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Summary We derive a theory of thin linearly quasicrystalline plates by studying the limit behavior of a three-dimensional flat body as its thickness tends to zero. We exhibit the existence of a surprisingly high number of models, each of them linked to a specific set of boundary conditions. As such, these results show that quasicrystals behave as smart materials.

INTRODUCTION

Here we perform an asymptotic modeling of linear quasicrystalline plates by regarding its thickness as a small parameter denoted by ε. We study the behavior of the solution of the physical problem when ε tends to 0. We show that depending on the type of boundary conditions, 26 different models indexed by a triplet $p = (p_1, p_2, p_3) \in \{0, 1, 2\}^3$ appear at the limit. Comparing to our previous studies devoted to the mathematical modeling of thin plates in the framework of multi-physical couplings [1]-[3], this number is stunning. This multiplication of models, however, has its roots in the very structure of quasicrystals. As shown in [4], the constitutive law of quasicrystalline media present a coupling between two different kinds of displacement fields: the *phonon* field, denoted by u, and the *phason* field, denoted by w (see (1) below). Whereas phonons are related to translation of atoms and therefore to classical elastic displacements, phasons are associated with atomic rearrangements and appear in the constitutive equations only through their (noon symmetrized) gradients. This is the main cause of the huge amount of models we obtain for a single quasicrystalline thin plate. Provided that we manage to control and/or measure the phason fields, quasicrystals may be used to design new kinds of smart structures.

SETTING THE PROBLEM

We will denote phonon fields by the letters u, v and v' while the phason fields will be denoted by w, ψ and ψ'. Depending on the nature of our formulation, these letters and some other symbols may be indexed by ε which stands for the thickness of the plate, seen as a parameter. The reference configuration of a linearly quasicrystalline thin plate is the closure in $\mathbb{R}^3$ of the set $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ whose outward unit normal is n^ε and where ω is a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary $\partial\omega$. Used as indices, letters i and j take their values in $\{1, 2, 3\}$ while α and β take their values in $\{1, 2\}$. With classical notations and a suitable partition of the plate (see [5] and [6]), the equations determining the physical state $s^\varepsilon = (u^\varepsilon, w^\varepsilon)$ at equilibrium are:

$$\mathcal{P}(\Omega^\varepsilon) \begin{cases} \text{div}\, \sigma^\varepsilon + f^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad \sigma^\varepsilon n^\varepsilon = F^\varepsilon \text{ on } \Gamma^\varepsilon_{uN}, \quad u^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{uD}, \\ \text{div}\, \tau^\varepsilon + g^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad \tau^\varepsilon_{ij} n^\varepsilon_j = G^\varepsilon_i \text{ on } \Gamma^\varepsilon_{wN,i}, \quad w^\varepsilon_i = w^\varepsilon_{0_i} \text{ on } \Gamma^\varepsilon_{wD,i}, \quad i = 1, 2, 3, \\ (\sigma^\varepsilon, \tau^\varepsilon) = Q^\varepsilon(x)(e(u^\varepsilon), \nabla w^\varepsilon) \text{ in } \Omega^\varepsilon, \end{cases}$$

where u^ε, w^ε, σ^ε and τ^ε respectively stand for the phonon field, the phason field, the phonon stress tensor and the phason stress tensor and where the operator Q^ε is such that:

$$\sigma^\varepsilon_{ij} = C^\varepsilon_{ijkl}\, e_{kl}(u^\varepsilon) + R^\varepsilon_{ijkl}\, (\nabla w^\varepsilon)_{kl}, \quad \tau^\varepsilon_{ij} = R^\varepsilon_{klij}\, e_{kl}(u^\varepsilon) + K^\varepsilon_{ijkl}\, (\nabla w^\varepsilon)_{kl}. \tag{1}$$

In these constitutive equations, C^ε_{ijkl}, K^ε_{ijkl} and R^ε_{ijkl} stand for the phonon, phason and phonon-phason coupling coefficients, respectively.

Without specifying the functional framework, we introduce a bilinear form m^ε:

$$m^\varepsilon(\xi, \zeta) = m^\varepsilon((v, \psi), (v', \psi')) = \int_{\Omega^\varepsilon} Q^\varepsilon(e(v), \nabla\psi) \cdot (e(v'), \nabla\psi')\, dx^\varepsilon, \tag{2}$$

and a linear form L^ε:

$$L^\varepsilon(\xi) = L^\varepsilon((v, \psi)) = \int_{\Omega^\varepsilon} (f^\varepsilon \cdot v + g^\varepsilon \cdot \psi)\, dx^\varepsilon + \int_{\Gamma^\varepsilon_{uN}} F^\varepsilon \cdot v\, ds^\varepsilon + \sum_{i=1}^{3} \int_{\Gamma^\varepsilon_{wN,i}} G^\varepsilon_i\, \psi_i\, ds^\varepsilon. \tag{3}$$

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The coupled physical problem then takes the form

$$\mathcal{P}(\Omega^\varepsilon) \quad : \quad \text{Find } s^\varepsilon = (u^\varepsilon, w^\varepsilon) \text{ such that } m^\varepsilon(s^\varepsilon, \xi) \,=\, L^\varepsilon(\xi), \quad \forall\, \xi.$$

With the classical assumptions of regularity of the loading and of uniform ellipticity and boundedness of elastic operators, it is possible to show that this problem has a unique solution. To derive simplified and accurate models, the true question is to study the behavior of s^ε when ε, *regarded as a parameter*, tends to zero.

THE MODELS

It is possible to show that 26 different limit behaviors (indexed by a triplet $p = (p_1, p_2, p_3)$ of $\{1, 2, 3\}^3 \setminus \{0, 0, 0\}$) appear, according to both the type and magnitude of the boundary conditions in $\mathcal{P}(\Omega^\varepsilon)$. Classicaly (see [7]) we come down to a fixed open set $\Omega = \omega \times (-1, 1)$ through the mapping π^ε:

$$x = (x_1, x_2, x_3) \in \overline{\Omega} \,\mapsto\, \pi^\varepsilon x = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega}^\varepsilon. \tag{4}$$

To get physically meaningful results, we have to make various kinds of assumptions. The more significant deals with the magnitude of the electromechanical loading:

$$\begin{cases} f^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon\, f_\alpha(x),\ f^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^2\, f_3(x),\ \forall\, x \in \Omega, \\ F^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon^2\, F_\alpha(x),\ F^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^3\, F_3(x),\ \forall\, x \in \Gamma_{uN} \cap \Gamma_\pm, \\ F^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon\, F_\alpha(x),\ F^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^2\, F_3(x),\ \forall\, x \in \Gamma_{uN} \cap \Gamma_{lat} \\ g^\varepsilon_i(\pi^\varepsilon x) = \varepsilon^{2-p_i}\, g_i(x),\ \forall\, x \in \Omega, \\ G^\varepsilon_i(\pi^\varepsilon x) = \varepsilon^{3-p_i}\, G_i(x),\ \forall\, x \in \Gamma_{wN,i} \cap \Gamma_\pm, \\ G^\varepsilon_i(\pi^\varepsilon x) = \varepsilon^{2-p_i}\, G_i(x),\ \forall\, x \in \Gamma_{wN,i} \cap \Gamma_{lat} \\ w^\varepsilon_{0_i}(\pi^\varepsilon x) = \varepsilon^{p_i}\, w_{0_i}(x),\ \forall\, x \in \Omega. \end{cases}$$

Also, with the true physical state $s^\varepsilon = (u^\varepsilon, w^\varepsilon)$ defined on Ω^ε, we associate a *scaled* physical state $s_p(\varepsilon) = (u(\varepsilon), w_p(\varepsilon))$ defined on Ω by:

$$u^\varepsilon_\alpha(x^\varepsilon) = \varepsilon\, u(\varepsilon)_\alpha(x), \quad u^\varepsilon_3(x^\varepsilon) = u(\varepsilon)_3(x), \quad w^\varepsilon_i(x^\varepsilon) = \varepsilon^{p_i}\, w_p(\varepsilon)_i(x), \quad \forall\, x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon, \tag{5}$$

We therefore build a family of variational problems indexed by ε. The asymptotic analysis of this family is processed classically: a priori estimates, week convergence, strong convergence. Without spelling out the topology, one of the main strong convergence result of this analysis is:

$$s_p(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \overline{s}_p. \tag{6}$$

To get physically meaningful results it is necessary to define a physical state $\overline{s}^\varepsilon_p$ over the genuine plate Ω^ε through the descaling $\overline{s}^\varepsilon_p(\pi^\varepsilon x) = \overline{s}_p(x)$, for all x in Ω. This physical state living in the genuine quasicrystalline plate is the unique solution of a problem posed over the reference configuration, namely:

$$\overline{\mathcal{P}}(\Omega^\varepsilon)_p : \text{Find } \overline{s}^\varepsilon_p \text{ such that } \int_{\Omega^\varepsilon} \widetilde{Q}^\varepsilon_p(x)\, k(\overline{s}^\varepsilon_p)^0_p \cdot k(\xi)^0_p\, dx = L^\varepsilon(\xi),\ \forall\, \xi.$$

The detailed definition of $k(\overline{s}^\varepsilon_p)^0_p$ cannot be given here. However, it corresponds to the effective components of the linearized tensor of small phononic strains and of the gradient of the phason field. The algebraic operations that are associated with these components lead to the explicit expression of $\widetilde{Q}^\varepsilon_p$ which gives the limit constitutive law.

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