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A thermomechanical finite element for the analysis of rectangular laminated beams

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Abstract

A new three-noded thermomechanical beam finite element is derived for the analysis of laminated beams. The mechanical part is based on a refined model. The representation of the transverse shear strain by cosine function allows avoiding shear correction factors. This kinematics accounts for the interlaminar continuity conditions at the interfaces between the layers, and the boundary conditions at the upper and lower surfaces of the beam. A conforming FE approach is carried out using Lagrange and Hermite interpolations. Concerning the thermal part, a layerwise approach through the thickness is used. Moreover, the continuity of the transverse component of the thermal flux is constrained between the layers. It presents the advantage of reducing the number of temperature unknowns, and improve the results at the interface between two adjacent layers.

Mechanical, thermal and thermomechanical tests are presented in order to evaluate the capability of this new finite element to give accurate results with respect to elasticity or finite element reference solutions. Both convergence velocity and accuracy are discussed.

Keywords: Thermomechanical coupling; Multilayered beam; Finite element; Higher order transverse shear; Layerwise approach

1. Introduction

Composite and sandwich structures are widely used in industrial field due to their excellent mechanical properties. In this context, they can be submitted to severe conditions which imply to take into account thermal effects. In fact, they can play an important role on the behaviour of structures in services, which leads to evaluate precisely their influence on stresses, particularly at the interface of layers.

The aim of this paper is to construct a finite element for analyzing laminated beams including thermomechanical effects in elasticity for small displacements.

From the literature, various theories developed in mechanics for composite or sandwich structures were extended to include thermal effects. They can be classified as

- the equivalent single layer (ESL): the number of unknowns are independent of the number of layers, but the shear stress

continuity at the interfaces of layer are often violated. We can distinguish the classical laminate theory [1], the first order shear deformation theory, and higher order theories [2], which analyse thermal stresses for beams and plates.

- the discrete layer theory or layer-wise approach (DLT): this theory aims at overcoming the restriction of the ESL about the discontinuity of in-plane displacement at the interface layers. In this framework, Kapuria used a zigzag theory for displacements with continuity of shear stresses in the thickness for beams [3] and plates [4]. See also [5]. It should be noted that Reddy [6] has already shown the efficiency of this approach in the framework of mechanics. Unfortunately, the cost increases with the number of plies.

Carrera [7,8] has presented various ESL and DL theories with mixed and displacement-based approaches for assumed distribution of temperatures (uniform, linear, localized).

For an overview about this subject, see for example [2,9,10].

Thus, we propose a new thermomechanical finite element for rectangular laminated beam analysis, so as to have a low cost

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tool simple to use and efficient. In fact, for the mechanical part, our approach is associated to ESL theory. This element is fully free of shear locking and is based on a refined shear deformation theory [11] avoiding the use of shear correction factors for laminates. It has only the three usual independent generalized displacements: 2 displacements and 1 rotation. The element is C^0 -continuous except for the transverse displacement associated with bending which is C^1 . For the thermal part, the layer-wise approach is carried out with a quadratic variation of the temperatures through the thickness. Moreover, interface conditions between two adjacent layers are prescribed, which leads to only one temperature unknown per layer as in [12].

Finally, all the interface and boundary conditions are exactly satisfied both for displacements, stresses, temperatures, and heat flux. So, this approach takes into account physical meaning. It should be noted that this finite element has already shown its capability in the framework of mechanics [13].

In this article, first, we describe the mechanical formulations. Then, the thermal part is developed. Finally, the thermomechanical coupling can be described, the thermal problem being solved. For each of these analyses, the associated finite element is given. These three parts are illustrated by numerical cases which have been performed upon various laminated beams. A parametric study is given to show the effects of divers parameters such as length-to-thickness ratio, and number of degrees of freedom. The accuracy of computations are also evaluated by comparisons with an exact three-dimensional theory for laminates in bending [14,5], and also two-dimensional finite element computations using a commercial code.

Table of principal notation

$[\sigma]$	stress tensor
$[\varepsilon]$	strain tensor
θ	temperature
$[K_{uu}^e]$	elementary stiffness matrix
$[K_{\theta}^{e(k)}]$	elementary conduction matrix for the layer (k)
$[F_{thmech}^{e(k)}]$	elementary thermomechanical coupling vector for the layer (k)
N	number of elements
sin	sinus model without continuity of the shear stress
sin-c	sinus model with continuity of the shear stress

2. Resolution of the mechanical problem

2.1. The governing equations for mechanics

Let us consider a beam occupying the domain $\mathcal{B} = [0, L] \times [-h/2 \leq z \leq h/2] \times [-b/2 \leq x_2 \leq b/2]$ in a cartesian coordinate (x_1, x_2, z) . The beam has a rectangular uniform cross section of height h , width b and is assumed to be straight. The beam is made of k layers of different linearly elastic materials. Each layer may be assumed to be transversely isotropic in the beam

Fig. 1. The laminated beam and co-ordinate system.

axes. The x_1 axis is taken along the central line of the beam whereas x_2 and z are the two axes of symmetry of the cross section intersecting at the centroid, see Fig. 1. As shown in this figure, the x_2 axis is along the width of the beam. This work is based upon a displacement approach for geometrically linear elastic beams.

2.1.1. Constitutive relation

Using matrix notations, the one dimensional constitutive equations of an orthotropic material are given by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & 0 \\ 0 & \bar{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{13} \end{bmatrix} \text{ i.e. } [\sigma] = [\bar{C}][\varepsilon]. \quad (1)$$

where we denote: the stress tensor $[\sigma]$; the strain tensor $[\varepsilon]$. Furthermore, in Eq. (1), the constitutive unidimensional laws are given by the elastic stiffness tensor $[\bar{C}]$.

Taking into account of the classic assumption $\sigma_{22} = \sigma_{33} = 0$ (transverse normal stresses are negligible), the longitudinal modulus is expressed from the three dimensional constitutive law by

$$\bar{C}_{11} = C_{11} - 2C_{12}^2/(C_{23} + C_{33}), \quad (2)$$

where C_{ij} are orthotropic three-dimensional elastic moduli. We also have $\bar{C}_{55} = C_{55}$.

2.1.2. The weak form of the boundary value problem

Using the above matrix notations and for admissible virtual displacement $\vec{u}^* \in U^*$, the variational principle is given by

Find $\vec{u} \in U$ (space of admissible displacements) such that

$$\begin{aligned} & - \int_{\mathcal{B}} [\varepsilon^*(\vec{u}^*)]^T [\sigma(\vec{u})] d\mathcal{B} + \int_{\mathcal{B}} [u^*]^T [f] d\mathcal{B} \\ & + \int_{\partial\mathcal{B}_F} [u^*]^T [F] d\partial\mathcal{B} = 0 \quad \forall \vec{u}^* \in U^* \end{aligned} \quad (3)$$

where $\vec{f}$ and $\vec{F}$ are the prescribed body and surface forces applied on $\partial\mathcal{B}_F$. $\varepsilon^*(\vec{u}^*)$ is the virtual strain.

Eq. (3) is a classical starting point for finite element approximations.

2.2. The displacement field for laminated beams

In this particular case based on various works on beams, plates and shells, Refs. [11,15,16], the displacement field is

assumed to be of the following form (with $w_{,1} = \partial w / \partial x_1$):

$$\begin{cases} u_1(x_1, x_2, z) = u(x_1) - zw(x_1)_{,1} \\ \quad + (\bar{f}(z) + g^{(k)}(z))(\omega_3(x_1) + w(x_1)_{,1}), \\ u_3(x_1, x_2, z) = w(x_1), \end{cases} \quad (4)$$

where in the classic manner w is bending deflection following the z direction. u is associated with the uniform extension of the cross section of the beam along the central line. And, ω_3 is the shear bending rotation around the z axis.

In the context of the sinus model with continuity, we have

$$\bar{f}(z) = f(z) - \frac{h}{\pi} b_{55} f_{,3}(z) \quad \text{and} \quad f(z) = \frac{h}{\pi} \sin \frac{\pi z}{h} \quad (5)$$

and this function will represent the transverse shear strain distribution due to bending by its derivative. The coefficient b_{55} and the function of the k th layer $g^{(k)}$ are given in Appendix A.

From Eq. (5), classical beam models can be deduced

- Navier Bernouilli

$$\bar{f}(z) = 0 \quad \text{and} \quad g^{(k)}(z) = 0.$$

- Timoshenko

$$\bar{f}(z) = z \quad \text{and} \quad g^{(k)}(z) = 0.$$

- Sinus model

$$\bar{f}(z) = \frac{h}{\pi} \sin \frac{\pi z}{h} \quad \text{and} \quad g^{(k)}(z) = 0.$$

Hence, it is obvious that lateral boundary conditions are satisfied in bending and it is not necessary to introduce transverse shear correction factors. It should be noted that it satisfies not only the lateral boundary conditions, but also the interface continuity conditions between layers. And, the procedure is given in Appendix A.

Matrix notations can be easily defined using a generalized displacement vector as

$$[u]^T = [F_u(z)][\mathcal{E}_u]$$

with

$$[\mathcal{E}_u]^T = \begin{bmatrix} u : w : w_{,1} : \omega_3 \end{bmatrix} \quad (6)$$

and where $[F_u(z)]$ is depending on the normal coordinate z . Its expression is given below

$$[F_u(z)] = \begin{bmatrix} 1 & 0 & -z + f(z) & f(z) \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (7)$$

From Eq. (A.5) and Eq. (A.2), the strains for the symmetric laminated beam element are

$$\begin{aligned} \varepsilon_{11} &= u_{,1} - zw_{,11} + \left(\bar{f}(z) + g^{(k)}(z) \right) (w_{,11} + \omega_{3,1}), \\ \gamma_{13} &= \left(\bar{f} + g^{(k)} \right) (w_{,1} + \omega_3). \end{aligned} \quad (8)$$

These expression can be described using a matrix notation

$$[\varepsilon] = [F_s(z)][\mathcal{E}_s]$$

with

$$[\mathcal{E}_s]^T = \begin{bmatrix} u_{,1} : w_{,1} : w_{,11} : \omega_{3,1} \end{bmatrix} \quad (9)$$

and where $[F_s(z)]$ is depending on the normal coordinate z . Its expression is given below

$$[F_s(z)] = \begin{bmatrix} 1 & 0 & -z + (\bar{f}(z) + g^{(k)}(z)) & (\bar{f}(z) + g^{(k)}(z)) \\ 0 & (\bar{f}_{,3} + g_{,3}^{(k)}) & 0 & 0 \end{bmatrix}. \quad (10)$$

2.2.1. Matrix expression for the weak form

From the weak form of the boundary value problem Eq. (3), and using Eqs. (9) and (10), an integration throughout the cross-section is performed in order to obtain an unidimensional formulation. Therefore, first left term of Eq. (3) can be written under the following form:

$$\int_{\mathcal{B}} [\varepsilon^*(\vec{u}^*)]^T [\sigma(\vec{u})] d\mathcal{B} = \int_0^L [\mathcal{E}_s^*]^T [k][\mathcal{E}_s] dx_1$$

with

$$[k] = \int_{\Omega} [F_s(z)]^T [\bar{C}][F_s(z)] d\Omega, \quad (11)$$

where $[\bar{C}]$ is the constitutive unidimensional law given in Section 2.1.1, and Ω represents the cross-section $[-h/2 \leq z \leq h/2] \times [-b/2 \leq x_2 \leq b/2]$.

In Eq. (11), the matrix $[k]$ is the integration throughout the cross-section of the material characteristics of the beam.

2.3. The finite element approximation for the mechanical part

This section is dedicated to the finite element approximation of the generalized displacement, see matrix $[\mathcal{E}_s]$ and $[\mathcal{E}_s^*]$, Eq. (9). It is briefly described, and the reader can obtain detailed description in [13].

2.3.1. The geometric approximation

Given the displacement field constructed above for sandwich beams, a corresponding finite element is developed to analyze the behaviour of laminated beam structures under combined loads. Let us consider the e th element L_e^h of the mesh $\bigcup L_e^h$. This element has three nodes, denoted by $(g_j)_{j=1,2,3}$, see Fig. 2. A point with coordinate x_1 on the central line of the beam is such that:

$$x_1(\xi) = \sum_{j=1}^2 Nl_j(\xi)x_1^e(g_j) \quad (12)$$

Fig. 2. Description of the laminated beam finite element.

where $Nl_j(\xi)$ are Serendipity linear interpolation functions and $x_1^e(g_j)$ are Cartesian coordinates (measured along the x_1 axis) of the node g_j of the element L_e^h . ξ is an isoparametric or reduced coordinate and its variation domain is $[-1, 1]$.

2.3.2. Interpolation for the bending-traction beam element

The finite element approximations of the assumed displacement field components are hereafter symbolically written as $u_i^h(x_1, x_2, z)$ where the superscript h refers to the mesh $\bigcup L_e^h$.

From the kinematics (see Eq. (A.6)), the transverse displacement w^h must be C^1 -continuous; whereas the rotation ω_3^h and the extension displacement u^h can be only C^0 -continuous. Therefore, the generalized displacement w^h are interpolated by the Hermite cubic functions $Nh_j(\xi)$.

According to the transverse shear locking phenomena, the other shear bending generalized displacements, rotation ω_3^h , are interpolated by Serendipity quadratic functions denoted $Nq_j(\xi)$. This choice allows having the same order of interpolation for both w_1^h and ω_3^h in the corresponding transverse shear strain components due to bending, and permits avoiding transverse shear locking using the field compatibility approach, see [17].

Finally, traction u^h is interpolated by Serendipity quadratic functions.

2.3.3. Elementary stiffness matrix

In the previous section, all the finite element mechanical approximations were defined and elementary rigidity $[K_{uu}^e]$ matrix can be deduced from Eq. (11). It has the following expression:

$$[K_{uu}^e] = \int_{L_e} [B]^T [k] [B] dL_e, \quad (13)$$

where $[B]$ is deduced expressing the generalized displacement vectors, see Eq. (9), from the elementary vector of degrees of freedom (dof) denoted $[q_e]$ by

$$[\mathcal{E}_s] = [B][q_e]. \quad (14)$$

The matrix $[B]$ contains only the interpolation functions, their derivatives and the jacobian components.

The same technique can be used defining the elementary mechanical load vector, denoted $[B_u^e]$, but it is not detailed here.

2.4. Results and discussions

The aim of the present investigation is to study the efficiency of this new element for analyzing the flexural behaviour of the sandwich/laminated beam. The problem considered here, for

Fig. 3. Variation of the non-dimensional maximum displacement ($w_m = 100w(L/2, 0)E_T h^3/(q_0 L^4)$) with respect to aspect ratio S —three layers ($0^\circ/90^\circ/0^\circ$).

evaluating the performance of the element in bending, are given below as

2.4.1. Flexural analysis

Problem 1. Simply supported composite cross-ply beam ($0^\circ, 90^\circ, 0^\circ$) and ($0^\circ, 90^\circ$) subjected to sinusoidal load $q = q_0 \sin(\pi x_1/L)$, Ref. [14]. The material properties used here are

$$E_L = 172.4 \text{ GPa}, \quad E_T = 6.895 \text{ GPa}, \quad G_{LT} = 3.448 \text{ GPa}, \\ G_{TT} = 1.379 \text{ GPa}, \quad \nu_{LT} = \nu_{TT} = 0.25,$$

where L refer to fiber direction, T refer to normal direction and all layers are equal in thickness.

Before proceeding to the detailed analysis, numerical computations are carried out for the rank of the element (spurious mode), convergence properties and the effect of aspect ratio (shear locking).

This element has a proper rank without any spurious energy modes when exact integration is applied to obtain all the stiffness matrices (see [13]). There is also no need to use shear correction factors here, as the transverse strain is represented by a cosine function. Further, based on progressive mesh refinement, eight element idealization is found to be adequate to model the laminated beam for a bending analysis (see Table 1).

The normalized displacement obtained at the middle of the simply supported composite beam, as mentioned in Problem 1 using the present element by considering various values for aspect ratio, is shown in Fig. 3 along with the exact solution [14], and they are found to be in excellent agreement. It is also inferred from Fig. 3 that the present element is free from shear locking phenomenon as the element is developed using a field compatibility approach. The numerical results for the normalized in-plane and inter-laminar shear stresses for ($0^\circ/90^\circ/0^\circ$) ($S = L/h = 4$ and $S = 10$) and ($0^\circ/90^\circ$) ($S = 4$) are presented in

Table 1

 $\bar{\sigma}_{13}(0, 0)$, w_m for different number of dofs (sin-c): mesh convergence study— $S = 20$

N	Number dof	$w_m = \frac{100w(L/2,0)E_T h^3}{q_0 L^4}$		$\bar{\sigma}_{13}(0, 0) = \sigma_{13}/q_0$			
			Error	Direct	Error	Equil. Eq	Error
1	6	0.6177	< 0.1%	9.3497	7%	5.8090	33%
2	12	0.6174	< 0.1%	9.1163	4%	7.9540	9%
4	24	0.6173	< 0.1%	9.1002	4%	8.5406	2%
8	48	0.6173	< 0.1%	9.0995	4%	8.6908	< 1%
16	96	0.6173	< 0.1%	9.0995	4%	8.7286	< 1%
Exact		0.6172		8.7490

Fig. 4. Distribution of $\bar{\sigma}_{11}$ along the thickness— $S = 4$ (left)— $S = 10$ (right); three layers ($0^\circ/90^\circ/0^\circ$). Fig. 5. Distribution of $\bar{\sigma}_{13}$ along the thickness— $S = 4$ (left)— $S = 10$ (right); three layers ($0^\circ/90^\circ/0^\circ$).

Figs. 4, 5 and 6, respectively, for further comparison. It is seen from these figures that the new element performs quite well for moderately thick beams as well as thin beam. It should be

noted that the transverse shear stress which is obtained by integration of equilibrium equation (denoted Equilibrium equation) improves results, especially for $S = 4$ (see Figs. 5 and 6 (left)).

Fig. 6. Distribution of $\bar{\sigma}_{13}$ (left) and $\bar{\sigma}_{11}$ (right) along the thickness— $S = 4$ —two layers ($0^\circ/90^\circ$).

3. Heat conduction problem

3.1. Thermal constitutive relation

The physical problem considered here involves the two-dimensional linear heat conduction with thermal conductivity components λ_1, λ_3 in the x_1, z directions, respectively. The constitutive relation is given by the Fourier law

$$[Q] = \begin{bmatrix} q_1 \\ q_3 \end{bmatrix} = [\bar{\lambda}][grad(\theta)], \quad (15)$$

where

$$[\bar{\lambda}] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{bmatrix} \quad \text{and} \quad [grad(\theta)] = \begin{bmatrix} \theta_{,x_1} \\ \theta_{,z} \end{bmatrix}$$

q_i , ($i = 1, 3$) is the heat flux components with respect to the coordinate system x_1, z respectively. $\theta_{,x_1}, \theta_{,z}$ stand for the derivative of θ with respect to x_1 and z , respectively.

It is assumed that the layers have a orthotropic symmetry. We also consider thermal conductivities as constant. However, this is not a limitation. The approach can be generalized in order to include complete conductivity matrix, interface thermal resistance, or temperature dependent coefficients.

3.2. The governing equation

For admissible virtual temperature $\theta^* \in \Theta^*$, the variational principle is given by

Find $\theta \in \Theta$ such that:

$$-\int_{\mathcal{B}} [grad(\theta^*)]^T [Q(\theta)] d\mathcal{B} + \int_{\mathcal{B}} \theta^* r_d d\mathcal{B} + \int_{\partial\mathcal{B}_h} \theta^* h_d d\partial\mathcal{B} = 0 \quad \forall \theta^* \in \Theta^*, \quad (16)$$

where r_d and h_d are the prescribed volume heat source and surface heat flux applied on $\partial\mathcal{B}_h$ respectively. Θ is the space of admissible temperatures. Again, Eq. (16) is the starting point for thermal finite element approximations.

3.3. Resolution of thermal problem for the laminated beam

3.3.1. Quadratic layerwise model

The laminate being subdivided into N discrete layers, continuous temperature field is written in the following form:

$$\theta(x_1, x_2, z) = \sum_{k=1}^N \theta^{(k)}(x_1, x_2, z) \chi_{[z_k, z_{k+1}[}, \quad (17)$$

where $\chi_{[z_k, z_{k+1}[} = 1$ if $z \in [z_k, z_{k+1}[$, 0 if not.

Temperature variations are independent of the $(0, x_2)$ direction. Then, this direction is not kept in the following notations.

The temperature variation in the x_1 direction depends on the 1D discretization. For the z direction, the layerwise technique is used.

A quadratic variation of the temperature along the thickness is considered. Introducing the non-dimensionalized thickness coordinate in the z direction, denoted ζ , for the (k) th layer, we have

$$\zeta(z) = 2 \frac{[z - (z_k + z_{k+1})/2]}{z_{k+1} - z_k}.$$

In the layerwise approach, the temperature per layer may be written as for $\zeta \in [-1, 1]$

$$\theta^{(k)}(x_1, \zeta) = \frac{1}{2} \zeta(\zeta - 1) \theta_{\text{Bottom}}^{(k)}(x_1) + \frac{1}{2} \zeta(\zeta + 1) \theta_{\text{Top}}^{(k)}(x_1) + (1 - \zeta^2) \theta_{\text{Middle}}^{(k)}(x_1), \quad (18)$$

where $\theta_{\text{Bottom}}^{(k)}, \theta_{\text{Top}}^{(k)}, \theta_{\text{Middle}}^{(k)}$ are the temperatures at the bottom, top, and middle, of the (k) th layer respectively, see Fig. 7.

Therefore, some matrices can be introduced in order to prepare the one-dimensional weak form for the heat conduction problem. For the (k) th layer, we can define

$$\theta^{(k)}(x_1, \zeta) = [Z(\zeta)][Cst] \begin{bmatrix} \theta_{\text{Bottom}}^{(k)}(x_1) \\ \theta_{\text{Middle}}^{(k)}(x_1) \\ \theta_{\text{Top}}^{(k)}(x_1) \end{bmatrix} \quad (19)$$

Fig. 7. Unknown temperatures in the (k) th layer.

with

$$[Z(\zeta)] = [1 \quad \zeta \quad \zeta^2] \quad \text{and} \quad [Cst] = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}.$$

For the gradient matrix, we can write an expression in the same way, i.e.:

$$[grad(\theta^{(k)}(x_1, \zeta))] = [Z_{g\theta}(\zeta)][Cst_{g\theta}][Der \theta^{(k)}] \quad (20)$$

with

$$[Z_{g\theta}(\zeta)] = \begin{bmatrix} 0 & 0 & 1 & \zeta & \zeta^2 \\ \frac{d\zeta}{dz} & \zeta \frac{d\zeta}{dz} & 0 & 0 & 0 \end{bmatrix}$$

and

$$[Cst_{g\theta}] = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & -1 & 0 & \frac{1}{2} \end{bmatrix}$$

$$[Der \theta^{(k)}] = \begin{bmatrix} \theta_{\text{Bottom}}^{(k)}(x_1) \\ \theta_{\text{Bottom}}^{(k)}(x_1),1 \\ \theta_{\text{Middle}}^{(k)}(x_1) \\ \theta_{\text{Middle}}^{(k)}(x_1),1 \\ \theta_{\text{Top}}^{(k)}(x_1) \\ \theta_{\text{Top}}^{(k)}(x_1),1 \end{bmatrix},$$

where

$$\frac{d\zeta}{dz} = \frac{2}{z_{k+1} - z_k}.$$

The indices Top, Middle, and Bottom are substituted by t , m , and b respectively, for the sake of simplicity.

The continuity of the temperature fields at the layer interfaces is automatically satisfied

$$\theta_b^{(k+1)}(x_1) = \theta_t^{(k)}(x_1).$$

In the following, the notations $\theta_b^{(k)}$ and $\theta_m^{(k)}$ are chosen. Then, the unknowns of the thermal conduction problem are the temperatures denoted $\theta_b^{(k)}$ and $\theta_m^{(k)}$ for $k = 1, \dots, N$ as indicated in Fig. 8. Thus, for a laminate comprising N layers, such a model taking a quadratic approximation through the thickness of each

layer requires $2N + 1$ independent temperatures when using only one subdivision per layer. It is possible to subdivide more any layer to improve accuracy of the computation, as it will be shown in the numerical examples.

3.3.2. Continuity of the heat flux

In the above approach, the continuity of the temperatures at the layer interfaces is automatically satisfied, which is not the case for the heat flux in the z direction. To respect physical meaning, this continuity at the interfaces is imposed as in [12], which allow to reduce the number of independent unknowns.

So, resulting conditions at interfaces between layers (k) and $(k + 1)$ are stated as follows:

$$[-\lambda_3^{(k)} \theta_{,z}^{(k)}]_{z=z_{k+1}} = [-\lambda_3^{(k+1)} \theta_{,z}^{(k+1)}]_{z=z_{k+1}}, \quad k = 1, \dots, N - 1. \quad (21)$$

Conditions on the temperatures may be deduced in the form

$$4\beta^{(k)} \theta_m^{(k)} + 4\beta^{(k+1)} \theta_m^{(k+1)} = \beta^{(k+1)} (3\theta_b^{(k+1)} + \theta_b^{(k+2)}) + \beta^{(k)} (\theta_b^{(k)} + 3\theta_b^{(k+1)}), \quad k = 1, \dots, N - 1 \quad (22)$$

with

$$\beta^{(k)} = \frac{\lambda_3^{(k)}}{z_{k+1} - z_k}.$$

These continuity conditions furnish a system of $N - 1$ equations, allowing to eliminate the unknowns: $\theta_b^{(2)}, \dots, \theta_b^{(N)}$. Then, we have only $N + 2$ unknowns $\theta_b^{(1)}, \theta_m^{(1)}, \theta_m^{(2)}, \dots, \theta_m^{(N)}, \theta_b^{(N+1)}$.

Now, boundary conditions on top and bottom faces of the laminates must be taken into account. If temperatures are prescribed, then the temperatures $\theta_b^{(1)}$ and $\theta_b^{(N+1)}$ are known. Similarly, a normal heat flux on top and bottom of the laminate may be prescribed as:

$$-\beta^{(1)} (-3\theta_b^{(1)} + 4\theta_m^{(1)} - \theta_b^{(2)}) = \bar{q}_3^{\text{inf}}, \quad (23)$$

$$-\beta^{(N)} (\theta_b^{(N)} - 4\theta_m^{(N)} + 3\theta_b^{(N+1)}) = \bar{q}_3^{\text{sup}}, \quad (24)$$

which allows to deduce $\theta_b^{(1)}$ and $\theta_b^{(N+1)}$. Combination of these boundary conditions are possible as well.

Finally, taking into account the interface continuity conditions of the heat flux between the layers (cf. Eq. (22)), and the top and bottom boundary conditions (cf. Eqs. (23) and (24)), it implies only N unknowns, i.e. one independent temperature per layer. It should be noted that the quadratic variation of temperature allows us to write the continuity conditions with a non constant heat flux across the thickness.

Remark. To impose these conditions of continuity in the framework of the Finite Element, a usual technique of condensation is carried out.

3.3.3. Matrices expressions for the weak form

From the weak form of the boundary value problem Eq. (16), and using Eqs. (20) and (15), an integration throughout the

Fig. 8. Description of unknown temperatures along the thickness.

cross-section is performed in order to obtain the unidimensional formulation. The first term of Eq. (16) can be written under the following form for one layer

$$\int_{\mathcal{B}} [\text{grad}(\theta^*)]^T [Q(\theta)] d\mathcal{B} = \int_0^L [\text{Der} \theta^{(k)*}]^T [k_\theta^{(k)}] [\text{Der} \theta^{(k)}] dx_1 \quad (25)$$

with

$$[k_\theta^{(k)}] = \int_{z_k}^{z_{k+1}} [Z_{g\theta}(z)]^T [C_{stg\theta}]^T [\bar{\lambda}] [C_{stg\theta}] [Z_{g\theta}(z)] d\Omega^{(k)}.$$

In Eq. (25), the matrix $[k_\theta^{(k)}]$ is the integration throughout the cross-section of one layer (k) of the thermal characteristics of the beam.

$$[N_T] = \begin{bmatrix} Nq_1(\xi) & Nq_2(\xi) & Nq_3(\xi) & 0 & 0 & 0 & 0 & 0 & 0 \\ Nq_1(\xi)_{,1} & Nq_2(\xi)_{,1} & Nq_3(\xi)_{,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Nq_1(\xi) & Nq_2(\xi) & Nq_3(\xi) & 0 & 0 & 0 \\ 0 & 0 & 0 & Nq_1(\xi)_{,1} & Nq_2(\xi)_{,1} & Nq_3(\xi)_{,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Nq_1(\xi) & Nq_2(\xi) & Nq_3(\xi) \\ 0 & 0 & 0 & 0 & 0 & 0 & Nq_1(\xi)_{,1} & Nq_2(\xi)_{,1} & Nq_3(\xi)_{,1} \end{bmatrix}$$

3.3.4. FE approximation for the thermal part

3.3.4.1. Interpolation for the thermal layerwise beam element

As seen in the mechanical part (Section 2.3), a finite element corresponding to the temperature field is developed to analyze the behaviour of laminated beam structures under thermal loads. The element with three nodes is used.

The generalized temperature is interpolated by Lagrange quadratic functions. Thus, in 1D, it can be written as

$$\theta(\xi) = \sum_{j=1}^3 Nq_j(\xi) \theta_j. \quad (26)$$

Eq. (26) is used to define the discretization of the matrix $[\text{Der} \theta^{(k)}]$.

The vector of degrees of freedom of the three nodes element L_e^h for the (k) th layer is then (see Fig. 9)

$$[q_{T_e}^{(k)}]^T = [\theta_{1b}^{(k)} \ \theta_{2b}^{(k)} \ \theta_{3b}^{(k)} \mid \theta_{1m}^{(k)} \ \theta_{2m}^{(k)} \ \theta_{3m}^{(k)} \mid \theta_{1t}^{(k)} \ \theta_{2t}^{(k)} \ \theta_{3t}^{(k)}] \quad (27)$$

with indices T_e for temperature.

Fig. 9. Description of the laminated beam finite element.

Then, in the layerwise approach, the generalized temperatures for the (k) th layer are approximated by the following matrix expressions:

$$[\text{Der} \theta^{(k)}] = [N_T] [q_{T_e}^{(k)}] \quad (28)$$

with

and

$$Nq_i(\xi)_{,1} = Nq_i(\xi)_{,\xi} \frac{d\xi}{dz}, \quad i = 1, 2, 3.$$

3.3.4.2. Elementary conduction matrix For one layer (k) , the conduction elementary matrix denoted $[K_\theta^{e(k)}]$ is deduced from Eqs. (25) and (28)

$$[K_\theta^{e(k)}] = \int_{L_e} [N_T]^T [k_\theta^{(k)}] [N_T] dL_e. \quad (29)$$

The same technique can be used defining the elementary thermal load vector, denoted $[B_\theta^e]$, but it is not detailed here.

3.4. Numerical examples: problem of heat conduction in a sandwich media

In this section, several cases are presented to evaluate the efficiency of our thermal approach. These examples cover a wide range of temperature distribution, with continuity of the component q_3 , and discontinuity of the component q_1 , at the interface between the layers. In the following, we denote “cont”

Fig. 10. Rectangular sandwich media.

Fig. 11. Boundary conditions.

if we constrain the continuity of the component q_3 of the heat flux, and “wtcont” if not.

3.4.1. Problem 1

A rectangular symmetric sandwich media is considered (see [12]). It is represented in Fig. 10.

The sandwich is composed by an isotropic core of thickness $h - 2e$, and isotropic skins of equal thickness $e = 0.2h$. We have $L/h = 2$. $k_{\text{skin}} = 0.5 \text{ W/(Km)}$ and $k_{\text{core}} = 50.5 \text{ W/(Km)}$ are the conductivity coefficients. The boundary conditions are given on Fig. 11. The structure is submitted at its top edge to a simply sinusoidal temperature loading given by $\bar{\theta}_t(x_1) = \theta_c \sin(\pi x_1 / L)$ with $\theta_c = 1$, while zero temperature is prescribed on the three other edges.

Temperature distributions along the thickness at station $x_1 = L/2$ are identical for exact solution, layerwise approach with and without continuity of the heat flux q_3 . Similarly, the distributions of the components q_3 and q_1 of the heat flux along the thickness are compared with the exact solution in Tables 2 and 3. In these tables, -/- indicates the number of numerical subdivisions in each layer. The mesh is composed of eight elements.

Results are in good agreement with the exact solution, even when using only one subdivision in each layer (see Figs. 12 and 13).

It should be noted that the constrained continuity of the heat flux improves results at the interfaces between the layers, which is very interesting. Moreover, the error on $q_{3 \text{ max}}$ is less than 1% for all the cases, especially with the constrained approach, which allows to save computational cost.

The layerwise technique allows us to add numerical layers through the thickness, i.e. subdivide the three layers. The fourth, fifth, sixth columns of Tables 2 and 3 show the results obtained with three different subdivisions: $1/2/1$, $2/3/2$, and $4/6/4$ lay-

Fig. 12. Temperatures across the thickness at $x_1 = L/2$.

ers, respectively. The convergence to the exact solution is very pronounced, especially for the heat flux q_3 .

It should be noted that the mesh has a very low influence on the temperatures and the component q_3 of the heat flux. On the other hand, the convergence of the maximal component q_1 of the heat flux is illustrated on Fig. 14. The error on $q_{1 \text{ max}}$ with a $N = 9$ mesh is less than 1%.

3.4.2. Problem 2

A three-layer sandwich beam with graphite-epoxy faces and a soft core with thickness $e = 0.1h$ is considered (Cf. [3]). The orientation of all the plies is: $\gamma_k = 0$, $k = 1, 2, 3$. The thermal material properties used here are

- for the Face
 $\lambda_L = 1.5 \text{ W m}^{-1} \text{ K}^{-1}$, $\lambda_T = 0.5 \text{ W m}^{-1} \text{ K}^{-1}$
- for the core
 $\lambda_1 = \lambda_3 = 3.0 \text{ W m}^{-1} \text{ K}^{-1}$

Two thermal load cases are considered:

- constrained temperature on the top and the bottom surfaces of the beam such as:
 $\theta(x_1, \pm h/2) = \theta_{\text{max}} \sin(\pi x_1 / L)$
- constrained opposite temperature on the top and the bottom surfaces of the beam such as: $\theta(x_1, h/2) = -\theta(x_1, -h/2) = \theta_{\text{max}} \sin(\pi x_1 / L)$

The $N = 8$ mesh is used.

The distributions of the temperature across the thickness are depicted in Fig. 15 for the two thermal load cases. Again, the results are close to the exact solution of the heat conduction problem with only one subdivision per layer. However, distribution of heat flux q_3 (Fig. 16 left) is different from the reference (see Fig. 16 right). Again, as in Section 3.4.1, the thickness is subdivided into more than three layers. A $2/6/2$ subdivision is depicted in Fig. 16 (right) and shows a very good result. To illustrate the convergence of this subdivision, an indicator is

Table 2
Heat flux q_3 across the thickness

	z	cont	wtcont				Exact
		1/1/1	1/1/1	1/2/1	2/3/2	4/6/4	
First layer	0.00	-0.066	-0.064	-0.064	-0.064	-0.064	-0.065
	0.10 ⁻	-0.064	-0.065	-0.065	-0.065	-0.065	-0.066
	0.10 ⁺				-0.065	-0.065	
	0.20 ⁻	-0.063	-0.067	-0.067	-0.067	-0.067	-0.068
Second layer	0.20 ⁺	-0.063	-0.000	-0.059	-0.064	-0.066	-0.068
	0.30 ⁻	-0.441	-0.377	-0.398	-0.397	-0.395	-0.396
	0.30 ⁺					-0.394	
	0.40 ⁻	-0.820	-0.753	-0.738	-0.730	-0.733	-0.733
	0.40 ⁺				-0.724	-0.731	
	0.50 ⁻	-1.198	-1.129	-1.077	-1.094	-1.087	-1.089
	0.50 ⁺			-1.060		-1.087	
	0.60 ⁻	-1.576	-1.505	-1.481	-1.463	-1.469	-1.472
	0.60 ⁺				-1.457	-1.469	
	0.70 ⁻	-1.955	-1.881	-1.903	-1.899	-1.887	-1.891
	0.70 ⁺					-1.886	
	0.80 ⁻	-2.333	-2.257	-2.324	-2.340	-2.354	-2.356
Third layer	0.80 ⁺	-2.333	-2.338	-2.338	-2.352	-2.356	-2.356
	0.90 ⁻	-2.400	-2.401	-2.401	-2.386	-2.390	-2.390
	0.90 ⁺				-2.386	-2.390	
	1.00	-2.467	-2.465	-2.465	-2.479	-2.483	-2.483
d.o.f.		45	75	105	195	405	—

Table 3
Heat flux q_1 across the thickness.

	z	cont	wtcont				Exact
		1/1/1	1/1/1	1/2/1	2/3/2	4/6/4	
First layer	0.00	0.000	0.000	0.000	0.000	0.000	0
	0.10 ⁻	0.010	0.010	0.010	0.010	0.010	0.01
	0.10 ⁺				0.010	0.010	
	0.20 ⁻	0.020	0.021	0.021	0.021	0.021	0.021
Second layer	0.20 ⁺	2.050	2.081	2.081	2.081	2.081	2.072
	0.30 ⁻	2.090	2.111	2.117	2.117	2.117	2.109
	0.30 ⁺					2.117	
	0.40 ⁻	2.189	2.200	2.206	2.206	2.206	2.197
	0.40 ⁺				2.206	2.206	
	0.50 ⁻	2.348	2.348	2.349	2.349	2.349	2.340
	0.50 ⁺			2.349		2.349	
	0.60 ⁻	2.566	2.556	2.550	2.551	2.551	2.541
	0.60 ⁺				2.551	2.551	
	0.70 ⁻	2.845	2.823	2.816	2.815	2.815	2.804
	0.70 ⁺					2.815	
	0.80 ⁻	3.182	3.149	3.149	3.149	3.149	3.137
Third layer	0.80 ⁺	0.032	0.031	0.031	0.031	0.031	0.031
	0.90 ⁻	0.404	0.405	0.405	0.405	0.405	0.403
	0.90 ⁺				0.405	0.405	
	1.00	0.788	0.788	0.788	0.788	0.788	0.785
d.o.f.		45	75	105	195	405	—

chosen as follows:

$$error(N_{\text{core}}) = 100 * \frac{|q_3(x_1 = L/2, z = 0, N_{\text{core}}) - q_3^{\text{ref}}|}{|q_3^{\text{ref}}|}$$

where N_{core} is the number of numerical layers of the core.

It is represented in Fig. 17. Few sublayers are necessary to improve results to the reference.

Finally, these different examples show the efficiency of the layerwise approach with constrained continuity of the component q_3 of heat flux. So, physical meaning is kept. The number

Fig. 13. Distribution of heat flux q_3 and q_1 along the thickness at $x_1 = L/2$ (left) and $x_1 = L$ (right).

Fig. 14. Error $\left(100 * \left| \frac{q_{1e \max} - q_{1e \max}^{\text{ref}}}{q_{1e \max}^{\text{ref}}} \right| \right)$ versus the number of dofs.

of unknowns remains low. Moreover, we have the possibility to improve precision with additional numerical layers across the thickness.

4. Thermomechanical coupling

In this section, we describe a thermomechanical coupling finite element. First, we solve the heat conduction problem. Then, we use the thermal results (temperatures), as a load for the mechanical part.

Temperature and displacement fields of Sections 2.1 and 3 are used.

4.1. Thermomechanical constitutive law

As in Section 2.2, the classic assumption $\sigma_{11} = \sigma_{33} = 0$ implies the following form of the constitutive law with thermo-mechanical coupling:

$$[\sigma] = [\bar{C}]([\varepsilon] - [\bar{\alpha}](\theta - \theta_0)) \quad \text{with} \quad [\bar{\alpha}] = \begin{bmatrix} \bar{\alpha}_1 \\ 0 \end{bmatrix}, \quad (30)$$

where $\bar{\alpha}_1$ can be expressed in term of thermal expansion coefficient α_1 , orthotropic three dimensional elastic moduli, and the orientation of the principal material axes. θ_0 is the temperature at which stresses and strains do not exist.

4.2. FE approximation for the coupling

4.2.1. Matrix expression for the weak form

From the weak form of the boundary value problem Eq. (3), and using Eqs. (30) and (9), an integration throughout the cross-section is performed in order to obtain an unidimensional formulation. Therefore, first left term of Eq. (3) which takes into account the thermomechanical coupling in the constitutive law, can be written under the following form:

$$\int_0^L [\mathcal{E}_s^*]^T [C \alpha^{(k)}] [Cst] \begin{bmatrix} \theta_{\text{Bottom}}^{(k)}(x_1) \\ \theta_{\text{Middle}}^{(k)}(x_1) \\ \theta_{\text{Top}}^{(k)}(x_1) \end{bmatrix} dx_1 - \int_0^L [\mathcal{E}_s^*]^T [C \alpha_{\theta_0}^{(k)}] \theta_0 dx_1$$

with

$$[C \alpha^{(k)}] = \int_{z_k}^{z_{k+1}} [F_s(z)]^T [\bar{C}] [\bar{\alpha}] [Z(\zeta)] d\Omega^{(k)}$$

and

$$[C \alpha_{\theta_0}^{(k)}] = \int_{z_k}^{z_{k+1}} [F_s(z)]^T [\bar{C}] [\bar{\alpha}] d\Omega^{(k)}. \quad (31)$$

Fig. 15. Distribution of temperature across the thickness for load case 1 (left)—2 (right).

Fig. 16. Distribution of heat flux q_3 along the thickness with and without continuity (left), and with sub-layers and exact solution (right).

In Eq. (31), the matrices $[C\alpha^{(k)}]$ and $[C\alpha_{\theta_0}^{(k)}]$ are the integration throughout the cross-section of one layer (k) of the thermomechanical characteristics of the beam.

4.3. Interpolation for the thermomechanical coupling: elementary coupling matrix

In Section 3.3.4, we have defined the interpolation of the temperature field, i.e.:

$$\begin{bmatrix} \theta_{\text{Bottom}}^{(k)}(x_1) \\ \theta_{\text{Middle}}^{(k)}(x_1) \\ \theta_{\text{Top}}^{(k)}(x_1) \end{bmatrix} = [N_\theta][q_{Te}^{(k)}] \quad (32)$$

with

$$[N_\theta] = \begin{bmatrix} Nq_1(\xi) & Nq_2(\xi) & Nq_3(\xi) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Nq_1(\xi) & Nq_2(\xi) & Nq_3(\xi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Nq_1(\xi) & Nq_2(\xi) & Nq_3(\xi) \end{bmatrix}.$$

In the previous section, all the finite element mechanical approximations were defined and elementary thermomechanical coupling $[F_{\text{thmecha}}^e]^{(k)}$ vector for one layer can be deduced from Eq. (31). It has the following expression:

$$[F_{\text{thmecha}}^e]^{(k)} = \left[\int_{L_e} [B]^T [C\alpha^{(k)}] [Cst] [N_\theta] dL_e \right] [q_{Te}^{(k)}] - \int_{L_e} [B]^T [C\alpha_{\theta_0}^{(k)}] \theta_0 dL_e. \quad (33)$$

It should be noted that the $[q_{Te}^{(k)}]$ vector comes from the thermal calculus.

Fig. 17. Error versus number of layers N_{core} .

4.4. Numerical results

4.4.1. Problem 1

The three-layer sandwich beam with graphite-epoxy faces and a soft core of Section 3.4.2 is considered. The mechanical material properties used here are

- for the face:
 $Y_L = 131.1 \text{ GPa}$, $Y_T = 6.9 \text{ GPa}$, $G_{LT} = 3.588 \text{ GPa}$,
 $G_{TT} = 2.3322 \text{ GPa}$, $\nu_{LT} = 0.32$, $\nu_{TT} = 0.49$, $\alpha_L = 0.0225 \times 10^{-6} \text{ K}^{-1}$, $\alpha_T = 22.5 \times 10^{-6} \text{ K}^{-1}$,
- for the core:
 $E_1 = 0.2208 \text{ MPa}$, $E_2 = 0.2001 \text{ MPa}$, $E_3 = 2760 \text{ MPa}$, $G_{12} = 16.56 \text{ MPa}$, $G_{23} = 455.4 \text{ MPa}$, $G_{13} = 545.1 \text{ MPa}$, $\nu_{12} = 0.99$,
 $\nu_{13} = 3 \times 10^{-5}$, $\nu_{23} = 3 \times 10^{-5}$, $\alpha_1 = \alpha_3 = 30.6 \times 10^{-6} \text{ K}^{-1}$.

The second thermal load case is considered, i.e., the constrained opposite temperature on the top and the bottom surfaces of the beam. It corresponds to a thermal bending problem.

Fig. 18. $\bar{u}_1(3h, z)$, $\bar{u}_3(0.5L, z)$ for $S = 50$.

Fig. 19. $\bar{\sigma}_{11}(0.5L, z)$, $\bar{\sigma}_{13}(3h, z)$ for $S = 50$.

Table 4

 $\bar{\sigma}_{13}(0, \pm h/6)$, $\bar{\sigma}_{11}(L/2, \pm h/6)$ for different number of dofs (sin-c): mesh convergence study— $S = 50$

N	Number dof	$\bar{\sigma}_{11}(L/2, \pm h/6)$	Error	$\bar{\sigma}_{13}(0, \pm h/6)$ Equil. Eq	Error
2	12	± 374.04	$< 1\%$	0.491	17%
4	24	± 374.07	$< 1\%$	0.574	3.5%
8	48	± 374.08	$< 1\%$	0.596	$< 1\%$
10	60	± 374.08	$< 1\%$	0.598	$< 1\%$
12	72	± 374.08	$< 1\%$	0.599	$< 1\%$
Exact		± 371.40		0.595	

Table 5

 $\bar{\sigma}_{13}(0, \pm h/6)$, $\bar{\sigma}_{11}(L/2, \pm h/6)$ for different values of S

S	$\bar{\sigma}_{11}(L/2, \pm h/6)$				$\bar{\sigma}_{13}(0, \pm h/6)$			
	sin-c	sin	exact	error	sin-c	sin	exact	error
20	± 374.0	± 374.1	± 371.5	$< 1\%$	1.509	1.504	1.441	4%
50	± 374.1	± 374.1	± 371.4	$< 1\%$	0.600	0.600	0.595	$< 1\%$
100	± 374.1	± 374.1	± 371.4	$< 1\%$	0.300	0.300	0.299	$< 1\%$

A thermomechanical analysis for this highly inhomogeneous simply-supported beam is carried out with a $N = 16$ mesh. The transverse shear stress is obtained by integration of the equilibrium equation. The reference solution is given by ANSYS software with a very refined mesh.

The results ($\bar{u}_1$, $\bar{u}_3$, $\bar{\sigma}_{11}$, $\bar{\sigma}_{13}$) are nondimensionalised, using

$$\bar{u}_1 = \frac{100u_1}{\alpha_T S h \theta_{\max}}, \quad \bar{u}_3 = \frac{100u_3}{\alpha_T S^2 h \theta_{\max}},$$

$$\bar{\sigma}_{11} = \frac{\sigma_{11}}{\alpha_T E_T \theta_{\max}}, \quad \bar{\sigma}_{13} = \frac{S \sigma_{13}}{\alpha_T E_T \theta_{\max}}. \quad (34)$$

The results are in good agreement with the reference calculation for the displacement $\bar{u}_1$, and the stresses $\bar{\sigma}_{11}$ and $\bar{\sigma}_{13}$. The errors for the maximal values of stresses are less than 1%. On the other end, the transverse displacement $\bar{u}_3$ is not correct due to our kinematics (Figs. 18 and 19).

4.4.2. Problem 2 (Cf. [7])

In order to see the capability of the refined model, the accuracy of the present theory is assessed by comparison with the exact 2D thermo-elasticity solution [18]. We focus on stresses which are the quantities of interest in the field of design.

A temperature field is applied to the beam, as follows

$$\theta(x_1, z) = \theta_{\max} \frac{2z}{h} \sin\left(\frac{\pi x_1}{L}\right). \quad (35)$$

It corresponds to a cylindrical bending. The beam is simply supported. The mechanical properties of the lamina are

$$\frac{E_T}{E_L} = 25, \quad \frac{G_{LT}}{E_T} = 0.5, \quad \frac{G_{TT}}{E_T} = 0.2,$$

$$\nu_{LT} = \nu_{TT} = 0.25, \quad \frac{\alpha_T}{\alpha_L} = 1125.$$

A cross-ply laminated case ($0^\circ/90^\circ/0^\circ$) is considered. A $N = 16$ mesh is used. The transverse stress is obtained by the integration of the equilibrium equations.

As in Ref. [7], we define the dimensionless quantities:

$$\bar{\sigma}_{11} = \frac{\sigma_{11}}{\alpha_L E_T \theta_{\max_2}}, \quad \bar{\sigma}_{13} = \frac{\sigma_{13}}{\alpha_L E_T \theta_{\max_2}}. \quad (36)$$

First, we show the convergence of the finite element for the stresses $\bar{\sigma}_{11}$ and the transverse shear one $\bar{\sigma}_{13}$ in Table 4. For $S = 50$, the results obtained are in good agreement with the exact values with few elements. Only four elements (24 dofs) drive to an error which is less than 3.5%. For the axial stress, the results are excellent with only two elements.

Then, we study the influence of the length to thickness ratio in Table 5. It permits to evaluate the performance of the element from moderately thick ($S = 20$) to very thin beam ($S = 100$). It should be noted that the results about the stress $\bar{\sigma}_{11}$ are in good agreement with respect to the exact solution. The error is less than 1% whenever the value of S . Good results are obtained for the transverse stress (error of 4% for $S = 20$, and less than 1% for $S > 20$).

Fig. 20 shows the comparison of transverse stress through the thickness for $S = 20$, and proves again the validity of our element. The transverse stress is obtained by the integration of the equilibrium equation. The stress $\bar{\sigma}_{11}$ through the thickness is also excellent (see Fig. 21).

5. Conclusion

A new three-node multilayered beam finite element has been presented to analyse the thermomechanical behaviour of sandwich and laminated beams. It is based on a refined kinematic model for the mechanical part, avoiding the use of transverse shear correction factors, and on the layerwise approach for the thermal part. It should be noted that all the interface and bound-

Fig. 20. $\bar{\sigma}_{13}(0, z)$ for $S = 20$.

Fig. 21. $\bar{\sigma}_{11}(L/2, z)$ for $S = 20$.

ary conditions are exactly satisfied both for the mechanical part (displacements, transverse shear stresses), and for the thermal part (temperatures, transverse heat flux). So, this approach has a strong physical meaning. It also has the advantage of reducing the number of thermal unknowns which makes this finite element simple and efficient for a low cost.

Several numerical evaluations have proved that this finite element has very good properties in the field of finite elements. Convergence velocity is high not only for displacements or temperatures but also for stresses and heat flux. Furthermore, accurate results are obtained and have been compared to exact solutions for mechanical, thermal and thermomechanical problems. In particular, we can stress the emphasis on the good results at the interface between two adjacent layers for transverse

heat flux and transverse shear stress. Moreover, the distribution of the stresses across the thickness are correctly described.

Appendix A. Continuity condition for the transverse shear stress

The lamination scheme for the beam is such that layers are parallel to the x_1 - x_2 plane, see Fig. 1.

In this case, both free traction conditions on plane $z = \pm h/2$ and interface continuity conditions must be satisfied. The bending part in the x_1 - z plane has therefore to be modified relative to Eq. (4).

To reach objectives about continuity and boundary conditions, we assume that the transverse shear stress $\sigma_{13}^{(k)}$ (where we denote $k = 1, 2, 3$ the layers from the bottom to the top of the beam) takes the following form:

$$\sigma_{13}^{(k)}(x_1, x_2, z) = \left(G_{13}^{(k)} \left(f(z)_{,3} - \frac{h}{\pi} b_{55} f(z)_{,33} \right) + a_{55}^{(k)} \right) \times (\omega_3(x_1) + w(x_1)_{,1}) \quad (\text{A.1})$$

still assuming both normal transverse strain $\varepsilon_{33}^{(k)}$ and stress $\sigma_{33}^{(k)}$ negligible, which in particular involves $u_3^{b3(k)} = w$ for the x_1 - z plane bending part. For small displacements, the transverse shear strain associated with the bending in the x_1 - z plane is then defined as

$$\varepsilon_{13}^{(k)} = u_{1,3}^{b3(k)} + u_{3,1}^{b3(k)} = u_{1,3}^{b3(k)} + w_{,1}. \quad (\text{A.2})$$

From the orthotropic constitutive law, we also find

$$\varepsilon_{13}^{(k)} = \frac{1}{G_{13}^{(k)}} \sigma_{13}^{(k)}(x_1, x_2, z). \quad (\text{A.3})$$

So, between Eqs. (A.2) and (A.3), we have

$$u_{1,3}^{b3(k)} + w_{,1} = \frac{1}{G_{13}^{(k)}} \sigma_{13}^{(k)}(x_1, x_2, z). \quad (\text{A.4})$$

Taking into account of Eq. (A.1), this latest becomes

$$u_1^{b3(k)}(x_1, x_2, z)_{,3} = -w(x_1)_{,1} + \left(f(z)_{,3} - \frac{h}{\pi} b_{55} f(z)_{,33} + \frac{a_{55}^{(k)}}{G_{13}^{(k)}} \right) \times (\omega_3(x_1) + w(x_1)_{,1}). \quad (\text{A.5})$$

Integrating this equation with respect to the z coordinate gives the shear-bending part of the displacement field associated with the bending in the x_1 - z plane

$$\begin{cases} u_1^{b3(k)}(x_1, x_2, z) = -zw(x_1)_{,1} + (\bar{f}(z) + g^{(k)}(z))(\omega_3(x_1) + w(x_1)_{,1}), \\ u_2^{b3(k)}(x_1, x_2, z) = 0, \\ u_3^{b3(k)}(x_1, x_2, z) = w(x_1), \end{cases} \quad (\text{A.6})$$

where we have introduced the following notations:

$$\begin{aligned}\bar{f}(z) &= f(z) - \frac{h}{\pi} b_{55} f(z)_{,3} \\ &= \frac{h}{\pi} \sin \frac{\pi z}{h} - \frac{h}{\pi} b_{55} \cos \frac{\pi z}{h}, \\ g^{(k)}(z) &= \frac{a_{55}^{(k)}}{G_{31}^{(k)}} z + d^{(k)}.\end{aligned}\quad (\text{A.7})$$

In Eq. (A.7), coefficients $d^{(k)}$ are determined such that the component $u_1^{b3(k)}$ of the displacement is continuous at the interface of adjacent layers and the coefficient b_{55} allows the component $u_1^{b3(k)}$ to disappear at the midplane. Finally, coefficients $a_{55}^{(k)}$ in Eq. (A.7) are computed from the requirement that the transverse shear stress $\sigma_{13}^{(k)}$ is continuous at the interface of adjacent layers and disappears at the top and bottom surfaces of the beam (faces $z = \pm h/2$, see Fig. 1).

We then obtain for a sandwich composite beam the following coefficients appearing in Eq. (A.6) due to symmetries of the problem:

$$\begin{aligned}b_{55} &= \frac{\sum_{k=1,2} (G_{13}^{(k)} - G_{13}^{(k+1)}) f(z^{(k+1)})_{,3}}{G_{13}^{(1)} + \sum_{k=1,2} (G_{13}^{(k)} - G_{13}^{(k+1)}) (\pi/h) f(z^{(k+1)}) + G_{13}^{(3)}}, \\ a_{55}^{(1)} &= G_{13}^{(1)} b_{55}, \\ a_{55}^{(k)} &= a_{55}^{(k-1)} + \left(G_{13}^{(k-1)} - G_{13}^{(k)} \right) f(z^{(k)})_{,3} \\ &\quad + \frac{\pi}{h} (G_{13}^{(k-1)} - G_{13}^{(k)}) b_{55} f(z^{(k)}), \quad k = 2, 3, \\ d^{(1)} &= d^{(2)} - z^{(2)} \left(\frac{a_{55}^{(1)}}{G_{13}^{(1)}} - \frac{a_{55}^{(2)}}{G_{13}^{(2)}} \right), \\ d^{(2)} &= \frac{h}{\pi} b_{55}, \\ d^{(3)} &= d^{(2)} + z^{(3)} \left(\frac{a_{55}^{(2)}}{G_{13}^{(2)}} - \frac{a_{55}^{(3)}}{G_{13}^{(3)}} \right).\end{aligned}\quad (\text{A.8})$$

Remark. Note that this approach for multilayered structures has been successfully used for shells [19] with angle-ply and cross-ply schemes.

Finally, the displacement field for symmetric sandwich beams made of transversely isotropic materials is given by the field shown by Eq. (A.6), in which we add the longitudinal displacement due to traction represented by the function $u(x_1)$, as in Eq. (4).

Of course, all displacement and stress conditions are exactly satisfied for shear-bending.

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