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Abstract. In this note, we establish an integral representation for a special function $E_{s,\alpha}(z)$ and apply it to some generalized poly-Cauchy numbers $c_{n,\alpha}^{(s)} := n![t^n]E_{s,\alpha}(\log(1+t))$. We recover, in the special case $\alpha = s = 1$, the integral representation of the Bernoulli numbers of the second kind $b_n = c_{n,1}^{(1)}/n!$ obtained by Feng Qi by quite different methods.

1. INTRODUCTION

Let α and s be parameters, with α real and postive and s complex with $\Re s > 0$. We define some generalized poly-Cauchy numbers, see [3] for $s = k$ integer, by the generating function

$$E_{s,\alpha}(\log(1+t)) = \sum_{n=0}^{\infty} c_{n,\alpha}^{(s)} \frac{t^n}{n!}, \quad (|t| < 1),$$

where $E_{s,\alpha}$ is the function, defined for any complexe number z, by

$$E_{s,\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!(n+\alpha)^s}.$$

This function is called poly-exponnential in [1] and, for $s = k$ a positive integer, the extended polylogarithm factorial function, see [3] for instance.

In the sequel, we

- obtain a first integral representation for $E_{s,\alpha}(z)$ and use it to obtain the main integral representation,
- apply it to get an integral representation for the generalized poly-Cauchy number $c_{n,\alpha}^{(s)}$,
- recover, by specialisation, the F. Qi's integral representation for the Cauchy number $c_{n,1}^{(1)}$.

2. MELLIN TYPE INTEGRAL REPRESENTATIONS

The firts integral representation is basic and, more or less well known.

Proposition 1. *The function $E_{s,\alpha}(z)$ admits the following integral representation,*

$$E_{s,\alpha}(z) = \frac{1}{\Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{zx} dx. \tag{1}$$

Proof. We firts obtain a Mellin type integral representation. As α and $\Re s$ are positive, we have for any non negative integer n,

$$\frac{\Gamma(s)}{(n+\alpha)^s} = \int_0^{\infty} t^{s-1} e^{-(n+\alpha)t} dt,$$

so

$$E_{s,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n+\alpha)^s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^{\infty} t^{s-1} e^{-(n+\alpha)t} dt.$$

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The convergence modes of the series and integral permit to exchange Σ and $\int$ and obtain

$$E_{s,\alpha}(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\alpha t} \left(\sum_{n=0}^{\infty} \frac{z^n e^{-nt}}{n!} \right) dt$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\alpha t} e^{ze^{-t}} \, dt \ .$$

At the end, the change $e^{-t} = x$ gives the desired formula. □

3. Main integral representation

Here is our main result.

Theorem 2. *Let a and b be real numbers and $z = a + ib$, then*

$$E_{s,\alpha}(z) = \frac{e^a}{2i\pi} \int_0^\infty \frac{E_{s,\alpha}(z_1(u)) - E_{s,\alpha}(z_2(u))}{u(u+e^a)} \, du \ , \tag{2}$$

where $z_1(u) = \log u + i(b+\pi)$ and $z_2(u) = \log u + i(b-\pi)$.

Proof. As mentionned bellow, we shall use the first integral representation (1),

$$E_{s,\alpha}(z) = \frac{1}{\Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{zx} dx$$
$$= \frac{1}{\Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} e^{ax} dx \ .$$

We put $\beta = e^a$, write

$$E_{s,\alpha}(z) = \frac{\beta}{\Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} \beta^{x-1} dx \ ,$$

and use the lemma

Lemma 3.

$$\beta^{x-1} = \frac{\sin(\pi x)}{\pi} \int_0^\infty \frac{u^{x-1}}{u+\beta} du \quad (0 < x < 1) \ . \tag{3}$$

Indeed, we find in many tables of integrals, as [2], the Mellin integral transform expression

$$\int_0^\infty \frac{t^{s-1}}{t+1} dt = \frac{\pi}{\sin(\pi s)} \qquad (0 < \Re s < 1) \ ,$$

and just transform it, by the change of variable $u = \beta t$, to get the formula (3) for any real number $x \in (0,1)$.

Hence, we have

$$E_{s,\alpha}(z) = \frac{\beta}{\pi\Gamma(s)} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} \sin(\pi x) \left(\int_0^\infty \frac{u^{x-1}}{u+\beta} du \right) dx \ ,$$

and, by the Fubini's theorem, that

$$E_{s,\alpha}(z) = \frac{\beta}{\pi\Gamma(s)} \int_0^\infty \left(\int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} u^x \sin(\pi x) dx \right) \frac{du}{u(u+\beta)} du \ .$$

We after consider the integral inside,

$$\begin{aligned}
&\int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} u^x \sin(\pi x) dx \\
&= \frac{1}{2i\pi} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{ibx} u^x (e^{i\pi x} - e^{-i\pi x}) dx \\
&= \frac{1}{2i\pi} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{(\log u + i(b+\pi))x} dx - \frac{1}{2i\pi} \int_0^1 (-\log x)^{s-1} x^{\alpha-1} e^{(\log u + i(b-\pi))x} dx \\
&= \frac{1}{2i\pi} \left(E_{s,\alpha}(\log u + i(b+\pi)) - E_{s,\alpha}(\log u + i(b-\pi)) \right) ,
\end{aligned}$$

and we are done . $\square$

4. THE GENERALIZED POLY-CAUCHY NUMBER INTEGRAL REPRESENTATION

Let $t \in (-1,1)$ and put $z = \log(1+t)$. Then $z_1(u) = \log u + i\pi$, $z_2(u) = \log u - i\pi$ and the integral representation (4) writes,

Corollary 4.

$$E_{s,\alpha}(\log(1+t)) = \frac{1+t}{2i\pi} \int_0^\infty \frac{E_{s,\alpha}(\log u + i\pi) - E_{s,\alpha}(\log u - i\pi)}{u(u+1+t)}\, du \ , \tag{4}$$

Remark 1. *If s is a real positive number, then*

$$E_{s,\alpha}(\log(1+t)) = \frac{1+t}{\pi} Im \int_0^\infty \frac{E_{s,\alpha}(\log u + i\pi)}{u(u+1+t)}\, du \ , \tag{5}$$

where Im stands for the imaginary part.

In order to get successive derivatives, in t, under the sign $\int$, that is legitimate, we write that $(1+t)(u+1+t)^{-1} = 1 - u(u+1+t)^{-1}$ and obtain for any positive integer n,

$$\frac{d^n}{dt^n} E_{s,\alpha}(\log(1+t)) = \frac{(-1)^{n-1} n!}{2i\pi} \int_0^\infty \frac{E_{s,\alpha}(\log u + i\pi) - E_{s,\alpha}(\log u - i\pi)}{(u+1+t)^{n+1}}\, du \ .$$

Letting $t = 0$, we obtain the integral representation for the generalized poly-Cauchy numbers,

Corollary 5.

$$\frac{c_{n,\alpha}^{(s)}}{n!} = \frac{(-1)^{n-1}}{2i\pi} \int_0^\infty \frac{E_{s,\alpha}(\log u + i\pi) - E_{s,\alpha}(\log u - i\pi)}{(u+1)^{n+1}}\, du \ . \tag{6}$$

5. THE QI INTEGRAL REPRESENTATIONS

For the special case $\alpha = s = 1$, we have

$$E_{1,1}(z) = \sum_{n=0}^\infty \frac{z^n}{(n+1)!} = \frac{e^z - 1}{z} \ ,$$

then

$$\mathrm{Im}\, E_{1,1}(\log u + i\pi)) = \mathrm{Im}\, \frac{-u-1}{\log u + i\pi} = \frac{\pi(u+1)}{\log^2 u + \pi^2} \ ,$$

and finally by (5),

$$E_{1,1}(\log(1+t)) = \frac{t}{\log(1+t)} = (1+t) \int_0^\infty \frac{u+1}{u(\log^2 u + \pi^2)(u+1+t)}\, du \ , \tag{7}$$

and by (6),

$$\frac{c_{n,1}^{(1)}}{n!} = (-1)^{n-1} \int_0^\infty \frac{du}{(\log^2 u + \pi^2)(u+1)^n} \,. \tag{8}$$

The formulae (7) and (8) above are the integral representations found in [4] by different methods.

6. Final remark

When the parameter s is real, so are the numbers $c_{n,\alpha}^{(s)}$, one could investigate, as done by Feng Qi, the complete monotonicity, the log-convexity and, may be, more other proprieties of these numbers by using the integral representations (5) and (6).

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