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DIMENSION OF CHARACTER VARIETIES FOR 3-MANIFOLDS

E. FALBEL, A. GUILLOUX

Abstract. Let M be an orientable 3-manifold, compact with boundary and Γ its fundamental group. Consider a complex reductive algebraic group G. The character variety $X(\Gamma, G)$ is the GIT quotient $\mathrm{Hom}(\Gamma, G)//G$ of the space of morphisms $\Gamma \to G$ by the natural action by conjugation of G. In the case $G = \mathrm{SL}(2, \mathbb{C})$ this space has been thoroughly studied.

Following work of Thurston [Thu80], as presented by Culler-Shalen [CS83], we give a lower bound for the dimension of irreducible components of $X(\Gamma, G)$ in terms of the Euler characteristic $\chi(M)$ of M, the number t of torus boundary components of M, the dimension d and the rank r of G. Indeed, under mild assumptions on an irreducible component X_0 of $X(\Gamma, G)$, we prove the inequality

$$\dim(X_0) \geq t \cdot r - d\chi(M).$$

Introduction

Representation varieties of fundamental groups of 3-manifolds have been studied for a long time. They are important on one hand, to understand geometric structures on 3-manifolds and their topology as Thurston's work [Thu80] has shown and, on the other hand, to obtain more refined topological information on 3-manifolds as in the Culler-Shalen theory [CS83].

The dimension of the character variety of $\pi_1(M)$ on $\mathrm{PSL}(2, \mathbb{C})$ (where M is a cusped hyperbolic manifold) at the representation corresponding to a complete finite volume hyperbolic structure is exactly the number of cusps (see [NZ85] where it is shown that, in fact, that representation is a smooth point of the character variety). A general bound was established by Thurston and made more precise by Culler-Shalen: the dimension of irreducible components containing irreducible representations with non-trivial boundary holonomy is bounded below by the number of cusps. In fact there are examples of irreducible representations with arbitrary large dimensions associated to hyperbolic manifolds with even only one cusp.

The proof of the bound given in both Thurston and Culler-Shalen uses trace identities for $\mathrm{SL}(2, \mathbb{C})$ which are difficult to work with in the higher dimensional case.

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In this paper we obtain a bound on the dimension of irreducible components which contain representations satisfying reasonable genericity conditions. They include a notion of irreducibility and one of boundary regularity. The bound is expressed in terms of the Euler characteristic $\chi(M)$ of M, the number t of torus boundary components of M, the dimension d and the rank r of G. Indeed, for an irreducible component X_0 of $X(\Gamma, G)$ satisfying the assumptions, we prove the inequality

$$\dim(X_0) \geq t \cdot r - d\chi(M).$$

In the case $G = \mathrm{SL}(n, \mathbb{C})$, thoroughly discussed below, the component containing the geometric representation in $\mathrm{SL}(n, \mathbb{C})$ has the minimal bound in our estimate (see for example [MFP12]). But we don't know if our bound is sharp for other reductive groups.

An important remark is that the bound obtained may be described as half the dimension of the character variety of ∂M. This is directly related to a description of the image (under restriction) of X_0 in the character variety of ∂M. In certain cases this image is a Lagrangian submanifold, as discussed in [Sik12, Thm. 61].

In Section 1 we explain our result for $\mathrm{SL}(n, \mathbb{C})$. Although a particular case of our general result, it is simpler to state and contains the essential idea. We prove the bound for any component containing a representation satisfying two conditions. We assume that its image is Zariski dense and that the image of the fundamental group of the boundary is regular. The later means that each image of a boundary fundamental group has centralizer of minimal dimension (equal to the rank $n-1$ of $\mathrm{SL}(n, \mathbb{C})$). Note that the boundary regularity condition in the $\mathrm{SL}(2, \mathbb{C})$ case simply means that the image is not central for each boundary component.

We consider next representations into complex reductive affine algebraic groups. We use the general definitions of irreducibility as in Sikora [Sik12] and the notion of regularity in reductive groups following Steinberg [Ste65] and prove the main Theorem 5.

A special case of the theorem occurs when M is a oriented complete cusped hyperbolic manifold and $\rho : \pi_1(M) \to \mathrm{SL}(n, \mathbb{C})$ is the representation obtained by composing a representation $\pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$ obtained from the hyperbolic structure with the irreducible embedding $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SL}(n, \mathbb{C})$. That representation is called a geometric representation and the dimension of the irreducible component of the character variety $X(\Gamma, \mathrm{SL}(n, \mathbb{C}))$ containing it was first described in [MFP12] (see also [BFG$^+$13, Gui13]). Evidence that this bound was valid for all components containing irreducibles was obtained in computations of the character variety for the figure eight knot fundamental group into $\mathrm{SL}(3, \mathbb{C})$ [FGK$^+$14, HMP15]. In this case, irreducible components containing irreducible representations are all two dimensional. Moreover each component contains Zariski-dense and boundary regular unipotent representations already computed in [Fal08] (see also [FKR14, CUR] for other examples).

One can use the results on local rigidity [BFG$^+$13] in order to prove that a component seen by the Deformation variety of an ideal triangulation (defined in [BFG$^+$13]) verifies this bound. We do not discuss this here, going straight to the proof which does not use triangulations.

As a final remark, we should note that an a priori bound on the dimension of irreducible components certainly simplifies effective computations as in [FGK$^+$14] by eliminating the need to checking the existence of lower dimensional components.

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1. A simple case

Before going to precise definitions in the framework of algebraic groups, let us sketch the proof in a simple yet interesting case. Namely, in this section we assume $G = \mathrm{SL}(n, \mathbb{C})$. It contains the original statement of Thurston (for the group $\mathrm{SL}(2, \mathbb{C})$). Recall that its dimension d is $n^2 - 1$ and its rank r is $n - 1$: this is the minimal dimension of the centralizer of an element.

Let M be an orientable 3-manifold, compact with boundary. In the following we suppose that there are no boundary component is a 2-sphere. Although the bound we obtain is valid in this case, if we simply fill the 2-sphere with a ball we obtain a better estimate. Let t be the number of torus boundary components. We fix for each torus boundary component an injection (denoted by an inclusion) $\pi_1(T) \subset \Gamma$. In other terms, for each of these torus boundary component, we choose a lift in the universal covering of M.

We want a lower bound on the dimension of components of $X(\Gamma, G) = \mathrm{Hom}(\Gamma, G)//G$. So consider an irreducible component R_0 of $\mathrm{Hom}(\Gamma, G)$ and X_0 its projection in $X(\Gamma, G)$. We work with two assumptions on an element $\rho_0 \in R_0$. One assumption should be a form of irreducibility for the whole representation ρ_0. The second assumption is a regularity assumption for the image under ρ_0 of the fundamental groups of the torus boundaries. In the first version of our theorem, we assume:

- Zariski-density: $\rho_0(\Gamma)$ is Zariski-dense in G.
- Boundary regularity: For any torus boundary component $T \subset \partial M$, ρ_0 maps $\pi_1(T)$ to a regular subgroup of $\mathrm{SL}(n, \mathbb{C})$, i.e., every subspace on which elements in $\rho_0(\pi_1(T))$ act by homothety is a line.

The second assumption is for example satisfied if the image of $\pi_1(T)$ is diagonalizable and every global eigenspace is a line, or if this image is unipotent and fixes a unique flag. It implies that the number of invariant subspaces is finite. It implies moreover that the centralizer of the image of $\pi_1(T)$ has dimension $n - 1$.

We get the following particular case of the general theorem:

Theorem 1. *Let X_0 be the projection in $X(\Gamma, G)$ of an irreducible component R_0 of $\mathrm{Hom}(\Gamma, G)$ containing a representation ρ_0 which is Zariski-dense and boundary regular.*

Then we have:

$$\dim(X_0) \geq (n-1)t - (n^2-1)\chi(M).$$

Note that the case of ρ_0 being the geometric representation is not handled by this theorem, as the representation is not Zariski-dense. It will be handled by the general case, see theorem 5.

Proof. Let us sketch the proof, inspired by Thurston and Culler-Shalen [CS83]. It works by induction on the number t of boundary tori.
Initialization: $t = 0$. The inequality $\dim(X_0) \geq -(n^2-1)\chi(M)$ in this case is very general and was already known by Thurston. We give a proof in section 3.2. Note that if $\chi(M)$ is non-negative the formula does not carry any information.
Propagation: $t-1 \to t$. Let T be a torus boundary in ∂M. We first use

Lemma 2. *There exists an element γ of Γ such that $\rho_0(\pi_1(T))$ and $\rho_0(\gamma)$ generate an irreducible representation of the subgroup of Γ generated by $\pi_1(T)$ and γ.*

Proof. By the boundary regular assumption, $\rho_0(\pi_1(T))$ has a finite number of stable subspaces. By Zariski-density of $\rho_0(\Gamma)$, we find an element $\gamma \in \Gamma$ such that $\rho_0(\gamma)$ does not stabilize any of these subspaces. Indeed, the union of all elements in the image of ρ which preserve one of the subspaces is contained in a Zariski closed subset.

Hence $\rho_0(\gamma)$ and $\rho_0(\pi_1(T))$ spans an irreducible subgroup of G. □

Remark 1.1. Using section 4.4 in [Tit72], one may prove the above lemma (and then the theorem) with a mildest irreducibility assumption and a different kind of regularity: namely that the Zariski-closure of ρ_0 is almost simple and irreducible, as e.g. the image of $SL_2(\mathbb{C})$ under the irreducible representation and the boundary torus is mapped on a non trivial diagonal subgroup. This would be applicable to the geometric representation.

However we will not precisely discuss this, as this will be handled by the general theorem.

Now, following Thurston and [CS83], we drill along γ in M. We get a new 3-manifold M' with a new genus 2 boundary component denoted by Σ_2 and $t-1$ torus boundary components. Let δ be a meridian of the new handle of Σ_2 (δ is an element of $\pi_1(\Sigma_2)$ whose normal closure $N(\delta)$ in Γ' is the kernel of the map $\Gamma' \to \Gamma$). Denote by Γ' its fundamental group. We have an inclusion $\pi_1(\Sigma_2) \subset \Gamma'$ and a surjective homomorphism $\Gamma' \to \Gamma$. Summarizing: