Viability analysis and minimal time problems for the Lotka-Volterra prey-predator model

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Abstract

In this work, we consider two approaches for the control of the classical Lotka-Volterra prey-predator model, with a controlled mortality on the predators. Our aim is to maintain the system *as much as possible* in a subset $K(\underline{x})$, which is defined as a prey density above a given threshold $\underline{x}$. We first establish an analytic description of the viability kernel of $K(\underline{x})$ when it is not empty (depending on the value $\underline{x}$), and then determine an optimal feedback control for the minimum time problem to reach this set, as a first control strategy. We also study the so-called *minimal time crisis problem*, which consists in minimizing the total time spent outside the set $K(\underline{x})$, providing a second control strategy. Finally, we compare these two strategies, showing that for a large subset of initial conditions the optimal trajectories and the length of time spent outside $K(\underline{x})$ are slightly different for the two problems. To do so, an equivalence of the time crisis problem with a finite horizon problem is provided in a general setting.

Keywords. Optimal Control, Viability Theory, Pontryagin Maximum Principle, Lotka-Volterra system.

1 Introduction

We consider the classical Lotka-Volterra prey-predator model

$$\left\{\begin{array}{lcl} \dot{x} & = & rx - axy, \\ \dot{y} & = & -my + bxy - uy, \end{array}\right.$$

in the positive orthant domain $\mathcal{D} := (\mathbb{R}_+ \setminus \{0\})^2$. Here, x and y denote respectively the prey and predator densities, and a, b, m, r are five positive parameters. The additional mortality term $-uy$ represents a *mortality* action on the predators (by chemical or biological means), where the control variable $u \in [0, \bar{u}]$ represents the removal effort (here $\bar{u} > 0$).

Many works on the control of Lotka-Volterra and more general prey-predator models are available in the literature since the seventies (see for instance the text book [19]). They provide efficient strategies to protect valuable living organisms (bacteria, crops, animals...) from invasions or damage by pests (phages, insects, predators...). Several situations depending on the way the control acts on the dynamics (on predators only, preys only or both [20]) or directly on the predation term [34], have been considered. Many approaches consist in driving the state of the system to a target usually defined as the equilibrium point of the dynamics without control, minimizing a criterion such as the time to reach the target [29], or an integral cost as a combination of the use of (non-null) control and the pest damage [20]. Other approaches study the optimal control over a prescribed time interval, possibly large as in [24], or the maximization of the population densities at the final time [1, 34]. For crops protection, stabilizing controllers such as in [19, 22] or impulsive controls [21, 26] have been also proposed.

These models have been also considered for harvesting purposes (such as fisheries) maximizing yields at steady state [6] or a discounted utility function over an infinite horizon [32]. More recently, it has been shown

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that discontinuous feedback controllers can stabilize the system about some desired steady states, that are unstable under constant harvesting [27].

Comparatively, there exist very few results concerning accessibility and viability properties of the Lotka-Volterra model with positive controls (apart the works [9, 15, 16] with different contexts and objectives).

In the present work, we address the problem of *preserving* the preys from the predators, maintaining *as much as possible* their density above a given threshold $\underline{x} > 0$, which amounts to have the state belonging to the set

$$K(\underline{x}) := \{(x, y) \in \mathcal{D} \; ; \; x \geq \underline{x}\}.$$

Although the mathematical model predicts that the preys cannot be extinct in finite time, one may consider that practically having a small density of preys expose them to a *danger* of disappearance that should be avoided as much as possible.

Throughout the paper, time varying controls $u(\cdot)$ will be sought among the set $\mathcal{U}$ defined by:

$$\mathcal{U} := \{u : [0, +\infty) \to [0, \bar{u}] \; ; \; u(\cdot) \text{ meas.}\}.$$

For simplicity and without any loss of generality, we choose the coefficients a, b, c equal to 1 and consider then the system:

$$\left\{ \begin{array}{lcl} \dot{x} & = & rx - xy, \\ \dot{y} & = & -my + xy - uy. \end{array} \right. \tag{1.1}$$

For any initial condition $z_0 = (x_0, y_0) \in \mathcal{D}$ and control law $u(\cdot) \in \mathcal{U}$, we shall denote by $z_u(\cdot, z_0) = (x_u(\cdot), y_u(\cdot))$ the unique solution of (1.1) defined over $\mathbb{R}_+$ such that $(x(0), y(0)) = z_0$.

Our objective is first to study the *intrinsic* compatibility of the threshold $\underline{x}$ with the range of control action $[0, \bar{u}]$ for maintaining the state of the system in the set $K(\underline{x})$ (independently to any other criterion). This question falls precisely in the field of the Viability Theory [2, 3, 14]. To our knowledge, the determination of the viability kernel for sets such as $K(\underline{x})$, which is defined as

$$Viab(\underline{x}) := \{z_0 \in \mathcal{D} \; ; \; \exists u(\cdot) \in \mathcal{U}, \; z_u(t, z_0) \in K(\underline{x}), \; \forall t \geq 0\},$$

has not been yet tackled in the literature for prey-predator models. Next, we shall examine control strategies to reach and stay *at best* in the set $K(\underline{x})$, when the initial state is outside this set. Minimizing the time appears to be the most natural way to achieve this objective, but there are two possibilities for defining this time:

1. the time spent outside the viability kernel. This choice guarantees that it is possible to stay in the set $K(\underline{x})$ for any future time, and amounts to solve the minimal time control problem with $Viab(\underline{x})$ as the target:
$$\inf_{u(\cdot) \in \mathcal{U}} T_u \quad \text{s.t. } z_u(T_u, z_0) \in Viab(\underline{x}). \tag{1.2}$$

2. the length of time spent outside the set $K(\underline{x})$. This choice does not ensure *a priori* that the state necessarily reaches first the viability kernel *i.e.* the optimal trajectory could enter and exit several times the set $K(\underline{x})$. This amounts to consider the so-called *minimal time crisis problem* introduced in [17]:
$$\inf_{u(\cdot) \in \mathcal{U}} J(u) := \int_0^\infty \mathbb{1}_{K(\underline{x})^c}(z_u(t, z_0)) \, \mathrm{d}t, \tag{1.3}$$
where $\mathbb{1}_{K(\underline{x})^c}$ denotes the characteristics function of $K(\underline{x})^c$ (*i.e.* the complementary of $K(\underline{x})$) defined by
$$\mathbb{1}_{K(\underline{x})^c}(x, y) := \left\{ \begin{array}{ll} 1 & \text{if} \quad (x, y) \notin K(\underline{x}), \\ 0 & \text{if} \quad (x, y) \in K(\underline{x}). \end{array} \right.$$

Let us underline that both problems are defined over an infinite horizon, so that the corresponding value functions and optimal controls are independent of the initial time.

For practitioners, the condition of non-vacuity of the viability kernel and its size (when it is not empty) provide information of the ability of the controlled system to sustain the threshold $\underline{x}$ on the preys density. For states in the interior of the viability kernel, it is known from the Viability Theory that a simple strategy to remain inside the viability kernel is to constraint control only when the trajectory hits the boundary of the viability such that the velocity vector points inward [2, 3]. For states outside the viability kernel, minimizing the time to reach the viability kernel appears to be a natural choice for the practitioners. Nevertheless, when trajectories enter and exit several times the set $K(\underline{x})$, as it can be the case for the Lotka-Volterra model, one may wonder if the minimal time strategy to reach the viability kernel could indeed let the trajectory spend significantly more time outside the set $K(\underline{x})$ than a strategy minimizing the total length of time spent outside would do. This is why we consider in this work both minimal time criteria, which provide two different *measures* of the *risk* of having the prey density below the threshold, and compare them.

Technically, both optimal control problems present several issues that we investigate in the present work.

- The first problem needs to determine first the viability kernel as a target. Its boundary might be non-smooth, opening the possibility of having several trajectories reaching a common boundary point with different velocity vectors.

- The minimal time crisis problem possesses a discontinuous Hamiltonian, preventing the use of classical necessary conditions (and standard numerical codes). In [17] a characterization of the value function as a generalized solution of an Hamilton-Jacobi equation has been proposed, but necessary conditions have not been studied. To overcome this difficulty, the authors have proposed recently an approximation scheme based of the Moreau-Yosida regularization of the characteristic function of the constraint set (here $K(\underline{x})$ plays the role of the constraint set) [4], for the finite horizon case. This regularization allows to use the available numerical methods (direct or indirect), that require a continuous integrand, to determine approximate optimal trajectories [5]. Let us also mention references [7, 8] for the study of a similar regularization scheme in the context of parabolic equations. Following [4], the hybrid maximum principle (see *e.g.* [12, 18, 23]) is well adapted to derive necessary optimality conditions on the time crisis problem over a finite horizon, as a theoretical or analytic tool.

The paper is organized as follows:

i. In Section 2, we provide an exact analytic description of the viability kernel $Viab(\underline{x})$ of $K(\underline{x})$.

ii. In Section 3, we determine the optimal synthesis for the minimal time problem to reach $Viab(\underline{x})$ and show that optimal controls are "bang-bang" using the Pontryagin Maximum Principle [28]. Switching curves are computed by a numerical integration of the state-adjoint system backward in time.

iii. In Section 4, we first provide an equivalent formulation of the time crisis problem over a finite horizon for a general control system (see Proposition 4.1). Thanks to this result, we can apply the hybrid maximum principle on (1.3). We then show that an optimal control for this problem is "bang-bang" and we depict the switching curves numerically. Optimal strategies for both problems (1.2)-(1.3) are then compared. When $Viab(\underline{x})$ is empty, we show that the time crisis function is necessarily equal to $+\infty$ and that no chattering occurs at the boundary of $K(\underline{x})$.

iv. The appendix provides details on the numerical scheme that has been used to depict optimal trajectories and the switching curves for both problems (1.2)-(1.3).

2 Determination of the viability kernel

We start by giving some definitions and recall some classical properties about the Lotka-Volterra prey-predator model.

Given a non-empty subset A of $\mathbb{R}^2$, we will denote by $\mathrm{Int}(A)$ its interior and by ∂A its boundary. Next, we also denote by $\|(x,y)\|$ the euclidean norm of a vector $(x,y) \in \mathbb{R}^2$. For a fixed $u \in [0,\bar{u}]$, we define the function $W_u : \mathcal{D} \to \mathbb{R}$ by:

$$W_u(x,y) := x - (m+u)\ln x + y - r\ln y, \quad (x,y) \in \mathcal{D},$$

together with the number $\underline{c}(u) \in \mathbb{R}$ defined by

$$\underline{c}(u) := W_u(m+u,r) = (m+u)(1-\ln(m+u)) + r(1-\ln r),$$

and the positive equilibrium point $E^\star(u)$ for (1.1)

$$E^\star(u) := (x^\star(u), y^\star) = (m+u, r).$$

For a given number $c \geq \underline{c}(u)$, we denote by $L_u(c)$, resp. by $S_u(c)$, the level set, resp. the sub-level set of W_u defined by

$$L_u(c) := \{(x,y) \in \mathcal{D},\ W_u(x,y) = c\}, \text{ resp. } S_u(c) := \{(x,y) \in \mathcal{D},\ W_u(x,y) \leq c\}.$$

We recall in the two following Lemmas classical results about the model (1.1) with constant control.

Lemma 2.1. *For a constant control u, a trajectory of (1.1) belongs to a level set $L_u(c)$ with $c \geq \underline{c}(u)$. The sets $L_u(c)$ are closed curves that surround the steady state $E^\star(u)$.*

Proof. By differentiating W_u w.r.t. x and y, one finds $\partial_x W_u(x,y) = 1 - \frac{m+u}{x}$ and $\partial_x W_u(x,y) = 1 - \frac{r}{y}$ for $(x,y) \in \mathcal{D}$. If $(x(\cdot), y(\cdot))$ is a solution of (1.1) with the constant control u, a direct computation gives

$$\frac{d}{dt} W_u(x(t), y(t)) = \partial_x W_u \dot{x} + \partial_y W_u \dot{y} = 0.$$

So, any solution of (1.1) with the constant control u belongs to a level set of the function W_u. As $W_u(x,y) \to +\infty$ when $\|(x,y)\| \to +\infty$, each level set $L_u(c)$ is bounded. For a constant control u, one can check that the single equilibrium of the dynamics in $\mathcal{D}$ is $E^\star(u)$, and that the level set $L_u(\underline{c}(u))$ is the singleton $\{E^\star(u)\}$. Therefore, for any initial condition in $\mathcal{D} \setminus \{E^\star(u)\}$, the trajectory belongs to a level set $L_u(c)$ with $c > \underline{c}(u)$ (recall that $W_u(x,y) \to +\infty$ when $\|(x,y)\| \to +\infty$). As $L_u(c)$ is a compact set that does not contain any equilibrium point, Poincaré-Bendixon Theorem allows to state that the trajectory converges to a limit cycle that belongs to the same level set $L_u(c)$. Therefore, the trajectory is periodic and $L_u(c)$ is a closed curve which surrounds $E^\star(u)$. □

For $u \in [0, \bar{u}]$, we define two functions $\phi_u : \mathbb{R}_+ \to \mathbb{R}$ and $\psi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\phi_u(x) := x - (m+u)\ln x,\ x \in \mathbb{R}_+ \quad \text{and} \qquad \psi(y) := y - r\ln y,\ y \in \mathbb{R}_+.$$

Lemma 2.2. *Given $u \in [0, \bar{u}]$, one has the following properties*

(i) *For any $x > \phi_u(m+u)$, there exists unique $x_u^+(x) \in (m+u, +\infty)$ and $x_u^-(x) \in (0, m+u)$ such that $\phi_u(x_u^+(x)) = \phi_u(x_u^-(x)) = x$.*

(ii) *If $p > \psi(r)$, the equation $\psi(y) = p$ has exactly two roots $y^-(p)$, $y^+(p)$ that satisfy $y^-(p) < r < y^+(p)$.*

Proof. One can easily check that $\lim_{x\to+\infty} \phi_u(x) = \lim_{x\to 0} \phi_u(x) = +\infty$. Moreover, by differentiating ϕ_u w.r.t. x, one finds $\phi_u'(x) = 1 - \frac{m+u}{x}$. So the function ϕ_u is decreasing from $+\infty$ down to $\phi_u(m+u)$ and increasing up to $+\infty$. Therefore, for any $x > \phi_u(m+u)$, the equation $\phi_u(z) = x$ has exactly two solutions $x_u^-(x)$, $x_u^+(x)$, with $x_u^-(x) < m+u$ and $x_u^+(x) > m+u$ which proves (i). Similarly, the function ψ is decreasing from $+\infty$ down to $\psi(r)$ and increasing up to $+\infty$, which gives (ii). □

For $c \in \mathbb{R}$, we consider the subsets of $\mathcal{D}$, $L_u^+(c)$, $L_u^-(c)$, $S_u^+(c)$ and $S_u^-(c)$ defined by:

$$L_u^+(c) := L_u(c) \cap \{y \geq r\}, \quad L_u^-(c) := L_u(c) \cap \{y \leq r\},$$

and

$$S_u^+(c) := S_u(c) \cap \{y \geq r\}, \quad S_u^-(c) := S_u(c) \cap \{y \leq r\},$$

and let $r_- \in (0, r]$ be defined by:

$$r_- := y^-(W_0(x_{\bar{u}}^+(\underline{x}), r) - \phi_0(\underline{x})).$$

The next Proposition provides a description of the viability kernel $Viab(\underline{x})$ of $K(\underline{x})$ for (1.1).

Proposition 2.1. *One has the following characterization of the viability kernel:*

(i) *If $m + \bar{u} < \underline{x}$, the set $Viab(\underline{x})$ is empty.*

(ii) *If $\bar{u} \geq \underline{x} - m$, the viability kernel is non-empty and is given by*

$$Viab(\underline{x}) = S^+_{\bar{u}}(W_{\bar{u}}(\underline{x}, r)) \bigcup \left(S^-_0(W_0(x^+_{\bar{u}}(\underline{x}), r)) \cap K(\underline{x})\right),$$

where $x^+_{\bar{u}}(\underline{x})$ is given by Lemma 2.2. Its boundary is the union of the three curves

$$\begin{array}{rcl} B^+(\underline{x}) & := & L^+_{\bar{u}}(W_{\bar{u}}(\underline{x}, r)), \\ B^-(\underline{x}) & := & L^-_0(W_0(x^+_{\bar{u}}(\underline{x}), r)) \cap \{x \geq \underline{x}\}, \\ B^0(\underline{x}) & := & \{\underline{x}\} \times [r_-, r]. \end{array}$$

Proof. Let us assume that $\bar{u} < \underline{x} - m$ and let $\varepsilon > 0$ be such that $m + \underline{u} < \underline{x} - \varepsilon$. Consider a trajectory $(x(\cdot), y(\cdot))$ that stays in the set $K(\underline{x})$ for any time $t \geq 0$. As $0 \leq u(t) \leq \bar{u}$, we deduce that

$$\dot{y} = y(x - m - u) \geq y(x - \underline{x} + \varepsilon) \geq \varepsilon y,$$

using that $x(t) \geq \underline{x}$ for any time $t \geq 0$. Therefore $y(\cdot)$ is increasing, unbounded and thus there exists $t_1 > 0$ such that $y(t) > r$ for any time $t \geq t_1$. It follows that

$$\dot{x}(t) = x(t)(r - y(t)) < x(t)(r - y(t_1)) \quad \forall t > t_1,$$

implying that for $t > t_1$ one has $\dot{x}(t) < -Cx(t)$ with $C := y(t_1) - r > 0$. Thus, there exists $t_2 > t_1$ such that $x(t_2) < \underline{x}$. So the trajectory $(x(\cdot), y(\cdot))$ must escape the set $K(\underline{x})$ in finite time, and we have a contradiction. Thus, the viability kernel $Viab(\underline{x})$ is empty which proves (i).

Assume now that one has $\bar{u} \geq \underline{x} - m$ and let us prove (ii). Notice first that the three curves $B^+(\underline{x})$, $B^-(\underline{x})$ and $B^0(\underline{x})$ belong to the set $K(\underline{x})$ and that their union $U(\underline{x})$ defines the boundary of a compact subset $Z(\underline{x})$ of $K(\underline{x})$, which is such that

$$Z(\underline{x}) = S^+_{\bar{u}}(W_{\bar{u}}(\underline{x}, r)) \bigcup \left(S^-_0(W_0(x^+_{\bar{u}}(\underline{x}), r)) \cap K(\underline{x})\right).$$

When $\bar{u} = \underline{x} - m$, the set $Z(\underline{x})$ is reduced to the single point $E^\star(\bar{u})$ that is an equilibrium of (1.1) for the constant control $\bar{u}$. Thus, $Z(\underline{x})$ is a viable set.

When $\bar{u} > \underline{x} - m$, we first show that for any initial condition in $U(\underline{x})$, there exists a trajectory that stays in $K(\underline{x})$ for any time $t \geq 0$. Consider an initial condition in the set $B^+(\underline{x})$. With the control $u = \bar{u}$, the corresponding solution of (1.1) remains on the level set $L_{\bar{u}}(W_{\bar{u}}(\underline{x}, r))$ which is contained in $K(\underline{x})$ as its extreme left point is $(\underline{x}, r)$. Take now an initial condition in $B^-(\underline{x})$. With the control $u = 0$, the corresponding solution of (1.1) remains on $B^-(\underline{x})$ until it reaches in finite time the boundary point $(x^+_{\bar{u}}(\underline{x}), r)$ that belongs to $B^+(\underline{x})$. From this point, we come back to the previous case. Finally, take an initial condition in $\mathrm{Int}(B^0(\underline{x}))$ (if not empty). On $\mathrm{Int}(B^0(\underline{x}))$, one has $\dot{x} > 0$ for any control as one has $r - y > 0$. Thus the trajectory enters the subset $Z(\underline{x})$ and cannot evade from $K(\underline{x})$ on $\mathrm{Int}(B^0(\underline{x}))$. If the trajectory touches $B^+ \cup B^-$, we face one of the two previous cases. So we conclude that $Z(\underline{x})$ is a viable domain.

We now show that $Z(\underline{x})$ is the largest viable domain included in $K(\underline{x})$, that is, the viability kernel $Viab(\underline{x})$, or equivalently that any trajectory with initial condition in $K(\underline{x}) \setminus Z(\underline{x})$ leaves the set $K(\underline{x})$ in a finite horizon. For convenience we consider the two subsets of $K(\underline{x}) \setminus Z(\underline{x})$, $C^+(\underline{x})$ and $C^-(\underline{x})$ defined by:

$$C^+(\underline{x}) = \left(K(\underline{x}) \setminus Z(\underline{x})\right) \cap \{y \geq r\} \quad \text{and} \quad C^-(\underline{x}) = \left(K(\underline{x}) \setminus Z(\underline{x})\right) \cap \{y \leq r\}.$$

Consider now an initial condition $(x_0, y_0) \in C^+(\underline{x})$, and let $(x(\cdot), y(\cdot))$ a solution of (1.1) starting from (x_0, y_0). One has $W_{\bar{u}}(x_0, y_0) > W_{\bar{u}}(\bar{x}, r)$, and by differentiating w.r.t t one finds:

$$\frac{d}{dt} W_{\bar{u}}(x(t), y(t)) = (y(t) - r)(\bar{u} - u(t)) \geq 0.$$

Therefore no trajectory can reach the level set $L_{\bar{u}}(W_{\bar{u}}(\bar{x}, r))$ from $K(\underline{x}) \setminus Z(\underline{x})$. When $y(t) = r$, one has $x(t) > m + \bar{u}$ and thus $\dot{y}(t) = r(x(t) - m - u(t)) > 0$ as $u(t) \leq \bar{u}$. We deduce that if there exists a trajectory with an initial condition (x_0, y_0) in $C^+(\underline{x})$ that stays in $K(\underline{x})$, it has to stay in $C^+(\underline{x})$. By differentiating w.r.t t, we obtain on $C^+(\underline{x})$ that

$$\frac{d}{dt} W_0(x(t), y(t)) = -u(t)(y(t) - r) \leq 0,$$

and thus $W_0(x(t),y(t)) \leq W_0(x_0,y_0)$. It follows that the trajectory is bounded. Moreover, one has $\dot{x} \leq 0$. *i.e.* the function $t \mapsto x(t)$ is non-increasing and thus converges to a certain $x_\infty > 0$. By Barbalat's Lemma (see for instance [25]), $\dot{x}(t)$ converges to 0 which implies that $y(t)$ tends to r. Then $W_{\bar{u}}(x(t),y(t))$ converges to $W_{\bar{u}}(x_\infty,r) > W_{\bar{u}}(\bar{x},r)$ which implies that $x_\infty < \bar{x}$. Thus, the trajectory necessarily leaves the set $K(\underline{x})$ and we have a contradiction.

Consider now an initial condition $(x_0,y_0) \in C^-(\underline{x})$. Similarly, one can show that no trajectory can reach the level set $L_0(W_0(x^+(\underline{x}),r))$ from $K(\underline{x}) \setminus Z(\underline{x})$. It follows that a trajectory with an initial condition in $C^-(\underline{x})$ that stays in $K(\underline{x})$ has to stay in $C^-(\underline{x})$ (otherwise, it reaches $C^+(\underline{x})$ and we have shown above that its has to escape $K(\underline{x})$). As previously, one can show that a trajectory that stays in $C^-(\underline{x})$ is bounded and as $t \mapsto x(t)$ is increasing, one obtains the convergence of $x(t)$ to a certain $x_\infty > \underline{x}$. As before, by Barbalat's Lemma, $\dot{x}(t)$ converges to 0 when $t \to +\infty$, which implies that $y(t) \to r$, and thus $x_\infty \geq x^+_{\bar{u}}(\underline{x}) > m + \bar{u}$. Therefore there exists $\varepsilon > 0$ and $t_0 > 0$ such that $\dot{y}(t) = (x(t) - m - u)y(t) > \varepsilon y(t)$ for any $t > t_0$. This gives a contradiction with the convergence of $y(t)$ to r when $t \to +\infty$. So, the trajectory has to enter $C^+(\underline{x})$ and then leaves $K(\underline{x})$. This ends the proof of (ii). □

The viability kernel is depicted on Fig. 1 in case (ii) of Proposition 2.1 together with the three curves $B^+(\underline{x})$, $B^-(\underline{x})$,$B^0(\underline{x})$ that define its boundary (see the Appendix for the numerical values of the parameters). It is worth noting that $B^+(\underline{x})$ is a semi-orbit of (1.1) with $u = \bar{u}$ passing trough the point $(\underline{x},r)$. Similarly, $B^-(\underline{x})$ is a semi-orbit of (1.1) with $u = 0$ passing though the point $(\underline{x},r_-)$.

Note that whenever $m + \bar{u} = \underline{x}$, then the viability kernel is reduced to one point *i.e.*

$$m + \bar{u} = \underline{x} \quad \Rightarrow \quad Viab(\underline{x}) = \{(\underline{x},r)\},$$

as the semi-orbit $B^+(\underline{x})$ is then reduced to the positive equilibrium point of (1.1) with $u = \bar{u}$.

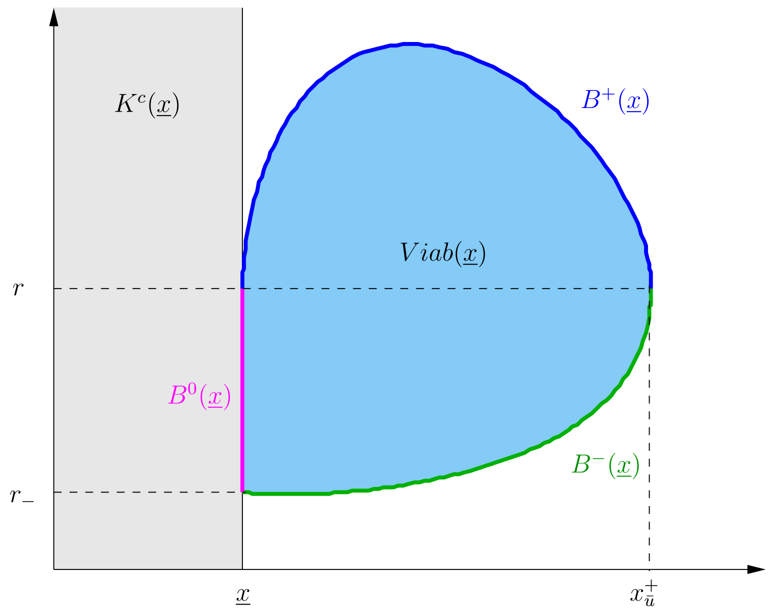

Figure 1: Viability kernel when $\bar{u} > \underline{x} - m$ (numerical values can be found in the appendix).

Definition 2.1. *Given a solution $z(\cdot,z_0)$ of* (1.1)*, we say that a time $t_c > 0$ is a crossing time for $z(\cdot)$ from $K(\underline{x})$ to $K(\underline{x})^c$ if the control u is left- and right-continuous at t_c, $z(t_c) \in \partial K$, and there exists $\eta > 0$ such that for any time $t \in [t_c - \eta, t_c]$, resp. $t \in (t_c, t_c + \eta]$, $z(t,z_0) \in K$, resp. $z(t,z_0) \in K^c$. Similarly, we define the notion of crossing time from $K(\underline{x})^c$ to $K(\underline{x})$.*

The viability kernel of $K(\underline{x})$ enjoys the following properties.

Proposition 2.2. *Suppose that $\bar{u} > \underline{x} - m$.*

(i) *Consider the unique solution of* (1.1) *backward in time with* $u=0$ *from* $(x_{\bar{u}}^{+}(\underline{x}),r)$*, and let* $t'>0$ *be the first time where this trajectory intersects the line* $\{y=r\}$*. Then, we have:*

$$x(t')\leq \underline{x}. \tag{2.1}$$

(ii) *The set* $Viab(\underline{x})$ *is a compact convex set with non-empty interior.*

(iii) *Suppose that* $t_1<t_2$ *are two consecutive crossing times from* $K(\underline{x})$ *to* $K(\underline{x})^c$ *and from* $K(\underline{x})^c$ *to* $K(\underline{x})$ *respectively and that* $(x(t_1),y(t_1))\notin Viab(\underline{x})$, $(x(t_2),y(t_2))\notin Viab(\underline{x})$*. Then, we have the following inequality:*

$$t_2-t_1\geq \frac{\ln\left(\frac{r}{r_-}\right)}{m+\bar{u}}. \tag{2.2}$$

Proof. To prove (i), suppose by contradiction that $x(t')>\underline{x}$. Denote by $(x_0(\cdot),y_0(\cdot))$ the unique solution of (1.1) with $u=0$ and such that $(x_0(0),y_0(0))=(x(t'),r)$. By construction, the point $(x_{\bar{u}}^{+}(\underline{x}),r)$ is on the graph of the parameterized curve $(x_0(\cdot),y_0(\cdot))$. Thus we denote by $t_0:=\inf\{t\geq 0\ ;\ (x_0(t),y_0(t))=(x_{\bar{u}}^{+}(\underline{x}),r)\}$, and let γ_0 be the parametrized curve $(x_0(\cdot),y_0(\cdot))$ on the interval $[0,t_0]$.

Now, consider the unique solution $(x_1(\cdot),y_1(\cdot))$ of (1.1) with $u=\bar{u}$ starting from the point $(x(t'),r)$ and let $t_1:=\inf\{t\geq 0\ ;\ y_1(t)=r\}$. We define γ_1 as the parametrized curve $(x_1(\cdot),y_1(\cdot))$ on the interval $[0,t_1]$. From (1.1), we know that the graph of γ_1 is below the graph of γ_0.

To conclude, we consider the unique solution $(\tilde{x}_1(\cdot),\tilde{y}_1(\cdot))$ of (1.1) with $u=\bar{u}$ starting from $(\underline{x},r)$ until the first time $t_1'>0$ where it reaches the segment line $\{y=r\}$. It can be noticed that this curve passes through the point $(x_{\bar{u}}^{+}(\underline{x}),r)$ and that its graph $\tilde{\gamma}_1$ is also below the graph of γ_0. Hence, both graphs γ_1 and $\tilde{\gamma}_1$ should intersect *i.e.* there must exist a time $\tau\in(0,t_0)$ such that $(x_1(\tau),y_1(\tau))=(\tilde{x}_1(\tau),\tilde{y}_1(\tau))$. By Cauchy-Lipschitz's Theorem, both solutions $(x_1(\cdot),y_1(\cdot))$ and $(\tilde{x}_1(\cdot),\tilde{y}_1(\cdot))$ should coincide everywhere which is a contradiction as $(x_1(0),y_1(0))\neq(\tilde{x}_1(0),\tilde{y}_1(0))$.

Let us now show (ii). From Proposition 2.1 and the previous point, we know that $Viab(\underline{x})$ is a compact subset of $\mathbb{R}^2$ with non-empty interior. Now, one can easily verify that solutions of (1.1) in the plane (x,y) satisfy:

$$\frac{d^2y}{dx^2}\bigg|_{u=0}(x)=\frac{y(r(x-m)^2+m(r-y)^2)}{(r-y)^3x^2}\quad\text{and}\quad\frac{d^2y}{dx^2}\bigg|_{u=\bar{u}}(x)=\frac{y(r(x-m-1)^2+(m+1)(r-y)^2)}{(r-y)^3x^2}.$$

It follows that for $y<r$, resp. $y>r$, one has $\frac{d^2y}{dx^2}_{|u=0}(x)>0$, resp. $\frac{d^2y}{dx^2}_{|u=\bar{u}}(x)<0$, which guarantees that the set $Viab(\underline{x})$ is convex.

Finally, to prove (iii), we integrate the equation $\dot{y}(t)=y(t)(x(t)-m-u(t))$ over $[t_1,t_2]$, which gives:

$$(m+\bar{u})(t_2-t_1)\geq\int_{t_1}^{t_2}(m+u(t))\ \mathrm{d}t=\int_{t_1}^{t_2}x(t)\ \mathrm{d}t-\int_{y(t_1)}^{y(t_2)}\frac{\mathrm{d}y}{y}\geq-\int_{y(t_1)}^{y(t_2)}\frac{\mathrm{d}y}{y}\geq\ln\left(\frac{r}{r_-}\right),$$

using that $y(t_1)>r$ and $y(t_2)<r_-$. This ends the proof. □

Remark 2.1. (i) *Whereas when* $m<\underline{x}$*, it is clear that* (2.1) *holds true (as the equilibrium point for* (1.1) *with* $u=0$ *is* (m,r)*), the previous Proposition shows that this property remains valid whenever* $m>\underline{x}$*. Note also that in the latter case, the set* $Viab(\underline{x})$ *is always non-empty as* $\bar{u}>0$*.*
(ii) *We know that for a given initial state in* $Viab(\underline{x})$*, any control* u *can be chosen until that the corresponding trajectory reaches the boundary of* $Viab(\underline{x})$ *(see e.g.* [2, 3]*). If* $(x_0,y_0)\in B^-(\underline{x})$*, resp.* $(x_0,y_0)\in B^+(\underline{x})$*, then only the control* $u=0$*, resp.* $u=\bar{u}$ *is admissible in order to stay in* $Viab(\underline{x})$*.*
(iii) *If* $\underline{x}>m$*, then any point of the segment* $[\underline{x},\bar{u}+m]\times\{r\}$ *is a steady-state point for* (1.1) *with a prescribed constant control whereas if* $\underline{x}\leq m$*, then any point of the segment* $[m,\bar{u}+m]\times\{r\}$ *is a steady-state point for* (1.1) *with a prescribed constant control.*
(iv) *Inequality* (2.2) *gives a lower bound between two consecutive crossing times and will be used in Section 4.*

3 Minimal time problem to reach the viability kernel

In this Section, we consider that the condition $\bar{u} \geq \underline{x} - m$ is fulfilled, which guarantees that the viability kernel $Viab(\underline{x})$ is non empty (see Proposition 2.1). We first study the attainability of $Viab(\underline{x})$ and then derive necessary conditions for the minimal time problem. Finally, we give the optimal synthesis.

3.1 Attainability of the viability kernel

We recall from the Viability Theory (see *e.g.* [2, 30, 31]) that the viability kernel $Viab(\underline{x})$ can be reached from outside only at its boundary in common with the boundary of $K(\underline{x})$, that is, accordingly to Proposition 2.1 at the line-segment $B^0(\underline{x}) = \{\underline{x}\} \times [r_-, r]$ (possibly reduced to a singleton when $\bar{u} = \underline{x} - m$).

In order to show the attainability of the target set, it is convenient to introduce the following feedback control.

Definition 3.1. *The myopic*[1] *state feedback is defined as*

$$\mathbf{u}[x, y] := \begin{cases} \bar{u} & \text{if} \quad y \geq r, \\ 0 & \text{if} \quad y < r. \end{cases} \tag{3.1}$$

Given an initial condition $(x_0, y_0) \in (\mathbb{R}_+^* \times \mathbb{R}_+^*) \backslash Viab(\underline{x})$, we denote by $(x_m(\cdot), y_m(\cdot))$ the unique solution of (1.1) starting from (x_0, y_0) at time 0 and associated to the control $u_m(\cdot)$ defined by $u_m(t) := \mathbf{u}[x_m(t), y_m(t)]$.

Proposition 3.1. *For any initial condition $(x_0, y_0) \in \mathcal{D} \backslash Viab(\underline{x})$, there exists a control $u \in \mathcal{U}$ steering (x_0, y_0) to the viability kernel $Viab(\underline{x})$.*

Proof. Suppose first that one has $m + \bar{u} > \underline{x}$, hence $Viab(\underline{x})$ has a non-empty interior.
First step. We show that it is enough to prove the result for any initial condition of type (x_0, r) with $x_0 > x_{\bar{u}}^+(\bar{x})$. If the initial condition (x_0, y_0) is such that $y_0 < r$, then it is enough to replace x_0 by $x_m(t_c)$ where t_c is the first time $t > 0$ such that $y_m(t_c) = r$. If $(x_m(t_c), y_m(t_c)) \in Viab(\underline{x})$, then the result is proved. Otherwise, we have $x_m(t_c) > x_{\bar{u}}^+(\bar{x})$. If now $y_0 > r$, we apply the control u_m until the first time $t_c' > 0$ such that $y_m(t_c') = r$. Then, for $t > t_c'$ (close to t_c') one has $y_m(t_c') < r$, and we conclude by the previous case.
Second step. We now show the Proposition for any initial condition (x_0, r) with $x_0 > x_{\bar{u}}^+(\bar{x})$. By applying the feedback control u_m, we can define two sequences of time $(t_n)_{n \geq 0}$ and $(t_n')_{n \geq 0}$ such that:

$$y_m(t_n) = y_m(t_n') = r \quad \text{and} \quad x_n' := x_m(t_n') < \underline{x} < x_n := x_m(t_n).$$

Moreover, the trajectory is such that for any $n \in \mathbb{N}$:

$$t \in (t_n, t_n') \Rightarrow y_m(t) > r \quad \text{and} \quad t \in (t_n', t_{n+1}) \Rightarrow y_m(t) < r.$$

We have $x_1 < x_0$. Indeed, consider the two solutions of (1.1), $\hat{x}_0(\cdot)$, resp. $\hat{x}_1(\cdot)$ with the control $u = 0$, resp. $u = \bar{u}$ starting from the point (x_0', r). We then have $\hat{x}_0(t) > \hat{x}_1(t)$ for any $t \in (0, \hat{t}]$ where $\hat{t}$ is such that $\hat{x}_0(\hat{t}) = r$. Now, as $\hat{x}_1(\cdot)$ passes though the point (x_0, r), we deduce that $x_1 < x_0$. Now, the two solutions of (1.1) with $u = \bar{u}$ starting from (x_0, r) and (x_1, r) cannot intersect, thus we deduce that $x_1' > x_0'$. By induction, we obtain that $(x_n)_{n \geq 0}$ is decreasing and that $(x_n')_{n \geq 0}$ is increasing.

Now, integrating (1.1) on the interval (t_0, t_0'), resp. (t_0', t_1) with $u = \bar{u}$, resp. with $u = 0$ yields:

$$\begin{cases} -x_0 + (m + \bar{u}) \ln(x_0) = -x_0' + (m + \bar{u}) \ln(x_0'), \\ -x_1 + m \ln x_1 = -x_0' + m \ln x_0'. \end{cases}$$

Thus we obtain the relation $x_0 - x_1 - m \ln \left(\frac{x_0}{x_1} \right) + \bar{u} \ln \left(\frac{x_0'}{x_0} \right) = 0$ and by induction we get:

$$\forall n \in \mathbb{N}^*, \quad x_{n-1} - x_n - m \ln \left(\frac{x_{n-1}}{x_n} \right) + \bar{u} \ln \left(\frac{x_{n-1}'}{x_{n-1}} \right) = 0.$$

[1] This terminology was introduced in [4] in the case where a control policy acts separately in two components of the state domain.

As $x_{n-1} < x_n$, we deduce that one has

$$x_{n-1} - x_n \geq \bar{u} \ln \left(\frac{x_{n-1}}{x'_{n-1}} \right). \tag{3.2}$$

To conclude, we suppose by contradiction that the trajectory always stays outside the set $Viab(\underline{x})$. By noticing that inequalities $x_{n-1} - x'_{n-1} \geq x^+_{\bar{u}}(\underline{x}) - \underline{x}$ and $x'_{n-1} \leq \underline{x}$ are fulfilled for any $n \geq 1$, one obtains $\frac{x_{n-1}}{x'_{n-1}} \geq \frac{x^+_{\bar{u}}(\bar{x})}{\underline{x}}$ which implies

$$x_{n-1} - x_n \geq \beta,$$

where $\beta := \bar{u} \ln \left(\frac{x^+_{\bar{u}}(\bar{x})}{\underline{x}} \right) > 0$ (recall that the interior of $Viab(\underline{x})$ is non-empty). Thus, one has for each $n \in \mathbb{N}$ $x_n \leq x_{n-1} - \beta$. Therefore we obtain a contradiction and the trajectory necessary enters the set $Viab(\underline{x})$ which ends the proof in the case where $m + \bar{u} > \underline{x}$.

Consider now the case $m + \bar{u} = \underline{x}$. Then $Viab(\underline{x})$ is reduced to a single point $(\underline{x}, r)$ that belongs to the periodic orbit O_r defined as the unique solution of (1.1) with $u = 0$ passing through this point. The second intersection point of this orbit with the line $\{y = r\}$ is denoted by (ν, r) with $0 < \nu < \underline{x}$. Now, if the initial condition is in the interior of O_r, then the control $u = \bar{u}$ steers (1.1) in finite time to O_r and the result follows. Following the proof of the result in the case where $m + \bar{u} > \underline{x}$, we can suppose that the initial condition (x_0, r) is such that $x_0 > \underline{x}$. Similarly, let us define two sequences of points (x_n), (x'_n) such that (x_n) is increasing and (x'_n) is decreasing. Moreover inequality (3.2) also holds true. To conclude, we suppose by contradiction that the trajectory does not intersect O_r. Hence, for any $n \in \mathbb{N}$ one has $x_n - x'_n \geq \underline{x} - \nu$ and $x'_n \leq \nu$. We then find that for any $n \in \mathbb{N}^*$

$$\frac{x_{n-1}}{x'_{n-1}} = \frac{x_{n-1} - x'_{n-1}}{x'_{n-1}} + 1 \geq \frac{\underline{x}}{\nu} > 0,$$

and for any $n \in \mathbb{N}$ we obtain

$$x_{n-1} - x_n \geq \beta',$$

with $\beta' := \bar{u} \ln (\underline{x}/\nu) > 0$. We can then conclude as in the previous case.

□

Remark 3.1. *Observe indeed that along trajectories of* (1.1) *one has*

$$\forall t \geq 0, \quad \frac{d}{dt} W_0(x(t), y(t)) = -u(t)(y(t) - r), \tag{3.3}$$

Finding a control strategy that makes the value of W_0 decreasing with time appears to be a an efficient strategy to steer (1.1) *from a given initial condition to $Viab(\underline{x})$. From* (3.3)*, we obtain the following inequalities*

$$\forall t \geq 0, \quad y(t) \geq r \ \Rightarrow \frac{d}{dt} W_0(x(t), y(t)) \geq -\bar{u}(y(t) - r) \quad \text{and} \quad y(t) \leq r \ \Rightarrow \frac{d}{dt} W_0(x(t), y(t)) \geq 0, \tag{3.4}$$

which allow to state that the myopic feedback $\mathbf{u}$ *is the control strategy that gives the maximal decreasing of the function W_0 along the trajectories. However, there is no evidence that this strategy corresponds to the optimal feedback control for the minimal time strategy to reach $Viab(\underline{x})$ that we study in the next sections.*

3.2 Necessary optimality conditions

Let us define the value function V associated to the minimum time control problem to reach $Viab(\underline{x})$. For a given initial condition $z_0 = (x_0, y_0) \in \mathcal{D}$, the function V is defined as

$$V(z_0) := \inf_{u(\cdot) \in \mathcal{U}} T_u \quad \text{s.t. } z_u(T_u, z_0) \in Viab(\underline{x}), \tag{3.5}$$

where $z_u(\cdot, z_0) := (x_u(\cdot), y_u(\cdot))$ is the unique solution of (1.1) associated to the control $u(\cdot) \in \mathcal{U}$ and T_u the first entry time of $z_u(\cdot, z_0)$ into the target set. From Proposition 3.1, the set $Viab(\underline{x})$ can be reached from any initial condition $(x_0, y_0) \in \mathcal{D}$. Therefore V is finite everywhere in $\mathcal{D}$ and the existence of an optimal control follows from standard argumentation based on Filipov's Theorem (see for instance [11]). Recall also that $Viab(\underline{x})$ can be reached from $K(\underline{x})^c$ only through $\partial K(\underline{x})\backslash\partial Viab(\underline{x})$ (see for instance [2]), that is here the line-segment $B^0(\underline{x})$.

Recall that given a non-empty closed convex subset $K \subset \mathbb{R}^n$, $n \geq 1$, the normal cone to K at a point $x \in K$ is defined as $N_K(x) := \{p \in \mathbb{R}^n \ ; \ p \cdot (y-x) \leq 0, \ \forall y \in K\}$ where $a \cdot b$ denotes the standard scalar product of two vectors $a, b \in \mathbb{R}^n$. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated to (3.5) defined by:

$$H = H(x, y, p, q, p_0, u) = px(r-y) + qy(x-m-u) + p_0.$$

We now apply the Pontryagin Maximum Principle (PMP) to (3.5) to derive necessary optimality conditions for Problem (3.5). Let $u \in \mathcal{U}$ be an optimal control defined over a certain time interval $[0, T_u]$ with $T_u < +\infty$ and let $z_u := (x_u, y_u)$ be the associated trajectory. Then, there exists an absolutely continuous map $\lambda := (p, q) : [0, T_u] \to \mathbb{R}^2$ and a number $p_0 \leq 0$ such that the following conditions are satisfied:

- The pair $(\lambda(\cdot), p_0)$ is non-zero.
- The adjoint vector satisfies the adjoint equation $\dot{\lambda}(t) = -\frac{\partial H}{\partial z}(z_u(t), \lambda(t), p_0, u(t))$ for a.e. $t \in [0, T_u]$ that is:
$$\left\{ \begin{array}{ll} \dot{p} & = p(y-r) - qy, \\ \dot{q} & = px + q(u+m-x). \end{array} \right. \tag{3.6}$$
- As $Viab(\underline{x})$ is a non-empty compact convex subset of $\mathbb{R}^n$, the transversality condition can be expressed as $\lambda(T_u) \in -N_{Viab(\underline{x})}(z(T_u))$ (see *e.g.* [33]).
- The control u satisfies the maximization condition:
$$u(t) \in \arg\max_{0 \leq \omega \leq \bar{u}} H(z_u(t), \lambda(t), p_0, \omega) \quad \text{a.e. } t \in [0, T_u]. \tag{3.7}$$

An *extremal* is a triplet $(z_u(\cdot), \lambda(\cdot), u(\cdot))$ satisfying (1.1)-(3.6)-(3.7). Moreover, as the system is autonomous and T_u is free, the Hamiltonian is equal to zero along any extremal trajectory. We say that the extremal is *normal* if $p_0 \neq 0$ and *abnormal* if $p_0 = 0$. Whenever an extremal trajectory is normal, we can always assume that $p_0 = -1$ (using that H and (3.6) are homogeneous). In view of the maximization condition in (3.7), we define the *switching function* ϕ as

$$\phi := -qy,$$

and we obtain the following optimal control law:

$$\left\{ \begin{array}{lll} \phi(t) > 0 & \Rightarrow & u(t) = \bar{u}, \\ \phi(t) < 0 & \Rightarrow & u(t) = 0, \\ \phi(t) = 0 & \Rightarrow & u(t) \in [0, \bar{u}]. \end{array} \right. \tag{3.8}$$

We call *switching time* t_c a time such that the switching function ϕ has non-constant sign in any neighborhood of t_c (and *switching point* for the corresponding state $z(t_c)$). From (3.8), we deduce that any switching time satisfies $\phi(t_c) = 0$. A direct computation shows that we have:

$$\dot{\phi}(t) = -p(t)x(t)y(t) \quad \text{a.e. } t \in [0, T_u].$$

Let us now explicit the transversality condition. Taking into account that $Viab(\underline{x})$ is non-smooth at the point $(\underline{x}, r_-)$, let w be the unit vector defined by $w := (\sin\psi, -\cos\psi)$ where $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is defined by

$$\tan\psi := \frac{(\underline{x}-m)r_-}{(r-r_-)\underline{x}},$$

and let $e_1 := (1, 0)$.

Lemma 3.1. *Suppose that $\bar{u} > \underline{x} - m$, i.e. that $Viab(\underline{x})$ has a non-empty interior. If $(x, y) \in B^0(\underline{x})$, we have:*

$$\begin{array}{lll} y \in (r_-, r] & \Rightarrow & N_{Viab(\underline{x})}(x, y) = \mathbb{R}_- \times \{0\}, \\ y = r_- & \Rightarrow & N_{Viab(\underline{x})}(x, y) = \{\alpha(\beta w - [1-\beta]e_1) \ ; \ (\alpha, \beta) \in \mathbb{R}_+ \times [0, 1]\}. \end{array}$$

Proof. The result is straightforward for $y \in (r_-, r]$ (note that for $y = r$, then $B^+(\underline{x})$ has a vertical tangent at the point $(\underline{x}, r)$).

Now, at the boundary point $(\underline{x}, r_-)$ of $Viab(\underline{x})$, the tangent cone is generated by the vectors $(0, 1)$ and $(\cos\psi, \sin\psi)$. The geometric computation of $N_{Viab(\underline{x})}(x, y)$ follows using that $Viab(\underline{x})$ is convex and that the normal cone to $Viab(\underline{x})$ at $(x, y) \in B^0(\underline{x})$ is the dual cone to the tangent cone to $Viab(\underline{x})$ at (x, y). □

Thanks to Pontryagin Principle, we can derive the following properties.

Proposition 3.2. *Let $u \in \mathcal{U}$ be an optimal control for (3.5) and $(z_u(\cdot), \lambda(\cdot), u(\cdot))$ the corresponding extremal trajectory defined over a time interval $[0, T_u]$. Then, the following properties hold true:*

(i) *The control u is bang-bang i.e. it satisfies $u(t) \in \{0, \bar{u}\}$ for a.e. $t \in [0, T_u]$ and:*

$$u(t) = \frac{\bar{u}}{2}\left(1 + \text{sign}(\phi(t))\right) \quad \text{a.e. } t \in [0, T_u]. \tag{3.9}$$

(ii) *The transversality condition on the adjoint vector at time T_u reads as follows (in the case where $\bar{u} > \underline{x} - m$ only):*

$$\begin{array}{lll} (x(T_u), y(T_u)) \in \{\underline{x}\} \times (r_-, r) & \Rightarrow & (p(T_u), q(T_u)) \in \mathbb{R}_+ \times \{0\}, \\ (x(T_u), y(T_u)) = (\underline{x}, r_-) & \Rightarrow & (p(T_u), q(T_u)) \in \{\alpha(-\beta w + (1-\beta)e_1)\ ;\ (\alpha, \beta) \in \mathbb{R}_+ \times [0, 1]\}. \end{array} \tag{3.10}$$

(iii) *If the extremal trajectory reaches the target at some point in $\{\underline{x}\} \times (r_-, r)$, then it is normal i.e. $p_0 \neq 0$.*

(iv) *If the extremal trajectory is abnormal, then any switching point lies on the line $\{y = r\}$.*

Proof. To prove (i), suppose that $\phi = 0$ on some time interval $[t_1, t_2]$. By differentiating ϕ w.r.t the time t, we obtain $\dot{q} = q = 0$ over $[t_1, t_2]$ implying $p = 0$ over $[t_1, t_2]$. From (3.6), we deduce that the adjoint vector λ is zero over $[0, T_u]$. We thus obtain a contradiction with the PMP using $H = 0$. This proves that (3.9) holds almost everywhere.

Lemma 3.1 together with the transversality condition $\lambda(T_u) \in -N_{Viab(\underline{x})}(z(T_u))$ straightforwardly implies (3.10) which proves (ii).

Let us now show (iii). Suppose by contradiction that $p_0 = 0$. Using that $H = 0$ and that $y(T_u) \neq r$, one obtains $p(T_u) = 0$. Thus we would have $p(T_u) = q(T_u) = 0$ and then $\lambda \equiv 0$ using (3.6). This contradicts the PMP as the pair $(\lambda(\cdot), p_0)$ would be zero.

Finally, suppose that the extremal is abnormal and let t_0 be a switching point implying $\phi(t_0) = q(t_0) = 0$. It follows that $p(t_0) \neq 0$ (otherwise the vector λ would be zero on $[0, T_u]$ and this would contradict the PMP). Now, suppose that $y(t_0) \neq r$, then we find that $p(t_0)x(t_0)(r - y(t_0)) \neq 0$ which again contradicts the PMP as one has $p_0 = 0$. Hence, we necessarily have $y(t_0) = r$ which proves (iv). □

Remark 3.2. *From (3.6) and the fact that $(\lambda(\cdot), p_0)$ is non-zero, the mapping $t \longmapsto (p(t), q(t))$ is always non-zero. Using a similar argument as in the proof of the first point of Proposition 3.2, one can prove that the zeros of ϕ are isolated.*

3.3 Optimal synthesis

We first analyze the behavior of the switching function $t \mapsto \phi(t)$, which is crucial in order to find an optimal control policy.

Lemma 3.2. *A normal extremal trajectory $(z(\cdot), \lambda(\cdot), u(\cdot))$ defined over $[0, T_u]$ satisfies the following properties:*

(i) *The switching function is solution of the ordinary differential equation (ODE):*

$$\dot{\phi}(t) = \frac{y(t)(m + u(t) - x(t))}{r - y(t)}\phi(t) - \frac{y(t)}{r - y(t)}, \quad \text{a.e. } t \in [0, T_u], \tag{3.11}$$

(ii) *At a time t_0 where $y(t_0) = r$, we have $\phi(t_0) \neq 0$ and:*

$$\phi(t_0) = \frac{1}{u(t_0) + m - x(t_0)}. \tag{3.12}$$

Proof. Let us first show that the set $S := \{t \in [0, T_u] \; ; \; y(t) = r\}$ is finite. If $t_0 \in S$, we have $q(t_0)y(t_0)(x(t_0) - m - u(t_0)) = 1$ which implies that $\dot{y}(t_0) \neq 0$, hence t_0 is isolated and thus S is finite. Using that $\phi = -qy$, $\dot{\phi} = -pxy$, and that $H = 0$, we get that (3.11) holds almost everywhere which proves (i). The proof of (ii) is straightforward combining $H = 0$ and $y(t_0) = r$. □

This Lemma leads to the following Proposition.

Proposition 3.3. *Let $(z(\cdot), \lambda(\cdot), u(\cdot))$ be a normal extremal trajectory defined over $[0, T_u]$. Then, the following properties hold true.*

(i) *If there exist two consecutive times $t_2 > t_1 > 0$ such that $y(t_1) = y(t_2) = r$, then the control u has exactly one switching time $t_c \in (t_1, t_2)$.*

(ii) *If, in addition, one has $x(t_1) > x(t_2)$, resp. $x(t_1) < x(t_2)$, then an optimal control satisfies $u = 0$, resp. $u = \bar{u}$ on (t_1, t_c) and then $u = \bar{u}$, resp. $u = 0$ on (t_c, t_2).*

Proof. From (3.12), the sign of $\phi(t_i)$, $i = 1, 2$ depends on the value of $x(t_i)$ compared to $u(t_0) + m$. Whenever the trajectory satisfies $y(t_1) = r$ with $x(t_1) > x^+_{\bar{u}}(\underline{x})$, we thus have $\phi(t_1) < 0$ implying $u = 0$. Using the inequality $x(t_2) < m$, we deduce that $\phi(t_2) > 0$, hence the trajectory necessarily has a switching point at some time $t_c \in (t_1, t_2)$. Now, from (3.11), one has $\dot{\phi}(t_c) = -\frac{y(t_c)}{r - y(t_c)} > 0$. Thus, the only possibility for the trajectory is to switch from $u = 0$ to $u = \bar{u}$. This shows the uniqueness of t_c in (t_1, t_2). If now $x(t_1) < x^+_{\bar{u}}(\underline{x})$, the same argumentation shows that there exists a unique switching time from $u = \bar{u}$ to $u = 0$ in (t_1, t_2). This ends the proof of the Proposition. □

We denote by γ the graph of the unique solution $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ of (1.1) backward in time starting from the point $(\underline{x}, r)$ associated to the myopic feedback control (3.1). Let τ_1 be the first time where $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ exits $K(\underline{x})$ and $\tau_2 > \tau_1$ be the first exit time of $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ of the set $\{(x, y) \in \mathcal{D} \; ; \; y \leq r\}$. Finally, let γ_1 be the restriction of $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ to the interval $[\tau_1, \tau_2]$. The optimal synthesis of the problem then reads as follows (see also Fig. 2).

Theorem 3.1. *Let (x_0, y_0) be an initial condition in $\mathcal{D} \backslash Viab(\underline{x})$.*

(i) *If $(x_0, y_0) \in \gamma$, then any optimal trajectory of (1.1) steering (x_0, y_0) to the target set is abnormal. The corresponding control is given by $u_m(\cdot)$ and switching points occur on the line $\{y = r\}$.*

(ii) *If $(x_0, y_0) \notin \gamma$, then any optimal trajectory of (1.1)steering (x_0, y_0) to the target set is normal. Moreover, if $u(\cdot)$ denotes the optimal control, there exists $p \in \mathbb{N}^*$, $s \in \{0, 1\}$, and a sequence of times $(\tau_k)_{0 \leq k \leq p}$ such that:*

- *We have $\tau_0 = 0 < \tau_1 < \cdots < \tau_{p-1} < \tau_p = T_u$ and τ_k is a switching time of u for $1 \leq k \leq p$.*
- *The optimal control u is given by*

$$u(t) = \frac{\bar{u}}{2}(1 + (-1)^{p-k-s}) \quad t \in (\tau_k, \tau_{k+1}), \quad 0 \leq k \leq p - 1. \tag{3.13}$$

- *If $y(T_u) \in (r_-, r)$, resp. $y(T_u) = r_-$, then $s = 1$, resp. $s = 0$.*

Proof. Let us prove (i) and take an initial condition on the curve γ. We already know from Proposition 3.2 that any trajectory starting on the curve γ and associated to the control u_m corresponds to an abnormal extremal trajectory. Indeed, such a trajectory remains on the curve γ, therefore switching points only occur on the axis $\{y = r\}$ implying that the trajectory is abnormal. We must prove that such an extremal is optimal. To do so, let us choose (x_0, y_0) on the curve γ, and let $(x(\cdot), y(\cdot), \lambda(\cdot), u(\cdot))$ be an optimal extremal trajectory steering (x_0, y_0) to $Viab(\underline{x})$. Let t_0 be defined as follows:

$$t_0 := \inf\{t \geq 0 \; ; \; \exists \varepsilon > 0 \, \forall \tau \in (t, t + \varepsilon) \quad (x(\tau), y(\tau)) \notin \gamma\}.$$

Suppose by contradiction that $t_0 < T_u$. As $(x(\cdot), y(\cdot), u(\cdot))$ is extremal, (3.7) implies that t_0 is necessarily a switching time from $u = 0$ to $u = \bar{u}$ or from $u = \bar{u}$ to $u = 0$. We argue that $y(t_0) \neq r$. Indeed, otherwise, we would have a contradiction with the definition of t_0 (as by definition, u_m switches on the line $\{y = r\}$). Hence, either we have $y(t_0) < r$ or $y(t_0) > r$. Now, the fact that $y(t_0) \neq r$ implies that the extremal trajectory is a normal one. Indeed, we cannot have $p(t_0) = 0$ by Cauchy-Lipschitz's Theorem, but as $y(t_0) \neq r$, we obtain that $p(t_0)x(t_0)(r - y(t_0)) \neq 0$ and thus p_0 must be non null. Suppose for instance that $y(t_0) < r$. By construction of t_0, this point is a switching time from $u = 0$ to $u = \bar{u}$ and we necessarily have $\dot{\phi}(t_0) \geq 0$. On the other hand, we obtain from (3.11) that

$$\dot{\phi}(t_0) = -\frac{y(t_0)}{r - y(t_0)} < 0,$$

and thus a contradiction. If $y(t_0) > r$, we obtain a similar contradiction with the sign of $\dot{\phi}$ at time t_0. This shows that $t_0 \geq T_u$, thus we have proved that the abnormal extremal trajectory starting from (x_0, y_0) with the control $u_m(\cdot)$ drives a solution of (1.1) optimally to the target.

Let us prove (ii). The first two properties follow from Proposition 3.2 and from the fact that the number of switching times of an optimal control is finite. Given an extremal trajectory $(x(\cdot), y(\cdot), \lambda(\cdot), u(\cdot))$ driving optimally $(x_0, y_0) \notin \gamma$ to $Viab(\underline{x})$, we consider two cases depending if $y(T_u) \in (r_-, r)$ or $y(T_u) = r_-$.

First case: $y(T_u) \in (r_-, r)$. From Proposition 3.2, we have $p_0 \neq 0$ *i.e.* the trajectory is normal. Now, as $\phi(T_u) = 0$ and $\dot{\phi}(T_u) = -\frac{y(T_u)}{r-y(T_u)} < 0$, we obtain that $u = \bar{u}$ in a left neighborhood of T_u. By using Proposition 3.3, we obtain that the extremal has exactly one switching time t_c between two consecutive instants $t_1 < t_2$ such that $y(t_1) = y(t_2) = r$. We thus obtain (3.13) by considering (1.1) backward in time from $t = T_u$ and by counting the number of times (denoted by $p - 1$ with $p \geq 1$) where the trajectory surrounds $Viab(\underline{x})$ before reaching (x_0, y_0). When $k = p - 1$, we obtain $u(t) = \frac{\bar{u}}{2}(1 + (-1)^{1-s})$, thus $s = 1$ as was to be proved.

Second case: $y(T_u) = r_-$. Suppose that the extremal is abnormal *i.e.* $p_0 = 0$. It follows that $q(T_u) > 0$. Otherwise, the transversality condition would imply $q(T_u) = 0$ and using $H = 0$ we would have $p(T_u) = 0$ and a contradiction with the PMP. We deduce that $\phi(T_u) < 0$ thus $u = 0$ in a left neighborhood of T_u. As the extremal is abnormal, switching points occur only on the line $\{y = r\}$. This shows that we have $u = 0$ on $[t_c, T_u]$ where t_c is the last time such that $y(t_c) = r$ before reaching $B^0(\underline{x})$. Thus, by integrating backward in time (1.1) from $(\underline{x}, r_-)$, we find that $u(\cdot) = u_m(\cdot)$ and that $(x_0, y_0) \in \gamma$ which is a contradiction. We have thus proved that the extremal optimal trajectory is normal. Finally, we have two cases depending on whether the optimal trajectory reaches $(\underline{x}, r_-)$ with either the control $u = 0$ or $u = \bar{u}$:

- The case where we have $u = \bar{u}$ at the terminal time T_u is similar to the first case $y(T_u) \in (r_-, r)$ above. Thus, the conclusion is obtained similarly as above.
- Now, suppose that we have $u = 0$ at the terminal time T_u. The trajectory necessary has a switching time on γ_1 (as it is normal). We thus obtain (3.13) by considering (1.1) backward in time from $t = T_u$ and by counting the number of times (denoted by $p - 1$ with $p \geq 1$) where the trajectory surrounds $Viab(\underline{x})$ before reaching (x_0, y_0). As $u = 0$ in a left neighborhood of T_u, we obtain $s = 0$.

□

When the viability kernel is reduced to the singleton $\{(\underline{x}, r)\}$ *i.e.* when $\bar{u} = \underline{x} - m$, Theorem 3.1 still holds true, even though there is no transversality condition on the terminal adjoint vector. This can be interpreted as the limiting case when the two extreme points of $B^0(\underline{x})$ collapse.

3.4 Discussion

Typical optimal trajectories are depicted on Fig. 2 (see Appendix for details on the numerical simulations).

We highlight the results of Theorem 3.1 and properties of optimal trajectories illustrated on Fig. 2 by the following remarks.

- From Theorem 3.1, any normal trajectory reaching $B_0(\underline{x})$ in its interior satisfies:

$$p - (2j + 1) \geq 0 \quad \text{and} \quad p - 2j \geq 0 \quad \Rightarrow \quad y(\tau_{p-2j+1}) > r \quad \text{and} \quad y(\tau_{p-2j}) < r.$$

- Any normal trajectory either reaches $B_0(\underline{x})$ with $u = \bar{u}$, or it reaches the point $(\underline{x}, r_-)$ with $u = 0$. In this latter case, an optimal trajectory switches from $u = \bar{u}$ to $u = 0$ on γ_1.
- Abnormal trajectories are contained in the curve γ and they are the only extremal trajectories for which switching points occur exactly on the line $\{y = r\}$.

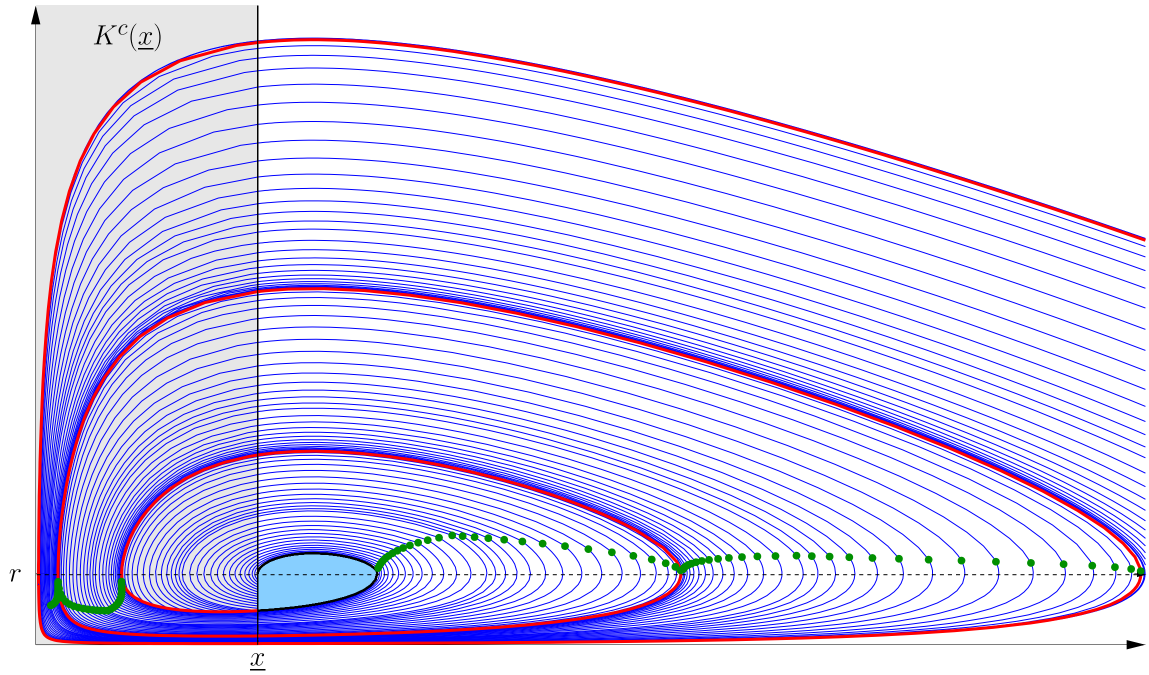

Figure 2: Examples of optimal trajectories for the minimal time to reach $Viab(\underline{x})$ (see Appendix for the numerical values). In blue, *normal* optimal trajectories reaching the target set at $B_0(\underline{x})$. In red, the *abnormal* optimal trajectory reaching $B_0(\underline{x})$ at the extreme point $(\underline{x}, r_-)$. Switches (from 0 to $\bar{u}$ or from $\bar{u}$ to 0) are represented by the green dots.

- Between two consecutive times $t_1 < t_2$ for which a normal extremal trajectory satisfies $y(t_1) = y(t_2) = r$, the extremal has exactly one switching time.
- When the initial condition z_0 is far away from the target, optimal trajectories have to surround the target set a number of times that is increasing with the distance of z_0 (see Fig. 2).
- The optimal control provided by Theorem 3.1 (ii) can be interpreted as a delayed perturbation of the myopic strategy (3.1): instead of switching on the line $\{y = r\}$, switching times are delayed and the corresponding switching points occur after the last intersection between the corresponding trajectory and the line $\{y = r\}$ (see the switching curves in red on Fig 2).

4 The minimal time crisis problem

To compare with the minimal time problem studied in Section 3, we first consider viability kernels $Viab(\underline{x})$ with non-empty interior, that is when the condition $\bar{u} > \underline{x} - m$ is fulfilled (see Proposition 2.1), and define the value function associated to the minimal time crisis problem, as

$$\theta(z_0) := \inf_{u \in \mathcal{U}} \int_0^\infty \mathbb{1}_{K(\underline{x})^c}(z_u(t, z_0)) \, \mathrm{d}t. \tag{4.1}$$

We first provide a general result that gives sufficient conditions for which the optimal solutions of the minimal time crisis problem (in infinite horizon) minimize the crisis time to reach the viability kernel in finite time. This result is of interest on itself and applies to the Lotka-Volterra prey predator model. This allows us to apply the Hybrid Maximum Principle (in finite time) and characterize optimal trajectories of the minimal time crisis problem. We then discuss and compare the optimal solutions with the ones of the minimal time to reach $Viab(\underline{x})$. Finally, we consider the case of empty viability kernels.

4.1 Equivalence of the time crisis problem with a finite horizon problem

In a general setup, we consider a control system:

$$\dot{y} = f(y, v), \tag{4.2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the dynamics, y is the state, and v the control that takes values in a non-empty closed subset Ω of $\mathbb{R}^m$. The admissible control set is classically

$$\mathcal{V}_{ad} := \{v : [0, +\infty) \to \Omega \; ; \; v \text{ meas.}\}.$$

We assume the usual regularity assumptions on the dynamics (see *e.g.* [13]):

(H1) The dynamics f is continuous w.r.t. (y, v), of class C^1 w.r.t. y and satisfies the linear growth condition: there exist $c_1 > 0$ and $c_2 > 0$ such that for all $y \in \mathbb{R}^n$ and all $v \in \Omega$, one has:

$$|f(y, v)| \leq c_1|y| + c_2, \tag{4.3}$$

where $|\cdot|$ is the euclidean norm in $\mathbb{R}^n$.

(H2) For any $y \in \mathbb{R}^n$, the velocity set $F(y) := \{f(y, v) \; ; \; v \in \Omega\}$ is a non-empty compact convex set.

It follows, by the Cauchy-Lipschitz's Theorem, that for any initial condition $y_0 \in \mathbb{R}^n$ and any $\tau \geq 0$, there exists a unique absolutely continuous solution of (4.2) defined over $[0, \tau]$ such that $y(0) = y_0$, denoted by $y_u(\cdot, y_0)$ hereafter.

Let K be a given closed subset of $\mathbb{R}^n$ and $K^c := \mathbb{R}^n \backslash K$ its complementary, the minimal time crisis problem states as

$$T(y_0) := \inf_{v \in \mathcal{V}_{ad}} \int_0^{+\infty} \mathbb{1}_{K^c}(y_v(t, y_0)) \, \mathrm{d}t, \tag{4.4}$$

where $\mathbb{1}_{K^c}$ is the characteristic function of K^c:

$$\mathbb{1}_{K^c}(x) := \begin{cases} 0 & \text{if} \quad x \in K, \\ 1 & \text{if} \quad x \notin K. \end{cases}$$

Under the additional hypothesis:

(H3) The viability kernel of K under the dynamics f, $Viab(K)$[2], is non-empty and for any initial condition in $\mathbb{R}^n$, there exists a control $v \in \mathcal{V}_{ad}$ steering y_0 to $Viab(K)$ in finite time,

one can consider the following optimal control problem:

$$\hat{T}(y_0) := \inf_{\tau \geq 0, v \in \mathcal{V}_{ad}} \left\{ \int_0^{\tau} \mathbb{1}_{K^c}(y_v(t, y_0)) \, \mathrm{d}t \; ; \text{ s.t. } y_v(\tau, y_0) \in Viab(K) \right\}. \tag{4.5}$$

The existence of an optimal control for both problems (4.4) and (4.5) follows from standard argumentation (see [17, 4]).

We introduce now an hypothesis on the crossing times:

(H4) There exists a number $\eta > 0$ such that for any pair of consecutive crossing times[3] $t_1 < t_2$ from K to K^c or from K^c to K, one has $t_2 - t_1 \geq \eta$,

which allows us to state the following equivalence result.

Proposition 4.1. *Suppose that assumptions* (H1)-(H4) *are satisfied. Then, for any $y_0 \in \mathbb{R}^n$ one has*

$$T(y_0) = \hat{T}(y_0).$$

Furthermore, for any y_0, the infimum in (4.5) *is reached for a finite τ.*

[2] The viability kernel of K under the dynamics f is defined as $Viab(K) := \{y_0 \in K \; ; \; \exists v(\cdot) \in \mathcal{V}_{ad}, \; y_v(t, z_0) \in K, \; \forall t \geq 0\}$.

[3] The definition of a crossing time in this setting is the same as the one introduced in Section 2 with K in place of $K(\underline{x})$.

Proof. As the value functions T and $\hat{T}$ are clearly identically equal to zero in $Viab(\underline{x})$, we consider $y_0 \notin Viab(K)$. It is known (see [17]) that one has $T(y_0) \leq \tilde{V}(y_0)$, where

$$\tilde{V}(y_0) := \inf_{v \in \mathcal{V}_{ad}} \{T_v \text{ s.t. } y_v(T_v, y_0) \in Viab(K)\}.$$

Moreover, Hypothesis (H3) implies $\tilde{V}(y_0) < +\infty$. Let $u^*(\cdot)$ be an optimal control for $T(y_0)$ and $y^*(\cdot, y_0)$ the associated solution starting from y_0. Define the time $\tau(y_0) \in \mathbb{R}_+ \cup \{+\infty\}$ by:

$$\tau(y_0) := \sup\{t \geq 0 \; ; \; y^*(t, y_0) \in K^c\},$$

and suppose by contradiction that $\tau(y_0) = +\infty$. As $T(y_0) < +\infty$, there exists $t_0 \geq 0$ such that $y^*(t_0, y_0) \in K$. Now, as $\tau(y_0) = +\infty$, there exists $t_1 \geq t_0$ such that t_1 is a crossing time from K to K^c. We now define t_2 as the first entry time $t > t_1$ of $y^*(\cdot, y_0)$ from K^c into K (t_2 exists as $T(y_0) < +\infty$). From (H4), we deduce that $t_2 - t_1 \geq \eta > 0$. By the Dynamic Programming Principle, $y^*(\cdot)$ is also optimal from $y^*(t_2)$ and one has $T(y^*(t_2)) \leq T(y_0) < +\infty$, $\tau(y^*(t_2)) = \tau(y_0) = +\infty$. Therefore, the same argument can be applied from $(t_2, y^*(t_2))$ and we obtain two increasing sequences of times $(t_{i,n})_n$, $i = 1, 2$, such that one has $t_{2,n} - t_{1,n} \geq \eta > 0$ for any $n \in \mathbb{N}$. This implies that one has $T(y_0) = +\infty$ and thus a contradiction. Therefore, we necessary have $\tau(y_0) < +\infty$ which implies that $y^*(t, y_0) \in K$ for any time $t \geq \tau(y_0)$ *i.e.* $y^*(\tau(y_0), y_0) \in Viab(K)$. It follows that

$$T(y_0) = \int_0^{\tau(y_0)} \mathbb{1}_{K^c}(y^*(t, y_0)) \, \mathrm{d}t \geq \hat{T}(y_0),$$

using that $y^*(\tau(y_0), y_0) \in Viab(K)$. On the other hand, let $(\hat{\tau}, \hat{v}(\cdot)) \in \mathbb{R}_+ \times \mathcal{U}_{ad}$ be an optimal pair for $\hat{T}(y_0)$. If $\hat{y}(\cdot, y_0)$ denotes the associated trajectory, we then have $\hat{y}(\hat{\tau}, y_0) \in Viab(K)$ with $\hat{v}$ defined over $[0, \hat{\tau}]$. Hence, we can extend $\hat{v}$ to a control function $\bar{u} \in \mathcal{V}_{ad}$ defined on $[0 + \infty)$ such that the associated trajectory $\bar{y}(\cdot, y_0)$ satisfies $\bar{y}(t, y_0) \in Viab(K)$ for any time $t \geq \hat{\tau}$. We then have

$$\hat{T}(y_0) = \int_0^{\hat{\tau}} \mathbb{1}_{K^c}(\hat{y}(t, y_0)) \, \mathrm{d}t = \int_0^{+\infty} \mathbb{1}_{K^c}(\bar{y}(t, y_0)) \, \mathrm{d}t \geq T(y_0),$$

and the conclusion follows.

Finally, to prove that the infimum in (4.5) is reached for a finite τ, we suppose by contradiction that this is not the case. As previously, we can then find two increasing sequences of times $(t_{i,n})_n$, $i = 1, 2$, such that one has $t_{2,n} - t_{1,n} \geq \eta$ for any $n \in \mathbb{N}$. Similarly, we find a contradiction with the fact that $\hat{T}(y_0)$ is finite, which ends the proof. □

In connection with Hypothesis (H4), it is relevant to recall the *chattering phenomenon*[4], following for instance [10, 35, 36].

Definition 4.1. *Given $y_0 \in \mathbb{R}^n$ and a solution $y(\cdot, y_0)$ of (4.2) defined over $[0, +\infty)$, we say that a chattering phenomenon occurs if there exist two sequences of times $(t_n^{out})_{n \geq 0}$, $(t_n^{in})_{n \geq 0}$ satisfying:*

- *For any $n \in \mathbb{N}$, t_n^{out} and t_n^{in} are two consecutive crossing time for $y(\cdot, y_0)$ from K to K^c and from K^c to K respectively.*
- *For any $n \in \mathbb{N}$, one has $t_n^{out} - t_n^{in} > 0$, and $t_n^{out} - t_n^{in} \to 0$ when $n \to +\infty$.*

Let us also recall the definition of *transverse* crossing times [4] (for convex sets K).

Definition 4.2. *Given an admissible trajectory $y_u(\cdot, y_0)$, a crossing time t_c (from K to K^c or from K^c to K) is transverse when there exists $\nu \in N_K(y_u(t_c, y_0))$ such that $\dot{y}_u(t_c, y_0) \cdot \nu \neq 0$.*

In other words, a transverse crossing time t_c is such that the trajectory does not hit the boundary of K tangentially while crossing K.

[4]In general, this terminology is used when an optimal control has an infinite number of switching points in a finite horizon (in presence of a singular arc of order 2), see Fuller's example [35].

4.2 Necessary optimality conditions

It has been proved in Section 3 that $Viab(\underline{x})$ can be reached from any initial condition $(x_0, y_0) \in \mathcal{D}$ (see Proposition 3.1). Therefore, as for (4.1), the Problem

$$\hat{\theta}(z_0) := \inf_{T\geq 0, u\in\mathcal{U}} \left\{ \int_0^T \mathbb{1}_{K(\underline{x})^c}(z_u(t,z_0))\ \mathrm{d}t,\ \text{s.t. } z_u(T,z_0) \in Viab(\underline{x}) \right\}, \tag{4.6}$$

has a solution. Moreover, Proposition 2.2 implies that for any pair of consecutive crossing times $t_1 < t_2$ from K to K^c and from K^c to K, one has

$$t_2 - t_1 \geq \delta := \frac{\ln\left(\frac{r}{r_-}\right)}{m + \bar{u}} > 0.$$

Hence, we can apply Proposition 4.1 and we get the equality

$$\theta(z_0) = \hat{\theta}(z_0),$$

for any $z_0 \in \mathcal{D}$. We are now in position to apply the Hybrid Maximum Principle [18, 23, 12] (in finite time) to Problem (4.6). As in the former work [4], we tackle the discontinuity of the integrand $\mathbb{1}_{K(\underline{x})^c}$ by considering the partition of the state space $\mathcal{D} = K(\underline{x}) \cup K(\underline{x})^c$. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated to (4.6) defined by:

$$H := H(x,y,p,q,p_0,u) = px(r-y) + qy(x-m-u) + p_0\mathbb{1}_{K(\underline{x})^c}(x,y).$$

If u is an optimal control of (4.6) and $(x_u(\cdot), y_u(\cdot))$ is the associated solution of (1.1), then the following optimality conditions are satisfied:

- There exist numbers $T \geq 0$, $p_0 \leq 0$ and a measurable function $\lambda(\cdot) := (p(\cdot), q(\cdot)) : [0,T] \to \mathbb{R}^2$ that is almost everywhere absolutely continuous, satisfying a.e. on $[0,T]$:

$$\left\{ \begin{array}{ll} \dot{p} & = p(y-r) - qy, \\ \dot{q} & = px + q(u+m-x). \end{array} \right. \tag{4.7}$$

- The control u satisfies the maximization condition:

$$u(t) \in \arg\max_{\omega\in[0,\bar{u}]} H(x(t), y(t), p(t), q(t), p_0, \omega) \quad \text{a.e. } t \in [0,T]. \tag{4.8}$$

- The Hamiltonian H is constant equal to zero along any extremal trajectory $(x(\cdot), y(\cdot), p(\cdot), q(\cdot), p_0, u(\cdot))$ satisfying (1.1)-(4.7)-(4.8) (we recall that the terminal time is free).
- The function $\lambda(\cdot)$ is discontinuous only at a crossing time t_c (from $K(\underline{x})$ to $K(\underline{x})^c$ or from $K(\underline{x})^c$ to $K(\underline{x})$) and one has

$$\lambda(t_c^+) - \lambda(t_c^-) \in N_{K(\underline{x})}(x(t_c), y(t_c)).$$

- The triplet $(p_0, p(\cdot), q(\cdot))$ is non identically null.
- As $(x(T), y(T)) \in Viab(\underline{x})$, one has $\lambda(T) \in -N_{Viab(\underline{x})}(x(T), y(T))$ *i.e.* $\lambda(T)$ satisfies (3.10).

As for the minimal time control problem, the switching function $\phi := -qy$ provides information on the optimal control. One obtains the following control law:

$$\left\{ \begin{array}{lll} \phi(t) > 0 & \Rightarrow & u(t) = \bar{u}, \\ \phi(t) < 0 & \Rightarrow & u(t) = 0, \\ \phi(t) = 0 & \Rightarrow & u(t) \in [0, \bar{u}]. \end{array} \right. \tag{4.9}$$

Moreover, the adjoint equation implies $\dot{\phi} = -pxy$.

As in Pontryagin's Principle, we say that an extremal $(x(\cdot), y(\cdot), p(\cdot), q(\cdot), p_0, u(\cdot))$ trajectory is *normal* if $p_0 \neq 0$ and is *abnormal* if $p_0 = 0$. Whenever an extremal trajectory is normal, we can always assume that $p_0 = -1$.

Following Definition 4.2 for the set $K(\underline{x})$, a crossing time t_c is *transverse* when one has $\dot{z}(t_c, z_0).e_1 \neq 0$, where e_1 denotes the vector $(1,0)$. We begin by two Lemmas that characterize crossing times.

Lemma 4.1. *Given a solution $z(\cdot, z_0)$ of (1.1), any crossing time t_c of $z(\cdot, z_0)$ such that $z(t_c, z_0) \notin Viab(\underline{x})$ is transverse.*

Proof. Suppose that a solution $z(\cdot, z_0) = (x(\cdot), y(\cdot))$ of (1.1) hits tangentially the boundary of $K(\underline{x})$ at some point $(\underline{x}, y)$. Then, we must have $\dot{z}(t_c, z_0) \cdot e_1 = 0$ which implies that $y(t_c) = r$. Hence, we obtain that $(x(t_c), y(t_c)) = (\underline{x}, r) \in Viab(\underline{x})$ which contradicts the hypothesis of the Lemma. Hence, any crossing time is transverse. □

Thanks to this Lemma, we can write the jump condition on the adjoint vector as follows (see [4]). Let t_c be a crossing time. Then, one has:

$$\begin{cases} x(t_c^-) > \underline{x} \text{ and } x(t_c^+) < \underline{x} \quad \Rightarrow \quad p(t_c^+) - p(t_c^-) = \dfrac{q(t_c)y(t_c)(u(t_c^+) - u(t_c^-)) - p_0}{\underline{x}(r - y(t_c))}, \\ x(t_c^-) < \underline{x} \text{ and } x(t_c^+) > \underline{x} \quad \Rightarrow \quad p(t_c^+) - p(t_c^-) = \dfrac{q(t_c)y(t_c)(u(t_c^+) - u(t_c^-)) + p_0}{\underline{x}(r - y(t_c))}, \end{cases} \tag{4.10}$$

and the function $q(\cdot)$ is (absolutely) continuous over $[0, T]$ whereas p is piece-wise (absolutely) continuous. We then obtain the following characterization of the jumps.

Lemma 4.2. *Let us consider an extremal trajectory $(x(\cdot), y(\cdot), p(\cdot), q(\cdot), p_0, u(\cdot))$.*

(i) *If the extremal is abnormal (i.e. $p_0 = 0$), then the adjoint vector $(p(\cdot), q(\cdot))$ is absolutely continuous.*

(ii) *If the extremal is normal (i.e. $p_0 < 0$), then a crossing time t_c is such that*

$$\begin{cases} x(t_c^-) > \underline{x} \text{ and } x(t_c^+) < \underline{x} \quad \Rightarrow \quad p(t_c^+) - p(t_c^-) = \dfrac{1}{\underline{x}(r - y(t_c))}, \\ x(t_c^-) < \underline{x} \text{ and } x(t_c^+) > \underline{x} \quad \Rightarrow \quad p(t_c^+) - p(t_c^-) = \dfrac{-1}{\underline{x}(r - y(t_c))}. \end{cases} \tag{4.11}$$

Proof. Let us first show that $q(t_c)y(t_c)(u(t_c^+) - u(t_c^-))$ is zero at any crossing time. The result is obvious if $q(t_c) = 0$. Now, if $q(t_c) < 0$, then $\phi > 0$ in a neighborhood of t_c, thus $u = \bar{u}$ in a neighborhood of t_c (recall that ϕ is continuous) so that $u(t_c^+) - u(t_c^-) = 0$. The same conclusion follows if $q(t_c) > 0$. Using the equality $q(t_c)y(t_c)(u(t_c^+) - u(t_c^-)) = 0$, one obtains straightforwardly (i) and (ii) from (4.10). □

We can now state our main result that characterize the optimal solutions of Problem (4.1).

Proposition 4.2. *Consider an optimal solution of Problem (4.1) defined over $[0, T]$.*

(1) *If $(x_0, y_0) \in \gamma$, then the optimal trajectory is abnormal and the optimal control is given by u_m. Switching points occur on the line $\{y = r\}$.*

(2) *If $(x_0, y_0) \notin \gamma$, then the optimal trajectory is normal. Moreover, the following properties hold true:*

  (i) *If τ is the last instant for which $y(\tau) = r$, then one has $u = \bar{u}$ over $[\tau, T]$.*

  (ii) *Any switching point in $K(\underline{x})^c \cap \{y > r\}$ is from $u = 0$ to $u = \bar{u}$.*

  (iii) *Any switching point in $K(\underline{x})^c \cap \{y < r\}$ is from $u = \bar{u}$ to $u = 0$.*

  (iv) *If a switching point occurs in $K(\underline{x})$ at an instant t_s, then we must have $y(t_s) = r$.*

Proof. First, notice that $\mathbb{1}_{K(\underline{x})^c}$ is zero in $K(\underline{x})$, hence any switching time t_s that occurs in the set $K(\underline{x})$ necessarily satisfies $y(t_s) = r$.

To prove (1), we suppose by contradiction (as in the proof of the first point in Theorem 3.1) that the trajectory starting from $(x_0, y_0) \in \gamma$ contains a switching point t_s such that $q(t_s) = 0$ and $y(t_s) \neq r$. We may suppose that t_s is the first one satisfying $y(t_s) \neq r$. Hence, the trajectory is normal (otherwise, the condition $H = 0$, $q(t_s) = 0$ and $y(t_s) \neq r$ would imply a contradiction). Finally, suppose that $y(t_s) > r$. Thus, t_s is a switching point from $u = \bar{u}$ to $u = 0$, *i.e.* $\phi(t_s) = 0$ implying also $\dot{\phi}(t_s) \leq 0$. From (3.11) (which remains valid in $K(\underline{x})^c$) we deduce that $\dot{\phi}(t_s) < 0$ which is a contradiction. If now $y(t_s) < r$, then t_s is by construction a switching point from $u = 0$ to $u = \bar{u}$, and we obtain a similar contradiction. Hence, we deduce that the optimal control is u_m and that the corresponding trajectory is abnormal.

To prove (2), we use the transversality condition (3.10) which implies that either $p(T) > 0$ and $q(T) = 0$, thus $\dot{\phi}(T^-) \leq 0$ (when $y(t) \in (r_-, r)$) or $q(T) < 0$ (when $y(t) = r_-$). Suppose that the trajectory reaches $B^0(\underline{x})$ in its interior. Then, one must have $p(T)\underline{x}(r - y(T)) + p_0 = 0$, hence $p_0 < 0$ and the trajectory is normal (otherwise we would have $p(\cdot)$ and $q(\cdot)$ identically equal to zero which contradicts the hybrid maximum principle). Suppose now that the trajectory reaches the point $(\underline{x}, r_-)$ at time $t = T$. Then, if the trajectory is abnormal, it must coincide with the curve γ (as switching points only occur on the line $\{y = r\}$. Thus we obtain a contradiction with $(x_0, y_0) \notin \gamma$.

It follows that one has $\phi > 0$ in a left neighborhood of T and thus there exists $\tau > 0$ such that the control satisfies $u = \bar{u}$ over $[\tau, T]$ which gives (i). Now, thanks to (3.11), we obtain that any switching point in $K(\underline{x})^c \cap \{y > r\}$, resp. $K(\underline{x})^c \cap \{y < r\}$ is from $u = 0$ to $u = \bar{u}$, resp. from $u = \bar{u}$ to $u = 0$ implying (ii)-(iii). Finally, note that in the set $K(\underline{x})$, the Hamiltonian writes

$$H = px(r - y) + qy(x - m - u) = 0,$$

implying that $y(t_s) = r$ whenever $q(t_s) = 0$. This ends the proof. □

We then deduce the following result.

Corollary 4.1. *Let $(x(\cdot), y(\cdot))$ be a normal extremal trajectory defined over a time interval $[t_0, t_2]$ such that :*

- *At time t_0, one has $y(t_0) = r$, $(x(t_0), y(t_0)) \in K(\underline{x})\backslash Viab(\underline{x})$, and t_0 is a switching point from $u = 0$ to $u = \bar{u}$.*
- *There exists $t_1 \in (t_0, t_2)$ such that $x(t_1) = x(t_2) = \underline{x}$ with $(x(t_2), y(t_2)) \notin Viab(\underline{x})$ and $t_1 < t_2$ are two consecutive crossing times from $K(\underline{x})$ to $K(\underline{x})^c$ and from $K(\underline{x})^c$ to $K(\underline{x})$ respectively.*

Then, the trajectory has exactly one switching time $t_s \in (t_1, t_2)$ from $u = \bar{u}$ to $u = 0$ such that $y(t_s) < r$.

Proof. As t_0 is a switching point such that $(x(t_0), y(t_0)) \in K(\underline{x})\backslash Viab(\underline{x})$, we necessarily have $u = \bar{u}$ over $[t_0, t_1)$. Now, using that $q(\cdot)$ is continuous and that no switching points occur in the interval $(t_0, t_1]$, we must have $q(t_1) < 0$ thus $\phi(t_1) > 0$, and consequently one has $\phi > 0$ in a right neighborhood of t_1. From Proposition 4.2, the trajectory cannot switch from $u = \bar{u}$ to $u = 0$ in the set $K(\underline{x})^c \cap \{y > r\}$. Recall that one has $\phi > 0$ when the trajectory crosses the line $\{y = r\}$. Suppose now that the trajectory does not switch in the set $K(\underline{x})^c \cap \{y < r\}$. Then, one has $\phi > 0$ until $t = t_2$. At this time, the trajectory enters $K(\underline{x})$ with $\phi > 0$ (as q is continuous), thus we have $u = \bar{u}$ until that the trajectory again reaches the point $(x(t_0), r)$. Indeed, recall that switching points in $K(\underline{x})$ only occur on the axis $\{y = r\}$. This contradicts the optimality of the trajectory. Hence, there must exist a switching point in the set $K(\underline{x})^c \cap \{y < r\}$ as was to be proved. □

4.3 Discussion

Typical optimal trajectories are depicted on Fig. 3 (see Appendix for details on the numerical simulations). Switching points are represented in black. Switching curves consist of the collection of these points.

Let us underline the following features of the optimal trajectories:

- Switching points in $K(\underline{x})\backslash Viab(\underline{x})$ only occur on the line $\{y = r\}$.
- Between two consecutive crossing times from K to K^c and from K^c to K, there exists at least one switching point in $K(\underline{x})$. Notice that the terminal time (*i.e.* the first time where an optimal trajectory enters $Viab(\underline{x})$) is not a crossing time.
- Abnormal trajectories are contained in the curve γ and they are the only extremal trajectories for which switching points occur exactly on the line $\{y = r\}$ (both in $K(\underline{x})$ and in $K(\underline{x})^c$).

In order to compare solutions of Problems (3.5) and (4.1), we consider the subset $\mathcal{E} \subset \mathcal{D}$ containing all the points of $\mathcal{D}$ that can reach $Viab(\underline{x})$ with the constant control $u = \bar{u}$ and such that the corresponding optimal trajectory does not contain any switching point. Moreover, let $\mathcal{F}$ be defined by:

$$\mathcal{F} := \mathcal{E} \cup \gamma.$$

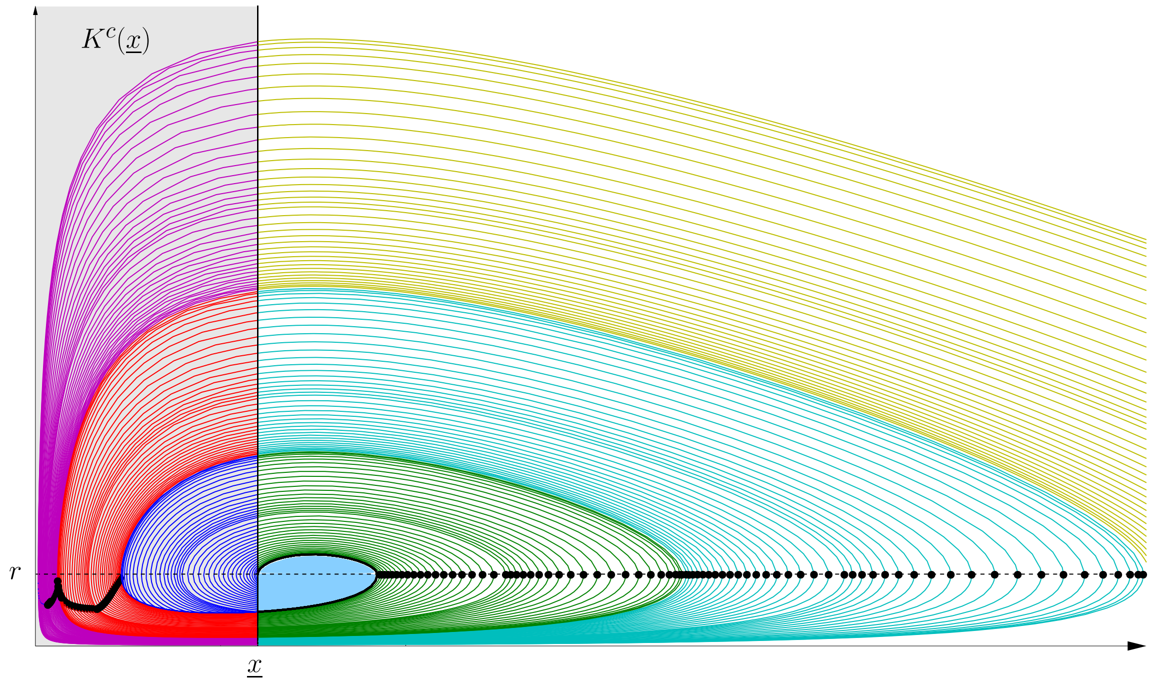

Figure 3: Examples of normal optimal trajectories for the minimal time crisis (see Appendix for the numerical values). Color of the trajectories are changed at each crossing time. Switching points are represented by the black dots.

Proposition 4.3. *Given an initial condition $z_0 = (x_0, y_0) \in \mathcal{D}$, we have the two following cases:*

(i) *If $z_0 \in \mathcal{F}$, optimal solutions for Problems* (3.5) *and* (4.1) *coincide.*

(ii) *If $z_0 \in \mathcal{D}\backslash\mathcal{F}$, then one has*

$$\theta(z_0) < J(u^\star, z_0),$$

where $u^\star$ denotes an optimal control for (3.5).

Proof. The proof of (i) is immediate from Theorem 3.1 and Proposition 4.2. Take now an initial condition $z_0 = (x_0, y_0) \in \mathcal{D}\backslash\mathcal{F}$ and let $u^\star$ be an optimal control for (3.5). As $u^\star \in \mathcal{U}$ is admissible for (4.1), we have $\theta(z_0) \leq J(u^\star, z_0)$. If we have $\theta(z_0) = J(u^\star, z_0)$, then $u^\star$ is necessarily an optimal control for (4.1). The switching points in $K(\underline{x})$ of the associated trajectory occur on the line $\{y = r\}$ only from Proposition 4.2. As $z_0 \in \mathcal{D}\backslash\mathcal{F}$ the trajectory necessarily has at least one switching point in the set $K(\underline{x})$ from Proposition 3.3 at some time t_c. At time t_c, we must have $y(t_c) > r$ as the corresponding trajectory is a normal extremal (see Theorem 3.1). This gives a contradiction and proves (ii). □

4.4 The case of empty viability kernel

We study here properties of the minimal time crisis problem when the viability kernel $Viab(\underline{x})$ is empty, that is when the condition $m + \bar{u} < \underline{x}$ is fulfilled (see Proposition 2.1).

Proposition 4.4. *When $m + \bar{u} < \underline{x}$, the following properties hold true.*

(i) *There is no chattering phenomenon for Problem 4.1 in the sense of Definition 4.1 for system* (1.1).

(ii) *For any $z_0 \in \mathcal{D}$, one has $\theta(z_0) = +\infty$.*

Proof. First, suppose that there exist two sequences of times (t_n^1) and (t_n^2) satisfying:

- both sequences (t_n^1) and (t_n^2) are increasing with $t_n^1 < t_n^2$ for any $n \in \mathbb{N}$ and such that $t_n^2 - t_n^1 \to 0$ when $n \to +\infty$.

- for any $n \in \mathbb{N}$, t_n^1, resp. t_n^2 is a crossing time from $K(\underline{x})$ to $K(\underline{x})^c$, resp. from $K(\underline{x})^c$ to $K(\underline{x})$,
- for any time $t \in (t_n^1, t_n^2)$, one has $(x(t), y(t)) \in K(\underline{x})^c$.

As $m + \bar{u} < \underline{x}$, there exists $\varepsilon > 0$ such that $m + \bar{u} + \varepsilon < \underline{x}$. Let us now integrate (1.1) over $[t_n^1, t_n^2]$. Since $u(t) \leq \bar{u}$ for any time t, one has:

$$\int_{y_n^1}^{y_n^2} \frac{\mathrm{d}y}{y} = \int_{t_n^1}^{t_n^2} (x(t) - m - u(t))\ \mathrm{d}t \geq \int_{t_n^1}^{t_n^2} (x(t) - m - \bar{u})\ \mathrm{d}t > \int_{t_n^1}^{t_n^2} (x(t) - \underline{x} + \varepsilon)\ \mathrm{d}t,$$

where $y_n^1 := y(t_n^1)$ and $y_n^2 := y(t_n^2)$, or equivalently

$$\varepsilon(t_n^2 - t_n^1) + \ln\left(\frac{y_n^1}{y_n^2}\right) < \int_{t_n^1}^{t_n^2} (\underline{x} - x(t))\ \mathrm{d}t. \tag{4.12}$$

As one should have at the crossing times $\dot{x}(t_n^1) > 0$ and $\dot{x}(t_n^2) < 0$, one immediately obtains from equations (1.1) the inequalities $y_n^1 > r > y_n^2$, and thus $\ln\left(\frac{y_n^1}{y_n^2}\right)$ is a positive number, for any n. We then deduce from (4.12) that $\ln\left(\frac{y_n^1}{y_n^2}\right)$ has to tend to 0 when n tends to $+\infty$, which implies that y_n^1 and y_n^2 both tend to r. Therefore, $y(\cdot)$ is uniformly bounded on the intervals $[t_n^1, t_n^2]$, say by a number $C > 0$. It follows that one has

$$\forall n \in \mathbb{N},\ \forall t \in [t_n^1, t_n^2], \quad |\dot{x}(t)| \leq x(t)(r + y(t)) \leq A,$$

where $A := \underline{x}(r + C)$. We then deduce that $|x(t) - \underline{x}| \leq A|t - t_n^1|$ for any $n \in \mathbb{N}$ and any time $t \in [t_n^1, t_n^2]$. Finally one has from (4.12)

$$\varepsilon(t_n^2 - t_n^1) + \ln\left(\frac{y_n^1}{y_n^2}\right) < \frac{A}{2}(t_n^2 - t_n^1)^2$$

which gives a contradiction for large values of n. This concludes the proof (i).

To prove (ii), suppose by contradiction that there exists $z_0 \in \mathcal{D}$ such that $\theta(z_0) < +\infty$ and let $(x(\cdot), y(\cdot), u(\cdot))$ be an optimal solution. Then, $x(\cdot)$ has an infinite number of crossing times. Otherwise, either $x(\cdot)$ remains in $K(\underline{x})$ after a certain time $\tau \geq 0$, which is a contradiction with $Viab(\underline{x}) = \emptyset$, or it remains in $K(\underline{x})^c$ after a certain time $\tau \geq 0$, which is a contradiction with $\theta(z_0) < +\infty$. Without any loss of generality, we can then suppose that there exist two sequences of times (t_n^1), (t_n^2) as above. We then obtain a contradiction as previously, which ends the proof. □

Remark 4.1. (i) *This result shows that even if $\theta(z_0) = +\infty$, then no chattering phenomenon occurs.*
(ii) *Proposition 4.4 implies that when $m + \bar{u} < \underline{x}$ one has $J(u, z_0) = +\infty$ for any $u \in \mathcal{U}$ and $z_0 \in \mathcal{D}$. However, it is possible to study the minimal time crisis problem restricted to a given finite horizon (see* [4, 5]*) and to characterize optimal controls in the same way.*

5 Conclusion

In this work, we have provided an exact determination of the viability kernel for the Lotka-Volterra prey-predator model when the control acts as a mortality term on the predators. We have characterized optimal controls for both minimum time to reach the viability kernel, and minimal time crisis problems. We have also shown that no chattering occurs even if the time crisis function is unbounded. To overcome the difficulties of the infinite horizon, we have also proposed in a general framework an equivalent formulation of the time crisis problem over a finite horizon when the viability kernel is non-empty under additional hypotheses (see Proposition 4.1). For the prey-predator model that we consider, it appears that optimal trajectories for both Problems (1.2) and (1.3) may enter and leave the set $K(\underline{x})$ an arbitrary large number of times (depending on the initial condition) before reaching $Viab(\underline{x})$. Thanks to the Pontryagin and Hybrid Maximum principles, we have then characterized a subset of the state space such that any optimal trajectory steering an initial condition from this set to the viability kernel spends more time outside $K(\underline{x})$ than an optimal trajectory for the minimal time crisis problem does. The complementary of this set contains initial conditions for which Problems (1.2)-(1.3) are equivalent, and in particular it contains the set of initial conditions for which optimal trajectories are abnormal. The methodology we have deployed here could be applied to other controls of the Lotka-Volterra equations or more general prey-predator models.

In terms of state constraints (here the constraint is expressed as $z_u(t, z_0) \in K(\underline{x})$), the optimal control for the minimum time crisis problem guarantees *less* violation of the state constraint than the one for the minimal time problem to reach $Viab(\underline{x})$. Therefore, the minimal time crisis function appears to be an interesting alternative to the strategy which consists in steering a system in minimal time to the viability kernel, although it presents some technicalities due to the discontinuity of the Hamiltonian.

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7 Appendix: numerical simulations

The parameters that have been used to perform Fig. 1, 2 and 3 are given in Table 1. The numerical simulations

r	m	$\bar{u}$	$\bar{x}$
1	1	0.5	1.2

Table 1: Parameters for Fig. 1, 2 and 3.

for obtaining Fig. 2 and 3 have been conducted as follows. Given a terminal time $T > 0$, we consider the state-adjoint system backward in time:

$$\left\{\begin{array}{ll} \dot{x} & = -x(r-y), \\ \dot{y} & = -y(x-m-u), \\ \dot{p} & = -p(y-r)+qy, \\ \dot{q} & = -px-q(u+m-x), \end{array}\right. \tag{7.1}$$

over $[0, T]$ together with the control law $u(t) = \text{sign}(-q(t))$ for a.e. $t \in [0, T]$ obtained from (4.8). The initialization makes use of the transversality condition (3.10) and is explained below.

Numerical determination of the extremal trajectories for the minimum time problem (3.5). Let $y_0 \in [r_-, r)$.
First case. If $y_0 \in (r_-, r)$, then $q(0) = 0$ and $p(0) = \frac{1}{\underline{x}(r-y_0)}$ (thanks to $H = 0$). Thus, (7.1) is initialized by the quadruple:

$$\left(\underline{x}, y_0, \frac{1}{\underline{x}(r-y_0)}, 0\right). \tag{7.2}$$

Second case. If $y_0 = r_-$, then there exists $\alpha \geq 0$ and $\beta \in [0, 1]$ such that

$$(p(0), q(0)) = \alpha(1-\beta(1+w_1), -\beta w_2) \quad \text{and} \quad \alpha = \frac{1}{\underline{x}(r-r_-)(1-\beta(1+w_1)) - \beta w_2 r_-(\underline{x}-m)}, \tag{7.3}$$

using the fact that the Hamiltonian is zero along any extremal trajectory. The system (7.1) is then initialized by the quadruple:

$$(\underline{x}, r_-, p(0), q(0)), \tag{7.4}$$

with $(p(0), q(0))$ and α given by (7.3). Notice that in this case, the value of $\beta \in [0, 1]$ is a parameter (as $K(\underline{x})$ is non-smooth at $(\underline{x}, r_-)$, there exist infinitely many extremal trajectories arising from $(\underline{x}, r_-)$).

Numerical determination of the extremal trajectories for the minimum time crisis problem (4.1).

The initialization of (7.1) is the same as for Problem (3.5). Moreover, the equation (7.1) remains valid as long as the trajectory does not belong to the boundary of $K(\underline{x})$. We thus impose the following condition:

$$\begin{cases} x(t_c^-) < \underline{x} \text{ and } x(t_c^+) > \underline{x} & \Rightarrow \quad p(t_c^+) - p(t_c^-) = \frac{1}{\underline{x}(r-y(t_c))}, \\ x(t_c^-) > \underline{x} \text{ and } x(t_c^+) < \underline{x} & \Rightarrow \quad p(t_c^+) - p(t_c^-) = \frac{-1}{\underline{x}(r-y(t_c))}, \end{cases} \tag{7.5}$$

at each crossing time t_c (recall that according to the Hybrid Maximum Principle applied on Problem (4.8) only p is discontinuous). Finally, the plots of optimal trajectories for (3.5) or (4.1) have been obtained as follows:

- Take $N \in \mathbb{N}^*$ and let $y_0^k := r_- + \frac{(k-1)}{N}(r - r_-)$ for $k = 1 \cdots N$.
- If $k = 1$ then choose $\beta_i = \frac{(i-1)}{N}$ for $i = 1 \cdots N$ and initialize (7.1) with (7.4) where β is replaced by β_i.
- If $k > 1$ initialize (7.1) with (7.2) where y_0 is replaced by y_0^k.

For both problems, the numerical integration of (7.1) is stopped when $t = T$ (with T chosen sufficiently large). Any zero of the switching function ϕ (or equivalently q) during the numerical integration is marked by a dot point on the picture. These points correspond to switching points in the state space and to the switching curves (*i.e.* the loci where the control switches either from $u = 0$ to $u = \bar{u}$ or from $u = \bar{u}$ to $u = 0$).

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