



**HAL**  
open science

## D1.1 -Overview of Lyapunov methods for time-delay systems

Alexandre Seuret, Frédéric Gouaisbaut, Lucie Baudouin

► **To cite this version:**

Alexandre Seuret, Frédéric Gouaisbaut, Lucie Baudouin. D1.1 -Overview of Lyapunov methods for time-delay systems. [Research Report] Rapport LAAS n°16308, LAAS-CNRS. 2016. hal-01369516

**HAL Id: hal-01369516**

**<https://hal.science/hal-01369516>**

Submitted on 21 Sep 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# D1.1 - Overview of Lyapunov methods for time-delay systems

A. Seuret<sup>a</sup> F. Gouaisbaut<sup>a</sup> L. Baudouin<sup>a</sup>

<sup>a</sup>LAAS - CNRS, Université de Toulouse, CNRS, UPS, 7 avenue du Colonel Roche, 31077 Toulouse, France.  
(e-mail: aseuret,fgouaisb,baudouin@laas.fr).

---

## Abstract

This manuscript presents a recent overview of time-delay systems as described in the proposal of the ANR project SCIDIS. After a brief description of the various problems arising in the study of such a class of infinite dimensional systems, we recall the main tools employed in the literature to address stability using a time domain approach, mainly the Lyapunov-Krasovskii theorem. The report lists, for instance, the class of delay systems that have been mainly dealt with in the literature, possible selections for the Lyapunov-Krasovskii functional candidate, and some integral and matrix inequalities that have been mainly provided by the participant to the SCIDIS project. Finally, several examples of time-delay systems are considered and demonstrate the potentials of the recent advances in this field.

---

## 1 Introduction

### 1.0.1 Particularities of time-delay systems

Unlike more classical systems governed by ordinary differential equations, time-delay systems represent a particular class of infinite dimensional systems that can be modeled for instance by the coupling of an ordinary differential equation and a partial differential equation. This particularity has several implications on the properties of the time-delay system under consideration

This section introduces some of the basic particularities of time-delay systems in terms of mathematical considerations through a simple example. More particularly, this section exposes some reasons for which researches are still investigating in the topic. To have a better understanding and reading of this section, we will focus on a simple example. The goal is to help the reader to understand the most relevant aspects of time-delay systems. Let  $x \in \mathbb{R}$  be a variable whose evolution is governed by:

$$\forall t > t_0, \quad \dot{x}(t) = -x(t-h) \tag{1}$$

where  $h > 0$  is a positive scalar which represents a constant delay. If one considers the delay-free case, i.e.  $h = 0$ , it is well known that the solutions of the system are stable and are of the form  $x(t) = x(t_0)e^{t_0-t}$ . In the following, particular aspects of this equation with delay will allow us pointing out the major difficulties of time-delay systems and the difference with the delay-free case.

**Initial conditions and functional state:** Consider the case where  $h = -\pi/2$ . The two functions  $x_1(t) = \sin(t)$  and  $x_2(t) = \cos(t)$  are trivial solutions of (1), which are depicted in Figure 1. In this figure, one can find a contradiction with the Cauchy theorem. In the delay free case, if two solutions of this linear differential equation cross, then the two solutions are the same. In this simple example, it is clear that the two solutions  $x_1$  and  $x_2$  cross each other infinitely many times but are, by definition, not equal. This problem comes from the fact that the state of a time-delay system is not only a vector considered at an instant  $t$ , as it is in the delay free case, but is function taken over an interval (or a window) of the form  $[t-h, t]$ . Consequently, it is not sufficient to initialize the state of the system by only

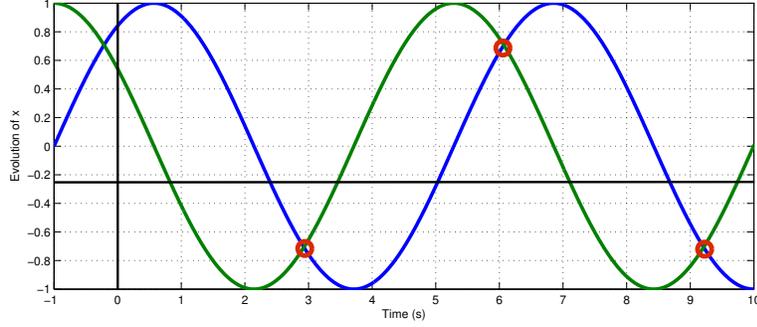


Fig. 1. Possible solution for  $h = \pi/2$

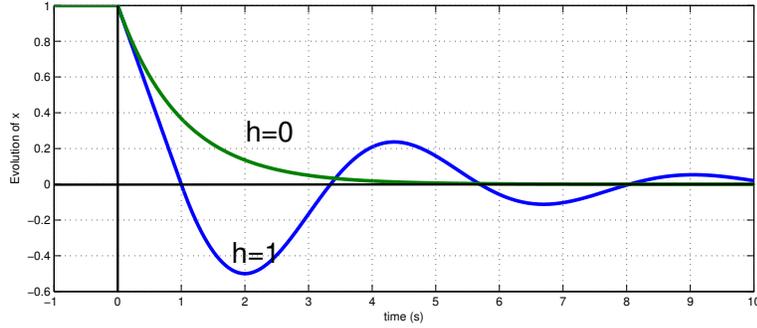


Fig. 2. Solution for  $h = 0, 1$  and constant initial conditions

including the initial position of the state at time  $t = 0$ . It is required to define a vector function  $\phi : [-h, 0] \rightarrow R$  such that  $x(\theta) = \phi(\theta)$  for all  $\theta$  lying in the first delay interval  $[-h, 0]$ .

Note, however, that the Cauchy theorem still holds. It is rewritten as follows: If two solutions are equals over an interval of length  $h$ , then the solutions are equals over the whole simulation time.

**Infinite dimensional systems:** Consider  $h = 1$  and the initial conditions  $\phi(\theta) = 1$ , for all  $\theta$  in  $[t_0 - h, t_0]$ . The solutions are shown in Figure 2.

As expected, in the non delay case, the solution is a exponential decreasing function. In the delay case, the solution are not always of this form anymore. First the solution have an oscillatory behavior around 0. Those oscillations are the usual and expected effects when introducing a delay in a dynamical system. For small values of the delay, those oscillations can of very low amplitude and thus negligible. However, for greater values of  $h$  (for instance  $h = 2$ ), the oscillations become of large amplitude and the solution are unstable.

Considering  $h = 1$  and the same initial conditions, it is possible to construct the solution of the system by integrating interval by interval:

$$\begin{aligned}
 t \in [-1, 0], & \quad x(t) = 1, \\
 t \in [0, 1], & \quad x(t) = 1 - t, \\
 t \in [1, 2], & \quad x(t) = 1/2 - t + t^2, \\
 & \dots
 \end{aligned}$$

Thus, the solution of the system is a polynomial functions whose degree increases with time. One can then see the time-delay system as an infinite dimensional system since the solutions of this time-delay system with a constant initial condition are a polynomial of infinite dimension.

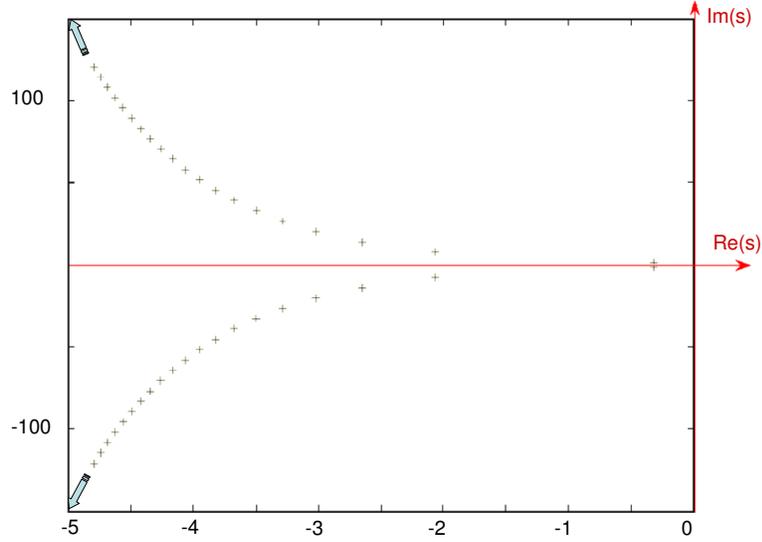


Fig. 3. Solution for  $h = 0, 1$  and constant and zero initial conditions

Another property of time-delay systems to understand that this class of systems is of infinite dimension, is to consider the Laplace transform of equation (1). The characteristic equation is

$$s + e^{-hs} = 0$$

This characteristic equation, even though it is quite simple, has an infinite number of complex roots as shown in Figure 3. This remark also illustrates that time-delay systems are infinite dimensional systems.

**Remark 1** *The stability conditions based on the location of roots of the characteristic equation still holds, i.e. the stability is ensured if its roots have a negative real part (see [GKC03] or [Nic01] for more detailed explanations). These methods will not be discussed in this manuscript.*

### 1.1 System and delay models

In this part, we will present the different types of delay systems encountered in the literature. We do not expose the Cauchy problem, initially studied by Mishkis [Mys51]. The reader is referred to the works [BC63], [KM99] or [Ric02] on the existence and uniqueness of solutions.

#### 1.1.1 General representation of delayed systems

As we have said, delayed systems are dynamical systems governed by functional differential equations bearing both on current and past values in time. If we assume that the derivative of the state vector can be expressed at each time  $t$ , such systems are governed by differential equations of the form:

$$\begin{cases} \dot{x}(t) = f(t, x_t, u_t), \\ x_{t_0} = \phi(\theta), \quad \forall \theta \in [t_0 - h, t_0], \\ u_{t_0} = \zeta(\theta), \quad \forall \theta \in [t_0 - h, t_0], \end{cases} \quad (2)$$

where  $h > 0$  and the functions  $x_t$  and  $u_t$  employ the Shimanov notation [Shi60], which consists of the following definition

$$x_t : \begin{cases} [-h, 0] \rightarrow \mathbb{R}^n, \\ \theta \mapsto x_t(\theta) = x(t + \theta), \end{cases} \quad (3)$$

$$u_t \quad : \quad \begin{cases} [-h, 0] \rightarrow \mathbb{R}^n, \\ \theta \mapsto u_t(\theta) = u(t + \theta). \end{cases} \quad (4)$$

We will denote in the sequel  $\mathcal{C} = \mathcal{C}^0([-h, 0], \mathbb{R}^n)$ , the set of continuous functions from  $[-h, 0]$  to  $\mathbb{R}^n$ . The function  $x_t \in \mathcal{C}$  represents the state of the delay system a time  $t$ ,  $u_t$  is the (control or disturbance) input of the system. The initial conditions, denoted as  $\phi$  and  $\zeta$  at time  $t_0$ , are functions from  $[t_0 - h, t_0]$  to  $\mathbb{R}^n$  and are generally assumed to be continuous or piecewise continuous. Due to these features, delay systems belongs to the class of infinite dimensional systems, since the state function  $x_t$  belongs to an infinite dimensional state.

### 1.1.2 Linear delay systems

In this chapter, we will focus on the case of linear delay systems. These systems can be seen as the linearized version of the general nonlinear class of systems presented in (2) around a local equilibrium. The most usual class of linear systems with delays is described by the following functional differential equation

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \quad (5)$$

where  $x(t) \in \mathbb{R}^n$  denotes the instantaneous state vector,  $\phi$  is a function defined on the interval  $[-h, 0]$  and represents the initial conditions of the system. The matrices  $A$  and  $A_d$  are, in the simplest case, assumed to be constant matrices.

Equation (5) refers to the case of linear continuous-time systems subject to a *discrete delay*, in the sense that only a discrete value of the state function  $x_t$ , i.e.  $x(t - h)$  affects the dynamics of the system. As mentioned in the previous chapter, this class of systems appears in *Networked Control Systems* [HNX07, Zam08], among many other application fields such as *Biology*, *Traffic Control* (see [Fri14, Nic01, Ric03, SNA<sup>+</sup>11] for more information). In this formulation, the delay  $h$  may be a constant scalar or a time-varying function. The usual assumption on the delay will be discussed in the next section. One may also face systems where several discrete values of the state function affect the current dynamics. In this situation, we are used to say that the system has multiple delays. The reader may refer to [CSC98, CGL05, DLSW09, OES05, SO05] to cite only few, in order to have an overview of the various methods for the stability analysis and control of this class of systems.

Another class of time-delay systems is the so-called distributed delay systems. It consists of dynamics where the whole state function  $x_t$  affects the current dynamics of the system. These systems can be modeled as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_{-h}^0 A_D(\theta)x(t + \theta)d\theta, & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \quad (6)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\phi \in \mathcal{C}([-h, 0] \rightarrow \mathbb{R}^n)$  is a continuous function, representing the initial conditions,  $A$  is constant matrix and  $A_D$  denotes a known continuous function of  $L_2([-h, 0] \rightarrow \mathbb{R}^{n \times n})$  and represents the kernel of the distributed delay. Several applications follows this clas of dynamics such as *combustion in rocket motor chambers* in [DLSW09], *population dynamics* with gamma-distribution [CG82], etc... This class of systems has been studied in a large number of papers (see [CZ07, FL12, FT09, Mor06, MNG07, SGA15, SF13, XFS01] to cite only few).

In order to cope with a larger class of systems, some uncertainties or disturbances may affect the previous dynamics. In this situation, the matrices  $A$  and  $A_d$  are not assumed to be know nor to be constant. The most common classes of uncertainties that have been considered in the literature are described below

**Norm Bounded Uncertainties:** In this situation, the matrices  $A$  and  $A_d$  are unknow and/or time-varying parameters but are assumed to satisfies the following condition

$$A = E\Theta(t)F, A_d = E_d\Theta(t)F_d,$$

where the matrices  $E, F, E_d, F_d$ , of appropriate dimension, are known and the uncertainties are captured in the uncertain matrix  $\Theta$ , which verifies

$$\Theta^\top(t)\Theta(t) \leq \epsilon I,$$

where  $\epsilon$  is a given parameter. This last equation justifies the name of norm-bounded uncertainties.

**Polytopic Uncertainties:** In this situation, the matrices  $A$  and  $A_d$  are assumed to belong to a polytope given by

$$\begin{bmatrix} A & A_d \end{bmatrix} \in \text{Co}_{i=1, \dots, m} \left\{ \begin{bmatrix} A^i & A_d^i \end{bmatrix} \right\}$$

where  $m$  is a positive integer, and the matrices  $A^i$  and  $A_d^i$ , where  $i = 1, \dots, m$  are constant and known. This formulation implicitly refers to the existence of weighting scalar functions  $\lambda_i$ , for  $i = 1, \dots, m$  that maps  $\mathbb{R}$  to  $[0, 1]$ , and such that  $\sum_{i=1}^m \lambda_i(t) = 1$  and

$$\begin{bmatrix} A & A_d \end{bmatrix} = \sum_{i=1}^m \lambda_i(t) \begin{bmatrix} A^i & A_d^i \end{bmatrix}.$$

In general, assessing stability of such classes of uncertain time-delay systems can be performed by simple tricks transforming stability conditions for a nominal linear delay system (5) to cope with these uncertainties. This manipulation relies generally on the convexity of the stability conditions with respect to the matrices  $A$  and  $A_d$ .

## 1.2 Usual assumptions on the delay functions

In this section, we will successively expose the various model of delays that can be found in the literature.

- a) Constant delay functions:** The first studies of time-delay systems concerned indeed the case of constant delay functions and was mainly carried out using frequency domain approaches. Indeed, one may look at several stability criteria applied to the Laplace transfer function (the reader may refer to the following books [GKC03, Nic01]). Concerning the time-domain approach and the second Lyapunov method, numerous studies have been provided, see for instance [Fri14, Kha12] to assess stability of linear systems with constant and known/unknown delays. Some of them are said delay-independent (see for instance [Bli01a, Bli01b, DLZ14, LGG16, NG12, XSW16] among many others), meaning that the conditions does not depend on the value of the delay, or are delay-dependent (see for instance [FL12, Fri01, Fri02, FS04, FS03, GP06a, HWLW05, MPKL01, XL05] among many others), meaning here that the condition may be guaranteed only for some values of the delay. Since the 90's, an explosion of the number of stability criteria within the time domain approach have been made possible thanks to the developments of semi-definite programming, allows to find solutions to Linear Matrix Inequalities in a simple and efficient manner, for instance on Matlab [EGN00, GA94, LPH02, Löf04, RZ00, HL03, Pac94, VB00]. For instance, one may have a look at [KNR99, LdS97] and [Nic01, GKC03] to find the first contributions in this direction.
- b) Bounded time-varying delays:** In practical application, the assumption of having constant delays becomes to restrictive. In particular in Networked Control Systems applications (see for instance [HNX07, LPA<sup>+</sup>06, Zam08]), delay may arise from the communication through unreliable (wireless) networks. For instance, congestion in the network and the packet loss phenomenon may lead to non negligible variations on the delay functions.

In such situations, researchers consider delay functions that verify the following assumption [J.H97]. There exists a positive scalar  $h_2 > 0$  such that :

$$0 \leq h(t) \leq h_2. \tag{7}$$

Some authors also include some additional conditions on the derivative of the delay functions in order to ensure causality and regularity. This will be described in the next paragraph. If no additional constraints on the derivative of the delay function is required, several authors denotes this class of delay function as *fast-varying delays* [FSR04]. This class of delay functions are of particular interests, notably for sampled-data and/or networked control systems.

- c) Interval or Non small delays:** Again, in the context of networked control systems or in some application such as in [ASRR07], the previous assumption may be too restrictive, since the delay functions are allowed to reach zero (i.e. system without delay), while, from the application point of view, this would be that the transport of the information or more generally of the quantity of interest may be achieved with a arbitrarily fast velocity. However in some applications, this velocity is limited and therefore, there is a minimal time before the current quantity of interest is available to the controller or to the system.

Hence, the assumption saying that the delay functions belong to an interval of the form  $[0, h_2]$  becomes too restrictive and the associated stability analysis may lead to inherent conservatism. One can then define interval

or non small delay functions that prevent the delay functions to be equal or too close to zero. The assumption on the delay functions is related to the existence of two positive scalars  $0 < h_1 \leq h_2$  such that:

$$0 < h_1 \leq h(t) \leq h_2. \quad (8)$$

The first results in this direction were developed in [Fri04, Fri06, FS00], or in [JH05, JH06]. The method proposed in these papers consists in rewriting the delay function as the sum of a constant and known delay equal, for instance  $h_1$  or  $(h_1 + h_2)/2$  and of the residual time-varying function. The first constant delay term can be seen as the nominal delay while the second term represents the variation or the disturbance with respect to this nominal constant delay.

**d) Delay functions with constrained derivatives** A large number of papers addressing the stability systems subject to time-varying delays require additional assumptions on the delay functions. It generally comes from the differentiation of integral terms considered over interval of the form  $[t, t - h(t)]$  or  $[t - h(t), t - h_2]$ . Historically, the first contributions in this direction require the following condition (see for instance [FS03, FS02b])

$$\dot{h}(t) \leq d < 1. \quad (9)$$

Looking more into the details of this assumption, the previous condition imposes that the function  $f(t) = t - h(t)$ , representing the evolution of the delayed information with respect to time, is strictly increasing and consequently bijective. From the engineering point of view, this assumption means that the delayed information or the quantity of interest affect the systems following a chronological order.

Another usual assumption arising from the recent developments on the stability analysis of systems subject to time-varying delays is provided below.

$$d_1 \leq \dot{h}(t) \leq d_2. \quad (10)$$

Here, the upper bound  $d_2$  is not necessarily strictly smaller than 1 [SG13]. This assumption is required when the resulting stability conditions depends linearly on the derivative of the delay functions. Thus this assumption is considered to include more constraints to the allowable delay function.

**e) Piecewise continuous delay functions:** This situation is relevant when one has to face with networked control systems where the information may travel among several communication channels. In this situation, the delay function may be affected by a brutal modification of channel, leading to a discontinuity in the delay function. Another relevant motivation of this class of delay functions is concerned with the stability analysis of sampled-data systems (see [FSR04]) where the effects of a periodic or aperiodic sampling can be modeled as a discontinuous delay function, which, in addition, verifies the following constraint on its derivative

$$\dot{h}(t) \leq 1. \quad (11)$$

Coming back to the comments provided on assumption d), this assumption makes that the function  $f(t) = t - h(t)$  can be constant. This means that the information is held while the delay verifies  $\dot{h}(t) = 1$ , which corresponds indeed to the effect of a sampling.

**f) State-dependent delay functions:** A less usual class of time-delay systems concerns the case of state dependent delay functions.

## 2 Stability of time-delay systems using the Second Lyapunov Method

In this section, we recall some results assessing the asymptotic stability of delay systems by focusing on a time-domain approach related to the second method of Lyapunov .

### 2.1 Second Lyapunov method

Let us consider the generic time-delay system given by

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x_t), & \forall t \geq 0 \\ x_0(\theta) &= \phi(\theta), & \forall \theta \in [-h, 0], \end{aligned} \quad (12)$$

for which we assume the existence and uniqueness of solutions and, without loss of generality, the solution  $x_t = 0$  is an equilibrium

## 2.2 Lyapunov-Razumikhin approach

In this approach, the goal is to consider a classical Lyapunov function  $V(t, x(t))$ , as the one employed for the delay-free case (i.e. for ordinary differential equations). The main idea of the Lyapunov-Krasovskii theorem is that it is not necessary to ensure the negative definiteness of  $\dot{V}(t, x(t))$  along all the trajectories of the system. Indeed, it is sufficient to ensure its negative definiteness only for the solutions that tend to escape the neighborhood of  $V(t, x(t)) \leq c$  of the equilibrium. This idea is formalized in the following theorem [KM99].

**Theorem 2** *let  $u, v$  and  $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be nondecreasing functions such that  $u(\theta)$  and  $v(\theta)$  are strictly positive for all  $\theta > 0$ . Assume that the vector field  $f$  of (12) is bounded for bounded values of its arguments.*

*If there exists a continuous and differentiable function  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that:*

$$a) \ u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|),$$

*b)  $\dot{V}(t, \phi) \leq -w(\|\phi(0)\|)$  for all trajectories of (12) satisfying:*

$$V(t + \theta, \phi(t + \theta)) \leq V(t, \phi(t)), \quad \forall \theta \in [-h, 0], \quad (13)$$

*then the solution  $x_t = 0$  is uniformly stable for (12).*

*Moreover, if  $w(\theta) > 0$  for all  $\theta$  and there exists a strictly increasing function  $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $p(\theta) > \theta$  for all  $\theta > 0$  and:*

$$i) \ u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|),$$

*ii)  $\dot{V}(t, \phi) \leq -w(\|\phi(0)\|)$ , for all trajectories of (12) verifying:*

$$V(t + \theta, x(t + \theta)) \leq p(V(t, x(t))), \quad \forall \theta \in [-h, 0], \quad (14)$$

*then such a function  $V$  Lyapunov-Razumikhin function and solution  $x_t = 0$  is uniformly asymptotically stable for system (12).*

In practice, the functions  $p$  are usually considered as  $p = q\theta$  where  $q$  is a constant strictly greater than 1. Moreover, the Lyapunov functions more commonly employed in the Razumikhin approach are of the forme:

$$V(t) = x^\top P x(t), \quad (15)$$

where  $P$  is a symmetric positive definite matrix of dimension  $n$ , the dimension of  $x(t)$ . Equation (14) thus becomes:

$$x^\top(t + \theta) P x(t + \theta) \leq q x^\top(t) P x(t), \quad \forall \theta \in [-h, 0], \quad \text{and} \quad q > 1. \quad (16)$$

Although the Lyapunov-Razumikhin approach generally leads to more conservative results than those based on the Lyapunov-Krasovskii theorem, presented in the next paragraph, it allows taking into account variable delays without restriction on derivative of the delay function (9) and generally leads to delay-independent stability conditions. It has also been shown that for constant delays, the existence of a Lyapunov-Razumikhin function is equivalent to the existence of a Lyapunov-Krasovskii functional [Dri77].

### 2.3 Lyapunov-Krasovskii approach

The Lyapunov-Krasovskii method is an extension of the second Lyapunov method dedicated to the stability analysis of functional differential equations. It consists in selecting “energy” functionals, i.e. (functions of the functional state  $x_t$ ) of the form  $\mathcal{V}(t, x_t)$ , that are positive definite and decreasing along the trajectories of system (12). The Lyapunov-Krasovskii theorem is stated below [KM99]

**Theorem 3** *let  $u, v$  and  $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be continuous and increasing functions such that  $u(\theta)$  et  $v(\theta)$  are strictly positive for all  $\theta > 0$  and  $u(0) = v(0) = 0$ . Assume that the vector field  $f$  of (12) is bounded for all bounded values of its arguments.*

*If there exists a continuous and differentiable functional  $\mathcal{V} : \mathbb{R} \times C \rightarrow \mathbb{R}_+$  such that :*

$$a) u(\|\phi(0)\|) \leq \mathcal{V}(t, \phi) \leq v(\|\phi\|),$$

*b)  $\dot{\mathcal{V}}(t, \phi) \leq -w(\|\phi(0)\|)$  for all  $t \geq t_0$  along the trajectories of (12), where  $\dot{\mathcal{V}}(t, \phi)$  denote here the derivative of  $\mathcal{V}$  in the Dini sense, i.e.  $\dot{\mathcal{V}}(t, \phi) = \lim_{\epsilon \rightarrow 0^+} \sup \frac{\mathcal{V}(t+\epsilon, x_{t+\epsilon}) - \mathcal{V}(t, x_t)}{\epsilon}$ .  
Then the solution  $x_t = 0$  of (12) is uniformly stable.*

*Moreover, if  $w(\theta) > 0$  for all  $\theta > 0$ , then the solution  $x_t = 0$  is uniformly asymptotically stable for system (12).*

*Such a functional is called a Lyapunov-Krasovskii functional.*

The main idea behind the statement of this theorem is to determine a positive definite functional  $\mathcal{V}$ , such that its derivative with respect to time along the trajectories of the system (12) is negative definite. The main problem within the application of this theorem is the design functional and then to provide some conditions that guarantee its positive definiteness and the negative definiteness of its derivative

The derivation of stability conditions using Lyapunov-Krasovskii functionals usually involves quite elaborate developments. To give an idea of the procedure involved in this approach and to provide a glimpse of its technical flavor, we present here some basics on the procedure to follow in order to derive asymptotic stability criteria for time-delay systems expressed in terms of Linear Matrix Inequality (LMI). Based on elementary considerations, we expose the main difficulties and the most relevant tools. The basic steps for deriving constructive stability conditions are illustrated as follows.

Step 1. Propose a candidate Lyapunov-Krasovskii functional  $V$ . The Lyapunov-Krasovskii functional that is necessary and sufficient for the stability of LTI systems with delay has a rather complex form, even for the case of constant delays [KM99]. Let us provide a non-exhaustive list of usual terms that are employed in the literature

- *Complete Lyapunov-Krasovskii functionals:*

$$\begin{aligned} \mathcal{V}(t, x_t) = & x_t^\top(0) P x_t(0) + 2x_t^\top(0) \left( \int_{-h}^0 Q(s) x_t(s) ds \right) \\ & + \int_{-h}^0 \int_{-h}^0 x_t^\top(s) \mathcal{T}(s, \theta) x_t(\theta) ds d\theta + \int_{-h}^0 x_t^\top(s) (S + (h+s)R) x_t(s) ds, \end{aligned} \quad (17)$$

where the matrices  $P = P^\top$ ,  $R = R^\top$  and  $S = S^\top$  and the matrix functions  $Q$  and  $T$  are matrices of appropriate dimension. The matrices function  $T$  also verifies  $T(s, \theta) = T^\top(\theta, s)$ .

Behind the complexity of the formulation, there are several interests of employing such a functionals. First of all, it is easy to see that when the delay  $h$  tends to zero, one recovers the classical quadratic function usually employed for linear time invariant systems. A second and notable interest of this functional has been demonstrated in [KZ03]. In this paper, it is shown that a linear system subject to a constant delay is asymptotically stable if and only if it admits a Lyapunov-Krasovskii functional that have exactly the same form as in (17), where the parameters  $P, Q, R, S, \mathcal{T}$  are derived from the solution of matrix partial differential equations. Unfortunately, this notable result only provides the existence of the Lyapunov-Krasovskii functional but does not provide a method

to construct those parameters from a numerical point of view. Therefore this method can not be directly applied to assess stability of time-delay systems.

An attempt in this direction was proposed by K. Gu in [Gu97, Gu01, GKC03] using a discretization process on the delay interval in which the parameters are, to stay short, affine functions of the integration variables. The resulting stability conditions are expressed in terms of LMI.

In [PPN07, PB11, PPL09], another method was proposed based on polynomial parameters, that successfully address the asymptotic stability of time-delay systems through the SoS of Squares framework [Par00, Par04, PPP02].

A notable aspect of these two methods is that the objective is finally to derive a numerical test to approximate the parameters of the Complete Lyapunov-Krasovskii functionals. Since in both methods, letting the number of discretization or the degree of the polynomials tend to infinity lead to a good approximation of those parameters. Notably, a discussion on the conservatism of the polynomial method was discussed in [PB11], leading potentially to a non conservative test.

Nevertheless, these two resulting stability conditions have drawbacks. Their complexity in terms of implementation and of number of decision variables made these method not “user-friendly” and many researchers of the fields were looking for simpler Lyapunov-Krasovskii functionals (see for instance [KS96]), leading to simplified stability conditions.

- *Delay-independent Lyapunov-Krasovskii functionals:*

$$V_0(x_t) = x_t^\top(0)Px_t(0) + \int_{-h}^0 x_t^\top(s)Sx_t(s)ds.$$

This functional can be compared with (17), by setting the parameters  $\mathcal{Q}$ ,  $\mathcal{T}$  and  $R$  to zero. The specifications delay-independent comes from the fact that, when deriving stability conditions from this functional, the resulting conditions does not depend on the value of the delay  $h$ . Hence, if the condition holds, it implies that the system remains stable for any values of the delay.

Of course this functional leads to conservative results since a large class of the delay systems may be stable only for some values of the delay. This is the reason why several study have considered functionals leading to delay-dependent conditions

- *Delay-dependent Lyapunov-Krasovskii functionals:*

$$V(x_t, \dot{x}_t) = x_t^\top(0)Px_t(0) + \int_{-h}^0 x_t^\top(s)Sx_t(s)ds + h \int_{-h}^0 \int_{\theta}^0 \dot{x}_t^\top(s)R\dot{x}_t(s)dsd\theta \quad (18)$$

or equivalently

$$V(x_t, \dot{x}_t) = x_t^\top(0)Px_t(0) + \int_{-h}^0 x_t^\top(s)Sx_t(s)ds + h \int_{-h}^0 (h+s)\dot{x}_t^\top(s)R\dot{x}_t(s)ds$$

where the matrices  $P$ ,  $S$  and  $R$  are symmetric positive definite. This class of functionals, introduced in in [FS02a], is richer than the delay-independent functional and includes many types of functionals. The novelty in the definition of this functional relies on the last terms, which depends on  $\dot{x}_t(s) = \frac{d}{dt}x(t+s)$ . This functional does not exactly meets the requirement of the Lyapunov-Krasovskii theorem since this theorem does not mention the possibility for a functional to have  $\dot{x}_t$  as an argument. Nevertheless, extensions of the Lyapunov-Krasovskii theorem including this particularity has been investigated and is now admitted (see, for instance [Fri14, Section 3.1.2]).

- *Multiple integral functionals:*

$$V_m(\dot{x}_t) = \frac{h^2}{2} \int_{-h}^0 \int_{\theta_1}^0 \int_{\theta_2}^0 \dot{x}_t^\top(s)R_m\dot{x}_t(s)dsd\theta_2d\theta_1 \quad (19)$$

or equivalently

$$V_m(\dot{x}_t) = \frac{h^2}{2} \int_{-h}^0 (h+s)^2 \dot{x}_t^\top(s)R_m\dot{x}_t(s)ds$$

where the matrix  $R_m$  is positive definite. This functional was considered. This class of functionals, introduced in [SLCR10] provided new possibilities to enrich the functional by many terms. In order to be efficient, from the numerical point of view, these functionals require generally the introduction of an additional quadratic term which

depends on  $\int_{-h}^0 x_t(s)ds$  or on  $\int_{-h}^0 \int_{\theta}^0 x_t(s)dsd\theta$ . Many extensions to more general multiple integral functional have been considered in the literature and have led to some numerical improvements (see, for instance, [FP13, PKPL11], among many others).

- *Delay partitioning/decomposition functionals:*

$$V_m(x_t) = h \int_{-h/2}^0 \begin{bmatrix} x_t(s) \\ x_t(s - h/2) \end{bmatrix}^\top R_m \begin{bmatrix} x_t(s) \\ x_t(s - h/2) \end{bmatrix} ds \quad (20)$$

Following the idea of the discretization method provided by K. Gu in [Gu97, Gu01, GKC03], several researches turns to enrich the functional with intermediate values of the state function. Indeed in the previous functional, one can see the introduction of the term  $x_t(-h/2)$ . As for the discretization method, this method allows to refine the resulting stability conditions by taking into account more and more information on the state function  $x_t$ . Several contributions towards this direction were proposed [GP07, Han08, ZH09, ZHWZ11, FLY15] among many other. It is worth mentioning that the analysis of reduction of the conservatism was studied in [Bri11, GP07].

To conclude on the selection of the candidate for being a good Lyapunov-Krasovskii functionals, the previous discussion show that much efforts have been dedicated to the construction of more and more evolved functionals. Except for the discretization and the delay partitioning methods, in most of the case, the main procedure is to follow the paradigm “try and check”, meaning that researchers were developing and introducing more and more terms to be included in the functional and have followed the procedure to derived “good” stability conditions. There was no clear vision and explanation on what is a good functional for a time-delay system. This question consisting in finding a good candidate still represents an open question, on which we are trying to provide an answer.

### Step 2. Compute the derivative of $V$ .

For the functional (18) this leads to

$$\begin{aligned} \dot{V}(x_t, \dot{x}_t) &= 2\dot{x}_t^\top(0)Px_t(0) + x_t^\top(0)Sx_t(0) - x_t^\top(-h)Sx_t(-h) \\ &\quad + h^2\dot{x}_t^\top(0)R\dot{x}_t(0) - h \int_{t-h}^t \dot{x}_t^\top(s)R\dot{x}_t(s)ds. \end{aligned} \quad (21)$$

The idea is then to rewrite this expression as a quadratic form expressed using all the relevant information on the state function, corresponding to the “LMIzation” of the expression of  $\dot{V}(x_t, \dot{x}_t)$ . The relevant information are in this situation composed by  $x_t(0)$ ,  $\dot{x}_t(0)$  and  $x_t(-h)$ . First, we note that there exists a redundancy in these three vectors by noting that  $\dot{x}_t(0) = Ax_t(0) + A_d x_t(-h)$ , in the case of a linear delay systems. One can either replace  $\dot{x}_t(0)$  by its expression or one may also keep this information and use the descriptor formulation [FS02a] or introduce slack variables [HWXL07b]. Note that in the case of constant and know matrices  $A$  and  $A_d$ , all these approaches lead to equivalent results (see [GP06a] for more details).

### Step 3. Over-approximate the integral terms.

Note that in (21), the last integral term cannot be straightforwardly converted in the quadratic formulation described above. Indeed the problem comes from the last negative integral term

$$- \int_{-h}^0 \dot{x}_t^\top(s)R\dot{x}_t(s)ds,$$

which is an impediment to the analysis of the sign of (21). Such terms are common in the derivative of Lyapunov-Krasovskii functionals and they need to be included using over-approximation methods. This procedure is applied in order to replace the integral terms by more simple expressions, that can be expressed in a quadratic form to be included in the previous formulation.

In the sequel, we will call this procedure as the use of *integral inequalities*. Unavoidably, using such integral inequalities introduces some conservatism in the analysis and consequently in the resulting stability conditions. In the next section, we will review the existing methods, which have been employed in the literature.

### 3 Integral inequalities and time-delay systems

Following the discussion on the methodology to derive stability conditions for time-delay systems and the steps of the procedure described in the previous section, the objective of this section is to provide generic tools that enable the “LMization” process, which consists, again, in transforming the previous expression in a more appropriate form to obtain an LMI formulation of the stability conditions. Indeed this step is crucial and, consequently, has to be studied carefully. In the following, we will consider the problem of providing integral inequalities which deliver a lower bound of an integral quadratic term of the form

$$\int_{-h}^0 x_t^\top(u) R x_t(u) du \quad \text{or} \quad \int_{-h}^0 \dot{x}_t^\top(u) R \dot{x}_t(u) du$$

where  $h$  is a positive scalar. In the sequel, a review of existing integral inequalities that have been recently employed in the context of time-delay systems will be provided.

**Remark 4** *For simplicity, the next developments will mainly focus on the derivation of lower bounds for the left-hand side integral of the previous equation. We will also show methods to extend these first results to the right-hand side integral.*

#### 3.1 Jensen’s inequality

The first method to treat this problem is based on the Jensen’s inequality formulated in the next lemma

**Lemma 5** *For a given  $n \times n$ -matrices  $R \succ 0$  and for any piecewise continuous function  $x$  in  $[-h, 0] \rightarrow \mathbb{R}^n$ , the following inequality holds:*

$$\int_{-h}^0 x^\top(u) R x(u) du \geq \frac{1}{h} \Omega_0^\top(x) R \Omega_0(x) \tag{22}$$

where

$$\Omega_0(x) = \int_{-h}^0 x(u) du.$$

The proof is omitted and can be found in several reference books [GKC03].

Naturally, Jensen’s inequality is likely to entail some inherent conservatism. Several works have been devoted to the reduction of associated conservatism using the discretization of the delay interval [Bri11, GP07].

In the next section we propose to use an alternative solution to reduce the inherent conservative of this inequality using two class of well-established inequalities. The first one refers to the so-called Wirtinger’s inequalities issues from the Fourier analysis. The second one is a particular interpretation of the Bessel’s inequality on Hilbert space.

#### 3.2 Wirtinger-based integral inequality

##### 3.2.1 Wirtinger inequalities

In the literature [Kam07], Wirtinger’s inequalities refer to inequalities which estimate the integral of the derivative function with the help of the integral of the function.

Wirtinger’s inequality have already been widely used in Automatic Control. To cite only few works, one may look at [Krs09, Chapter 15], and at [FO09] in the context of *Distributed Parameter Systems* or in [LF12] for *Sampled-Data Systems*.

Often proved using Fourier analysis, it exists several versions which depend on the characteristics or constraints we impose on the function. Let us focus on the following Wirtinger’s inequality adapted to our purpose.

**Lemma 6** Consider a given  $n \times n$ -matrix  $R \succ 0$ . Then, for all function  $z$  in  $\mathcal{C}^1([-h, 0] \rightarrow \mathbb{R}^n)$  which satisfies  $z(0) = z(-h) = 0$ , the following inequality holds

$$\int_{-h}^0 \dot{z}^\top(u) R \dot{z}(u) du \geq \frac{\pi^2}{h^2} \int_{-h}^0 z^\top(u) R z(u) du, \quad (23)$$

*Proof* : The proof is omitted but can be found in [Kam07]. ◇

It is worth noting that this inequality is not related to the Jensen's inequality in its essence. Indeed, the function  $z$  has to meet several constraints whereas the function  $x$  is assumed to be a continuous function in the Jensen's inequality. The next section shows how to create a relation between them.

### 3.2.2 Application of the Wirtinger's inequalities: First version [SG12]

The objective of this section is twofold. On the first hand, we aim at providing new tractable inequalities based on Lemma 6, which can be easily implemented into a convex optimization scheme. On the other hand, we propose an inequality which is proved to be less conservative than Jensen's one. Thus a first step consists in defining an appropriate function  $z$  such that this integral appears naturally in the developments. Thus a necessary condition is that the function  $z$  has the following form

$$z_W(u) = \int_{-h}^u x(s) ds - y(u), \quad (24)$$

where  $x$  is a continuous function in  $[-h, 0] \rightarrow \mathbb{R}^n$  as defined in the Jensen's inequality and  $y$  is a function of  $u$  to be defined and are chosen so that the function  $z$  meets the different constraints imposed by Lemma 6.

Following this idea, the next lemma is provided ([SG12]).

**Lemma 7** Let  $R$  be a positive definite matrix of  $\mathcal{S}^n$ . Then, for any continuous function  $x$  in  $[-h, 0] \rightarrow \mathbb{R}^n$  the following inequality holds:

$$\int_{-h}^0 x^\top(u) R x(u) du \geq \frac{1}{h} \begin{bmatrix} \Omega_0(x) \\ \Omega_1(x) \end{bmatrix}^\top \begin{bmatrix} R & \\ & \frac{\pi^2}{4} R \end{bmatrix} \begin{bmatrix} \Omega_0(x) \\ \Omega_1(x) \end{bmatrix} \quad (25)$$

where

$$\begin{aligned} \Omega_0(x) &= \int_{-h}^0 x(u) du \\ \Omega_1(x) &= \int_{-h}^0 x(u) du - \frac{2}{h} \int_{-h}^0 \int_{-h}^u x(s) ds du \end{aligned}$$

*Proof* : For any continuous function  $x$  from  $[-h, 0]$  to  $\mathbb{R}^n$ , define the function  $z_{W1}$  given by

$$z_{W1}(u) = \int_{-h}^u x(s) ds - \frac{u+h}{h} \int_{-h}^0 x(s) ds = \int_{-h}^u x(s) ds - \frac{u+h}{h} \Omega_0(x), \quad \forall u \in [-h, 0].$$

The second term is a polynomial of degree 1 which compensates the first term when  $u = 0$ . By construction, the function  $z_{W1}(u)$  meets the conditions of the Wirtinger's inequality given in Lemma 6, that is  $z_{W1}(-h) = z_{W1}(0) = 0$ . We also note that the function  $z_{W1}$  admits a continuous derivative with respect to the variable  $u$ , which is given by

$$\dot{z}_{W1}(u) = x(u) - \frac{1}{h} \Omega_0(x), \quad \forall u \in [-h, 0].$$

This function  $z_{W1}$  has also been defined such that the computation of  $\dot{z}_{W1}$  makes appear the original function  $x$  as suggested in equation (24). The computation of the left-hand-side of the inequality stated in Lemma 6 leads to:

$$\int_{-h}^0 \dot{z}_{W1}^\top(u) R \dot{z}_{W1}(u) du = \int_{-h}^0 x^\top(u) R x(u) du - \frac{1}{h} \Omega_0^\top(x) R \Omega_0(x) \quad (26)$$

**Remark 8** *At this step, one can already find an alternative proof of the Jensen's inequality. Indeed, since the matrix  $R$  is assumed to be symmetric positive definite, the left-hand-side of (26) is positive definite, which ensures the following inequality*

$$0 \leq \int_{-h}^0 \dot{z}_{W1}^\top(u) R \dot{z}_{W1}(u) du = \int_{-h}^0 x^\top(u) R x(u) du - \frac{1}{h} \Omega_0^\top(x) R \Omega_0(x).$$

We already note that this inequality is exactly the Jensen's inequality in Lemma 5.

Consider now the right-hand side of the inequality (23). Applying the Jensen's inequality, we have

$$\frac{\pi^2}{h^2} \int_{-h}^0 \dot{z}_{W1}^\top(u) R \dot{z}_{W1}(u) du \geq \frac{\pi^2}{h^3} \left( \int_{-h}^0 z_{W1}(u) du \right)^\top R \left( \int_{-h}^0 z_{W1}(u) du \right), \quad (27)$$

where simple calculations show that

$$\int_{-h}^0 z_{W1}(u) du = \left( \int_{-h}^0 \int_{-h}^u x(s) ds du - \frac{h}{2} \int_{-h}^0 x(u) du \right) = -\frac{h}{2} \Omega_1(x).$$

The proof is concluded by application of the Wirtinger's inequality provided in Lemma 6 which ensures

$$\int_{-h}^0 x^\top(u) R x(u) du - \frac{1}{h} \Omega_0^\top(x) R \Omega_0(x) \geq \frac{\pi^2}{4h} \Omega_1^\top(x) R \Omega_1(x),$$

as to be demonstrated. ◇

Several comments on this new inequalities can already be done. In light of Remark 8, the Wirtinger inequality provides a method to obtain a more accurate lower bound of the integral

$$\int_{-h}^0 x^\top(u) R x(u) du.$$

Indeed, since  $R$  is positive definite, the term  $\frac{\pi^2}{4h} \Omega_1^\top(x) R \Omega_1(x)$  is also positive, showing that Lemma 7 is less conservative than the Jensen's inequality. A second comment concerns the introduction of a new information  $\Omega_1(x)$  which depends on the function  $x$ . As shown in [SG12] (and latter on in this chapter), a particular attention has to be paid on this term when one wants to use in inequality to derive stability condition for time-delay systems.

### 3.2.3 Application of the Wirtinger's inequalities: Second version

In this section, we propose to refine and precise this first Wirtinger-based integral inequality. To do so, we will consider again a function  $z$  as defined in (24). This will lead to the following Lemma which was presented in [SG13].

**Lemma 9** *Consider a given matrix  $R \succ 0$ . Then, for all continuous function  $x$  in  $[-h, 0] \rightarrow \mathbb{R}^n$  the following inequality holds:*

$$\int_{-h}^0 x^\top(u) R x(u) du \geq \frac{1}{h} \begin{bmatrix} \Omega_0(x) \\ \Omega_1(x) \end{bmatrix}^\top \begin{bmatrix} R \\ 3R \end{bmatrix} \begin{bmatrix} \Omega_0(x) \\ \Omega_1(x) \end{bmatrix} \quad (28)$$

where

$$\begin{aligned}\Omega_0(x) &= \int_{-h}^0 x(u)du \\ \Omega_1(x) &= \int_{-h}^0 x(u)du - \frac{2}{h} \int_{-h}^0 \int_{-h}^u x(s)dsdu\end{aligned}$$

**Remark 10** This new lemma takes the same formulation as in Lemma 7 but only a coefficient in the right-hand side has been modified. Indeed compared to Lemma 7, the coefficient  $\pi^2/4$  is replaced by a greater value, 3. Therefore, the lower bound of the integral provided in this new version is less conservative.

*Proof :* Following the proof of Lemma 7, we introduce a new function  $z_{W2}$  defined for any continuous function  $x$ , given by define the function  $z$  given by

$$z_{W2}(u) = \int_{-h}^u x(s)ds - \frac{u+h}{h}\Omega_0(x) - \frac{u(u+h)}{h^2}\Theta, \quad \forall u \in [-h, 0]. \quad (29)$$

where  $\Theta$  is a constant vector of  $\mathbb{R}^n$  to be defined. The difference between  $z$  and the one proposed in [SG12] appears in the third term. This last term is a polynomial term of degree 2, which becomes zero when  $u = -h$  and  $u = 0$ . Again, by construction, the function  $z_{W2}(u)$  meets the requirements conditions of the Wirtinger's inequality given in Lemma 6, that is  $z(a) = z(b) = 0$ .

Then following the same procedure as in Lemma 9, we first note that the derivative of  $z$  with respect to  $u$  is given by

$$\dot{z}_{W2}(u) = x(u) - \frac{1}{h}\Omega_0(x) - \frac{h+2u}{h^2}\Theta, \quad \forall u \in [-h, 0].$$

Then we expand the expression of  $\int_{-h}^0 \dot{z}_{W2}^\top(u)R\dot{z}_{W2}(u)du$  and on the other side we apply the Jensen inequality to  $\int_{-h}^0 z_{W2}^\top(u)Rz_{W2}(u)du$ . After performing several calculations of integral and integrations by parts (the details can be found in [SG13]), the following inequality is derived

$$\int_{-h}^0 x^\top(u)Rx(u)du - \frac{1}{h}\Omega_0^\top(x)R\Omega_0(x) - \frac{3}{h}\Omega_1^\top(x)R\Omega_1(x) \geq \left(\frac{\pi^2-12}{36h}\right)(\Theta - 3\Omega_1(x))^\top R(\Theta - 3\Omega_1(x)).$$

Since  $12 \geq \pi^2$  and  $R \succ 0$ , the right-hand side of the previous inequality is non positive independently of the choice of  $\Theta$ . Hence, its maximum is reached and is zero when selecting  $\Theta = 3\Omega_1(x)$ , which concludes the proof.  $\diamond$

### 3.3 Extension of the Wirtinger-based inequalities

Compared to the existing contributions within the stability analysis of time-delay systems, the Wirtinger-based integral inequality can be seen as an alternative solution of the discretization or the partition of the delay interval. The main interests of this inequality are described below.

- **Wirtinger-based inequality instead of Jensen's inequality:** Since the Wirtinger-based integral inequality has a similar structure as the Jensen's inequality, it is somehow easy to extend the analysis of various classes of time-delay systems, for instance, *uncertain systems, fuzzy systems, Lur'e systems, neural networks, systems with input saturations*, or various classes of control problems such as *exponential stability, stabilization, optimal control*, etc...
- **Futher improvements on integral inequalities:** The conservatism of stability conditions for time-delay systems employing the Jensen's inequality was already pointed out in the 2000's. For a long time, the only method to reduce the conservatism of these analysis was to employ a discretization method, a delay-partitioning approach, to

augment the model a state augmentation or to introduce additional terms in the Lyapunov-Krasovskii functional in a heuristic manner (i.e. "try and check"). The Wirtinger-based integral inequality provides a new direction for the reduction of the conservatism. Several researches turn out to provide less conservative integral inequalities that refine this new results. On a first hand, one may look at the *Auxiliary-function based integral inequality* in [PLL15], the *free-matrix-based integral inequality* in [ZHWS15], the *Wirtinger-based double integral inequality* [PKP<sup>+</sup>15] or in [Kim16, CXC<sup>+</sup>16] and the *Bessel-Legendre inequality* [SG14, SG15, SGA15, GT16]. Details and explanation on the last method will be provided in the following section, that gives an alternative and more accurate vision of this inequality and also generalizes its concept.

- **Summation inequalities for discrete systems with delays:** On the other hand, a summation version of the Jensen's inequality has been widely employed in the literature to assess stability of discrete-time systems with time-delay. Therefore, the Wirtinger-based integral inequality gave also a motivation to translate this integral inequality into the discrete-time domain. Indeed new *summation inequalities* have been derived in [SGF15, ZH15, GKNT15, NTP15] or in [CLX16] and have led to less conservative stability conditions on various class of discrete-time-delay systems.

### 3.4 Bessel-Legendre inequality on Hilbert spaces

In the previous section, we have shown that increasing the degree of the polynomial function in  $z_{W1}$  allows a reduction of the conservatism of the resulting integral inequality. In this section, we aim at providing a method that pursues this idea of increasing the degree of the polynomial terms. However, at this stage, there is no obvious method that allows generalizing this method and to select in an accurate manner a "good" sequence of polynomials and the right "additional signals", which will be denoted as  $\Omega_i(x)$ , for  $i = 2, 3, \dots$

In this section, we aim at presenting a method to extend, in a generic manner, the Wirtinger-based integral inequality and to provide less and less conservative integral inequalities. This method is based on the Bessel's inequality on Hilbert Space and on the sequence of Legendre polynomials. A motivation of this solution is provided hereafter.

#### 3.4.1 Preliminaries

In this section, we will present an general integral inequality that can be interpreted as the Bessel inequality on Hilbert spaces. In order to fully understand this vision, let us recall a technical detail in the proof of the first Wirtinger-based integral inequality presented in Lemma 7. In this proof, a remark on equation (26) mentioned that an alternative proof of the Jensen's inequality than the usual one. Let us first recall this equation with the particular selection of  $R = I$  to simplify the next developments. After some calculations we obtained the following equation

$$\int_{-h}^0 |z_0(u)|^2 du = \int_{-h}^0 |x(u)|^2 du - \frac{1}{h} \left| \int_{-h}^0 x(u) du \right|^2 \quad (30)$$

where

$$z_0(u) = x(u) - \frac{1}{h} \int_{-h}^0 x(u) du, \quad \forall u \in [-h, 0].$$

The function  $z_0$  can be interpreted as the difference between the function  $x$  under consideration and its average over the interval  $[-h, 0]$ . It can be also rewritten in the following form

$$z_0(u) = x(u) - \mathcal{L}_0^*(u) \int_{-h}^0 \mathcal{L}_0^*(u) x(u) du, \quad \forall u \in [-h, 0],$$

where  $\mathcal{L}_0^*(u) = \left( \int_{-h}^0 1 ds \right)^{-1/2} = h^{-1/2}$  is the constant function of  $\mathcal{C}([-h, 0])$ , which is normalized with respect to the inner product defined on the set of  $\mathcal{C}([-h, 0], \mathbb{R}^n)$  by

$$\langle f, g \rangle := \int_{-h}^0 f(u)g(u) du$$

for any continuous functions  $f$  and  $g$  in  $\mathcal{C}([-h, 0], \mathbb{R}^n)$ . In light of this expression, one can see that the vector  $z_0$  represent the difference between the function  $x$  and its projection, in the sense of the integral inner product to the set of constant function. The norm associated to this inner product is defined by

$$\|x\|_{\mathcal{C}}^2 = \int_{-h}^0 x^\top(s)x(s)ds, \quad \forall x \in \mathcal{C}([-h, 0], \mathbb{R}^n). \quad (31)$$

and we can rewrite (30) as follows

$$\|x - \mathcal{L}_0^*(u) \langle \mathcal{L}_0^*, x \rangle\|_{\mathcal{C}}^2 = \|z_0\|_{\mathcal{C}}^2 = \|x\|_{\mathcal{C}}^2 - \langle \mathcal{L}_0^*, x \rangle^2 \geq 0 \quad (32)$$

with the light abuse of notation consisting in denoting

$$\langle \mathcal{L}_0^*, x \rangle = \int_{-h}^0 \mathcal{L}_0^*(u)x(u)du = \sqrt{\frac{1}{h}} \int_{-h}^0 x(u)du \quad \left( = \sqrt{\frac{1}{h}} \Omega_0(x) \right)$$

Following this framework, the Jensen's inequality, which can be summarized as follows

$$\|x\|_{\mathcal{C}}^2 - \langle \mathcal{L}_0^*, x \rangle^2 \geq 0,$$

can be interpreted in two manners. The first one is related to the Cauchy-Schwartz inequality. Another and more interesting interpretation relies on graphical considerations. It consists in noting that the inequality  $\|x\|_{\mathcal{C}}^2 \geq \langle \mathcal{L}_0^*, x \rangle^2$ , means that the norm of the infinite dimensional vector  $x$  is greater than its projection over the set of constant functions.

Let us see how this framework includes the Wirtinger-based integral inequality. In equation (29), we have considered the following function

$$z_1(u) = x(u) - \frac{1}{h} \Omega_0(x) - \frac{3}{h} \left( \frac{h+2u}{h} \right) \Omega_1(x), \quad \forall u \in [-h, 0].$$

where we already include the fact that the best selection for  $\Theta$  is  $\Theta = 3\Omega_1(x)$ . Let us denote  $\mathcal{L}_1$  the polynomial of degree 1 given by  $\mathcal{L}_1(u) = \frac{h+2u}{h}$  and its normalized version denoted as  $\mathcal{L}_1^*(u) = \mathcal{L}_1(u)/\sqrt{\langle \mathcal{L}_1, \mathcal{L}_1 \rangle}$  for all  $u$  in  $[-h, 0]$ . Simple calculations show that

$$\langle \mathcal{L}_1, \mathcal{L}_1 \rangle = \frac{h}{3}, \quad \langle \mathcal{L}_1, x \rangle = \Omega_1(x),$$

and that these two polynomials  $\mathcal{L}_0^*$  and  $\mathcal{L}_1^*$  interestingly satisfies the following equalities

$$\langle \mathcal{L}_0^*, \mathcal{L}_1^* \rangle = 0, \quad \langle \mathcal{L}_0^*, \mathcal{L}_0^* \rangle = \langle \mathcal{L}_1^*, \mathcal{L}_1^* \rangle = 1,$$

which means that the functions  $\mathcal{L}_0^*$  and  $\mathcal{L}_1^*$  represent an orthonormal sequence of  $\mathcal{C}([-h, 0], \mathbb{R}^n)$  associated to the inner product  $\langle \cdot, \cdot \rangle$ . Following the previous discussion, we can rewrite the Wirtinger-based integral inequality as follows

$$\|z_1\|_{\mathcal{C}}^2 = \left\| x - \sum_{k=0}^1 \mathcal{L}_k^* \langle \mathcal{L}_k^*, x \rangle \right\|_{\mathcal{C}}^2 = \|x\|_{\mathcal{C}}^2 - \sum_{k=0}^1 |\langle \mathcal{L}_k^*, x \rangle|^2 \geq 0,$$

which can be interpreted graphically as an inequality relating again the norm of the infinite dimensional function  $x$  to its projection over the set of polynomials of degree less than 1. This inequality can also be interpreted as the *Bessel's inequality* on Hilbert space with the *sequence of orthonormal polynomials*  $\{\mathcal{L}_0^*, \mathcal{L}_1^*\}$ .

This discussion drive us to find a direct solution to the problem stated at the beginning of this section. Indeed it suffices to design a sequence of orthonormal (or orthogonal) polynomials with respect to the inner product under

consideration and to apply gives the directions to derive less conservative integral inequalities. More generally, the problem has become the selection an appropriate orthonormal sequence (or basis) of the Hilbert space composed by  $\mathcal{C}([-h, 0], \mathbb{R}^n)$  associated to the inner product  $\langle \cdot, \cdot \rangle$ .

Hopefully, this problem has been widely investigated in Mathematics and there exists many works on orthogonal basis on this particular Hilbert space. One may look at trigonometric functions or at polynomials functions such as Legendre's polynomials among many other functions. As induced by the notation  $\mathcal{L}_0$  and  $\mathcal{L}_1$  to denote the polynomials of degree of 0 and 1, we will exploit, in the next paragraph, the properties of the Legendre polynomials in order to derive efficient and less conservative integral inequalities. Latter, we will show how these inequalities can be included in the stability analysis of time-delay systems.

### 3.4.2 Basics on Legendre polynomials

In the following, a brief recall of the Legendre polynomials and their relevant properties is proposed.

**Definition 1** *The Legendre polynomials considered over the interval  $[-h, 0]$  are defined by*

$$\forall k \in \mathbb{N}, \quad \mathcal{L}_k(u) = (-1)^k \sum_{l=0}^k p_l^k \left( \frac{u+h}{h} \right)^l.$$

with  $p_l^k = (-1)^l \binom{k}{l} \binom{k+l}{l}$ .

The sequence of Legendre polynomials  $\{\mathcal{L}_k, k \in \mathbb{N}\}$  forms an orthogonal sequence with respect to the inner product:

$$\langle f, g \rangle = \int_{-h}^0 f(u)g(u)du, \quad \forall f, g \in \mathcal{C}. \quad (33)$$

These polynomials satisfy the following properties:

**Property 2** *The Legendre polynomials verify the following properties*

**P1 Orthogonality:**

$$\forall (k, l) \in \mathbb{N}^2, \quad \int_{-h}^0 \mathcal{L}_k(u)\mathcal{L}_l(u)du = \begin{cases} 0, & k \neq l, \\ \frac{h}{2k+1}, & k = l. \end{cases} \quad (34)$$

**P2 Boundary conditions:**

$$\forall k \in \mathbb{N}, \quad \mathcal{L}_k(0) = 1, \quad \mathcal{L}_k(-h) = (-1)^k.$$

**P3 Differentiation:**

$$\frac{d}{du} \mathcal{L}_k(u) = \begin{cases} 0, & k = 0, \\ \sum_{i=0}^{k-1} \frac{(2i+1)}{h} (1 - (-1)^{k+i}) \mathcal{L}_i(u), & k \geq 1. \end{cases}$$

### 3.4.3 Bessel-Legendre inequality

Based on the Legendre polynomials, the following lemma, presented in [SG14, SG15] is derived.

**Lemma 11** *Let  $x \in \mathcal{C}([-h, 0], \mathbb{R}^n)$  and  $R \in \mathbb{S}_n^+$  and  $h > 0$ . Then, the inequality*

$$\int_{-h}^0 x^\top(u) R x(u) du \geq \frac{1}{h} \begin{bmatrix} \Omega_0(x) \\ \Omega_1(x) \\ \vdots \\ \Omega_N(x) \end{bmatrix}^\top \begin{bmatrix} R & & & \\ & 3R & & \\ & & \ddots & \\ & & & (2N+1)R \end{bmatrix} \begin{bmatrix} \Omega_0(x) \\ \Omega_1(x) \\ \vdots \\ \Omega_N(x) \end{bmatrix} \quad (35)$$

holds for all  $N \in \mathbb{N}$ , where  $\Omega_k(x) = \int_{-h}^0 \mathcal{L}_k(u)x(u)du$ ,  $k = 0, \dots, N$ .

*Proof* : Consider a function  $x$  in  $\mathcal{C}([-h, 0], \mathbb{R}^n)$ , a matrix  $R$  in  $\mathbb{S}_n^+$  and  $h > 0$ . Define the function  $z_N$  by

$$z_N(u) = x(u) - \sum_{k=0}^N \frac{\int_{-h}^0 \mathcal{L}_k(u)x(u)du}{\int_{-h}^0 \mathcal{L}_k^2(u)du} \mathcal{L}_k(u) = x(u) - \sum_{k=0}^N \frac{2k+1}{h} \Omega_k(x) \mathcal{L}_k(u)$$

Clearly,  $z_N$  is in  $\mathcal{C}([-h, 0], \mathbb{R}^n)$  and represents the approximation error between  $x$  and its projection to the polynomial set  $\{\mathcal{L}_k, k = 0, \dots, N\}$  with respect to the inner product (31). The integral  $\int_{-h}^0 z_N^\top(u)Rz_N(u)du$  exists and the orthogonal property **P1** yields

$$\begin{aligned} \int_{-h}^0 z_N^\top(u)Rz_N(u)du &= \int_{-h}^0 x^\top(u)Rx(u)du \\ &\quad - 2 \sum_{k=0}^N \frac{2k+1}{h} \left( \int_{-h}^0 \mathcal{L}_k(u)x(u)du \right)^\top R\Omega_k(x) \\ &\quad + \sum_{k=0}^N \left( \frac{2k+1}{h} \right)^2 \left( \int_{-h}^0 \mathcal{L}_k^2(u)du \right) \Omega_k^\top(x)R\Omega_k(x). \end{aligned}$$

From their definition, we have

$$\Omega_k(x) = \int_{-h}^0 \mathcal{L}_k(u)x(u)du, \quad \left( \frac{2k+1}{h} \right)^2 \int_{-h}^0 \mathcal{L}_k^2(u)du = \frac{2k+1}{h},$$

which yields

$$\int_{-h}^0 z_N^\top(u)Rz_N(u)du = \int_{-h}^0 x^\top(u)Rx(u)du - \sum_{k=0}^N \frac{2k+1}{h} \Omega_k^\top(x)R\Omega_k(x). \quad (36)$$

Finally, inequality (35) is obtained by noting that  $\int_{-h}^0 z_N^\top(u)Rz_N(u)du > 0$  since  $R \succ 0$ .  $\diamond$

**Remark 12** Considering the BL inequality with  $N = 0$  and  $N = 1$  leads to the celebrated Jensen's Inequality and the Wirtinger-based integral inequality [SG13] and the auxiliary function-based integral inequality [PLL15].

**Remark 13** It is well known that the set of polynomial is dense in  $\mathcal{C}([-h, 0], \mathbb{R}^n)$ , Therefore the sequence of polynomial  $\{\mathcal{L}_k\}_{k \geq 0}$  represents a basis of  $\mathcal{C}([-h, 0], \mathbb{R}^n)$ . Then, when  $N$  tends to infinity, the Bessel-Legendre inequality tends to an equality and

$$\int_{-h}^0 x^\top(u)Rx(u)du = \frac{1}{h} \sum_{k=0}^{\infty} (2k+1) \Omega_k^\top(x)R\Omega_k(x).$$

#### 3.4.4 Optimality of the Bessel-Legendre inequality

In the previous lemma, we have used the Legendre polynomials because of their orthogonal properties with respect to the integral inner product. This choice of polynomials has shown some interests since the results inequality has a diagonal structure.

The selection of the Legendre polynomials is interesting not only for this diagonal structure. From the approximation theory, the polynomial  $\sum_{k=0}^N \frac{2k+1}{h} \Omega_k(x) \mathcal{L}_k(u)$  is the best polynomial approximation of degree  $N$  of the function  $x$  in the sense of the inner product, since it minimizes the distance between  $x$  with the set of polynomials of degree less than or equal  $N$ . In other words, the minimization

$$\min_{P \in \mathbb{R}_N^n[u]} \int_{-h}^0 (x(u) - P(u))^\top R (x(u) - P(u)) du$$

where  $\mathbb{R}_N^n[u]$  is the set of polynomials of  $\mathbb{R}^n$  of degree less than  $N$ , is obtained with

$$P(u) = \sum_{k=0}^N \frac{\langle x, \mathcal{L}_k \rangle}{\langle \mathcal{L}_k, \mathcal{L}_k \rangle} \mathcal{L}_k(u) = \sum_{k=0}^N \frac{2k+1}{h} \Omega_k(x) \mathcal{L}_k(u).$$

where we used a light abuse of notation. Therefore, the Bessel-Legendre inequality provided in Lemma 11 is optimal. To see this optimality, let us consider the canonical basis of the polynomials. Following the same procedure as in the previous lemma, the following lemma is derived

**Lemma 14** *Let  $x \in \mathcal{C}([-h, 0], \mathbb{R}^n)$  and  $R \in \mathbb{S}_n^+$  and  $h > 0$ . Consider a sequence of polynomials  $\{p_k\}_{k=0, \dots, N}$  representing a basis of the set of polynomials of degree less than  $N$  and a sequence of real numbers  $\{\beta_k\}_{k=0, \dots, N}$ .*

*Then, the inequality*

$$\int_{-h}^0 x^\top(u) R x(u) du \geq \begin{bmatrix} \Theta_0(x) \\ \Theta_1(x) \\ \vdots \\ \Theta_N(x) \end{bmatrix}^\top \begin{bmatrix} \alpha_{00}R & \alpha_{01}R & \dots & \alpha_{0N}R \\ \alpha_{01}R & \alpha_{11}R & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0N}R & \dots & \dots & \alpha_{NN}R \end{bmatrix} \begin{bmatrix} \Theta_0(x) \\ \Theta_1(x) \\ \vdots \\ \Theta_N(x) \end{bmatrix} \quad (37)$$

holds for all  $N \in \mathbb{N}$ , where, for  $i, k = 0, \dots, N$

$$\alpha_{ik} = \begin{cases} -\beta_k \beta_i \int_{-h}^0 p_k(u) p_i(u) du, & \text{if } i \neq k, \\ 2\beta_k - \beta_k^2 \int_{-h}^0 p_k^2(u) du, & \text{if } i = k, \end{cases} \quad \text{and } \Theta_k(x) = \int_{-h}^0 p_k(u) x(u) du.$$

*Proof*: The proof is similar to the one of Lemma 11. Consider a function  $x$  in  $\mathcal{C}([-h, 0], \mathbb{R}^n)$ , a matrix  $R$  in  $\mathbb{S}_n^+$  and  $h > 0$ . Define the function  $z_N$  by

$$y_N(u) = x(u) - \sum_{k=0}^N \beta_k p_k(u) \int_{-h}^0 p_k(s) x(s) ds = x(u) - \sum_{k=0}^N \beta_k p_k(u) \Theta_k(x).$$

The integral  $\int_{-h}^0 y_N^\top(u) R y_N(u) du$  exists and we have

$$\begin{aligned} \int_{-h}^0 y_N^\top(u) R y_N(u) du &= \int_{-h}^0 x^\top(u) R x(u) du - 2 \sum_{k=0}^N \beta_k \left( \int_{-h}^0 p_k(u) x(u) du \right)^\top R \Theta_k(x) \\ &\quad + \sum_{k=0}^N \sum_{i=0}^N \beta_k \beta_i \int_{-h}^0 p_k(s) p_i(s) ds \Theta_k^\top(x) R \Theta_i(x). \end{aligned}$$

Noting that  $\Theta_k = \int_{-h}^0 p_k(u)x(u)du$ , it yields

$$\begin{aligned} \int_{-h}^0 y_N^\top(u)Ry_N(u)du &= \int_{-h}^0 x^\top(u)Rx(u)du - \sum_{k=0}^N (2\beta_k - \beta_k^2)\Theta_k^\top(x)R\Theta_k(x) \\ &\quad + 2\sum_{k=1}^N \sum_{i=k+1}^N \beta_k\beta_i \int_{-h}^0 p_k(s)p_i(s)ds\Theta_k^\top(x)R\Theta_i(x). \end{aligned} \quad (38)$$

Finally, inequality (37) is obtained by noting that  $\int_{-h}^0 y_N^\top(u)Ry_N(u)du > 0$  since  $R \succ 0$ .  $\diamond$

This lemma presents another integral inequality, which can be used to obtain a new stability condition. However, on the one hand, the matrix  $[\alpha_{ik}]_{i,k=0,\dots,N}$  is not necessarily positive definite. This means that this new inequality may be really conservative. On a second hand, following the discussion above, the polynomial  $\sum_{k=0}^N \frac{2k+1}{h}\Omega_k\mathcal{L}_k(u)$  minimizes the distance between the infinite dimensional function  $x$  and the set of polynomials of degree less than  $N$ . This means that, the inequality

$$\int_{-h}^0 y_N^\top(u)Ry_N(u)du \geq \int_{-h}^0 z_N^\top(u)Rz_N(u)du$$

and the equality occurs only if  $y_N = z_N$ . Finally, the two expressions provided in (36) and (38) implies the following inequality

$$\begin{bmatrix} \Theta_0 \\ \Theta_1 \\ \vdots \\ \Theta_N \end{bmatrix}^\top \begin{bmatrix} \alpha_{00}R & \alpha_{01}R & \dots & \alpha_{0N}R \\ \alpha_{01}R & \alpha_{11}R & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0N}R & \dots & \dots & \alpha_{NN}R \end{bmatrix} \begin{bmatrix} \Theta_0 \\ \Theta_1 \\ \vdots \\ \Theta_N \end{bmatrix} \leq \begin{bmatrix} \Omega_0 \\ \Omega_1 \\ \vdots \\ \Omega_N \end{bmatrix}^\top \begin{bmatrix} R & & & \\ & 3R & & \\ & & \ddots & \\ & & & (2N+1)R \end{bmatrix} \begin{bmatrix} \Omega_0 \\ \Omega_1 \\ \vdots \\ \Omega_N \end{bmatrix},$$

where the argument “ $(x)$ ” has been omitted for the sake of simplicity.

This demonstrates that the Bessel-Legendre inequality is the less conservative inequality that can be derived, of course, when one considers the projection of polynomials sets.

**Remark 15** *This results can also be related to the “sum of squares” framework that has been developed and employed in similar contexts of automatic control in [HLL09, Par00, PPP02, PPN07, PAV<sup>+</sup>13, VAP14]. In this framework, the objectives often relies on the optimization of the coefficients  $\beta_k$ ,  $k = 0, \dots, N$ , that delivers the best inequality. Lemma 11 only says that the optimal integral inequality for this inner product is related to the Legendre polynomials.*

#### 3.4.5 Further comments on the approximation of infinite dimensional state function.

Another interesting comments on this framework is related to the interpretation of the function  $z_N$  which is recalled for the sake of simplicity

$$z_N(u) = x(u) - \sum_{k=0}^N \frac{2k+1}{h}\Omega_k(x)\mathcal{L}_k(u).$$

As we mentioned earlier in this chapter, this function can be interpreted that error of approximation of the infinite dimensional function  $x$  by a finite dimensional polynomial function  $\sum_{k=0}^N \frac{2k+1}{h}\Omega_k(x)\mathcal{L}_k(u)$ . In other words, we have

$$x(u) \approx \sum_{k=0}^N \frac{2k+1}{h}\Omega_k(x)\mathcal{L}_k(u) = \frac{1}{h} \begin{bmatrix} \mathcal{L}_0(u)I & \mathcal{L}_1(u)I & \dots & \mathcal{L}_N(u) \end{bmatrix} \begin{bmatrix} \Omega_0(x) \\ \Omega_1(x) \\ \vdots \\ \Omega_N(x) \end{bmatrix}.$$

In general, the series  $\sum_{k=0}^N \frac{2k+1}{h} \Omega_k(x) \mathcal{L}_k(u)$  does not converge point to point to  $x(u)$ . It only converge to  $x$  in the sense of the norm associated to the inner product. However, if the function  $x$  is infinitely differentiable over the interval  $(-h, 0)$ , then we have

$$x(u) = \sum_{k=0}^{\infty} \frac{2k+1}{h} \Omega_k(x) \mathcal{L}_k(u), \quad \forall u \in (-h, 0).$$

Coming back to the Lyapunov analysis and the stability analysis of time-delay systems, the convergence of the projections  $\Omega_k(x_t)$ , for all  $k \geq 0$  and where  $x_t$  is the state of a time-delay system, to zero implies the convergence of  $x$  to zero as well. This will be taken into account in the Lyapunov analysis developed in the next section.

Moreover, we will show that if we can prove the converge to zero of  $\Omega_k(x_t)$ , for all  $k = 0, \dots, N$ , then the remainder of the projections,  $\Omega_k(x_t)$ , for all  $k \geq N + 1$ , also converge to zero.

To conclude, the Bessel-Legendre inequality introduces additional information on the delay system brought by the projection vectors  $\Omega_k(x_t)$ , for  $k = 0, \dots, N$ , where  $x_t$  is the state of a time-delay system. The objective, when Including these terms in a Lyapunov analysis finally consists in taking into account more and more information on the system state  $x_t$ .

#### 3.4.6 A suitable corollary for the stability analysis of time-delay systems

As it was mentioned in the introduction, the problem is often to derive a lower bound of  $\int_{-h}^0 \dot{x}^\top(u) R \dot{x}(u) du$ . The next corollary addresses this particular problem.

**Corollary 16** *Let  $x$  be such that  $\dot{x} \in \mathcal{C}$ ,  $R \in \mathbb{S}_n^+$  and  $h > 0$ . Then, the integral inequality*

$$\int_{-h}^0 \dot{x}^\top(u) R \dot{x}(u) du \geq \frac{1}{h} \xi_N^\top \left[ \sum_{k=0}^N (2k+1) \Gamma_N(k)^\top R \Gamma_N(k) \right] \xi_N, \quad (39)$$

holds for all integer  $N \in \mathbb{N}$  where

$$\begin{aligned} \xi_N &= \begin{cases} [x^\top(0) \ x^\top(-h)]^\top, & \text{if } N = 0, \\ \left[ x^\top(0) \ x^\top(-h) \ \frac{1}{h} \Omega_0^\top(x) \ \dots \ \frac{1}{h} \Omega_{N-1}^\top(x) \right]^\top, & \text{if } N > 0, \end{cases} \\ \Gamma_N(k) &= \begin{cases} \begin{bmatrix} I & -I \end{bmatrix}, & \text{if } N = 0, \\ \begin{bmatrix} I & (-1)^{k+1} I & \gamma_{Nk}^0 I & \dots & \gamma_{Nk}^{N-1} I \end{bmatrix}, & \text{if } N > 0. \end{cases} \\ \gamma_{Nk}^i &= \begin{cases} -(2i+1)(1 - (-1)^{k+i}), & \text{if } i \leq k, \\ 0, & \text{if } i > k. \end{cases} \end{aligned}$$

and where  $\Omega_k$  is defined in Lemma 11.

*Proof :* Applying Lemma 11 to the order  $N$  leads to

$$\int_{-h}^0 \dot{x}^\top(u) R \dot{x}(u) du \geq \frac{1}{h} \sum_{k=0}^N (2k+1) \Omega_k^\top(\dot{x}) R \Omega_k(\dot{x}), \quad (40)$$

where we recall that  $\Omega_k(\dot{x}) = \int_{-h}^0 \mathcal{L}_k(u)\dot{x}(u)du$ , for all  $k = 0, 1, \dots, N$ . An integration by parts ensures that, for all  $k \geq 0$

$$\Omega_k(\dot{x}) = \mathcal{L}_k(0)x(0) - \mathcal{L}_k(-h)x(-h) - \int_{-h}^0 \left( \frac{d}{du} \mathcal{L}_k(u) \right) x(u)du.$$

Thanks to properties **P2** and **P3** of the Legendre polynomials, the following expression is derived

$$\Omega_k(\dot{x}) = x(0) - (-1)^k x(-h) + \sum_{i=0}^{k-1} \frac{\gamma_{Nk}^i}{h} \Omega_i(x) = \Gamma_N(k) \xi_N. \quad (41)$$

Replacing  $\Omega_k(\dot{x})$  by its expression using  $\Omega_i(x)$ ,  $i = 0, \dots, k$  and the matrices  $\Gamma_N(k)$ ,  $k = 1, \dots, N$  leads to (39) and concludes the proof.  $\diamond$

## 4 Applications to linear systems with a single constant delay

### 4.1 Stability theorem

In this paragraph, a first stability result for time-delay systems is provided by the use of the Bessel-Legendre inequality developed in the previous section. We will study the stability of the following linear system subject to a constant delay

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h_2, 0], \end{cases} \quad (42)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\phi$  is the initial conditions and  $A$  and  $A_d$ , are constant matrices. The following stability theorem, presented in [SG15], is provided by the use of Corollary 16 with an arbitrary  $N$ .

**Theorem 17** *For a given integer  $N$  and a constant delay  $h$ , assume that there exist a matrix  $P_N \in \mathbb{S}_{(N+1)n}$  and two matrices  $S, R \in \mathbb{S}_n^+$  such that the LMI*

$$\begin{aligned} \Theta_N(h) &= \begin{cases} P_N \succ 0, & \text{if } N = 0, \\ P_N + \frac{1}{h} \begin{bmatrix} 0 & & & \\ & S & & \\ & & \ddots & \\ & & & (2N-1)S \end{bmatrix} \succ 0, & \text{if } N > 0, \end{cases} \\ \Phi_N(h) &= \Phi_{N0}(h) - \begin{bmatrix} \Gamma_N(0) \\ \vdots \\ \Gamma_N(N) \end{bmatrix}^\top \begin{bmatrix} R & & & \\ & 3R & & \\ & & \ddots & \\ & & & (2N+1)R \end{bmatrix} \begin{bmatrix} \Gamma_N(0) \\ \vdots \\ \Gamma_N(N) \end{bmatrix} \prec 0, \end{aligned} \quad (43)$$

hold, where  $\Gamma_N(k)$ , for all  $k = 0, \dots, N$ , are defined in Corollary 16 and

$$\begin{aligned}\Phi_{N0}(h) &= \text{He}(G_N^\top(h)PH_N) + \tilde{S}_N + h^2F_N^\top RF_N, \\ \tilde{S}_N &= \text{diag}\{S, -S, 0_{Nn}\}, \\ S_N &= \text{diag}\{S, 3S, \dots, (2N+1)S\}, \\ F_N &= \begin{bmatrix} A & A_d & 0_{n,nN} \end{bmatrix}, \\ G_N(h) &= \begin{bmatrix} I & 0_n & 0_{n,nN} \\ 0_{nN,n} & 0_{nN,n} & hI_{nN} \end{bmatrix}, \\ H_N &= \begin{bmatrix} F_N^\top & \Gamma_N^\top(0) & \Gamma_N^\top(1) & \dots & \Gamma_N^\top(N-1) \end{bmatrix}^\top.\end{aligned}$$

Then the time-delay system (42) is asymptotically stable for the constant delay  $h$ .

*Proof* : Guided by the B-L inequality (39) and the signals involved, we consider the following extra-states  $\tilde{x}_N(t)$  defined by:

$$\tilde{x}_N(t) = \begin{bmatrix} x_t(0) \\ \int_{-h}^0 \mathcal{L}_0(s)x_t(s)ds \\ \vdots \\ \int_{-h}^0 \mathcal{L}_{N-1}(s)x_t(s)ds \end{bmatrix} = \begin{bmatrix} x_t(0) \\ \Omega_0(x_t) \\ \vdots \\ \Omega_{N-1}(x_t) \end{bmatrix},$$

if  $N \geq 1$  and  $\tilde{x}_0(t) = x_t(0)$ , if  $N = 0$ . The augmented vector  $\tilde{x}_N$  is composed by the instantaneous state  $x_t(0)$  and the projections of the state function  $x_t$  to the  $N$  first Legendre polynomials. Following the proof of Corollary 16 and equation (41), an integration by parts allows expressing the time derivative of  $\tilde{x}_N$  as follows

$$\dot{\tilde{x}}_N(t) = H_N \xi_N(t), \quad (44)$$

where

$$\xi_N(t) = \begin{bmatrix} x_t(0) \\ x_t^\top(-h) \\ \frac{1}{h} \int_{-h}^0 \mathcal{L}_0(s)x_t(s)ds \\ \vdots \\ \frac{1}{h} \int_{-h}^0 \mathcal{L}_{N-1}(s)x_t(s)ds \end{bmatrix}, \quad N \geq 1,$$

and, if  $N = 0$ ,  $\xi_0^\top(t) = [x_t^\top(0) \ x_t^\top(-h)]$ . It appears that this augmented system is the interconnection of the original delay system and a LTI system defined by the states  $\int_{-h}^0 \mathcal{L}_k(s)x_t(s)ds$ , for  $k = 0, \dots, N-1$ . It is also worth mentioning that the only delayed term in (44) is  $x_t(-h)$ . Then, a natural choice for the Lyapunov-Krasovskii functional is

$$V_N(x_t, \dot{x}_t) = \tilde{x}_N^\top(t)P_N\tilde{x}_N(t) + \int_{t-h}^t x^\top(s)Sx(s)ds + h \int_{t-h}^t \int_{\theta}^\top \dot{x}^\top(s)R\dot{x}(s)dsd\theta, \quad (45)$$

On a first hand, following the procedure provided in [GKC03], the condition  $S \succ 0$  allows applying Lemma 11 to the second term of  $V_N$  to give a more accurate lower bound of the functional. In order to be consistent with the definition of  $\tilde{x}_N$ , Lemma 11 is considered with the order  $N-1$ . It thus yields

$$V_N(x_t, \dot{x}_t) \geq \tilde{x}_N^\top(t)\Theta_N(h)\tilde{x}_N(t) + h \int_{t-h}^t \int_{\theta}^\top \dot{x}^\top(s)R\dot{x}(s)dsd\theta.$$

Then the positive definiteness of  $V_N$  results from the conditions  $S \succ 0$ ,  $R \succ 0$  and  $\Theta_N(h) \succ 0$ . This also implies that there exists a sufficiently small  $\epsilon_1 > 0$ , such that  $\Theta_N(h) \succ \begin{bmatrix} \epsilon_1 I & 0 \\ 0 & 0 \end{bmatrix}$ . It follows that  $V_N(x_t, \dot{x}_t) \geq \epsilon_1 |x_t(0)|^2$ .

Furthermore, there exists a sufficiently large scalar  $\lambda > 0$  such that  $P_N \prec \lambda \text{diag}(I, I, 3I, 5I, \dots, (2N-1)I)$ . It thus holds

$$V_N(x_t, \dot{x}_t) \leq \lambda |x_t(0)|^2 + \lambda \sum_{i=0}^{N-1} (2i+1) \Omega_i^\top \Omega_i + \int_{t-h}^t x^\top(s) S x(s) ds + h \int_{t-h}^t \int_{\theta}^\top \dot{x}^\top(s) R \dot{x}(s) ds d\theta.$$

Thanks to Lemma 11, we obtain

$$V_N(x_t, \dot{x}_t) \leq \lambda |x_t(0)|^2 + \int_{t-h}^t x^\top(s) (\lambda h I + S) x(s) ds + h \int_{t-h}^t \int_{\theta}^\top \dot{x}^\top(s) R \dot{x}(s) ds d\theta,$$

which guarantees that there exists a scalar  $\epsilon_2 > 0$ , such that  $V_N(x_t, \dot{x}_t) \leq \epsilon_2 |\bar{x}_t|_h^2$ , for all  $t > h$ , where  $\bar{x}_t = \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}$ . Then it holds

$$\epsilon_1 |x_t(0)|^2 \leq V_N(x_t, \dot{x}_t) \leq \epsilon_2 |\bar{x}_t|_h^2. \quad (46)$$

Consider now the derivative of  $V_N$ , for all  $t \geq h$ . We obtain

$$\begin{aligned} \dot{V}_N(x_t, \dot{x}_t) &= 2\bar{x}_N^\top(t) P_N \dot{\bar{x}}_N(t) + x_t^\top(0) S x_t(0) - x_t^\top(-h) S x_t(-h) + h^2 \dot{x}_t^\top(0) R \dot{x}_t(0) \\ &\quad - h \int_{-h}^0 \dot{x}_t^\top(s) R \dot{x}_t(s) ds. \end{aligned} \quad (47)$$

By noting that  $\bar{x}_N(t) = G_N(h) \xi_N(t)$ ,  $\dot{\bar{x}}_N(t) = H_N \xi_N(t)$  and  $\dot{x}_t(0) = F_N \xi_N(t)$ , it yields

$$\dot{V}_N(x_t, \dot{x}_t) = \xi_N^\top(t) \Phi_{N0}(h) \xi_N(t) - h \int_{-h}^0 \dot{x}_t^\top(s) R \dot{x}_t(s) ds. \quad (48)$$

Finally, applying Corollary 16 to the order  $N$  and injecting the resulting inequality into (48) leads to  $\dot{V}_N(x_t, \dot{x}_t) \leq \xi_N^\top(t) \Phi_N(h) \xi_N(t)$ . Hence, if the LMI (43) are satisfied, there exists a scalar  $\epsilon_3 > 0$  such that  $\Phi_N(h) \prec \begin{bmatrix} -\epsilon_3 I & 0 \\ 0 & 0 \end{bmatrix}$ . We finally obtain

$$\dot{V}_N(x_t, \dot{x}_t) \leq -\epsilon_3 |x_t(0)|^2, \quad \forall t \geq h. \quad (49)$$

The end of the proof is taken from the proof of Theorem 1 from [FSR04]. Integrating (49) we have

$$V_N(x_t, \dot{x}_t) - V_N(x_h, \dot{x}_h) \leq -\epsilon_3 \int_h^\top |x_s(0)|^2 ds \quad (50)$$

and, hence, (46) yields

$$\epsilon_1 |x_t(0)|^2 \leq V_N(x_t, \dot{x}_t) \leq V_N(x_h, \dot{x}_h) \leq \epsilon_2 |\bar{x}_h|_h^2.$$

Since  $|x_h|_h \leq c_1 |\phi|_h$ ,  $c_1 > 0$  (cf. [HL93] p. 168) and  $\dot{x}$ , defined by the right-hand side of (42), satisfies  $|\dot{x}_h|_h \leq c_2 |\phi|_h$ ,  $c_2 > 0$ , we obtain that

$$|x_t(0)|^2 \leq V_N(x_h, \dot{x}_h) / \epsilon_1 \leq c_3 |\phi|_h^2, \quad c_3 > 0.$$

Hence, (42) is stable. To prove asymptotic stability we note that, for any initial condition  $\phi$ ,  $x$  is uniformly continuous on  $[0, \infty)$  (since  $\dot{x}$  defined by the right-hand side of (42) is uniformly bounded). Moreover, (50) yields that  $|x_t(0)|^2$  is integrable on  $[h, \infty)$ . Then, by Barbalat's lemma,  $x_t(0) \rightarrow 0$  as  $t \rightarrow \infty$ . Consequently, if the LMI of Theorem are satisfied, the delay system (42) is asymptotically stable for the constant delay  $h$ .  $\diamond$

**Remark 18** Taking  $N = 0$  in Theorem 17 allows retrieving one of the most classical delay-dependent stability conditions based on Jensen's inequality and LMI [GP06b]. Additionally, choosing  $N = 1$  leads to the stability conditions from [SG13].

**Remark 19** An interpretation of the BL inequality and the associated stability analysis is provided in the context of robust analysis in [GAS13, GASP16]. In this second paper, we have notably showed that the same LMI conditions presented in Theorem 17 imply

$$\det(sI - A - A_d e^{-hs}) \neq 0, \quad \forall s \in \mathbb{C}, \text{ s.t. } \operatorname{Re}(s) \geq 0,$$

which brings the link between this Lyapunov analysis and the frequency analysis of the time-delay system.

#### 4.2 Remark on the choice of the Lyapunov-Krasovskii functional

A comment on the Lyapunov-Krasovskii functional,  $V_N$ , and its relation with the class of functionals studied in [Gu01, PPL09] is highlighted here. Indeed by considering the functional (45) and by defining the polynomial matrix  $D(s) = \operatorname{diag}(0_n, \mathcal{L}_0(s)I, \mathcal{L}_1(s)I, \dots, \mathcal{L}_N(s)I)$  and the matrices

$$\begin{aligned} \tilde{P} &= \begin{bmatrix} I \\ 0_{nN,n} \end{bmatrix}^\top P_N \begin{bmatrix} I \\ 0_{nN,n} \end{bmatrix}, \quad S(s) = S \\ \mathcal{Q}(s) &= \begin{bmatrix} I \\ 0_{nN,n} \end{bmatrix}^\top P_N D(s), \quad \mathcal{T}(s, \xi) = D^\top(s) P_N D(\xi). \end{aligned}$$

Therefore, the functional  $V_N$  can be rewritten as

$$\begin{aligned} V_N(x_t, \dot{x}_t) &= x^\top(t) \tilde{P} x(t) + 2x^\top(t) \int_{-h}^0 \mathcal{Q}(s) x_t(s) ds + \int_{-h}^0 \int_{-h}^0 x_t^\top(s) \mathcal{T}(s, \xi) x_t(\xi) ds d\xi \\ &\quad + \int_{-h}^0 x_t^\top(s) S(s) x_t(s) ds + h \int_{-h}^0 \int_{\theta}^0 \dot{x}_t^\top(s) R \dot{x}_t(s) ds d\theta. \end{aligned} \tag{51}$$

The three first terms of  $V_N$  are similar to the ones presented in equation (17) and employed in [PPL09] and in [Gu01]. In [PPL09], the degree of freedom comes from the degree of the polynomial matrices  $Q(s)$ ,  $S(s)$  and  $R(s, \xi)$ , denoted as  $D_p$ . In [Gu01], the degree of the polynomial is always 1 but the degree of freedom comes from the degree of discretization, denoted latter on as  $D_d$ .

A first difference with respect to these two approaches is that in our setup, the polynomial matrix  $S(s)$  is constant. The consequence is that our method requires less parameters to define the functional when increasing  $D_p$  or  $D_d$ . Another difference relies on the last integral quadratic term of  $V_N$  which depends on  $\dot{x}_t$ . Finally, the previous theorem does not need to enter into the sum of squares framework which generally requires the use of additional decision relaxation variables when testing the stability conditions.

**Remark 20** Note that in [SG14] or in [SGA15], similar theorems were deployed based on the complete representation of Lyapunov-Krasovskii functionals, as employed in, for instance, [PPL09] and [Gu01]. It simply consists in a light modification in 51 where the last integral term depends on  $x_t$  instead of  $\dot{x}_t$ .

#### 4.3 Hierarchy of LMI stability conditions

This section is devoted to proving that the previous stability conditions form a hierarchy of LMI conditions. This is formulated in the following theorem based on the stability conditions of Theorem 17.

**Theorem 21** For any time-delay system (42), define the set  $\mathcal{H}_N$  by

$$\mathcal{H}_N := \left\{ h \in \mathbb{R}^+ \text{ s.t. } \Theta_N(h) \succ 0, \quad \Phi_N(h) \prec 0, P_N, S(N) \succ 0, R(N) \succ 0 \right\}$$

Then, it holds

$$\mathcal{H}_N \subset \mathcal{H}_{N+1}, \quad \forall N \geq 0.$$

*Proof* : Let  $N \in \mathbb{N}$ . If  $\mathcal{H}_N$  is empty, the inclusion is trivial. If  $\mathcal{H}_N$  is not empty, then consider an element  $h \in \mathcal{H}_N$ . From the definition of  $\mathcal{H}_N$ , there exist symmetric matrices  $P_N, S(N) \succ 0$  and  $R(N) \succ 0$  such that  $\Theta_N(h) \succ 0$  and  $\Phi_N(h) \prec 0$ . Taking advantages of the construction of the Lyapunov-Krasovskii functional (45), we suggest the matrices

$$P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{cases} S(N+1) = S(N) = S, \\ R(N+1) = R(N) = R, \end{cases}$$

where  $\epsilon$  is a scalar to be chosen. Then the matrix  $\Theta_{N+1}(h)$  can be rewritten as

$$\Theta_{N+1}(h) = \begin{bmatrix} \Theta_N(h) & 0 \\ 0 & \frac{2N+1}{h}S \end{bmatrix}.$$

The assumptions  $\Theta_N(h) \succ 0$  and  $S \succ 0$  thus ensures  $\Theta_{N+1}(h) \succ 0$ .

Concerning the second LMI condition  $\Phi_N$ , we first note that

$$\begin{aligned} H_{N+1} &= \begin{bmatrix} H_N & 0_{Nn,n} \\ \Gamma_{N+1}(N) \end{bmatrix}, \\ G_{N+1}(h) &= \begin{bmatrix} G_N(h) & 0_{Nn,n} \\ 0_{n,Nn} & hI \end{bmatrix}, \\ \tilde{S}_{N+1} &= \begin{bmatrix} \tilde{S}_N & 0_{Nn,n} \\ 0_{n,Nn} & 0_n \end{bmatrix}, \\ F_{N+1} &= \begin{bmatrix} F_N & 0_n \end{bmatrix}. \end{aligned}$$

From these expressions, the matrix  $\Phi_{N+1}(h)$  can be expressed using the matrix  $\Phi_N(h)$  as follows

$$\Phi_{N+1}(h) = \begin{bmatrix} I & 0 \\ \Gamma_{N+1}(N+1) \end{bmatrix}^\top \begin{bmatrix} \Phi_N(h) & 0 \\ 0 & -(2N+3)R \end{bmatrix} \begin{bmatrix} I & 0 \\ \Gamma_{N+1}(N+1) \end{bmatrix}.$$

Since  $\Phi_N(h) \prec 0$ ,  $R \succ 0$  and by noting that the matrix  $\begin{bmatrix} I & 0 \\ \Gamma_{N+1}(N+1) \end{bmatrix}$  is non singular, the first term of the previous expression is negative definite.

Therefore,  $h$  belongs to  $\mathcal{H}_{N+1}$ . Finally, since  $h$  is any element of  $\mathcal{H}_N$ , it implies that  $\mathcal{H}_N \subset \mathcal{H}_{N+1}$ .  $\diamond$

Theorem 21 proves that, the stability conditions provided in Theorem 17 at the order  $N+1$  delivers, at least, the same result the same condition taken at the order  $N$ . Since Theorem 17 only provides sufficient stability condition, the set  $\mathcal{H}_N$ , for a given  $N \in \mathbb{N}$  represents an inner approximation of the stability pockets.

Moreover, the previous proof also show that

$$\mathcal{H}_{N+1} \Big|_{P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & 0 \end{bmatrix}} = \mathcal{H}_N,$$

where the set on the left-hand-side stands for the restriction of  $\mathcal{H}_{N+1}$  to matrices  $P_{N+1}$  of the corresponding form. A brief recursive reasoning allows us to obtain that

$$\mathcal{H}_{N+1} \Big|_{P_{N+1} = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}} = \mathcal{H}_0, \quad P_0 \in \mathbb{S}^n$$

This equality of sets can be interpreted as follows. If one uses the Bessel-Legendre (or the Wirtinger-based) inequality, without augmenting the size of the matrix  $P_N$ , or, again, without enriching the functional by the projection  $\Omega_k$ , with  $k \leq N - 1$ , then the result is strictly equivalent to the one obtained using the Jensen's inequality (i.e.  $N = 0$ ). Therefore the augmentation of the size of the matrix  $P_N$  is crucial to derive less conservative numerical results.

Finally, Theorem 17 provides a hierarchy of LMI conditions for the analysis of time-delay systems. This means that increasing  $N$  potentially helps to reduce the conservatism of the conditions, and potentially to detect greater and greater stability regions. Nevertheless, Theorem 17 does not prove that the conditions of Theorem 17 will converge to the analytical bounds of the delay. In other words, Theorem 17 does not answer to the question

*“If the system is asymptotically stable for a given delay  $h$ ,  
does there exist a integer  $N$  such that the LMI conditions provided in Theorem 17 holds?”*

As it will be exposed in the next section, where numerical examples are treated, the conditions of Theorem 17 are very efficient from the numerical point of view and also its complexity is very competitive with respect to other stability conditions from the literature. Through these examples, we have always been able to obtain an encouraging result, even for non trivial systems.

We have not been able to answer to this question yet. However, we strongly believe that the properties of the Legendre polynomials together with the existence of a complete Lyapunov-Krasovskii functionals will lead the way to a positive answer.

## 5 Numerical applications and extensions

### 5.1 Examples of linear systems with a single discrete delay

The purpose of the following section is to illustrate on academic and non trivial examples how the inequalities given in Section 3 lead to a relevant reduction of conservatism in the stability condition.

#### 5.1.1 Example 1:

Consider the same linear time-delay system (42) presented in (70) with the matrices

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \quad (52)$$

This system is a well-known delay dependent stable system, that is the delay free system is stable and the maximum allowable delay  $h_{\max} = 6.1725$  can be easily computed by delay sweeping techniques. The results are reported in Table 1. Many recent papers give the same result since they are intrinsically based on the same Lyapunov-Krasovskii functional and use the same bounding cross terms technique i.e. Jensen inequality. Some papers [SLCR10],[SLC09] which use an augmented Lyapunov, based on the addition on a triple Integral term on the Lyapunov-Krasovskii functional can go further but with a numerically increasing burden, compared to our proposal. The partitioning approach proposed by [Han09] based on the discrete delay decomposition gives an upperbound which tends to the analytical value even if the numerical complexity remains important. The robust approach [KR07] gives a very good upper-bound with a similar computational complexity than our present result. The discretized Lyapunov-Krasovskii functional proposed by [Gu01] as well as the sum of square optimization scheme developed by Peet et al [PPL09] give a delay upperbound very closed to the maximum allowable delay with an increasing numerical complexity.

#### 5.1.2 Example 2:

This example is provided to illustrate Theorem 21. Note that this system has not been studied in the literature of time-delay system using the Lyapunov-Krasovskii Theorem and LMI conditions. It is extracted from the dynamics

Theorems	$h_{max}$	number of variables	Methods
Theoretical bound	6.1725	-	Frequency approach
[GP06b, HWXL07a]	4.472	$1.5n^2 + 1.5n$	L-K functional + Jensen
[SLCR10]	5.02	$18n^2 + 18n$	L-K functional + Jensen
[Kim11]	4.97	$69n^2 + 5n$	L-K functional + Jensen
[SLC09]	5.30	$8.5n^2 + 3.5n$	L-K functional + Jensen
[SG12]	5.901	$3n^2 + 2n$	L-K functional + Wirtinger-based
[KR07]	6.1107	$1.5n^2 + 9n + 9$	IQC + filter
[AGJ10]	5.120	$7n^2 + 4n$	QS + filter
[AG09a]	5.120	$6.5n^2 + 3.5n$	QS + Augmentation
[Han09], $D_d = 2$	5.71	$4n^2 + 2n$	L-K functional + Jensen + Partition
[Han09], $D_d = 3$	5.96	$6.5n^2 + 2.5n$	L-K functional + Jensen + Partition
[Han09], $D_d = 4$	6.05	$10n^2 + 3n$	L-K functional + Jensen + Partition
[Gu01], $D_d = 1$	6.053	$7.5n^2 + 3.5n$	L-K functional + Discretization
[Gu01], $D_d = 2$	6.165	$10.5n^2 + 4.5n$	L-K functional + Discretization
[Gu01], $D_d = 3$	6.171	$14.5n^2 + 4.5n$	L-K functional + Discretization
[Gu01], $D_d = 4$	6.171	$20.5n^2 + 5.5n$	L-K functional + Discretization
[PPL09], $D_p = 1$	5.19	$7n^2 + 3n$	L-K functional + SOS
[PPL09], $D_p = 2$	5.90	$12.5n^2 + 4.5n$	L-K functional + SOS
[PPL09], $D_p = 3$	6.10	$21n^2 + 6n$	L-K functional + SOS
Th.17, $N = 0$	4.472	$1.5n^2 + 1.5n$	L-K functional + Bessel
Th.17, $N = 1$	6.059	$3n^2 + 2n$	L-K functional + Bessel
Th.17, $N = 2$	6.168	$5.5n^2 + 2.5n$	L-K functional + Bessel
Th.17, $N = 3$	6.1725	$9n^2 + 3n$	L-K functional + Bessel

Table 1  
Results for Example (70) for constant delay  $h$ . The degree of discretization  $D_d$  and the degree of the polynomial  $D_p$  are defined in subsection 4.2.

of machining chatter [ZKT01, SNA<sup>+</sup>11] and is given by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top.$$

A delayed static output feedback controller is proposed:

$$u(t) = -Ky(t) + Ky(t-h),$$

where  $K$  is the gain of the controller and  $h$  is an unknown constant delay. The resulting dynamics is thus modeled by a time-delay system:

$$\dot{x}(t) = A_0x(t) + A_1x(t-h),$$

with  $A_0 = A - BKC$  and  $A_1 = BKC$ .

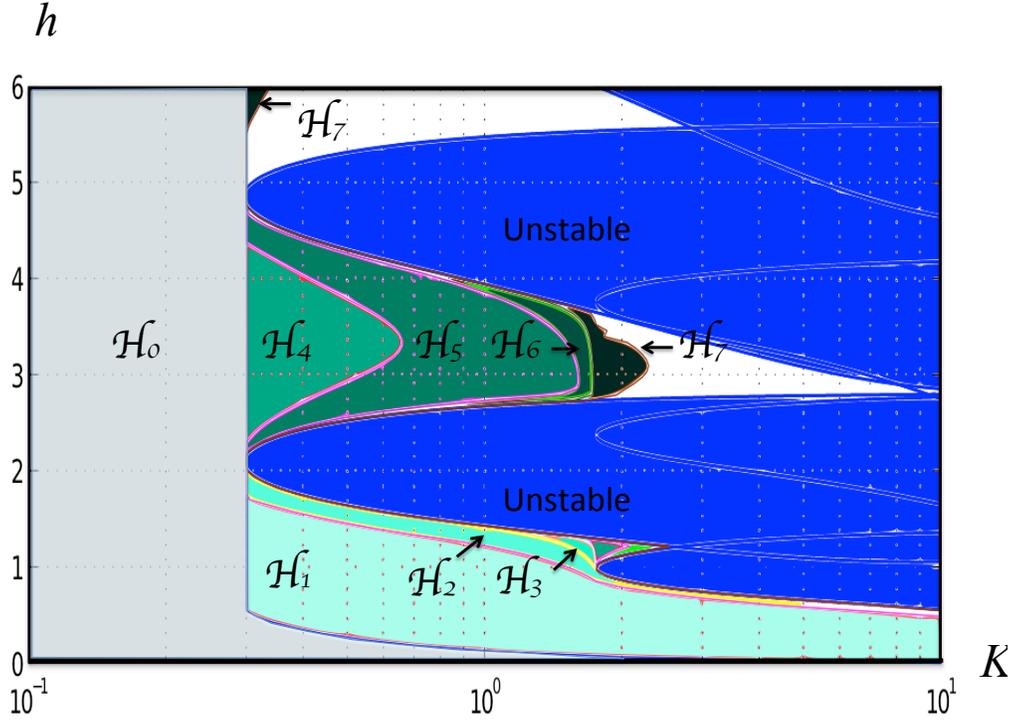


Fig. 4. Stability region in the plan  $(K, h)$ , obtained using Theorem 17 for  $N = 0, \dots, 7$ .

The results are summarized in Figure 4 which shows the stability regions in the  $(K, h)$  plane. The blue region represents the instability region which have been calculated using a gridding over  $K$  along with the `allmargin` function of the Control Toolbox of Matlab©. Then, Theorem 17 provides inner approximations of the stability region delimited by colored curves. The curve  $N = 0$  perfectly detects the independent of the delay stability region (for  $K \leq 0.3$ ) as well as a first delay dependent stability pocket. It corresponds to the maximal allowable delay when Jensen's Lemma is used when establishing the stability criterion. Taking  $N = 1$  allows retrieving the same results as in [SG13] which uses Wirtinger's Lemma. Clearly, increasing  $N$  ( $N = 0, 1, \dots, 7$ ) allows reducing the pessimism and discovering new stability pockets. This figure illustrates the implication of Theorem 21 on the inclusions  $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \dots \subset \mathcal{H}_7$ . Another important remark is that increasing  $N$  can improve significantly the inner approximations of the stability region. For instance,  $\mathcal{H}_4 \setminus \mathcal{H}_3$ ,  $\mathcal{H}_5 \setminus \mathcal{H}_4$  or even  $\mathcal{H}_7 \setminus \mathcal{H}_6$  are surprisingly large sets.

### 5.1.3 Example 3:

Consider the system with pointwise delay taken from [Fre00] given by

$$\dot{x}(t) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dot{x}(t-h) + \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} x(t) = 0$$

The analysis provided in [Fre00] based on a frequency method ensures that this systems has exactly three stable delay intervals  $[0.4108, 0.7509]$ ,  $[2.054, 2.252]$  and  $[3.697, 3.754]$ . Figure 5 shows the inner approximations obtained with similar conditions as the ones presented in Theorem 17 (according to the modifications explained in Remark 20) for several values of  $N$ . The first interval is detected at  $N = 4$  and a good<sup>1</sup> estimation of the first stable interval is first obtained at  $N = 7$ . First values of the delay  $h$  in the second stable interval are detected with  $N = 9$  and a good estimation of the whole interval is obtained at  $N = 11$ . Delay values in the third and last stable interval, which is more difficult to detect are found with  $N = 14$  and a good estimation of the interval is provided with  $N = 16$ .

<sup>1</sup> with a precision of  $10^{-4}$

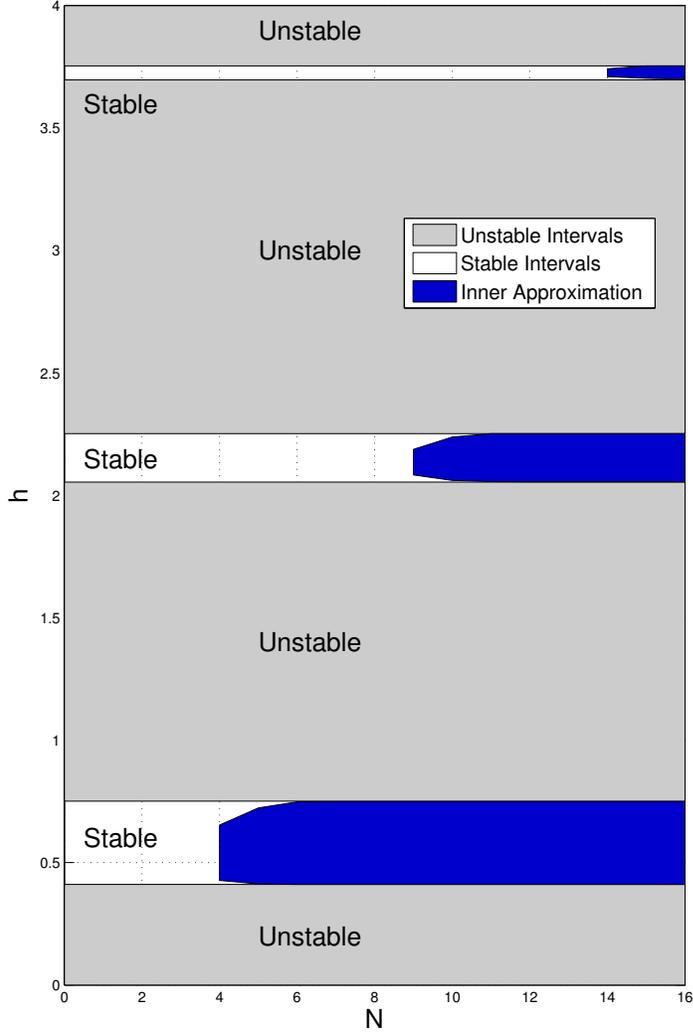


Fig. 5. Stability region in the plan  $(N, h)$ , obtained using Theorem 17 (and Remark 20 for Example 4).

This example shows the efficiency of our method to detect stability intervals even for systems with discrete delay and with dimension that are not small unlike usual examples. In addition, The example and Figure 5 also illustrate Theorem 21. Indeed, Figure 5 shows that the stability regions become larger when  $N$  increases. The stability conditions are able to assess stability of the system for all the values of the delay which belongs to the stable intervals.

## 5.2 Examples of linear systems with multiple discrete delays

Through this example we aim at showing that the proposed approach can address the case of linear systems with multiple delays. The associated analysis, presented in [SBS16a] is omitted here for simplicity. However, and to be short, the method consists in extending the Lyapunov-Krasoskii functionals to cope with the several delays using a coupled PDE-ODE systems and a homogeneous delay representation.

### 5.2.1 Example 4:

Let us consider one of the most classical example of systems with multiple delays, which was studied in [SNA<sup>+</sup>11], using a frequency domain analysis. This system is governed by the following equation

$$\dot{x}(t) = -1.3x(t) - x(t - h_1) - 0.5x(t - h_2)$$

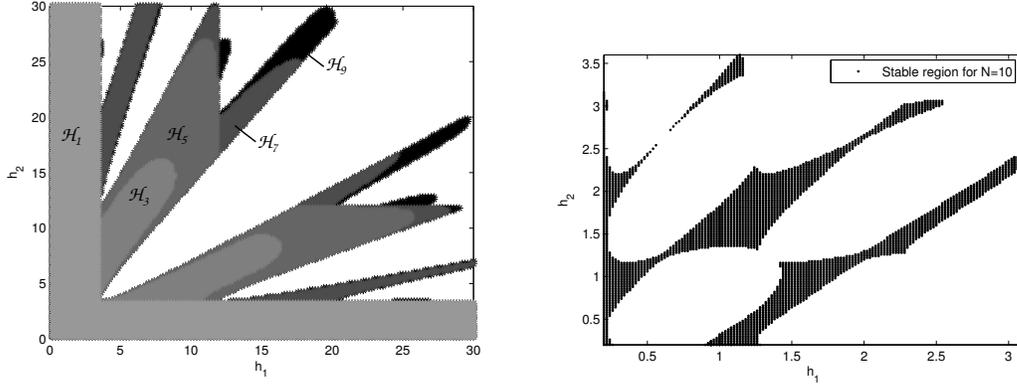


Fig. 6. Stability regions in the plan  $(h_1, h_2)$  obtained in [SBS16b] for Example 5 (Left) and Example 6 (right).

The conditions proposed in Theorem 17 can be extended to the case of multiple delays as proposed in [SBS16b] addresses also the stability of systems that may be unstable for a very high speed of transport as it is illustrated with this example. For this example, Fig. 6 gives the stability regions for different values of  $h_1$  and  $h_2$ . We can remark that increasing  $N$  allows us to broaden the stability region of the coupled system. This example prove also Theorem 21 where  $\mathcal{H}_1 \subset \mathcal{H}_3 \subset \dots \subset \mathcal{H}_9$  and if we have stability for  $N = 1$  with a given  $h_1$  and  $h_2$ , we ensure stability for the same pair of transport speed with  $N > 1$ .

### 5.2.2 Example 5

Consider the linear system with *cross-talking* multiple delays, studied in [SNA<sup>+</sup>11] and governed by the following delay differential equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -20 & -1 \end{bmatrix} x(t) - \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix} x(t - h_1) - \begin{bmatrix} 0 & 0 \\ 4 & 1 \end{bmatrix} x(t - h_2) - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - h_1 - h_2). \quad (53)$$

For this example, Fig. 6 gives the stability regions for different value of  $h_1$  and  $h_2$  with  $N = 10$ . Again, this figure shows that the method based on the Bessel-Legendre inequality is able to assess stability of linear systems with multiple delays in a efficient manner.

### 5.3 Examples of linear systems with a distributed delay

In [SGA15], we have extended the analysis, which have led to Theorem 17 to the class of distributed delay systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d \int_{-h}^0 f(\theta)x(t + \theta)d\theta, \quad \forall t \geq 0, \\ x(t) = \phi(t), \quad \forall t \in [-h, 0], \end{cases} \quad (54)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\phi$  is a continuous function, representing the initial conditions,  $A$  and  $A_d$  are constant matrices and  $f$  denotes a known continuous function of  $L_2([-h, 0] \rightarrow \mathbb{R})$  and represents the kernel of the distributed delay. The delay  $h$  is assumed to be constant. This class of systems can be seen as a particular case of (6), where the most general case of kernel matrix  $A_D(\theta)$  is replaced by  $f(\theta)A_d$ . The goal was here to provide a generic analysis which is able to assess stability of system (54) for any continuous scalar kernel  $f$  and to derive sufficient stability conditions based, again on the properties of the Legendre polynomials.

For the sake of simplicity, the detailed analysis will not be presented in this manuscript but the reader may refer to [SGA15] to have a better understanding of how the Bessel-Legendre inequality is applied to this problem.

Theorems	$h_{max}$	number of variables
[BMV09] (“analytical” bound)	1.498	–
[SF13]	1.03	3
Th.17 ( $N = 0$ )	0.95	5
Th.17 ( $N = 1$ )	1.45	8
Th.17 ( $N = 2$ )	1.497	12
Th.17 ( $N = 3$ )	1.498	17

Table 2  
Maximal allowable delay  $h_{max}$  for system (55).

Theorems	first interval	second interval	nv
[BMV09]	[0, 0.964]	[1.372, 2.105]	–
[SF13]	[0, 0.964]	–	3
Th.17 ( $N = 1$ )	[0, 0.91]	–	8
Th.17 ( $N = 2$ )	[0, 0.963]	[1.382, 2.100]	12
Th.17 ( $N = 3$ )	[0, 0.964]	[1.372, 2.103]	17
Th.17 ( $N = 4$ )	[0, 0.964]	[1.372, 2.105]	23

Table 3  
Intervals of stability w.r.t.  $h$  for system (56). The notation “nv” denotes the number of decision variables.

### 5.3.1 Example 7:

Let us consider the following scalar example

$$\dot{x}(t) = -x(t) + \int_{-h}^0 e^{-\theta} \sin(\theta)x(t+\theta)d\theta. \quad (55)$$

Using a numerical method [BMV05, BMV09], system (55) is shown to be asymptotically stable for all delays less than 1.498. Table 2 shows results obtained with Theorem 17. As expected, better results are obtained as the degree of the polynomial  $N$  increases. Moreover, the theoretical upper bound of the delay is recovered with  $N = 3$ .

### 5.3.2 Example 8:

Consider the distributed delay system

$$\dot{x}(t) = -2x(t) + \int_{-h}^0 (\theta + 3 \cos \theta)x(t+\theta)d\theta. \quad (56)$$

This example is interesting because it is stable for all  $h$  in  $[0, 0.964]$  and in  $[1.372, 2.105]$  and unstable otherwise [BMV09]. Table 3 shows results obtained with Theorem 17. Note that even if [SF13] provides a very good estimation of the first interval of stability, it is not able to detect the second one.

### 5.3.3 Example 9:

Consider the distributed delay system

$$\dot{x}(t) = -ax(t) - \int_{-h}^0 \gamma(k, \alpha, -\theta)x(t+\theta)d\theta, \quad (57)$$

where  $a$  is a positive scalar and  $\gamma$  the scalar kernel function of the truncated Gamma Distribution defined by

$$\gamma(k, \alpha, \theta) = \frac{\theta^{k-1} e^{-\theta/\alpha}}{(k-1)! \alpha^k}, \quad \forall (k, \alpha, \theta) \in \mathbb{N} \times (0, \infty) \times [-h, 0],$$

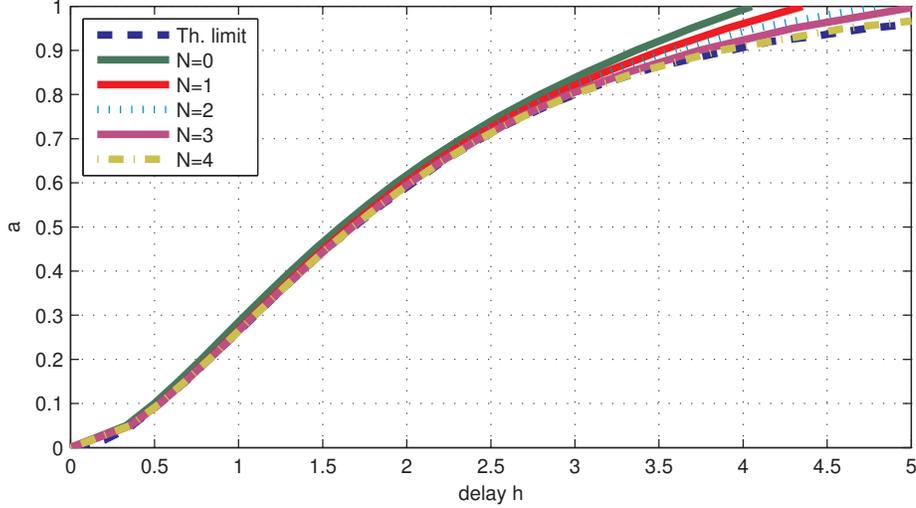


Fig. 7. Stability regions in the plan  $(a, h)$  for system (57), obtained using Theorem 17 for several values of  $N$ .

and such that  $\int_{-\infty}^0 \gamma(k, \alpha, \theta) d\theta = 1$ . From the theoretical point of view, Gamma distributions are often considered over the interval  $(-\infty, 0]$ . Since the kernel  $\gamma$  contains an exponential term, it is reasonable to consider the truncated interval  $[-h, 0]$  because the main contribution to the distributed term relies on this first interval. Figure 7 represents the stability regions obtained by solving Theorem 17 for several values of  $N$  when  $\alpha = 1$  and  $k = 1$ . The dashed black line represents the theoretical limits resulting from the eigenvalue analysis issued from [BMV09]. Figure 7 shows that from small values of  $a$  in  $(0, 0.6]$ , Theorem 17 with  $N = 0, 1$  delivers good estimations of the stability regions. However for larger values of  $a$ , Figure 7 shows that Theorem 17 with  $N = 0, 1$  is conservatism since the stability regions do not match with the theoretical limits drawn by the dashed blue line. However, increasing  $N$  in Theorem 17 allows reducing this conservatism and one can see that for  $N = 4$ , the estimation of the stability region is very close to the theoretical region. This example illustrates the potential of our hierarchical approach to reduce the conservatism by increasing the LMI parameter  $N$ , of course at the price of increasing the complexity of the conditions, showing the tradeoff between conservatism and complexity.

#### 5.4 Examples of linear systems with a time-varying delay

Consider a linear time-delay system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h(t)), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h_2, 0], \end{cases} \quad (58)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\phi$  is the initial condition and  $A, A_d \in \mathbb{R}^{n \times n}$  are constant matrices. There exist positive scalars  $h_2 \geq 0$  and  $d_1 \leq d_2 \leq 1$  such that

$$\begin{aligned} h(t) &\in [0, h_2], & \forall t \geq 0, \\ \dot{h}(t) &\in [d_1, d_2], & \forall t \geq 0, \end{aligned} \quad (59)$$

#### 5.5 From Constant to time-varying delays: an overview

In Section 4, a first stability analysis based on Legendre-based Lyapunov-Krasovskii functionals has been provided to cope with linear systems subject to constant time-delay, following the contribution of [SG15]. We recall that this

analysis is based on the following functional

$$\begin{aligned}
V_N(x_t, \dot{x}_t) &= \tilde{x}_N^\top(t) P_N \tilde{x}_N(t) + \int_{t-h}^t x^\top(s) S x(s) ds \\
&\quad + h \int_{t-h}^t \int_{\theta}^t \dot{x}^\top(s) R \dot{x}(s) ds d\theta,
\end{aligned} \tag{60}$$

where  $N$  is a non negative integer,  $P_N \in \mathcal{S}^{(N+2)n}$ ,  $S, R \in \mathcal{S}_+^n$  and where we recall that the augmented vector  $\tilde{x}_N$  is given by

$$\tilde{x}_N(t) = \begin{bmatrix} x_t(0) \\ \int_{-h}^0 \mathcal{L}_0(s) x_t(s) ds \\ \vdots \\ \int_{-h}^0 \mathcal{L}_{N-1}(s) x_t(s) ds \end{bmatrix},$$

if  $N \geq 1$  and  $\tilde{x}_0(t) = x_t(0)$ , if  $N = 0$ . Guided by this functional (60) dedicated to the analysis of systems with a constant delay, we propose to consider the following extension which aims at considering the case of time-varying delays. It consists in using the following Lyapunov-Krasovskii functional

$$\begin{aligned}
\mathcal{V}_N(x_t, \dot{x}_t) &= \bar{x}_N^\top(t) P_N \bar{x}_N(t) + \int_{t-h(t)}^t x^\top(s) S x(s) ds + \int_{t-h_2}^{t-h(t)} x^\top(s) Q x(s) ds \\
&\quad + h_2 \int_{t-h_2}^t \int_{\theta}^t \dot{x}^\top(s) R \dot{x}(s) ds d\theta,
\end{aligned} \tag{61}$$

where, in this case, the matrix  $P_N$  is now in  $\mathcal{S}^{(2N+3)n}$  and where  $S, Q, R$  are in  $\mathcal{S}_+^n$ . This new functional is defined using the augmented vector  $\bar{x}_N$  defined by

$$\bar{x}_N(t) = \begin{bmatrix} x_t(0) \\ \int_{-h(t)}^0 \mathcal{L}_0(s) x_t(s) ds \\ \vdots \\ \int_{-h(t)}^0 \mathcal{L}_{N-1}(s) x_t(s) ds \\ \int_{-h_2}^{-h(t)} \mathcal{L}_0(s) x_t(s) ds \\ \vdots \\ \int_{-h_2}^{-h(t)} \mathcal{L}_{N-1}(s) x_t(s) ds \end{bmatrix},$$

if  $N \geq 1$  and  $\bar{x}_0(t) = x_t(0)$ , if  $N = 0$ .

This functional generalizes the one that has been defined in (60) ([SG15]) for time-varying delay systems. Indeed selecting  $N = 0$  in (61) allows retrieving the same functional as in [SG15]. The proposed extension to time-varying delay consequently is not an easy task since the time-varying delay appears in the definition of  $\bar{x}_N(t)$ . Hence the following developments aim at providing stability conditions expressed in terms of LMIs using such a class of functionals.

**Remark 22** *It would also be possible to replace the last term of  $\mathcal{V}_N$ ,*

$$\mathcal{V}_{dN}(\dot{x}_t) = h_2 \int_{t-h_2}^t \int_{\theta}^t \dot{x}^\top(s) R \dot{x}(s) ds d\theta,$$

by

$$\bar{\mathcal{V}}_{dN}(\dot{x}_t) = h_2 \int_{t-h(t)}^t \int_{\theta}^t \dot{x}^\top(s) R \dot{x}(s) ds d\theta + h_2 \int_{t-h_2}^{t-h(t)} \int_{\theta}^t \dot{x}^\top(s) R \dot{x}(s) ds d\theta,$$

but this will not be described in this manuscript.

While the computation of the derivative of the functional is standard (even if highly technical), the last term generates some notable complication with respect to the constant case. Indeed, what is usually done in the literature is described by the following steps.

**Computation of the derivative:** Differentiating the last term of  $\mathcal{V}_N$  leads to

$$\dot{\mathcal{V}}_{dN}(\dot{x}_t) = h_2^2 \dot{x}^\top(t) R \dot{x}(t) - h_2 \int_{t-h_2}^t \dot{x}^\top(s) R \dot{x}(s) ds.$$

**Decomposition :** In order to include the intermediate information  $x_t(-h)$ , the integral is split into parts, which make appear the intermediate value  $t - h(t)$ :

$$\dot{\mathcal{V}}_{dN}(\dot{x}_t) = h_2^2 \dot{x}^\top(t) R \dot{x}(t) - h_2 \int_{t-h(t)}^t \dot{x}^\top(s) R \dot{x}(s) ds - h_2 \int_{t-h_2}^{t-h(t)} \dot{x}^\top(s) R \dot{x}(s) ds.$$

**Application of integral inequalities:** Then we can use, for instance and for simplicity, Corollary 16 with  $N = 1$ , which corresponds to the use of the Wirtinger-based integral inequality on this quadratic integral on  $\dot{x}_t$ , to obtain

$$\dot{\mathcal{V}}_{dN}(\dot{x}_t) \leq h_2^2 \dot{x}^\top(t) R \dot{x}(t) - \frac{h_2}{h} \xi_1^\top(t) \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \xi_1(t) - \frac{h_2}{h_2-h} \xi_2^\top(t) \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \xi_2(t)$$

where

$$\xi_1(t) = \begin{bmatrix} x(t) - x(t-h) \\ x(t) + x(t-h) - \frac{2}{h} \int_{t-h}^t x(s) ds \end{bmatrix},$$

$$\xi_2(t) = \begin{bmatrix} x(t-h) - x(t-h_2) \\ x(t-h) + x(t-h_2) - \frac{2}{h_2-h} \int_{t-h_2}^{t-h} x(s) ds \end{bmatrix}$$

or equivalently, in a more compact form

$$\dot{\mathcal{V}}_{dN}(\dot{x}_t) \leq h_2^2 \dot{x}^\top(t) R \dot{x}(t) - \xi^\top(t) \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{\alpha} \tilde{R} & 0 \\ 0 & \frac{1}{1-\alpha} \tilde{R} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \xi(t)$$

where

$$\alpha = \frac{h}{h_2} \in [0, 1], \quad \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix},$$

$$W_1 = \begin{bmatrix} I & -I & 0 & 0 & 0 \\ I & I & 0 & -2I & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & I & -I & 0 & 0 \\ 0 & I & I & 0 & -2I \end{bmatrix},$$

$$\xi(t) = \left[ x^\top(t) \ x^\top(t-h(t)) \ x^\top(t-h_2) \int_{t-h(t)}^t \frac{x^\top(s)}{h(t)} ds \int_{t-h_2}^{t-h(t)} \frac{x^\top(s)}{h_2-h(t)} ds \right]^\top.$$

**Remark 23** *In the previous developments we have use the Wirtinger-based integral inequality (Corollary 16 with  $N = 1$ ) but a similar reasoning can be achieved for any  $N$ . The general case  $N$  leads to several technical difficulties that may disturb the reader of this section. The reader may refer to [SG16b] to find the generalized version of the following developments.*

**Apply matrix inequality:** After applying the integral inequalities, one may see that the resulting term is not convex with respect to the delay  $h$  (or  $\alpha$ ). Several methods have then be deployed to encompass this difficulty and to allow us for some numerical tests.

Among the most popular methods, one may refer to the Moon *et al* inequality [MPKL01], presented in the following lemma

**Lemma 24** For given positive definite matrix  $\tilde{R}$  in  $\mathbb{S}^{2n}$  and matrices  $W_1, W_2$  in  $\mathbb{R}^{2n \times 5n}$ , the inequality

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{\alpha} \tilde{R} & 0 \\ 0 & \frac{1}{1-\alpha} \tilde{R} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \geq He \left\{ \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}^\top \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \right\} - \alpha N_1^\top \tilde{R}^{-1} N_1 - (1-\alpha) N_2^\top \tilde{R}^{-1} N_2,$$

holds for any matrices  $N_1, N_2 \in \mathbb{R}^{5n \times 2n}$  and for all  $\alpha$  in  $(0, 1)$ .

*Proof* : The proof of this lemma results from the expansion of the two following squares

$$(\tilde{R}W_1 - \alpha N_1)^\top \frac{\tilde{R}^{-1}}{\alpha} (\tilde{R}W_1 - \alpha N_1) + (\tilde{R}W_2 - (1-\alpha)N_2)^\top \frac{\tilde{R}^{-1}}{1-\alpha} (\tilde{R}W_2 - (1-\alpha)N_2) \geq 0.$$

◇

This lemma has been widely used in the literature in different (and some hidden) forms. It is numerically efficient but at the price of a notable increase of the number of decision variables, in this case  $20n^2$ .

Recently, a efficient method, called the reciprocally convex combination lemma has been proposed in [PKJ11], and has shown its efficiency on stability theorem based on the Jensen's inequality. Indeed it allows retrieving the same numerical results as the ones issued from the application of Lemma 24 but with a lower number of decision variables. This lemma is stated below

**Lemma 25** Given a matrix  $\tilde{R} \in \mathbb{S}_+^{2n}$ , and two matrices  $W_1$  and  $W_2$  in  $\mathbb{R}^{n \times m}$ , then the improved reciprocally convex combination guarantees that if there exists a matrix  $X$  in  $\mathbb{R}^{2n \times 2n}$  such that

$$\begin{bmatrix} \tilde{R} & X \\ X^\top & \tilde{R} \end{bmatrix} \succeq 0,$$

then the following matrix inequality holds for all  $\alpha \in (0, 1)$

$$\begin{bmatrix} \frac{1}{\alpha} \tilde{R} & 0 \\ 0 & \frac{1}{1-\alpha} \tilde{R} \end{bmatrix} \succeq \begin{bmatrix} \tilde{R} & X \\ X^\top & \tilde{R} \end{bmatrix}.$$

*Proof* : A proof can be found in [PKJ11] to consider a larger class of reciprocally convex combination. A simpler and more dedicated proof is provided here. Define the scalar  $\beta = \sqrt{\frac{1-\alpha}{\alpha}}$ . Then, we have, by congruence

$$0 \succeq \begin{bmatrix} \beta I & 0 \\ 0 & -\beta^{-1} I \end{bmatrix} \begin{bmatrix} \tilde{R} & X \\ X^\top & \tilde{R} \end{bmatrix} \begin{bmatrix} \beta I & 0 \\ 0 & -\beta^{-1} I \end{bmatrix} = \begin{bmatrix} \frac{1-\alpha}{\alpha} \tilde{R} & -X \\ -X^\top & \frac{\alpha}{1-\alpha} \tilde{R} \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha} \tilde{R} & 0 \\ 0 & \frac{1}{1-\alpha} \tilde{R} \end{bmatrix} - \begin{bmatrix} \tilde{R} & X \\ X^\top & \tilde{R} \end{bmatrix},$$

which concludes the proof. ◇

Note that in the case of the same matrix inequality but resulting from the Jensen inequality instead of the Wirtinger-based inequality, an interpretation of this lemma has been provided in [SGL16]. The combination of the Jensen inequality with the reciprocally convex combination lemma can be seen as a discretized version of Jensen's inequality.

While both Lemma 24 and 25 lead to the same level of conservatism when employing the Jensen inequality, it has been revealed in [] that the second one leads to more conservative result than the first one when employing the Wirtinger-based inequality. Therefore, a refined version of this lemma has been provided in [] and is stated below

**Lemma 26** Let  $\tilde{R}$  be a positive definite matrix in  $\mathbb{S}^{2n}$  for a given integer  $n > 0$ . If there exist two matrices  $X_1, X_2$  in  $\mathbb{S}^{2n}$  and  $Y_1, Y_2$  in  $\mathbb{R}^{2n \times 2n}$  such that

$$\begin{bmatrix} \tilde{R} & 0 \\ * & \tilde{R} \end{bmatrix} - \begin{bmatrix} X_1 & Y_1 \\ * & 0 \end{bmatrix} \succ 0, \quad \begin{bmatrix} \tilde{R} & 0 \\ * & \tilde{R} \end{bmatrix} - \begin{bmatrix} 0 & Y_2 \\ * & X_2 \end{bmatrix} \succ 0, \quad (62)$$

then the following inequality holds for all  $\alpha \in (0, 1)$

$$\begin{bmatrix} \frac{1}{\alpha} \tilde{R} & 0 \\ * & \frac{1}{1-\alpha} \tilde{R} \end{bmatrix} \succeq \begin{bmatrix} \tilde{R} & 0 \\ * & \tilde{R} \end{bmatrix} + \begin{bmatrix} (1-\alpha)X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & \alpha X_2 \end{bmatrix}. \quad (63)$$

*Proof :* If inequalities (62) are verified, then a convex combination of these two equations leads to the inequality

$$\begin{bmatrix} R & 0 \\ * & R \end{bmatrix} - \begin{bmatrix} \alpha X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & (1-\alpha)X_2 \end{bmatrix} \succ 0,$$

for all  $\alpha \in [0, 1]$ . Pre- and post-multiplying this inequality by the matrix  $\begin{bmatrix} \beta I & 0 \\ 0 & \beta^{-1} I \end{bmatrix}$ , where  $\beta = \sqrt{\frac{1-\alpha}{\alpha}}$ , leads to

$$\begin{bmatrix} \frac{1-\alpha}{\alpha} R & 0 \\ * & \frac{\alpha}{1-\alpha} R \end{bmatrix} - \begin{bmatrix} \frac{\alpha(1-\alpha)}{\alpha} X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & \frac{\alpha(1-\alpha)}{1-\alpha} X_2 \end{bmatrix} \succ 0,$$

for all  $\alpha \in (0, 1)$ . Finally noting that  $\frac{1-\alpha}{\alpha} = \frac{1}{\alpha} - 1$  and  $\frac{\alpha}{1-\alpha} = \frac{1}{1-\alpha} - 1$ , the previous inequality can be rewritten as

$$\begin{bmatrix} \frac{1}{\alpha} R & 0 \\ * & \frac{1}{1-\alpha} R \end{bmatrix} - \begin{bmatrix} R & 0 \\ * & R \end{bmatrix} - \begin{bmatrix} (1-\alpha)X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & \alpha X_2 \end{bmatrix} \succ 0,$$

which concludes the proof.  $\diamond$

The particular selection in the previous lemma with  $X_1 = X_2 = 0$  and  $Y_1 = Y_2 = Y$  leads to the original reciprocally convex combination lemma.

These two results will be employed in the remainder of the paper. As shown in [SG16a], Lemma 26 refines the original convex combination lemma since it allows obtaining a lower bound which depends explicitly on  $\alpha$  which refers to the time-varying delay  $h(t)$ .

Based on the previous discussions, an example of stability conditions for time-varying delay is provided below and exploit the Wirtinger-based inequality of Corollary ?? and the delay-dependent reciprocally convex combination lemma presented in Lemma26.

### 5.6 Examples of stability conditions

Based on the previous developments, the following stability theorem is provided.

**Theorem 27** Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{3n}$ ,  $S_1, S_2, R$  in  $\mathbb{S}_+^n$ ,  $X_1, X_2$  in  $\mathbb{S}^{2n}$  and two matrices  $Y_1, Y_2$  in  $\mathbb{R}^{2n \times 2n}$ , such that the conditions

$$\begin{bmatrix} \tilde{R} - X_1 & Y_1 \\ * & \tilde{R} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \tilde{R} & Y_2 \\ * & \tilde{R} - X_2 \end{bmatrix} \succeq 0, \quad (64)$$

$$\Phi(0,d_1) \prec 0, \Phi(h_2,d_1) \prec 0, \Phi(0,d_2) \prec 0, \Phi(h_2,d_2) \prec 0, \quad (65)$$

are satisfied where

$$\begin{aligned} \Phi(\theta, \eta) &= \Phi_0(\theta, \eta) - G_2^\top \Psi(\theta) G_2 \\ \Phi_0(\theta, \eta) &= \text{He} \{ G_1^\top(\theta) P G_0(\eta) \} + \hat{S}(\eta) + h_2^2 g_0^\top R g_0, \\ \hat{S}(\eta) &= \text{diag}(S_1, (1-\eta)(S_2 - S_1), -S_2, 0_{2n}), \\ \tilde{R} &= \text{diag}(R, 3R), \\ \Psi(\theta) &= \begin{bmatrix} \tilde{R} + \frac{h_2-\theta}{h_2} X_1 & \frac{\theta}{h_2} Y_1 + \frac{h_2-\theta}{h_2} Y_2 \\ * & \tilde{R} + \frac{\theta}{h_2} X_2 \end{bmatrix}, \end{aligned} \quad (66)$$

and where

$$\begin{aligned} g_0 &= \begin{bmatrix} A & A_d & 0 & 0 & 0 \end{bmatrix}, \\ G_0(\eta) &= \begin{bmatrix} A & A_d & 0 & 0 & 0 \\ I & -(1-\eta)I & 0 & 0 & 0 \\ 0 & (1-\eta)I & -I & 0 & 0 \end{bmatrix}, \\ G_1(\theta) &= \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta I & 0 \\ 0 & 0 & 0 & 0 & (h_2 - \theta)I \end{bmatrix}, \\ G_2 &= \begin{bmatrix} I & -I & 0 & 0 & 0 \\ I & I & 0 & -2I & 0 \\ 0 & I & -I & 0 & 0 \\ 0 & I & I & 0 & -2I \end{bmatrix}. \end{aligned} \quad (67)$$

Then, system (58) is asymptotically stable for all time-varying delay  $h$  satisfying (59).

### 5.7 Reduction of the number of decision variables

In the previous theorem, the number of decision variables can be reduced by introducing some constraints in the slack variables introduced by application of the reciprocally convex combination Lemma 25. This relaxation is proposed in the following corollary.

**Corollary 28** *Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{3n}$ ,  $S_1, S_2, R$  in  $\mathbb{S}_+^n$ ,  $X$  in  $\mathbb{S}^{2n}$  and a matrix  $Y$  in  $\mathbb{R}^{2n \times 2n}$ , such that conditions (64) and (65) are verified with*

$$X_1 = X_2 = X \text{ and } Y_1 = Y_2 = X.$$

Then system (58) is asymptotically stable for all time-varying delay  $h$  satisfying (59).

**Remark 29** *The reduction of the computational complexity of the resulting stability conditions leads obviously to an increase of the conservatism of the stability conditions, as it will be showed in the example section. This shows again the traditional tradeoff between computational complexity and conservatism.*

The success of the reciprocally convex combination lemma over the Moon inequality relies on the fact that when employing it for stability theorem based on the Jensen inequality, equivalent results were obtained with a significantly reduced number of decision variables. In the following paragraph we will present a similar result to Theorem 27, which is based on the application of Moon's inequality instead of Lemma 24. This leads to the following result:

Th.	No. of variables	Th.	No. of variables
[AG09b]	$480n^2 + 8n$	[FS02a]	$5.5n^2 + 1.5n$
[HWXL07a]	$3n^2 + 3n$	[PK07]	$11.5n^2 + 4.5n$
[SG13]	$10n^2 + 3n$	[ZHWS15]	$54n^2 + 9n$
[ZHWX13]	$17n^2 + 5n$	-	-
Th. 27	$18n^2 + 5n$	Cor. 28	$12n^2 + 4n$
Th. 30	$26n^2 + 3n$		

Table 4

Number of decision variables involved in several conditions from the literature and in Theorem 27 and its corollaries

**Theorem 30** Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{3n}$ ,  $S_1, S_2, R$  in  $\mathbb{S}_+^n$ , and a matrix  $Y$  in  $\mathbb{R}^{5n \times 4n}$ , such that the conditions

$$\bar{\Phi}(0, d_1) \prec 0, \bar{\Phi}(h_2, d_1) \prec 0, \bar{\Phi}(0, d_2) \prec 0, \bar{\Phi}(h_2, d_2) \prec 0, \quad (68)$$

are satisfied where

$$\bar{\Phi}(\theta, \eta) = \begin{bmatrix} \Phi_0(\theta, \eta) - He\{[Y_1 Y_2]G_2\} & \frac{\theta}{h_2}Y_1 & \frac{h_2 - \theta}{h_2}Y_2 \\ * & \frac{\theta}{h_2}\tilde{R} & 0 \\ * & * & \frac{h_2 - \theta}{h_2}\tilde{R} \end{bmatrix} \quad (69)$$

and where the matrices  $\Phi_0(\theta, \eta)$ ,  $\tilde{R}$  and  $G_2$  are given in (67). Then, system (58) is asymptotically stable for all time-varying delay  $h$  satisfying (59).

Recently, a novel contribution based on *Free-Weighting Matrix Inequality* was proposed in [ZHWS15]. It has been showed in this paper that an alternative presentation of the Wirtinger-based integral inequality (Lemma 9) can be presented by an efficient introduction of free-weighting-matrices leading to less conservative results than when using the Wirtinger-based inequality. In this paper, we will compare the various results presented here with Corollary 1 of [ZHWS15], which uses exactly the same Lyapunov-Krasovskii functional as the one presented in (45). This will allow for a fair comparison between the various inequalities employed in all these results. Note that the main stability theorem of [ZHWS15] exploits additional terms in the construction of the functional, leading to a reducing of the conservatism. We will not present the numerical results in the present paper since our goal is to show the conservatism of integral and matrix inequalities and their associated numerical complexity.

### 5.8 Numerical examples

In this section, we will consider two academic examples taken from the literature. Our goal is to illustrate and compare the efficiency of the conditions presented in Theorems 27 and 30 and Corollary 28 and for various conditions taken from the literature for the stability analysis of linear systems with time-varying delays. Before entering into the numerical results, we would like to point out in Table 4, the number of decision variables involved in the conditions presented in this paper and in existing results from the literature. For the two next examples, we expose in Tables 5 the maximal upper-bound,  $h_2$  of the delay function for various values bounds on the derivative of the delay function, i.e.  $d_2 (= -d_1)$ .

There exists a large number of paper dealing with the stability analysis of such a class of systems. Because of space limitations, we consider only few representative conditions from the literature. On a first side, conditions derived using Jensen's inequality ([PKJ11]), Wirtinger-based inequality ([SGF13]), auxiliary-based inequality [PLL15] and the recent free-matrix-based inequality ([ZHWS15]). On the other hand, we also discriminate conditions that are based on Young/Moon inequality [ZHWS15], or on the Reciprocally convex combination lemma [PKJ11, SGF13, PLL15]. A last comment on the contribution presented in [PLL15]. Indeed, the conditions proposed in [PLL15] is proven to be less conservative than the Wirtinger-based inequality together with the reciprocally convex combination lemma. Therefore, it is expected that the conditions presented in [PLL15] are less conservative than the one from Theorem 27.

$d_2$	0.0	0.1	0.2	0.5	0.8	1
[FS02a]	4.47	3.60	3.03	2.00	1.36	0.99
[HWXL07a]	4.47	3.60	3.03	2.04	1.49	1.34
[PK07]	4.47	3.65	3.16	2.33	1.93	1.86
[AG09b]	6.11	4.79	3.99	2.68	1.95	1.60
[ZHWX13] (N=3)	5.92	4.62	3.76	2.44	2.07	2.07
[SG13]*	6.05	4.70	3.83	2.42	2.13	2.12
[ZHWS15] (Cor.1)*	6.05	4.71	3.84	2.45	2.21	2.18
[ZHWS15] (Th.1)	6.05	4.78	4.06	3.05	2.61	-
Cor. 28*	6.05	4.70	3.83	2.42	2.20	2.20
Th. 27*	6.05	4.71	3.85	2.48	2.24	2.24
Th. 30*	6.05	4.71	3.85	2.48	2.30	2.30

Table 5

Example 1: Admissible upper bound of  $h_2$  for various values of  $d_2 = -d_1$ . The mark ‘\*’ means that the stability conditions are based on the same functional.

**Example 1:** Consider the following much-studied linear time-delay system (42) with

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}.$$

The results obtained by solving Theorem 27 and its corollary show a clear reduction of the conservatism. Moreover, the improvements due to the use of Lemma 26 and its corollary can be seen when comparing the results obtained with [SGF13] and the stability conditions provided in the present paper. Indeed the only difference between these two papers is the use of the delay-dependent reciprocally convex lemma. Moreover, it is worth noting that Theorem 27 and its corollaries provide less conservative results, on this example, than other conditions from the literature except for [PLL15] with  $h_1 = 3$ . This improvement of [PLL15] can be explained by the use of the auxiliary function integral inequality, which is less conservative than the Wirtinger inequality. It is also worth noting that Theorem 27 and its corollaries leads in general to the same results except for small lower bounds  $h_1 = 0$  even if the computational complexities of the stability conditions are different.

## 6 Conclusions

In this report, an overview of the recent developments in the area of time-delay system has been provided. It demonstrates that, despite the already 20 years of investigation, this field still attracts many researchers around the world. There are still many directions to investigate and several key aspects to be undertaken.

## References

- [AG09a] Y. Ariba and F. Gouaisbaut. An augmented model for robust stability analysis of time-varying delay systems. *Int. Journal of Control*, 82:1616–1626, September 2009.
- [AG09b] Y. Ariba and F. Gouaisbaut. Input-output framework for robust stability of time-varying delay systems. In *Conference on Decision and Control and Chinese Control Conference. (CDC - CCC’09)*, pages 274–279. IEEE, 2009.
- [AGJ10] Y. Ariba, F. Gouaisbaut, and K.H. Johansson. Stability interval for time-varying delay systems. In *49th IEEE Conference on Decision and Control (CDC’10)*, pages 1017–1022, 2010.
- [ASRR07] J. Anthonis, A. Seuret, J.-P. Richard, and H. Ramon. Design of a pressure control system with dead band and time delay. *IEEE Trans. on Control Systems Technology*, 15(6):1103–1111, 2007.
- [BC63] R. Bellman and S.L. Cooke. *Differential Difference Equations*. Academic Press, New York, 1963.
- [Bli01a] P.-A. Bliman. LMI characterization of the strong delay-independent stability of linear delay systems via quadratic lyapunov–krasovskii functionals. *Systems & Control Letters*, 43(4):263–274, 2001.

- [Bli01b] P.-A. Bliman. Lyapunov-Krasovskii functionals and frequency domain: Delay-independent absolute stability criteria for delay systems. *International Journal of Robust and Nonlinear Control*, 11(8):771–788, 2001.
- [BMV05] D. Breda, S. Maset, and R. Vermiglio. Pseudospectral differencing methods for characteristic roots of delay differential equations. *SIAM Journal on Scientific Computing*, 27(2):482–495, 2005.
- [BMV09] D. Breda, S. Maset, and R. Vermiglio. Trace-dde: a tool for robust analysis and characteristic equations for delay differential equations. In *Topics in Time Delay Systems*, volume 388 of *Lecture Notes in Control and Information Sciences*, pages 145–155. Springer Berlin Heidelberg, 2009.
- [Bri11] C. Briat. Convergence and equivalence results for the Jensen’s inequality - application to time-delay and sampled-data systems. *IEEE Trans. on Automatic Control*, 56(7):1660–1665, 2011.
- [CG82] K.L/ Cooke and Z. Grossman. Discrete delay, distributed delay and stability switches. *Journal of Mathematical Analysis and Applications*, 86(2):592–627, 1982.
- [CGL05] W.-H. Chen, Z.-H. Guan, and X. Lu. Delay-dependent exponential stability of uncertain stochastic systems with multiple delays: an LMI approach. *Systems & Control Letters*, 54(6):547–555, 2005.
- [CLX16] J. Chen, J. Lu, and S. Xu. Summation inequality and its application to stability analysis for time-delay systems. *IET Control Theory & Applications*, 2016.
- [CSC98] Y.-Y. Cao, Y.-X. Sun, and C. Cheng. Delay-dependent robust stabilization of uncertain systems with multiple state delays. *IEEE Trans. on Automatic Control*, 43(11):1608–1612, 1998.
- [CXC<sup>+</sup>16] J. Chen, S. Xu, W. Chen, B. Zhang, Q. Ma, and Y. Zou. Two general integral inequalities and their applications to stability analysis for systems with time-varying delay. *International Journal of Robust and Nonlinear Control*, 2016.
- [CZ07] W.-H. Chen and W.X. Zheng. Delay-dependent robust stabilization for uncertain neutral systems with distributed delays. *Automatica*, 43(1):95–104, 2007.
- [DLSW09] B. Du, J. Lam, Z. Shu, and Z. Wang. A delay-partitioning projection approach to stability analysis of continuous systems with multiple delay components. *IET Control Theory & Applications*, 3(4):383–390, 2009.
- [DLZ14] X. Ding, X. Liu, and S. Zhong. Delay-independent criteria for exponential admissibility of switched descriptor delayed systems. *Applied Mathematics and Computation*, 228:432–445, 2014.
- [Dri77] D.R. Driver. *Ordinary and delay differential equations*. Springer-Verlag, New York, 1977.
- [EGN00] L. El Ghaoui and S.-I. Niculescu. *Advances in linear matrix inequality methods in control*, volume 2. Siam, 2000.
- [FL12] Z. Feng and J. Lam. Integral partitioning approach to robust stabilization for uncertain distributed time-delay systems. *International Journal of Robust and Nonlinear Control*, 22(6):676–689, 2012.
- [FLY15] Z. Feng, J. Lam, and G.-H. Yang. Optimal partitioning method for stability analysis of continuous/discrete delay systems. *International Journal of Robust and Nonlinear Control*, 25(4):559–574, 2015.
- [FO09] E. Fridman and Y. Orlov. Exponential stability of linear distributed parameter systems with time-varying delays. *Automatica*, 45(1):194–201, 2009.
- [FP13] M. Fang and J.H. Park. A multiple integral approach to stability of neutral time-delay systems. *Applied Mathematics and Computation*, 224:714–718, 2013.
- [Fre00] P. Freitas. Delay-induced instabilities in gyroscopic systems. *SIAM Journal on Control and Optimization*, 39(1):196–207, 2000.
- [Fri01] E. Fridman. New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems and Control Letters*, 43:309–319, 2001.
- [Fri02] E. Fridman. Stability of linear descriptor systems with delay: A Lyapunov-based approach. *Journal of Mathematical Analysis and Applications*, 273(1):24–44, 2002.
- [Fri04] E. Fridman. Stability of linear functional differential equations: A new Lyapunov technique. In *Proceedings of Mathematical Theory of Networks and Systems*, September 2004.
- [Fri06] E. Fridman. A new Lyapunov technique for robust control of systems with uncertain non-small delays. *IMA Journal of Mathematical Control and Information*, 23(2):165–179, 2006.
- [Fri14] E. Fridman. *Introduction to time-delay systems: Analysis and control*. Springer, 2014.
- [FS00] E. Fridman and U. Shaked. Stability and guaranteed cost control of uncertain discrete delay systems. *International Journal of Control*, 45(12):2406–2411, 2000.
- [FS02a] E. Fridman and U. Shaked. A descriptor system approach to  $H^\infty$  control of linear time-delay systems. *IEEE Trans. on Automatic Control*, 47(2):253–270, 2002.
- [FS02b] E. Fridman and U. Shaked. An improved stabilization method for linear time-delay systems. *IEEE Trans. on Automatic Control*, 47(11):1931–1937, 2002.
- [FS03] E. Fridman and U. Shaked. Delay-dependent stability and  $H^\infty$  control: constant and time-varying delays. *International journal of control*, 76(1):48–60, 2003.

- [FS04] E. Fridman and U. Shaked. An improved delay-dependent  $H^\infty$  filtering of linear neutral systems. *IEEE Trans. on Signal and Processing*, 52(3):668 – 673, 2004.
- [FSR04] E. Fridman, A. Seuret, and J.-P. Richard. Robust sampled-data stabilization of linear systems: An input delay approach. *Automatica*, 40(8):1141–1446, 2004.
- [FT09] E. Fridman and G. Tsodik.  $H^\infty$  control of distributed and discrete delay systems via discretized Lyapunov functional. *European Journal of Control*, 15(1):84–94, 2009.
- [GA94] P. Gahinet and P. Apkarian. A linear matrix inequality approach to  $H^\infty$  control. *International Journal of Robust and Nonlinear Control*, 4(4):421–448, 1994.
- [GAS13] F. Gouaisbaut, Y. Ariba, and A. Seuret. Bessel inequality for robust stability analysis of time-delay system. In *IEEE Conference on Decision and Control (CDC'13)*, pages 928–933, 2013.
- [GASP16] F. Gouaisbaut, Y. Ariba, A. Seuret, and D. Peaucelle. Stability analysis of time-delay systems via Bessel's inequality : a quadratic separation approach. *submitted to Systems & Control Letters*, 2016.
- [GKC03] K. Gu, V.-L. Kharitonov, and J. Chen. *Stability of time-delay systems*. Birkhauser, 2003.
- [GKNT15] E. Gyurkovics, K. Kiss, I. Nagy, and T. Takacs. Multiple summation inequalities and their application to stability analysis of discrete-time delay systems. *arXiv preprint arXiv:1512.02052*, 2015.
- [GP06a] F. Gouaisbaut and D. Peaucelle. Delay-dependent stability analysis of linear time delay systems. In *IFAC Workshop on Time-Delay Systems (TDS'06)*, 2006.
- [GP06b] F. Gouaisbaut and D. Peaucelle. A note on stability of time delay systems. In *5<sup>th</sup> IFAC Symposium on Robust Control Design (ROCOND'06)*, Toulouse, France, July 2006.
- [GP07] F. Gouaisbaut and D. Peaucelle. Robust stability of time-delay systems with interval delays. In *IEEE Conference on Decision and Control (CDC'07)*, pages 6328–6333, New Orleans, USA, 2007.
- [GT16] E. Gyurkovics and T. Takacs. Multiple integral inequalities and stability analysis of time delay systems. *arXiv preprint arXiv:1602.07887*, 2016.
- [Gu97] K. Gu. Discretized LMI set in the stability problem of linear uncertain time-delay systems. *Int. J. of Control*, 68:923 – 934, 1997.
- [Gu01] K. Gu. A further refinement of discretized Lyapunov functional method for the stability of time-delay systems. *Int. Journal of Control*, 74(10):967 – 976, 2001.
- [Han08] Qing-Long Han. A delay decomposition approach to stability of linear neutral systems. In *17th IFAC World Congress, Seoul, South Korea*, 2008.
- [Han09] Q.-L. Han. A discrete delay decomposition approach to stability of linear retarded and neutral systems. *Automatica*, 45(2):517 – 524, 2009.
- [HL93] J. Hale and S. M.V. Lunel. *Introduction to Functional Differential Equations*. Springer-Verlag, New York, 1993.
- [HL03] D. Henrion and J.-B. Lasserre. GloptiPoly: Global optimization over polynomials with Matlab and SeDuMi. *ACM Trans. on Mathematical Software (TOMS)*, 29(2):165–194, 2003.
- [HLL09] D. Henrion, J.-B. Lasserre, and J. Löfberg. Gloptipoly 3: moments, optimization and semidefinite programming. *Optimization Methods & Software*, 24(4-5):761–779, 2009.
- [HNX07] J.P. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1):138–162, 2007.
- [HWLW05] Y. He, Q.-G. Wang, C. Lin, and M. Wu. Augmented lyapunov functional and delay-dependent stability criteria for neutral systems. *International Journal of Robust and Nonlinear Control*, 15(18):923–933, 2005.
- [HWXL07a] Y. He, Q.-G. Wang, L. Xie, and C. Lin. Further improvement of free-weighting matrices technique for systems with time-varying delay. *Automatic Control, IEEE Trans. on*, 52(2):293–299, 2007.
- [HWXL07b] Y. He, Q.G. Wang, L. Xie, and C. Lin. Further improvement of free-weighting matrices technique for systems with time-varying delay. *IEEE Trans. on Automatic Control*, 52(2):293–299, February 2007.
- [J.H97] J.Hale. *Theory of functional differential equations*. Springer-Verlag, 1997.
- [JH05] X. Jiang and Q.-L. Han. On  $H^\infty$  control for linear systems with interval time-varying delay. *Automatica*, 41:2099–2106, 2005.
- [JH06] X. Jiang and Q.-L. Han. Delay-dependent robust stability for uncertain linear systems with interval time-varying delay. *Automatica*, 42(6):1059–1065, 2006.
- [Kam07] W.D. Kammler. *A first Course in Fourier Analysis*. Cambridge University Press, Cambridge, 2007.
- [Kha12] V.L. Kharitonov. *Time-delay systems: Lyapunov functionals and matrices*. Springer Science & Business Media, 2012.
- [Kim11] J.H. Kim. Note on stability of linear systems with time-varying delay. *Automatica*, 47(9):2118–2121, 2011.
- [Kim16] J.-H. Kim. Further improvement of jensen inequality and application to stability of time-delayed systems. *Automatica*, 64:121–125, 2016.
- [KM99] V Kolmanovskii and A Myshkis. Introduction to the theory and applications of functional. *Differential*

- Equations*, 463, 1999.
- [KNR99] V.B. Kolmanovskii, S.-I. Niculescu, and J.-P. Richard. On the Lyapunov-Krasovskii functionals for stability analysis of linear delay systems. *Int. J. Control*, 72:374–384, 1999.
- [KR07] C.-Y. Kao and A. Rantzer. Stability analysis of systems with uncertain time-varying delays. *Automatica*, 43(6):959 – 970, 2007.
- [Krs09] M. Krstic. *Delay compensation for nonlinear, adaptive, and PDE systems*. Birkhäuser Boston, 2009.
- [KS96] V.B. Kolmanovskii and L.E. Shaikhet. *Control of systems with aftereffect*. American Mathematical Society, 1996.
- [KZ03] V.L. Kharitonov and A. Zhabko. Lyapunov-Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39(1):15–20, 2003.
- [LdS97] X. Li and C.-E. de Souza. Criteria for robust stability and stabilization of uncertain linear systems with state delay. *Automatica*, 33:1657–1662, 1997.
- [LF12] K. Liu and E. Fridman. Wirtinger’s inequality and Lyapunov-based sampled-data stabilization. *Automatica*, 48(1):102–108, 2012.
- [LGG16] X. Li, H. Gao, and K. Gu. Delay-independent stability analysis of linear time-delay systems based on frequency discretization. *Automatica*, 70:288–294, 2016.
- [Löf04] J. Löfberg. YALMIP: A toolbox for modeling and optimization in MATLAB. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, pages 284–289. IEEE, 2004.
- [LPA<sup>+</sup>06] I. Lopez, J.L. Piovesan, C.T. Abdallah, D. Lee, O. M. Palafox, M. Spong, and R. Sandoval. Practical issues in networked control systems. In *American Control Conference, Minneapolis, US*, June, 2006.
- [LPH02] Y. Labit, D. Peaucelle, and D. Henrion. SeDuMi interface 1.02: a tool for solving LMI problems with SeDuMi. In *Computer Aided Control System Design, 2002. Proceedings. 2002 IEEE International Symposium on*, pages 272–277. IEEE, 2002.
- [MNG07] C.-I. Morarescu, S.-I. Niculescu, and K. Gu. Stability crossing curves of shifted gamma-distributed delay systems. *SIAM Journal on Applied Dynamical Systems*, 6(2):475–493, 2007.
- [Mor06] C.-I. Morarescu. *Qualitative analysis of distributed delay systems: Methodology and algorithms*. PhD thesis, Ph. D. thesis, University of Bucharest/Université de Technologie de Compiègne, 2006.
- [MPKL01] Y.S. Moon, P.G. Park, W.H. Kwon, and Y.S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of control*, 74(14):1447–1455, 2001.
- [Mys51] A.D. Myshkis. General theory of differential equations with delay. *Transl AMS*, 55:1–62, 1955.
- [NG12] S.-I. Niculescu and K. Gu. *Advances in time-delay systems*, volume 38. Springer Science & Business Media, 2012.
- [Nic01] S.-I. Niculescu. *Delay Effects on Stability. A Robust Control Approach*. Springer-Verlag, 2001.
- [NTP15] P.T. Nam, H. Trinh, and P.N. Pathirana. Discrete inequalities based on multiple auxiliary functions and their applications to stability analysis of time-delay systems. *Journal of the Franklin Institute*, 352(12):5810–5831, 2015.
- [OES05] N. Olgac, A.F. Ergenc, and R. Sipahi. ‘delay scheduling’: A new concept for stabilization in multiple delay systems. *Journal of Vibration and Control*, 11(9):1159–1172, 2005.
- [Pac94] A. Packard. Gain scheduling via linear fractional transformations. *Systems & Control Letters*, 22(2):79–92, 1994.
- [Par00] P. A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, 2000.
- [Par04] P. A. Parrilo. Web site for SOSTOOLS. <http://www.cds.caltech.edu/sostools/>, 2004.
- [PAV<sup>+</sup>13] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. Parrilo. Sostools version 3.00 sum of squares optimization toolbox for Matlab. *arXiv preprint arXiv:1310.4716*, 2013.
- [PB11] M.M. Peet and P.-A. Bliman. On the conservatism of the sum-of-squares method for analysis of time-delayed systems. *Automatica*, 47(11):2406–2411, 2011.
- [PK07] P.G. Park and J.W. Ko. Stability and robust stability for systems with a time-varying delay. *Automatica*, 43(10):1855–1858, 2007.
- [PKJ11] P.G. Park, J.W. Ko, and C. Jeong. Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, 47(1):235–238, 2011.
- [PKP<sup>+</sup>15] M.J. Park, O.M. Kwon, J.H. Park, S.M. Lee, and E.J. Cha. Stability of time-delay systems via Wirtinger-based double integral inequality. *Automatica*, 55:204–208, 2015.
- [PKPL11] M.-J. Park, O.M. Kwon, J.H. Park, and S.-M. Lee. A new augmented lyapunov–krasovskii functional approach for stability of linear systems with time-varying delays. *Applied Mathematics and Computation*, 217(17):7197–7209, 2011.
- [PLL15] P.G. Park, W.I. Lee, and S.Y. Lee. Auxiliary function-based integral inequalities for quadratic functions

- and their applications to time-delay systems. *Journal of the Franklin Institute*, 352(4):1378–1396, 2015.
- [PPL09] M.M. Peet, A. Papachristodoulou, and S. Lall. Positive forms and stability of linear time-delay systems. *SIAM Journal on Control and Optimization*, 47(6):3227–3258, 2009.
- [PPN07] A. Papachristodoulou, M. M. Peet, and S.-I. Niculescu. Stability analysis of linear systems with time-varying delays: Delay uncertainty and quenching. In *46<sup>th</sup> IEEE Conference on Decision and Control (CDC'07)*, New Orleans, LA, USA, 2007.
- [PPP02] S. Prajna, A. Papachristodoulou, and P. A. Parrilo. Introducing SOSTOOLS: a general purpose sum of squares programming solver. *Proceedings of the IEEE Conference on Decision and Control*, 2002.
- [Ric02] J.-P. Richard. *Outils mathématiques pour les systèmes dynamiques*. Hermès - Lavoisier, 2002.
- [Ric03] J.-P. Richard. Time delay systems: an overview of some recent advances and open problems. *Automatica*, 39:1667–1694, 2003.
- [RZ00] M.A. Rami and X.Y. Zhou. Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls. *Automatic Control, IEEE Trans. on*, 45(6):1131–1143, 2000.
- [SBS16a] M. Safi, L. Baudouin, and A Seuret. Stability analysis of a system coupled to a transport equation using integral inequalities. *submitted to Automatica*, 2016.
- [SBS16b] M. Safi, L. Baudouin, and A. Seuret. Stability analysis of a system coupled to a transport equation using integral inequalities. *submitted to Automatica*, 2016.
- [SF13] O. Solomon and E. Fridman. New stability conditions for systems with distributed delays. *Automatica*, 49(11):3467 – 3475, 2013.
- [SG12] A. Seuret and F. Gouaisbaut. On the use of the Wirtinger’s inequalities for time-delay systems. In *10<sup>th</sup> IFAC Workshop on Time Delay Systems (IFAC TDS'12)*, Boston, MA, USA, 2012.
- [SG13] A. Seuret and F. Gouaisbaut. Wirtinger-based integral inequality: Application to time-delay systems. *Automatica*, 49(9):2860 – 2866, 2013.
- [SG14] A. Seuret and F. Gouaisbaut. Complete quadratic Lyapunov functionals using Bessel-Legendre inequality. *Proceeding of the 13th European Control Conference*, 2014.
- [SG15] A. Seuret and F. Gouaisbaut. Hierarchy of LMI conditions for the stability of time delay systems. *Systems & Control Letters*, 81:1–7, 2015.
- [SG16a] A. Seuret and F. Gouaisbaut. Delay-dependent reciprocally convex combination lemma. *submitted to Automatica*, 2016.
- [SG16b] A. Seuret and F. Gouaisbaut. Stability of linear systems with time-varying delays using Bessel-Legendre inequalities. *submitted to IEEE Trans. on Automatic Control*, 2016.
- [SGA15] A. Seuret, F. Gouaisbaut, and Y. Ariba. Complete quadratic Lyapunov functionals for distributed delay systems. *Automatica*, 62:168–176, 2015.
- [SGF13] A. Seuret, F. Gouaisbaut, and E. Fridman. Stability of systems with fast-varying delay using improved Wirtinger’s inequality. In *52nd IEEE Conference on Decision and Control (CDC'13)*, 2013.
- [SGF15] A. Seuret, F. Gouaisbaut, and E. Fridman. Stability of discrete-time systems with time-varying delays via a novel summation inequality. *IEEE Trans. on Automatic Control*, 60(10):2740–2745, 2015.
- [SGL16] A. Seuret, F. Gouaisbaut, and K. Liu. Discretized Jensen’s inequality: an alternative vision of the reciprocally convex combination lemma. In *14<sup>th</sup> IFAC Workshop on Time Delay Systems (IFAC TDS'16)*, Istanbul, Turkey, 2016.
- [Shi60] S.N. Shimanov. On stability in the critical case of a zero root for systems with time lag. *J. Appl. Math. Mech.*, 24:653 – 668, 1960.
- [SLC09] J. Sun, G.P. Liu, and J. Chen. Delay-dependent stability and stabilization of neutral time-delay systems. *International Journal of Robust and Nonlinear Control*, 19(12):1364–1375, 2009.
- [SLCR10] J. Sun, G.P. Liu, J. Chen, and D. Rees. Improved delay-range-dependent stability criteria for linear systems with time-varying delays. *Automatica*, 46(2):466 – 470, 2010.
- [SNA<sup>+</sup>11] R. Sipahi, S. Niculescu, C.T. Abdallah, W. Michiels, and K. Gu. Stability and stabilization of systems with time delay. *Control Systems, IEEE*, 31(1):38 –65, feb. 2011.
- [SO05] R. Sipahi and N. Olgac. Complete stability robustness of third-order LTI multiple time-delay systems. *Automatica*, 41(8):1413–1422, 2005.
- [VAP14] G. Valmorbidia, M. Ahmadi, and A. Papachristodoulou. Semi-definite programming and functional inequalities for distributed parameter systems. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pages 4304–4309. IEEE, 2014.
- [VB00] J.G. VanAntwerp and R.D. Braatz. A tutorial on linear and bilinear matrix inequalities. *Journal of Process Control*, 10(4):363–385, 2000.
- [XFS01] L. Xie, E. Fridman, and U. Shaked. Robust  $h^\infty$  control of distributed delay systems with applications to combustion control. *IEEE Trans. on Automatic Control*, 46(12):1930–1935, 2001.
- [XL05] S. Xu and J. Lam. Improved delay-dependent stability criteria for time-delay systems. *ieetac*, 50(3):384–387, March 2005.

- [XSW16] Q. Xu, G. Stepan, and Z. Wang. Delay-dependent stability analysis by using delay-independent integral evaluation. *Automatica*, 70:153–157, 2016.
- [Zam08] S. Zampieri. A survey of recent results in Networked Control Systems. In *Proc. 17<sup>th</sup> IFAC World Congress*, Seoul, Korea, July 2008.
- [ZH09] X.-M. Zhang and Q.-L. Han. A delay decomposition approach to delay-dependent stability for linear systems with time-varying delays. *International Journal of robust and nonlinear control*, 19(17):1922–1930, 2009.
- [ZH15] X.-M. Zhang and Q.-L. Han. Abel lemma-based finite-sum inequality and its application to stability analysis for linear discrete time-delay systems. *Automatica*, 57:199–202, 2015.
- [ZHWS15] H.-B. Zeng, Y. He, M. Wu, and J. She. Free-matrix-based integral inequality for stability analysis of systems with time-varying delay. *IEEE Trans. on Automatic Control*, 60(10):2768–2772, 2015.
- [ZHWX13] H.-B. Zeng, Y. He, M. Wu, and S.-P. Xiao. Less conservative results on stability for linear systems with a time-varying delay. *Optimal Control Applications and Methods*, 34(6):670–679, 2013.
- [ZHWW11] H.-B. Zeng, Y. He, M. Wu, and C.-F. Zhang. Complete delay-decomposing approach to asymptotic stability for neural networks with time-varying delays. *Neural Networks, IEEE Transactions on*, 22(5):806–812, 2011.
- [ZKT01] J. Zhang, C. R. Knopse, and P. Tsiotras. Stability of time-delay systems: Equivalence between Lyapunov and scaled small-gain conditions. *IEEE Trans. on Automatic Control*, 46(3):482–486, March 2001.