INTEGRAL REPRESENTATION FOR SOME GENERALIZED POLY-CAUCHY NUMBERS

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INTEGRAL REPRESENTATION FOR SOME GENERALIZED POLY-CAUCHY NUMBERS

J. CHIKHI

Abstract. In this note, we establish an integral representation for a special function $E_{s,\alpha}(z)$ and apply it to some generalized poly-Cauchy numbers $c_{n,\alpha}^{(s)} := n! [t^n] E_{s,\alpha}(\log(1+t))$. We recover, in the special case $\alpha = s = 1$, the integral representation of the Bernoulli numbers of the second kind $b_n = c_{n,1}^{(1)}/n!$ obtained by Feng Qi by quite different methods.

1. Introduction

Let $\alpha$ and $s$ be parameters, with $\alpha$ real and positive and $s$ complex with $\Re s > 0$. We define some generalized poly-Cauchy numbers, see [3] for $s = k$ integer, by the generating function

$$E_{s,\alpha}(\log(1+t)) = \sum_{n=0}^{\infty} c_{n,\alpha}^{(s)} \frac{t^n}{n!}, \quad (|t| < 1),$$

where $E_{s,\alpha}$ is the function, defined for any complex number $z$, by

$$E_{s,\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!(n+\alpha)^s}.$$

This function is called poly-exponential in [1] and, for $s = k$ a positive integer, the extended polylogarithm factorial function, see [3] for instance.

In the sequel, we
- obtain a first integral representation for $E_{s,\alpha}(z)$ and use it to obtain the main integral representation,
- apply it to get an integral representation for the generalized poly-Cauchy number $c_{n,\alpha}^{(s)}$,
- recover, by specialisation, the F. Qi’s integral representation for the Cauchy number $c_{n,1}^{(1)}$.

2. Mellin Type Integral Representations

The firts integral representation is basic and, more or less well known.

Proposition 1. The function $E_{s,\alpha}(z)$ admits the following integral representation,

$$E_{s,\alpha}(z) = \frac{1}{\Gamma(s)} \int_{0}^{1} (-\log x)^{s-1} x^{\alpha-1} e^{zx} dx.$$

Proof. We firts obtain a Mellin type integral representation. As $\alpha$ and $\Re s$ are positive, we have for any non negative integer $n$,

$$\frac{\Gamma(s)}{(n+\alpha)^s} = \int_{0}^{\infty} t^{n-1} e^{-(n+\alpha)t} dt,$$

so

$$E_{s,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+\alpha)^s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{0}^{\infty} t^{n-1} e^{-(n+\alpha)t} dt.$$
The convergence modes of the series and integral permit to exchange \( \Sigma \) and \( \int \) and obtain

\[
E_{s, \alpha}(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\alpha t} \left( \sum_{n=0}^{\infty} \frac{z^n e^{-\alpha t}}{n!} \right) dt
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\alpha t} e^{e^{-t}} dt .
\]

At the end, the change \( e^{-t} = x \) gives the desired formula. \( \square \)

3. MAIN INTEGRAL REPRESENTATION

Here is our main result.

**Theorem 2.** Let \( a \) and \( b \) be real numbers and \( z = a + ib \), then

\[
E_{s, \alpha}(z) = \frac{e^a}{2\pi} \int_0^\infty \frac{E_{s, \alpha}(z_1(u)) - E_{s, \alpha}(z_2(u))}{u(u + e^a)} \, du ,
\]

where \( z_1(u) = \log u + i(b + \pi) \) and \( z_2(u) = \log u + i(b - \pi) \).

**Proof.** As mentionned bellow, we shall use the first integral representation \([1]\),

\[
E_{s, \alpha}(z) = \frac{1}{\Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{e^{x}} dx
\]

\[
= \frac{1}{\Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} e^{ax} dx .
\]

We put \( \beta = e^a \), write

\[
E_{s, \alpha}(z) = \frac{\beta}{\Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} e^{ax} \beta^{-1} dx ,
\]

and use the lemma

**Lemma 3.**

\[
\beta^{s-1} = \frac{\sin(\pi x)}{\pi} \int_0^\infty \frac{u^{x-1}}{u + \beta} du \quad (0 < x < 1) .
\]

Indeed, we find in many tables of integrals, as \([2]\), the Mellin integral transform expression

\[
\int_0^\infty \frac{t^{s-1}}{t+1} dt = \frac{\pi}{\sin(\pi s)} \quad (0 < \Re s < 1) ,
\]

and just transform it, by the change of variable \( u = \beta t \), to get the formula \([3]\) for any real number \( x \in (0, 1) \). Hence, we have

\[
E_{s, \alpha}(z) = \frac{\beta}{\pi \Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} \sin(\pi x) \left( \int_0^\infty \frac{u^{x-1}}{u + \beta} du \right) dx ,
\]

and, by the Fubini’s theorem, that

\[
E_{s, \alpha}(z) = \frac{\beta}{\pi \Gamma(s)} \int_0^\infty \left( \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} u^x \sin(\pi x) dx \right) \frac{du}{u(u + \beta)} du .
\]
We after consider the integral inside,

\[ \int_0^1 (-\log x)^s \alpha^{-1} e^{ix} u \sin(\pi x) dx \]

\[ = \frac{1}{2i\pi} \int_0^1 (-\log x)^s \alpha^{-1} e^{ix} u (e^{ix} - e^{-ix}) dx \]

\[ = \frac{1}{2i\pi} \int_0^1 (-\log x)^s \alpha^{-1} e^{ix} (\log u + i(b + \pi)x) dx - \frac{1}{2i\pi} \int_0^1 (-\log x)^s \alpha^{-1} e^{ix} (\log u + i(b - \pi)x) dx \]

\[ = \frac{1}{2i\pi} (E_{s,A}(\log u + i(b + \pi)) - E_{s,A}(\log u + i(b - \pi))) , \]

and we are done. \qed

4. THE GENERALIZED POLY-CAUHY NUMBER INTEGRAL REPRESENTATION

Let \( t \in (-1, 1) \) and put \( z = \log(1 + t) \). Then \( z_1(u) = \log u + i\pi, z_2(u) = \log u - i\pi \) and the integral representation \([4]\) writes,

**Corollary 4.**

\[ E_{s,A}(\log(1 + t)) = \frac{1 + t}{2i\pi} \int_0^\infty \frac{E_{s,A}(\log u + i\pi) - E_{s,A}(\log u - i\pi)}{u(u + 1 + t)} \, du , \]

**Remark 1.** If \( s \) is a real positive number, then

\[ E_{s,A}(\log(1 + t)) = \frac{1 + t}{\pi} \text{Im} \int_0^\infty \frac{E_{s,A}(\log u + i\pi)}{u(u + 1 + t)} \, du , \]

where Im stands for the imaginary part.

In order to get successive derivatives, in \( t \), under the sign \( \int \), that is legitimate, we write that \((1 + t)(u + 1 + t)^{-1} = 1 - u(u + 1 + t)^{-1}\) and obtain for any positive integer \( n \),

\[ \frac{d^n}{dt^n} E_{s,A}(\log(1 + t)) = \frac{(-1)^{n-1}n!}{2i\pi} \int_0^\infty \frac{E_{s,A}(\log u + i\pi) - E_{s,A}(\log u - i\pi)}{(u + 1 + t)^{n+1}} \, du . \]

Letting \( t = 0 \), we obtain the integral representation for the generalized poly-Cauchy numbers,

**Corollary 5.**

\[ \frac{c_{s,A}^{(n)}}{n!} = \frac{(-1)^{n-1}n!}{2i\pi} \int_0^\infty \frac{E_{s,A}(\log u + i\pi) - E_{s,A}(\log u - i\pi)}{(u + 1 + t)^{n+1}} \, du . \]

5. THE QI INTEGRAL REPRESENTATIONS

For the special case \( \alpha = s = 1 \), we have

\[ E_{1,1}(z) = \sum_{n=0}^\infty \frac{z^n}{(n+1)!} = e^z - 1 , \]

then

\[ \text{Im} E_{1,1}(\log u + i\pi)) = \text{Im} \frac{-u - 1}{\log u + i\pi} = \frac{\pi(u + 1)}{\log^2 u + \pi^2} , \]

and finally by \([5]\),

\[ E_{1,1}(\log(1 + t)) = \frac{t}{\log(1 + t)} = (1 + t) \int_0^\infty \frac{u + 1}{u(\log^2 u + \pi^2)(u + 1 + t)} \, du , \]
and by \(6\),

\[
\frac{c^{(1)}_{n,1}}{n!} = (-1)^{n-1} \int_0^{\infty} \frac{du}{(\log^2 u + \pi^2)(u + 1)^n} .
\]

The formulae (7) and (8) above are the integral representations found in [4] by different methods.

6. **Final remark**

When the parameter \(s\) is real, so are the numbers \(c^{(s)}_{n,\alpha}\), one could investigate, as done by Feng Qi, the complete monotonicity, the log-convexity and, may be, more other properties of these numbers by using the integral representations (5) and (6).

**References**


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