volume constraints. This procedure may be also updated to be used to add material at grid points of domain $D \backslash \Omega$ [15,16].

5.4. *Topology and shape optimization problem*

Finally algorithm (A1) is employed to solve numerically structural optimization problem (16) considered as the simultaneous shape and topology optimization problem. In literature [14–17,20] many theoretical and numerical difficulties of incorporating the topological derivatives in the framework of level set based algorithms for solving structural optimization problems are reported. It is known that due to conditions imposed on time step ensuring the stability of the up-wind scheme to solve the level set equation, level set method lacks nucleation mechanism for new holes within existing shapes. The level set method is capable of performing topology changes in the evolving shape. Merging holes or breaking up of one hole into two are typical topology changes which can be treated using the level set method. Therefore, the level set based numerical algorithm can perform topology optimization either if the number of holes in the initial structure is sufficiently large [14,17,20] or if the additional source term taking into account topological changes is added as a velocity field to the Hamilton–Jacobi equation (32) [15,16] leading to switched topological and shape derivative algorithm. Recently, both approaches have been improved by using radial basis functions to approximate the level set function Φ [20] or improved source term based on the additional continuity conditions of the cost functional [16], respectively. Algorithm (A1) follows switched topological and shape derivative approach. First, the topology optimization problem is solved with material volume ratio constraint. Next the shape optimization step is performed using level set approach where the structure material volume calculated at the preceding step is being kept constant. The algorithm is stopped if the volume constraint is satisfied, i.e., the change of Lagrange multiplier μ_1 associated with this constraint is less than the prescribed tolerance. It means that the norm of the shape gradient is small enough.

6. Numerical implementation

In order to solve numerically structural optimization problem (16) we have to discretize it. Computational domain D, employed in solving this optimization problem, is divided into mesh of rectangles. Define the mesh grid of hold-all domain D. Let Δx_i, $i = 1, 2$, denote the space discretization steps in x_i, $i = 1, 2$, directions, respectively.

State (9)–(10) and adjoint (19)–(20) systems are discretized using finite element method [1]. Displacement and stress functions in state system (9)–(10) are approximated by piecewise bilinear functions in domain D and piecewise constant functions on the boundary Γ_2, respectively. Similar approximation is used to discretize the adjoint system (19)–(20) or (26)–(27). These systems are solved using the primal-dual algorithm with active set strategy [25]. In level set approach these state and adjoint systems are transferred from domain Ω into fixed hold-all domain D using the regularized Heaviside and Dirac functions. These functions are approximated by [17]

$$H(x) = \begin{cases} \alpha, & x < -\Delta, \\ \dfrac{3(1-\alpha)}{4}\left(\dfrac{x}{\Delta} - \dfrac{x^3}{3\Delta^3}\right) + \dfrac{1+\alpha}{2}, & -\Delta \leqslant x \leqslant \Delta, \\ 1, & x \geqslant \Delta, \end{cases} \tag{44}$$

$$\delta(x) = \begin{cases} \dfrac{3(1-\alpha)}{4\Delta}\left(1 - \dfrac{x^2}{\Delta^2}\right), & |x| \leqslant \Delta, \\ 0, & |x| > \Delta, \end{cases} \tag{45}$$

where $\alpha > 0$ is a small number ensuring the nonsingularity of the state equation (3) and $\Delta = \max(\Delta x_1, \Delta x_2)$ describes the width of numerical approximation for $\delta(x)$ and $H(x)$. The regularized Heaviside function (44) allows to evaluate numerically volume integrals transformed from Ω into D using a standard sampling technique. Similarly regularized Dirac function (45) allows to evaluate boundary integrals transformed into D. The embedding function Φ, describing the boundary of the optimized structure, may be represented in the form

$$\Phi(x,t) = \sum_i \Phi_i(t) N_i(x), \tag{46}$$

where $\Phi_i(t)$ are the nodal values of the level set function and $N_i(x)$ describe the standard interpolation functions [13]. The choice of these interpolation functions is governed by accuracy and computational costs requirements. The nodal values are updated during the optimization process.

Consider discretization of Hamilton–Jacobi equation (32). Let us denote $t^{n+1} = t^n + \Delta t$, $n = 1, 2, \ldots, N$, where Δt denotes the time step discretization. Moreover, $V_{ij}^n = V(x_{ij}, t^n)$, $x_{ij} = (x_1^i, x_2^j) \in D$ and $\Phi_{ij}^n = \Phi(x_{ij}, t^n)$. Following [13,17], the explicit up-wind scheme is used to solve the discrete Hamilton–Jacobi equation (32) with the following update equation:

$$\Phi_{ij}^{n+1} = \Phi_{ij}^n - \Delta t[\max(v_{ij}^n, 0)\nabla^+ + \min(v_{ij}^n, 0)\nabla^-], \tag{47}$$

where v_{ij}^n denotes extended normal velocity at point x_{ij} in time t^n and

$$\begin{aligned} \nabla^+ = {} & [\max(D_{ij}^{-x_1}\Phi^n, 0)^2 + \min(D_{ij}^{+x_1}\Phi^n, 0)^2 \\ & + \max(D_{ij}^{-x_2}\Phi^n, 0)^2 + \min(D_{ij}^{+x_2}\Phi^n, 0)^2]^{1/2}, \end{aligned}$$

$$\begin{aligned} \nabla^- = {} & [\max(D_{ij}^{+x_1}\Phi^n, 0)^2 + \min(D_{ij}^{-x_1}\Phi^n, 0)^2 \\ & + \max(D_{ij}^{+x_2}\Phi^n, 0)^2 + \min(D_{ij}^{-x_2}\Phi^n, 0)^2]^{1/2}. \end{aligned}$$

$D_{ij}^{+x_1}\Phi^n$, $D_{ij}^{+x_2}\Phi^n$ as well as $D_{ij}^{-x_1}\Phi^n$, $D_{ij}^{-x_2}\Phi^n$ are the forward and backward approximations of the x_1 and x_2 derivatives of function $\Phi^n = \Phi(x, t^n)$, respectively. To ensure stability of this scheme, time step Δt is required to satisfy the Courant–Friedrichs–Levy condition

$$\Delta t \max|V_{ij}^n| \leqslant \gamma_{min}. \tag{48}$$

Moreover, to increase the accuracy of the numerical results, the level set function $\Phi(x,t)$ is initialized as the signed distance function satisfying the eikonal equation

$$|\nabla\Phi(x,t)| = 1. \tag{49}$$

At each iteration of the scheme (47) we have to extend the normal velocity field to hold-all domain D according to (38), (39). The solution q_{ij}^k is computed based on the following up-wind approximation of (38) at each iteration n of the scheme (47)

$$\begin{aligned} q_{ij}^{k+1} = {} & q_{ij}^k - \Delta\tau[\max(s_{ij}n_1^{ij}, 0)D_{ij}^{-x_1}q^k + \min(s_{ij}n_1^{ij}, 0)D_{ij}^{+x_1}q^k \\ & + \max(s_{ij}n_2^{ij}, 0)D_{ij}^{-x_2}q^k + \min(s_{ij}n_2^{ij}, 0)D_{ij}^{+x_2}q^k], \end{aligned}$$

$s_{ij} = S(\Phi_{ij}^k)$ and $q^k = q(x, t^k)$. Central difference method is used to compute the approximations of the unit normal vector $n = (n_1, n_2)$, i.e., $n_1 = \nabla_{x_1}\Phi / \sqrt{\nabla_{x_1}\Phi^2 + \nabla_{x_2}\Phi^2}$, $n_2 = \nabla_{x_2}\Phi / \sqrt{\nabla_{x_1}\Phi^2 + \nabla_{x_2}\Phi^2}$ and $n_k^{ij} = n_k(x_{ij})$, $k = 1, 2$. The initial value of q_0 in (39) is equal to $V \cdot n$ at the grid points of D which distance from the interface is less than γ_{min} and equals 0 in other points.

7. Numerical examples and discussion

The discretized structural optimization problem (16) is solved numerically. The numerical algorithms described in the previous sections have been used. The algorithm is programmed in Matlab environment. As an example a body occupying 2D domain

$$\Omega = \{(x_1, x_2) \in R^2 : 0 \leqslant x_1 \leqslant 8 \wedge 0 < v(x_1) \leqslant x_2 \leqslant 4\}$$

is considered. The boundary Γ of the domain Ω is divided into three pieces

$$\begin{aligned} \Gamma_0 &= \{(x_1, x_2) \in R^2 : x_1 = 0, 8 \wedge 0 < v(x_1) \leqslant x_2 \leqslant 4\}, \\ \Gamma_1 &= \{(x_1, x_2) \in R^2 : 0 \leqslant x_1 \leqslant 8 \wedge x_2 = 4\}, \\ \Gamma_2 &= \{(x_1, x_2) \in R^2 : 0 \leqslant x_1 \leqslant 8 \wedge v(x_1) = x_2\}. \end{aligned}$$

The domain Ω and the boundary Γ_2 depend on the function v. This function is the variable subject to shape optimization. The initial position of the boundary Γ_2 is displayed in Fig. 1. The computations are carried out for the elastic body characterized by Poisson's ratio $\nu = 0.29$, Young's modulus $E = 2.1 \times 10^{11}\ \mathrm{N/m^2}$. The body is loaded by boundary traction $p_1 = 0, p_2 = -5.6 \times 10^6$ N along Γ_1, body forces $f_i = 0$, $i = 1, 2$. Auxiliary function ϕ is selected as piecewise constant (or linear) on D and is approximated by a piecewise constant (or bilinear) functions. The computational domain $D = [0, 8] \times [0, 4]$ is selected. Domain D is discretized with a fixed rectangular mesh of 24×12 and a time step size of $\Delta t = 10^{-3}$ is adopted. Regularization parameters of Heaviside and Dirac functions have values $\alpha = 10^{-5}$ and $\Delta = 2$, while error tolerance parameter in algorithm (A1) $\varepsilon_1 = \varepsilon_2 = 10^{-4}$. The penalty parameters in (40) have values $c_1 = 10^{-2}$, $c_2 = 10^{-2}$.

The obtained results are shown in Figs. 1–4. Fig. 1 displays the evolution of the zero level set in the computational domain D and the optimal domain obtained due to shape optimization algorithm (A2) only. The algorithm terminates at iteration 10 with the value of cost functional K equal to 1.4×10^{-3}. The zero level set describing the optimal boundary evolved from initially convex set to the nonconvex one at the optimal solution. The maximal normal contact stress has been reduced. The obtained optimal normal contact stress shown in Fig. 2 is almost constant along the optimal shape boundary $\Gamma_2^\star$. The cost functional value change in the iteration process is shown in Fig. 3.

Fig. 4 presents the optimal domain of structural optimization problem (16) obtained with using topological optimization procedure only. Material volume fraction $r_{fr} = 0.5$ is prescribed.

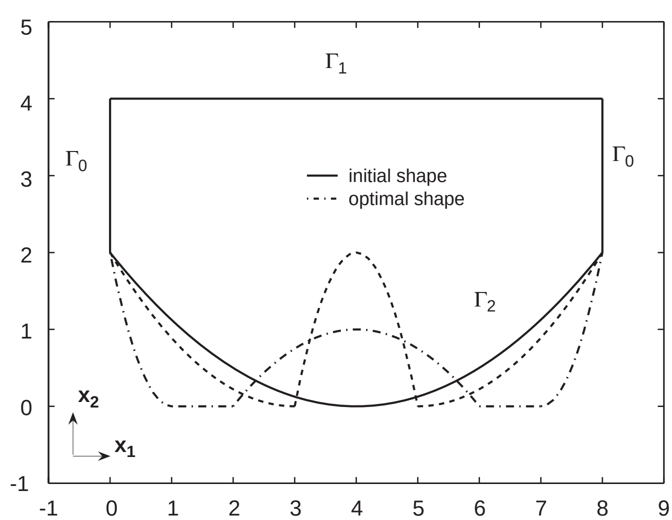

Fig. 1. Evolution of zero level set function Φ. Optimal domain—shape optimization.

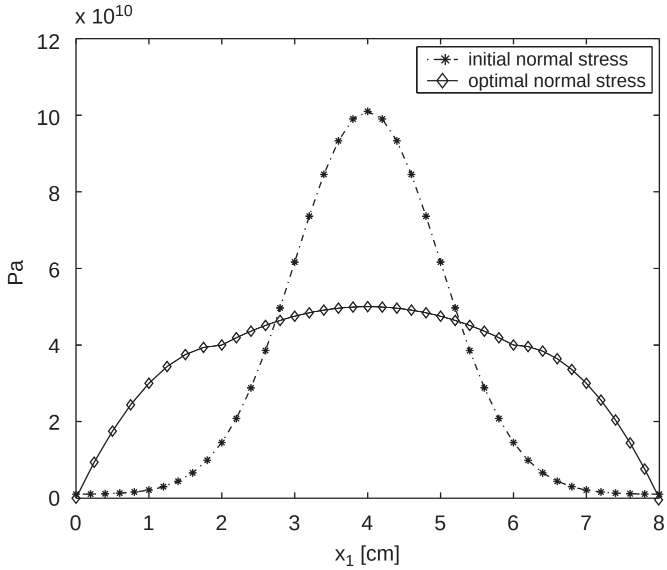

Fig. 2. Initial and optimal contact stress distribution—shape optimization.

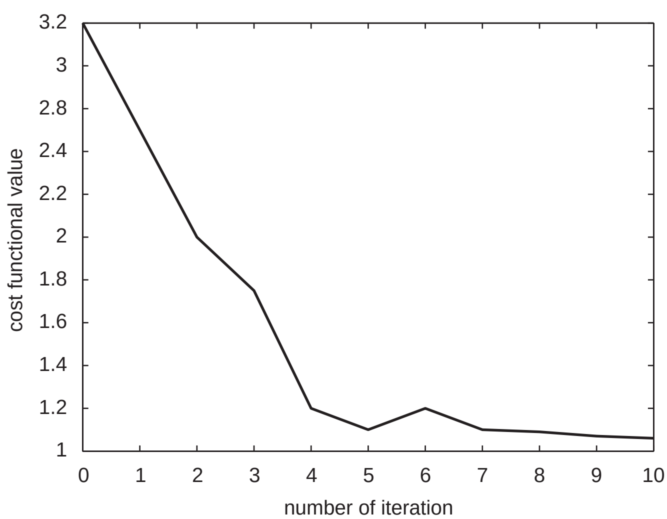

Fig. 3. Cost functional during the iteration process—shape optimization.

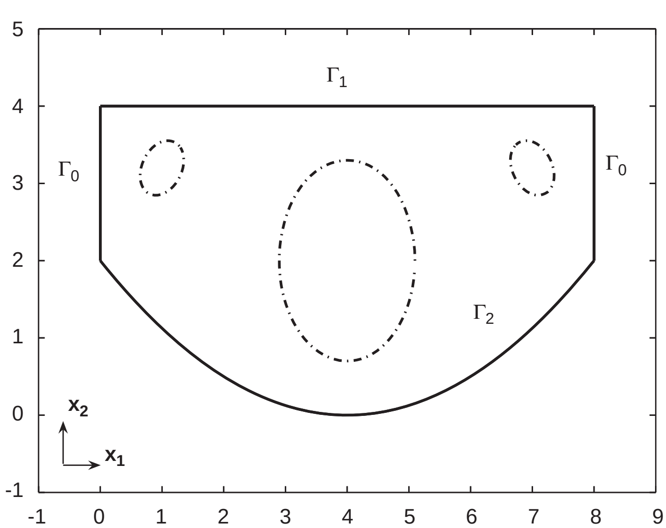

Fig. 4. The optimal domain—topology optimization.

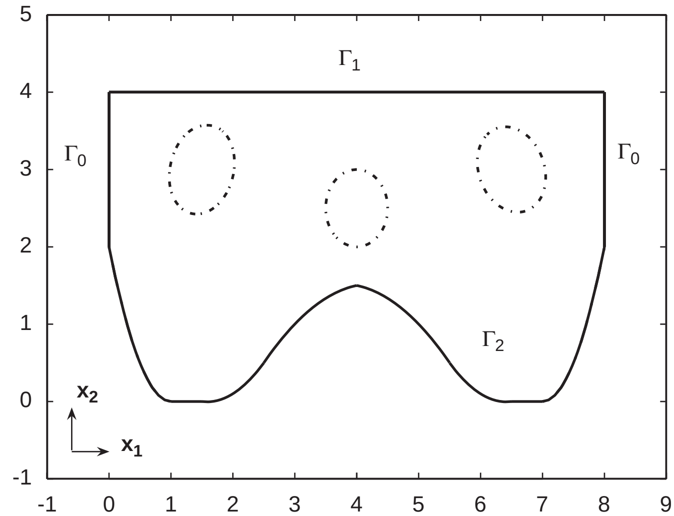

Fig. 5. The optimal domain—simultaneous topology and shape optimization.

Holes are denoted by dotted lines. The big hole appears in the central part of the optimal domain and two smaller ones near the fixed edges. Optimal normal contact stress distribution and cost functional value change during the iteration process are similar to shown in Figs. 2 and 3.

Fig. 5 presents the optimal domain obtained by solving topological and shape optimization problem in the computational domain D using algorithm (A1) and employing the optimality condition (21)–(23) with domain differential (28). As previously, the holes denoted also by dotted lines appear in the central part of the body and near the fixed edges. However, in this case, these holes are smaller than in the previous case of topology optimization only. Although the shape of the optimal contact boundary Γ_2 is similar to the optimal shape obtained in Fig. 1 but this shape is not so strongly changed as the optimal shape obtained in Fig. 1. The obtained normal contact stress is almost constant along the optimal shape boundary.

8. Conclusions

In the paper the topology and shape optimization problem for elastic contact problem with the prescribed friction is solved numerically using the topological derivative method as well as the level set approach combined with the shape gradient method. The friction term complicates both the form of the gradients of the cost or penalty functionals as well as numerical process. Obtained numerical results seem to be in accordance with physical reasoning. They indicate that the proposed numerical algorithm allows for significant improvements of the structure from one iteration to the next. They also indicate the future research direction aiming at better reconciliation in one algorithm procedures governing holes nucleation and shape evolution.

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