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► To cite this version:

O. Polit, Isabelle Bruant. Electric potential approximations for an eight node plate finite element. Computers & Structures, 2006, 84 (22-23), pp.1480-1493. 10.1016/j.compstruc.2006.01.032 . hal-01366938

HAL Id: hal-01366938

<https://hal.science/hal-01366938>

Submitted on 11 Jan 2019

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Electric potential approximations for an eight node plate finite element

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Abstract

The aim of this work is to develop a computational tool for multilayered piezoelectric plates: a low cost tool, simple to use and very efficient for both convergence velocity and accuracy, without any classical numerical pathologies. In the field of finite elements, two approaches were previously used for the mechanical part, taking into account the transverse shear stress effects and using only five unknown generalized displacements: C^0 finite element approximation based on first-order shear deformation theories (FSDT) [Polit O, Touratier M, Lory P. A new eight-node quadrilateral shear-bending plate finite element. *Int J Numer Meth Eng* 1994;37:387–411] and C^1 finite element approximations using a high order shear deformation theory (HSDT) [Polit O, Touratier M. High order triangular sandwich plate finite element for linear and nonlinear analyses. *Comput Meth Appl Mech Eng* 2000;185:305–24]. In this article, we present the piezoelectric extension of the FSDT eight node plate finite element. The electric potential is approximated using the layerwise approach and an evaluation is proposed in order to assess the best compromise between minimum number of degrees of freedom and maximum efficiency. On one side, two kinds of finite element approximations for the electric potential with respect to the thickness coordinate are presented: a linear variation and a quadratic variation in each layer. On the other side, the in-plane variation can be quadratic or constant on the elementary domain at each interface layer. The use of a constant value reduces the number of unknown electric potentials. Furthermore, at the post-processing level, the transverse shear stresses are deduced using the equilibrium equations.

Numerous tests are presented in order to evaluate the capability of these electric potential approximations to give accurate results with respect to piezoelectricity or finite element reference solutions. Finally, an adaptative composite plate is evaluated using the best compromise finite element.

Keywords: Finite element; Multilayered plates; Smart structures; Electric potential approximations; Equilibrium equations; Numerical results

1. Introduction

Due to different kinds of attractive applications like vibration control, noise attenuation, . . . , the study of multilayered plates with piezoelectric actuators and sensors is a very actual trend in the research community. Thus, the development of new numerical tools for analysing non-homogeneous structures has become an important research topic since a few years. A survey of beam, plate and shell finite elements available for this kind of coupled field problems are given in Saravanos and Heyliger [3] and Benjeddou [4]. A more recent state of the art is also given by Benjeddou [5].

From the literature on numerical methods for multilayered plates where some or all layers are made of piezoelectric materials, several approaches were developed and a simple classification can be given:

- three-dimensional finite elements, but the drawback is the computational cost for obtaining acceptable aspect ratios for refined meshes;
- full layerwise formulation for displacement and electric potential, see Saravanos et al. [6], with the same drawback as above: the cost increases rapidly with the number of plies;
- higher order displacement field for the mechanical part and layerwise technique for the electric potential; it is well-known that this can give good approximation for

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the transverse shear stresses and does not require the use of shear correction factor. But the number of unknown generalized displacements can be large, see for example recent papers by Correia et al. [7] and Araujo et al. [8] using until 11 generalized displacements;

- Carrera [9] approach including the zig-zag effect and the interlaminar equilibria for the transverse shear stresses. The drawbacks of this model are the use of two further unknowns for the displacement than the classical five generalized displacements and numerical pathologies as transverse shear locking and rank-deficiency;
- first-order shear deformation and layerwise technique for the electric potential; this is the well-known Reissner–Mindlin approach, with only five generalized displacements, requiring shear correction factors.

Our approach is clearly associated with the last item and in this field, we can point out a first family of electric potential approximations, see Suleman and Venkayya [10], Correia et al. [7], Bansal and Ramaswamy [11], Mukherjee and Joshi [12], using only one dof for each piezoelectric lamina. Unfortunately, this layerwise approximation is not efficient for all situations as it will be seen later. In a second family, a linear variation is assumed in each layer with respect to the thickness and two quadrilateral finite elements (FE) can be cited: a 9 nodes FE proposed by Sheikh et al. [13] and a 4 nodes FE by Cen et al. [14]. For the in-plane variation, they use one electric potential dof at each node. It must be noticed that Cen et al. [14] take into account the transverse shear locking in order to present a reliable FE while this problem is often treated inadequately using reduced or selective integration techniques in the literature.

In this work, a displacement approach without any numerical pathology is presented. The piezoelectric layers for both sensor and actuator can be surface bonded or embedded anywhere inside the core and the through-thickness variation of the potential may have different patterns. Various electric potential approximations are evaluated in order to assess their efficiency. In the thickness direction, a linear and a quadratic variation in each layer are implemented. For the in-plane variation, electric potential dof defined per node or per interface are used. A clear comparison of the electric potential approximations is given through several numerical evaluations.

The first section of this work deals with the governing equations and the introduction of the boundary value problem. The second part is dedicated to the displacement approximation, including some numerical evaluations. The third part gives the coupled equations and describes the approximation of the electric potential. In the last section, piezoelectric problems are presented.

2. The governing equations for piezoelectricity

This section is dedicated to composite laminates with embedded piezoelectric sensors and actuators.

Let us consider a plate occupying the domain $\mathcal{P} = \Omega \times [-\frac{e}{2} \leq z \leq \frac{e}{2}]$ in a Cartesian coordinate system $(x_1, x_2, x_3 = z)$ where e is the constant thickness of the plate $\mathcal{P}$ and Ω is an arbitrary region in the (x_1, x_2) plane. The boundary of the domain is denoted $\partial\mathcal{P}$.

The displacement field with respect to the basis vector $\mathbf{e}_i$ and the electric potential are denoted:

$$\begin{cases} \mathbf{u}(x_1, x_2, z, t) = \sum_{i=1}^3 u_i(x_1, x_2, z, t) \mathbf{e}_i \\ \phi(x_1, x_2, z, t) \end{cases} \quad (1)$$

belonging respectively to the space of admissible displacements U and the space of admissible electric potentials Φ .

2.1. The piezoelectric laminae

In matrix notations, the two-dimensional constitutive equations of a piezoelectric material are given by

$$\begin{cases} [T] = [\bar{C}][S] - [\bar{e}]^T[E] \\ [D] = [\bar{e}][S] + [\bar{\epsilon}][E] \end{cases} \quad (2)$$

where we denote the stress vector $[T]$, the strain vector $[S]$, the electric field vector $[E]$ and the electric displacement vector $[D]$. Moreover, in Eq. (2), the linear converse and direct piezoelectric constitutive bidimensional laws are given by the elastic stiffness matrix $[\bar{C}]$, the piezoelectric matrix $[\bar{e}]$ and the electric permittivity matrix $[\bar{\epsilon}]$. Definitions of those matrices are given in Appendix A from the three-dimensional components.

2.2. The weak form of the boundary value problem

Using the above matrix notations and for admissible virtual displacement $\mathbf{u}^* \in U^*$ and admissible electric potential $\phi^* \in \Phi^*$, the electric potential (or field)-based variational principle is given by: find $(\mathbf{u}, \phi) \in U \times \Phi$ such that:

$$\begin{aligned} \int_{\mathcal{P}} \rho [\mathbf{u}^*]^T [\ddot{\mathbf{u}}] d\mathcal{P} &= - \int_{\mathcal{P}} [S^*(\mathbf{u}^*)]^T [T(\mathbf{u})] d\mathcal{P} \\ &+ \int_{\mathcal{P}} [E^*(\phi^*)]^T [D(\phi)] d\mathcal{P} \\ &+ \int_{\mathcal{P}} [\mathbf{u}^*]^T [f] d\mathcal{P} + \int_{\partial\mathcal{P}_F} [\mathbf{u}^*]^T [F] d\partial\mathcal{P} \\ &- \int_{\mathcal{P}} q \phi^* d\mathcal{P} - \int_{\partial\mathcal{P}_Q} Q \phi^* d\partial\mathcal{P} \end{aligned} \quad (3)$$

$\forall (\mathbf{u}^*, \phi^*) \in U^* \times \Phi^*$

where $[f]$ and $[F]$ are the prescribed body and surface force vectors applied on $\partial\mathcal{P}_F$, q and Q are the prescribed body and surface charges applied on $\partial\mathcal{P}_Q$ and ρ is the mass density. Finally, $S^*(\mathbf{u}^*)$ and $E^*(\phi^*)$ are the virtual strain and virtual electric field.

Eq. (3) is a good starting point for finite element approximations using independent variables $\mathbf{u}$ and ϕ .

3. The finite element approximation for the mechanical part

This section is dedicated to the finite element approximations of the mechanical part. The displacement field is firstly introduced and finite element approximations of the generalized displacement are briefly described. The reader can obtain a detailed description in Polit [15] for isotropic plate.

3.1. Displacement field and strains components

The kinematics is based on the Reissner–Mindlin plate model:

$$\begin{cases} u_1(x_1, x_2, z, t) = v_1(x_1, x_2, t) + z\theta_2(x_1, x_2, t) \\ u_2(x_1, x_2, z, t) = v_2(x_1, x_2, t) - z\theta_1(x_1, x_2, t) \\ u_3(x_1, x_2, z, t) = v_3(x_1, x_2, t) \end{cases} \quad (4)$$

where v_α are the membrane displacements with respect to x_α directions, θ_α are the positive rotations of the fiber initially normal to the plate midsurface and v_3 is the transverse displacement in the normal direction.

Matrix notations can be easily defined using a generalized displacement vector as

$$[u]^T = [F_u(z)][\mathcal{E}_u] \quad \text{with } [\mathcal{E}_u]^T = [v_1 \quad v_2 \quad v_3 \quad \theta_1 \quad \theta_2] \quad (5)$$

where $[F_u(z)]$ is depending on the normal coordinate z . Its expression is

$$[F_u(z)] = \begin{bmatrix} 1 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (6)$$

For small strains, the following expressions are obtained for the strain components:

$$\begin{aligned} S_{11} &= v_{1,1} + z\theta_{2,1} \\ S_{22} &= v_{2,2} - z\theta_{1,2} \\ S_{12} &= v_{1,2} + v_{2,1} + z(\theta_{2,2} - \theta_{1,1}) \\ S_{23} &= v_{3,1} - \theta_1 \\ S_{13} &= v_{3,2} + \theta_2 \end{aligned} \quad (7)$$

As above for the displacement, the strain components can be described using matrix notation:

$$[S] = [F_s(z)][\mathcal{E}_s] \quad \text{with} \quad [\mathcal{E}_s]^T = [v_{1,1} \quad v_{1,2} \quad v_{2,1} \quad v_{2,2} \quad \theta_{1,1} \quad \theta_{1,2} \quad \theta_{2,1} \quad \theta_{2,2} \quad S_{23} \quad S_{13}] \quad (8)$$

and where $[F_s(z)]$ is depending on the normal coordinate z . Its expression is given below:

$$[F_s(z)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -z & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -z & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

3.2. Matrix expression for the weak form

From the weak form of the boundary value problem Eq. (3), and using Eqs. (8) and (9), an integration throughout the thickness is performed in order to obtain a bidimensional formulation. Therefore, first right term of Eq. (3) can be written in the following form:

$$\int_{\mathcal{P}} [S^*(\mathbf{u}^*)]^T [T(\mathbf{u})] d\mathcal{P} = \int_{\Omega} [\mathcal{E}_s^*]^T [ku] [\mathcal{E}_s] d\Omega \quad \text{with } [ku] = \int_{-e/2}^{e/2} [F_s(z)]^T [\bar{C}] [F_s(z)] dz \quad (10)$$

where $[\bar{C}]$ is the constitutive bidimensional law given in Appendix A.

Same calculations for the left member of Eq. (3) using Eqs. (5) and (6) give:

$$\int_{\mathcal{P}} \rho [u^*]^T [\ddot{u}] d\mathcal{P} = \int_{\Omega} [\mathcal{E}_u^*]^T [mu] [\ddot{\mathcal{E}}_u] d\Omega \quad \text{with } [mu] = \int_{-e/2}^{e/2} \rho [F_u(z)]^T [F_u(z)] dz \quad (11)$$

In Eqs. (10) and (11), the matrices $[ku]$ and $[mu]$ are the integrations throughout the thickness of the material characteristics of the plate.

3.3. The geometric approximation

The eight-node quadrilateral finite element is presented in Fig. 1. The in-plane co-ordinates (x_1, x_2) are approximated on the reference bi-unit domain with respect to the reduced coordinates (ξ, η) by

$$x_\alpha(\xi, \eta) = \sum_{i=1}^8 Nq_i(\xi, \eta) (x_\alpha)_i \quad \text{for } \alpha \in \{1, 2\} \quad (12)$$

where $Nq_i(\xi, \eta)$ are the classical Serendipity interpolation functions, see [1].

3.4. The displacement approximations

From Eq. (4), three displacement components v_i and two rotations θ_α have to be approximated.

An eight-node quadrilateral plate finite element, with five dof per node, was previously developed and all details are given in [1,16]. For the present element, the same

Fig. 1. The reference domain of the eight-node finite element.

displacement approximations are used and are briefly described below:

- for the membrane and bending part of this finite element, an isoparametric procedure is used and the displacements v_α and the rotations θ_α are approximated using the same functions as the geometry, see Eq. (12).
- for the transverse shear strains, a methodology named “field compatibility” has been developed in order to avoid the transverse shear locking in the thin plate domain:

- the transverse shear strains are defined in reduced coordinates:

$$\gamma_\xi = \beta_\xi + v_{3,\xi} \quad \gamma_\eta = \beta_\eta + v_{3,\eta} \quad (13)$$

where β_ξ, β_η are the rotations in reduced coordinates obtained from the positive rotations θ_α ($\alpha = 1, 2$) of Eq. (4).

In order to ensure the same polynomial approximation for the rotation and the transverse displacement in Eq. (13), v_3 is assumed to be cubic, introducing four supplementary dof at the mid-side nodes: $(v_{3,\xi})_5, (v_{3,\eta})_6, (v_{3,\xi})_7, (v_{3,\eta})_8$.

- A linear variation of the tangential transverse shear strain component is assumed on each side of the elementary domain, see Fig. 1. Thus, the supplementary dof introduced at the previous step can be expressed as a linear combination of the rotation and transverse displacement values. Therefore, a new finite element approximation is obtained for the transverse displacement v_3 .
- The interpolation of the reduced transverse shear strain components is defined in the following polynomial basis as the intersection sets of monomial terms from ξ and η :

$$\begin{aligned} \mathcal{B}(\gamma_\xi) &= \mathcal{B}(\beta_\xi) \cap \mathcal{B}(v_{3,\xi}) = \{1, \xi, \eta, \xi\eta, \eta^2\} \\ \mathcal{B}(\gamma_\eta) &= \mathcal{B}(\beta_\eta) \cap \mathcal{B}(v_{3,\eta}) = \{1, \xi, \eta, \xi\eta, \xi^2\} \end{aligned} \quad (14)$$

- According to the dimension of the polynomial basis, five points are needed for each reduced transverse shear strains. These points were chosen as indicated in Fig. 2 because this location gives the best results in case of distorted meshes (see [1]). The following finite element approximation is obtained for the reduced transverse shear strains:

Fig. 2. Point locations for the transverse shear strains evaluations.

$$\begin{aligned} \gamma_\xi(\xi, \eta) &= \sum_{I=1}^5 C_{\xi I}(\xi, \eta) \gamma_{\xi I} \\ \gamma_\eta(\xi, \eta) &= \sum_{J=1}^5 C_{\eta J}(\xi, \eta) \gamma_{\eta J} \end{aligned} \quad (15)$$

where $C_{\xi I}$ and $C_{\eta J}$ are interpolation functions [1].

- Using the Jacobian matrix, the physical transverse shear strains S_{13} and S_{23} are deduced from the reduced transverse shear strains of Eq. (15).

This methodology gives a finite element denoted CL8 for v_3 Cubic— γ tangential Linear—8 nodes.

3.5. The elementary matrices

In the previous section, the finite element approximations for the mechanical part were defined and elementary stiffness $[K_{uu}^e]$ and mass $[M_{uu}^e]$ matrices can be deduced from respectively Eqs. (10) and (11). They have the following expressions:

$$\begin{aligned} [K_{uu}^e] &= \int_{\Omega_e} [Bu]^T [ku] [Bu] d\Omega_e \\ [M_{uu}^e] &= \int_{\Omega_e} [Nu]^T [mu] [Nu] d\Omega_e \end{aligned} \quad (16)$$

where $[Bu]$ and $[Nu]$ are deduced expressing the generalized displacement vectors, see Eqs. (8) and (5), from the elementary vector of dof denoted $[qu_e]$ by

$$[\mathcal{E}_s] = [Bu][qu_e] \quad [\mathcal{E}_u] = [Nu][qu_e] \quad (17)$$

The matrices $[Bu]$ and $[Nu]$ contain only the interpolation functions, their derivatives and the Jacobian matrix components. In the same way and for the virtual part, the discretization gives:

$$[\mathcal{E}_s^*] = [Bu][q_e^*] \quad [\mathcal{E}_u^*] = [Nu][q_e^*] \quad (18)$$

The same technique can be used defining the elementary mechanical load vector, denoted $[L_u^e]$, and details can be found in [1].

3.6. Some numerical results

In this section, two results are presented in order to show the efficiency of the CL8 finite element. They deal with static mechanical problems. The first one is dedicated to the shear locking problem while the second one concerns a composite plate for which elasticity solutions were given by Srinivas and Rao [17]. Unless otherwise specified, all units used in this work refer to those of the International System (M, K, S, A). The meshes are presented in Fig. 3.

3.6.1. The transverse shear locking

The shear locking problem is characterized as a dependency of the convergence velocity with respect to the

Fig. 3. The meshes used for the numerical evaluations.

thickness of the plate. When the thickness of the plate tends towards 0, the convergence velocity becomes very slow and numerous elements are necessary in order to recover the reference value available from Kirchhoff–Love model. In order to evaluate the sensitivity of this finite element to the transverse shear locking, a very simple test is introduced:

- geometry:* square plate $a \times a$, with a length to thickness ratio a/e varying from 10 – 10^6 ; only a quarter of the plate is meshed;
- boundary conditions and loading:* the plate is simply supported at its edges, and is subjected to a uniform transverse load p_0 ;
- material properties:* Young modulus $E = 10^6$ and Poisson's ratio $\nu = 0.3$;
- mesh:* $N = 2$;
- results:* the central deflection $v_3(a/2, a/2)$ is given.

Fig. 4 gives the evolution of the central deflection with respect to the Kirchhoff–Love reference solution. A classical isoparametric approach, using a 3×3 integration points is presented and the reduced selective integration on the transverse shear strains with 2×2 is also given. It can be observed that the selective integration is not sufficient for the eight nodes to avoid the transverse shear locking, while CL8 finite element gives very good results with respect to the reference solution using exact full integration. Furthermore, CL8 has a proper rank, passes all patch tests and has

Fig. 4. A simply supported plate: the transverse shear locking.

a very quick convergence for the transverse displacement for all types of boundary conditions as it will be seen in the next sub-section.

3.6.2. The Srinivas and Rao [17] sandwich plate

This test is about a simply supported 3 plies sandwich plate under a uniform transverse load. For this test an elasticity solution is available in Srinivas and Rao [17]. Furthermore, it is a well-known benchmark in the field of sandwich plate evaluation in the field of composite structures. The test is detailed below:

geometry: square plate $a \times a$ with $a = 10$, and a length to thickness ratio $a/e = 10$; only a quarter of the plate is meshed; The core has a thickness $e_c = 0.8$, while the two faces are characterized by $e_f = 0.1$

boundary conditions and loading: the plate is simply supported at its edges, and is subjected to a uniform transverse load p_0 ;

material properties: the core properties are

$$\begin{aligned} E_1 &= 897949, & E_2 &= 471424, & G_{12} &= 262931, \\ G_{13} &= 159914, & G_{23} &= 266810, & \nu_{12} &= 0.44 \end{aligned}$$

and the skin to core property ratio is 15.

Table 1 gives the convergence of the CL8 finite element for the transverse displacement, the stresses T_{11} , T_{22} and the transverse shear one T_{13} . For this last component, two results are available: (a) from the constitutive law (direct value); (b) using the equilibrium equation at the post-processing level. The results obtained are in good agreement with the reference values with few elements.

4. The finite element approximation for the electric part

4.1. The electric potential and the electric field vector

The potential function $\phi(x_1, x_2, x_3 = z)$ is added in order to deal with coupled problems and a layerwise approximation is used in the thickness direction. The main advantage

Table 1

Deflection and stresses in three-ply orthotropic plates under an uniform load p_0 , using CL8

N	dof	$v_3 E_1 / p_0$	T_{11} / p_0	T_{22} / p_0	T_{13} / p_0	
					Direct value	Equilibrium equation
1	15	117.14	29.134	22.798	1.484	1.097
2	60	121.73	56.722	40.993	2.441	2.508
4	240	121.96	61.764	44.126	3.026	3.349
8	960	121.97	63.179	45.014	3.330	3.772
16	3840	121.97	63.678	45.324	3.638	3.950
Ref. [17]		121.72	66.787	46.424		3.964

of this choice is to define independently the finite element approximation for the mid-plane and for the thickness direction. Two kinds of approximations are implemented in this work. For a layer denoted (k) and a reduced normal coordinate $\zeta \in [-1, 1]$, we have:

- a linear variation using two potential values $(\phi_{\text{bot}}^{(k)}, \phi_{\text{top}}^{(k)})$ located at the bottom and the top of each layer:

$$\phi^{(k)}(x_1, x_2, z(\zeta)) = \frac{1}{2}(1 - \zeta)\phi_{\text{bot}}^{(k)}(x_1, x_2) + \frac{1}{2}(1 + \zeta)\phi_{\text{top}}^{(k)}(x_1, x_2) \quad (19)$$

- a quadratic variation using three potential values $(\phi_{\text{bot}}^{(k)}, \phi_{\text{mid}}^{(k)}, \phi_{\text{top}}^{(k)})$ located at the bottom, middle and top of each layer:

$$\begin{aligned} \phi^{(k)}(x_1, x_2, z(\zeta)) &= \frac{1}{2}\zeta(\zeta - 1)\phi_{\text{bot}}^{(k)}(x_1, x_2) + \frac{1}{2}\zeta(\zeta + 1)\phi_{\text{top}}^{(k)}(x_1, x_2) \\ &\quad + (1 - \zeta^2)\phi_{\text{mid}}^{(k)}(x_1, x_2) \end{aligned} \quad (20)$$

For a layer $e^{(k)}$ with $z \in [z_{\text{bot}}^{(k)}, z_{\text{top}}^{(k)}]$, the relation between the thickness coordinate z and the reduce coordinate ζ is given by

$$z(\zeta) = \frac{1}{2}(z_{\text{bot}}^{(k)} + z_{\text{top}}^{(k)}) + \frac{1}{2}ze^{(k)} \quad (21)$$

Therefore, some matrices can be introduced in order to prepare the two-dimensional weak form for the dielectric part and the coupling between electrical and mechanical effects. We can define for each layer:

- for the linear variation:

$$\begin{aligned} [\phi^{(k)}(x_1, x_2, z(\zeta))] &= [Z_{\phi_{\text{lin}}}(z)][Cst_{\text{lin}}] \begin{bmatrix} \phi_{\text{bot}}^{(k)}(x_1, x_2) \\ \phi_{\text{top}}^{(k)}(x_1, x_2) \end{bmatrix} \\ \text{with } [Z_{\phi_{\text{lin}}}(z)] &= \begin{bmatrix} 1 & \zeta \end{bmatrix} \quad \text{and} \quad [Cst_{\text{lin}}] = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \end{aligned} \quad (22)$$

- for the quadratic variation:

$$\begin{aligned} [\phi^{(k)}(x_1, x_2, z(\zeta))] &= [Z_{\phi_{\text{quad}}}(z)][Cst_{\text{quad}}] \begin{bmatrix} \phi_{\text{bot}}^{(k)}(x_1, x_2) \\ \phi_{\text{mid}}^{(k)}(x_1, x_2) \\ \phi_{\text{top}}^{(k)}(x_1, x_2) \end{bmatrix} \\ \text{with } [Z_{\phi_{\text{quad}}}(z)] &= \begin{bmatrix} 1 & \zeta & \zeta^2 \end{bmatrix} \\ \text{and } [Cst_{\text{quad}}] &= \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \end{aligned} \quad (23)$$

For the electric field vector, a relation can be obtained with the same technique and, omitting the coordinates for the right member, we have in the case of the linear approach:

$$\begin{aligned} -[E^{(k)}(x_1, x_2, z(\zeta))] &= \begin{bmatrix} \phi^{(k),1} \\ \phi^{(k),2} \\ \phi^{(k),3} \end{bmatrix} = [Z_{E_{\text{lin}}}(z)][CstE_{\text{lin}}][\mathcal{D}er_{\text{lin}}\phi^{(k)}] \\ \text{with } [Z_{E_{\text{lin}}}(z)] &= \begin{bmatrix} 0 & 1 & \zeta & 0 & 0 \\ 0 & 0 & 0 & 1 & \zeta \\ d\zeta/dz & 0 & 0 & 0 & 0 \end{bmatrix} \\ [CstE_{\text{lin}}] &= \begin{bmatrix} -1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 & 0 & 1/2 \end{bmatrix} \\ \text{and } [\mathcal{D}er_{\text{lin}}\phi^{(k)}] &= \begin{bmatrix} \phi_{\text{bot}}^{(k)}(x_1, x_2) \\ \phi_{\text{bot}}^{(k)}(x_1, x_2)_{,1} \\ \phi_{\text{bot}}^{(k)}(x_1, x_2)_{,2} \\ \phi_{\text{top}}^{(k)}(x_1, x_2) \\ \phi_{\text{top}}^{(k)}(x_1, x_2)_{,1} \\ \phi_{\text{top}}^{(k)}(x_1, x_2)_{,2} \end{bmatrix} \end{aligned} \quad (24)$$

where $\frac{d\zeta}{dz} = \frac{2}{e^{(k)}}$ is deduced from Eq. (21).

The same expressions can be obtained for the quadratic approach and is not specified here nor detailed thereafter. Eqs. (22) and (24) for the linear variation are very useful for obtaining the bidimensional form of the variational principle, which is the topic of the next sub-section.

4.2. Matrices expressions for the weak form

From the weak form of the boundary value problem Eq. (3), and using Eqs. (22) and (24) for the linear approach, an integration throughout the thickness is performed in order to obtain the bidimensional formulation. The second term of the right member of Eq. (3), introducing Eq. (2), can be written under the following form:

$$\begin{aligned} \int_{\mathcal{P}} [E^*(\phi^*)]^T [D(\phi)] d\mathcal{P} &= \int_{\mathcal{P}} [E^*(\phi^*)]^T [\bar{e}][S(\mathbf{u})] d\mathcal{P} \\ &\quad + \int_{\mathcal{P}} [E^*(\phi^*)]^T [\bar{\epsilon}][E(\phi)] d\mathcal{P} \end{aligned} \quad (25)$$

where $[\bar{e}]$ and $[\bar{\epsilon}]$ are the modified piezoelectric and dielectric matrices given in Appendix A.

The first term of the second member in Eq. (25) gives the coupling matrix between elastic mechanical and electrical effects while the second term gives the dielectric matrix. Therefore and for the linear variation, using Eq. (8) for the strain and Eq. (24) for the electric field vector, the first term becomes for each layer (k) :

$$\begin{aligned} \int_{\Omega} [\mathcal{D}er_{\text{lin}}\phi^{(k)*}]^T [CstE_{\text{lin}}]^T [k\phi u^{(k)}] [\mathcal{E}_s] d\Omega \\ \text{with } [k\phi u^{(k)}] = \int_{z_{\text{bot}}^{(k)}}^{z_{\text{top}}^{(k)}} [Z_{E_{\text{lin}}}(z)]^T [\bar{e}][F_s(z)] dz \end{aligned} \quad (26)$$

In the same way, introducing matrix notations given by Eq. (24), we obtain for the dielectric matrix and for each layer (k):

$$\int_{\Omega} [\mathcal{D}er_{lin}\phi^{(k)*}]^T [CstE_{lin}]^T [k\phi^{(k)}] [CstE_{lin}] [\mathcal{D}er_{lin}\phi^{(k)}] d\Omega$$

$$\text{with } [k\phi^{(k)}] = \int_{z_{bot}^{(k)}}^{z_{top}^{(k)}} [Z_{E_{lin}}(z)]^T [\bar{\epsilon}] [Z_{E_{lin}}(z)] dz \quad (27)$$

In Eqs. (26) and (27), the matrices $[k\phi u^{(k)}]$ and $[k\phi^{(k)}]$ result from the integration throughout the thickness for each layer of the piezoelectric and the dielectric characteristics of the plate.

The last step is about the finite element approximation with respect to the in-plane coordinates for $\phi_{bot}^k(x_1, x_2)$, $\phi_{top}^k(x_1, x_2)$, and their derivatives in $[\mathcal{D}er_{lin}\phi^{(k)}]$.

4.3. The electric potential approximations

For the in-plane variation with respect to the reference co-ordinates (ξ, η) , two approximations are implemented:

- a quadratic variation using Serendipity interpolation, see Fig. 1

$$\phi(\xi, \eta) = \sum_{i=1}^8 Nq_i(\xi, \eta) \phi_i \quad (28)$$

where ϕ_i is the electric potential at the i th node.

- a constant value in the elementary domain

$$\phi(\xi, \eta) = \phi_e \quad (29)$$

Eq. (28) is used to define the discretization of the matrix $[\mathcal{D}er_{lin}\phi^{(k)}]$, last step in order to obtain the elementary matrices for the coupling effect, given by Eq. (26), and for the dielectric effect, given by Eq. (27). For each layer, we can write:

$$[\mathcal{D}er_{lin}\phi^{(k)}] = [B_{lin}\phi] [q\phi_e^{(k)}] \quad (30)$$

where $[B_{lin}\phi]$ contains the interpolation functions (see Eq. (28)), their derivatives and the Jacobian matrix components and $[q\phi_e^{(k)}]$ is the vector of the electric dof associated with the layer (k). The same matrices are introduced for the virtual part.

At the layer level, the coupling elementary matrix $[K_{\phi u}^e(k)]$ is finally obtained after substituting Eqs. (17) and (30) in Eq. (26):

$$[K_{\phi u}^e(k)] = \int_{\Omega} [B_{lin}\phi]^T [CstE_{lin}]^T [k\phi u^{(k)}] [Bu] d\Omega \quad (31)$$

and from Eq. (27), the electric matrix $[K_{\phi\phi}^e(k)]$ is deduced:

$$[K_{\phi\phi}^e(k)] = \int_{\Omega} [B_{lin}\phi]^T [CstE_{lin}]^T [k\phi^{(k)}] [CstE_{lin}] [B_{lin}\phi] d\Omega \quad (32)$$

Using Eq. (29), it must be denoted that the matrix $[\mathcal{D}er_{lin}\phi^{(k)}]$ becomes very simple because all the in-plane derivatives vanish.

4.4. The elementary matrices

From Eqs. (31) and (32), a sum on the layers gives the following elementary matrices:

- the coupling matrices $[K_{\phi u}^e]$ from Eq. (31) and $[K_{u\phi}^e] = [K_{\phi u}^e]^T$ from the first right term of Eq. (3) using the coupled constitutive equation Eq. (2);
- the electric matrix $[K_{\phi\phi}^e]$ from Eq. (32);

The electric load vector, denoted $[L_{\phi}^e]$, which is null in absence of prescribed body and surface charges, is finally introduced.

5. The electro-mechanical system

From the weak formulation given by Eq. (3), the representation of the coupled dynamic system can be expressed in a very global compact form as follows:

$$\begin{bmatrix} [M_{uu}] & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [\ddot{q}_u] \\ [\ddot{q}_{\phi}] \end{bmatrix} + \begin{bmatrix} [K_{uu}] & [K_{u\phi}] \\ [K_{u\phi}]^T & [K_{\phi\phi}] \end{bmatrix} \begin{bmatrix} [q_u] \\ [q_{\phi}] \end{bmatrix} = \begin{bmatrix} [L_u] \\ [L_{\phi}] \end{bmatrix} \quad (33)$$

The displacement dof are in the matrix $[q_u]$, while $[q_{\phi}]$ contains the electric potential dof.

For the static analysis, Eq. (33) is solved with $[M_{uu}] = [0]$ and in the case of the modal analysis, Eq. (33) is used with $[L_u] = [L_{\phi}] = [0]$. A usual static condensation is performed from the second equation of Eq. (33) in order to obtain a relation between $[q_{\phi}]$ and $[q_u]$. The following expression is deduced:

$$[q_{\phi}] = [K_{\phi\phi}]^{-1} ([L_{\phi}] - [K_{u\phi}]^T [q_u]) \quad (34)$$

Substituting Eq. (34) in the first equation of Eq. (33), the following algebraic system has to be solved:

$$\begin{aligned} [M_{uu}] [\ddot{q}_u] + ([K_{uu}] + [K_{u\phi}] (-[K_{\phi\phi}]^{-1} [K_{u\phi}]^T) [q_u] \\ = [L_u] - [K_{u\phi}] [K_{\phi\phi}]^{-1} [L_{\phi}] \end{aligned} \quad (35)$$

The boundary conditions for the electric potential are imposed, as for the mechanical displacement, through penalty function method where a high stiffness value is added to the diagonal elements corresponding to the restrained dof.

6. The numerical results for piezoelectric coupling problems

In this section, several tests are presented validating our finite elements and evaluating their efficiency. From the previous sections, different electric potential

approximations are introduced while displacement approximation remains the same. Therefore, three finite elements are used for the numerical evaluation denoted as follows:

nodal dof:

- CL8NZ2 based on a quadratic variation in each layer and a quadratic variation in each interface (1 layer gives 24 dof in each element);
- CL8NZ based on a linear variation in each layer and a quadratic variation in each interface (1 layer gives 16 dof in each element);

element dof:

- CL8EZ based on a linear variation in each layer and a constant value in each interface (1 layer gives 2 dof in each element).

The first test deals with a monomorph plate with large range of length to thickness ratio. Convergence evaluation are presented in order to compare the electric potential approximations with respect to the thickness of the plate. Then simulations are presented for usual tests dealing with sandwich plates and laminated plates. Finally, the last test is about adaptive composite plates with piezoelectric actuators. In this last case, CL8EZ is very useful modeling piezoelectric patches.

6.1. The monomorph plate

A simply supported monomorph plate is first studied under electric or mechanical loads. The plate has the following characteristics:

geometry: a rectangular plate of length $a = 0.025$ m, width $b = 0.0125$ m and three length to thickness ratios $a/e = 5, 10, 50$; only a quarter of the plate is meshed;

boundary conditions and loading: the rectangular plate is simply supported at its edges, and is subjected to:

- a transverse normal uniform load $F_3 = 5.e4$ Pa with $\phi(x_1, x_2, \pm e/2) = 0$ V called short circuited electric boundary conditions;
- no mechanical load and the electric potential loads: $\phi(x_1, x_2, -e/2) = -100$ V and $\phi(x_1, x_2, e/2) = 100$ V;

material properties: PZT4 with

$C_{11} = C_{22} = 139$ GPa, $C_{12} = 77.8$ GPa, $C_{33} = 115$ GPa, $C_{23} = C_{13} = 74.3$ GPa, $C_{44} = C_{55} = 25.6$ GPa, $C_{66} = 30.6$ GPa, $e_{31} = e_{32} = -5.2$ C/m², $e_{33} = 15.1$ C/m², $e_{15} = e_{24} = 12.7$ C/m², $\epsilon_{11} = \epsilon_{22} = 13.06$ nF/m, $\epsilon_{33} = 11.51$ nF/m,

results: $v_3(a/2, b/2, 0)$, $v_1(a/2, b/2, 0)$ and $\phi(a/2, b/2, 0)$.

In the case of the semi thick plate $a/e = 50$, a convergence study with respect to the mid-plane mesh, see Fig. 3, is firstly conducted. Short circuited electric boundary conditions are considered and the results are given in

Table 2

Short circuited monomorph: mesh convergence study, $a/e = 50$.

N		1	2	4	8
		16	48	168	624
CL8NZ2	Mec. dof				
	Elec. dof	24	63	195	675
	v_3 (μ m)	-227.24	-227.24	-227.23	-227.23
CL8NZ	ϕ (V)	-140.46	-125.62	-121.88	-121.01
	Elec. dof	24	63	195	675
	v_3 (μ m)	-236.35	-236.35	-236.34	-236.34
CL8EZ	ϕ (V)	-146.14	-130.71	-126.81	-125.88
	Elec. dof	3	12	48	192
	v_3 (μ m)	-242.87	-238.02	-236.81	-236.51
Fernandes [18]	ϕ (V)	-83.74	-115.14	-122.99	-124.96
	$v_3 = -227.26$ μ m				$\phi = -120.73$ V

Table 2. In the thickness direction, electric potential is defined using 3 piezoelectric dof:

- two layers for the linear approximation in order to introduce the coupling effects between mechanics and piezoelectricity and to have one dof ϕ on the mid-plane $z = 0$;
- one layer for the quadratic approximation.

It must be noticed that the deflection is less sensitive to the mid-plane mesh than the electric potential. CL8NZ2 is very efficient and accurate because the very coarse mesh $N = 1$ gives the reference deflection. For the electric potential the mesh $N = 4$ is needed for obtaining an error of less than 1% with respect to the reference solution. Same convergence is observed with CL8NZ finite element but the error for the deflection is of 4% with $N = 1$ and of 5% with $N = 4$ for the electric potential. For CL8EZ, convergence is slower and the $N = 4$ mesh is needed also for the deflection for obtaining an error of less than 4%. Consequently, a reasonable accuracy seems to be obtained with the mesh $N = 4$. It must be denoted that the use of CL8EZ decreases strongly the number of electric dof (for $N = 8$, 192 instead of 675).

Using the $N = 4$ mesh, a convergence study is conducted as a function of the number of numerical layers. The results are presented in Table 3. CL8NZ2 is very accurate as previously seen with only one layer, and no variation is observed when increasing the number of numerical layers. For the linear case, deflection and electric potential are sensitive to the thickness discretization, but values with $N = 4$ and 4 layers (five values) are in good agreement with respect to the three-dimensional reference values. The errors are of less than 1.5%. For the ratio $a/e = 50$, Fig. 5 gives the distribution of the electric potential in the middle of the plate for 1, 2, 4, 6 and 8 numerical layers. The parabolic distribution is recovered by increasing the number of numerical layers.

In the case of the linear approximation, the electric potential with respect to the x_1 direction using $N = 8$ and 4 numerical layers is presented in Fig. 6 using CL8NZ and CL8EZ. As the electric potential is constant on each

Table 3

Short circuited monomorph: number of numerical layers convergence study, $a/e = 50$

No. of layers		1	2	4	6	8
	Mech. dof	168	168	168	168	168
CL8NZ2	Elec. dof	195	325	585	845	1105
	v_3 (μm)	-227.23	-227.23	-227.23	-227.23	-227.23
	ϕ (V)	-121.88	-121.87	-121.87	-121.87	-121.87
CL8NZ	Elec. dof	130	195	325	455	585
	v_3 (μm)	-268.67	-236.34	-229.45	-228.21	-227.78
	ϕ (V)	0.	-126.81	-123.07	-122.40	-122.21
CL8EZ	Elec. dof	32	48	80	112	144
	v_3 (μm)	-268.67	-236.81	-230.00	-228.79	-228.37
	ϕ (V)	0	-122.99	-119.41	-118.76	-118.54
Fernandes [18]		$v_3 = -227.26 \mu\text{m}$ $\phi = -120.73 \text{ V}$

Fig. 5. Short circuited monomorph: distribution of ϕ as a function of the number of numerical layers.

Fig. 6. Short circuited monomorph: distribution of ϕ (V) with respect to x_1 (m).

Table 4

Short circuited monomorph: length to thickness ratio influence

	$a/e = 5$		$a/e = 10$		$a/e = 50$	
	v_3 (μm)	ϕ (V)	v_3 (μm)	ϕ (V)	v_3 (μm)	ϕ (V)
CL8NZ2	-0.255	-12.77	-1.872	-24.54	-227.23	-121.87
CL8NZ	-0.257	-12.90	-1.891	-24.78	-229.45	-123.07
CL8EZ	-0.266	-11.94	-1.910	-23.88	-230.00	-119.41
Fernandes [18]	-0.252	-12.17	-1.867	-24.19	-227.26	-120.73

Table 5

Monomorph under an electric potential load: length to thickness ratio influence

	$a/e = 5$	$a/e = 10$	$a/e = 50$
	v_1 (μm)	v_1 (μm)	v_1 (μm)
CL8NZ2	0.082	0.164	0.822
CL8NZ	0.082	0.164	0.822
CL8EZ	0.082	0.164	0.822
Fernandes [18]	0.074	0.156	0.813

element using CL8EZ, the electric potential distribution is discontinuous.

From the above results, the $N = 4$ mesh with, respectively, 4 and 1 numerical layers for CL8NZ/CL8EZ and CL8NZ2 are used in the other tests. First of all, results varying the length to thickness ratio are presented in Table 4. The largest error is obtained for the $a/e = 5$ ratio and for the electric potential with (6%, 5%) for, respectively, (linear, quadratic) approximations. For the other ratios, each finite element provides reasonable accuracy, the quadratic approximation giving the best results with errors of less than 1.5%. Same order of accuracy are obtained for CL8NZ and CL8EZ.

The case of the electric potential load on the top and bottom faces with opposite signs, inducing a traction effect, is presented in Table 5. Same results are obtained for the in-plane displacement and the largest error is obtained for the $a/e = 5$ ratio with 10.8%. For this kind of load, CL8EZ must be used because same accuracy is obtained with less number of dof.

6.2. The sandwich plates

6.2.1. Sandwich plates without intermediary electric load

The first sandwich plate is made of two piezoelectric layers with the same thickness and one elastic core, and three length to thickness ratios are studied. The two piezoelectric layers (ZnO^+ and ZnO^-) have opposite polarization axes with respect to the z -direction.

This sandwich has the following characteristics:

geometry: a rectangular plate of length $a = 0.025 \text{ m}$, width $b = 0.0125 \text{ m}$ with ratios $a/e = 5, 10, 50$; the thickness of the skins is $e_s = 0.2e$ and the core thickness is $e_c = 0.6e$; only a quarter of the plate is meshed;

boundary conditions and loading: the rectangular plate is simply supported at its edges, and is subjected to:

- a transverse normal uniform load $F_3 = 5.e4$ Pa with $\phi(x_1, x_2, \pm e/2) = 0$ V called short circuited electric boundary conditions;
- no mechanical load and electric potential loads: $\phi(x_1, x_2, -e/2) = -100$ V and $\phi(x_1, x_2, e/2) = 100$ V;

material properties: $\text{ZnO}^+/\text{Si}/\text{ZnO}^-$ with

- ZnO^+ : $C_{11} = C_{22} = 209.7$ GPa, $C_{12} = 121.1$ GPa, $C_{33} = 210.9$ GPa, $C_{13} = C_{23} = 105.1$ GPa, $C_{44} = C_{55} = 42.5$ GPa, $C_{66} = 44.3$ GPa, $e_{31} = e_{32} = -.61$ C/m², $e_{33} = 1.14$ C/m², $e_{15} = e_{24} = -.59$ C/m², $\epsilon_{11} = \epsilon_{22} = .0738$ nF/m, $\epsilon_{33} = .0783$ nF/m.
- ZnO^- : same material properties as ZnO^+ except for $e_{31} = e_{32} = .61$ C/m², $e_{33} = -1.14$ C/m², $e_{15} = e_{24} = .59$ C/m².
- Si: $C_{11} = C_{22} = C_{33} = 166$ GPa, $C_{12} = C_{13} = C_{23} = 63.9$ GPa, $C_{44} = C_{55} = C_{66} = 79.6$ GPa, $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = .1045$ nF/m,

mesh used: $N = 8$ for the first load case, $N = 4$ for the second one, and one numerical layer in each physical layer;

results: $v_3(a/2, b/2, 0)$ and $\phi(a/2, b/2, 0)$.

First results concern the transverse uniform load and the short circuited boundary condition. The values are given in Table 6 with the reference three-dimensional solutions from Fernandes [18]. Accurate results are obtained for the $a/e = 50$ and $a/e = 10$ because error on the transverse displacement is less than 1% and error on the electric potential is less than 3.2%. For the ratio $a/e = 5$ error on displacement stays acceptable with 2.3% but for the electric potential, 13% is reached even for CL8NZ2. Therefore, this is not due to the electric potential approximation but to the ratio, which is known as the validity limit of classical plate models. For the ratio $a/e = 50$, the distribution of the electric potential in the middle of the plate is presented in Fig. 7 for the linear and quadratic approximations (CL8NZ and CL8NZ2). For the linear case, different numbers of numerical layers (1, 2, 3) for the two piezoelectric skins and (1, 2) for the elastic core are used. Variation in the core is linear while the distribution in the skins is slightly quadratic. In fact, no influence is noticed on the results in Table 6 for CL8NZ/CL8EZ because the same accuracy as CL8NZ2 is obtained.

Table 6
Short circuited sandwich plate: length to thickness ratio influence

	$a/e = 5$		$a/e = 10$		$a/e = 50$	
	v_3 (μm)	ϕ (V)	v_3 (μm)	ϕ (V)	v_3 (μm)	ϕ (V)
CL8NZ2	-0.1662	-35.04	-1.242	-70.95	-151.73	-356.70
CL8NZ	-0.1665	-35.10	-1.244	-71.06	-151.97	-357.28
CL8EZ	-0.1663	-35.46	-1.244	-70.93	-151.99	-354.66
Fernandes [18]	-0.1627	-31.16	-1.235	-68.85	-151.69	-354.74

Fig. 7. Short circuited sandwich plate: distribution of ϕ for different layer numbers, using CL8NZ and CL8NZ2, $a/e = 50$.

Table 7

Sandwich plate under electric potential load: length to thickness ratio influence

	$a/e = 5$ v_3 (μm)	$a/e = 10$ v_3 (μm)	$a/e = 50$ v_3 (μm)
CL8NZ2	0.0099	0.04	0.9919
CL8NZ	0.0099	0.04	0.9935
CL8EZ	0.0099	0.04	0.9935
Fernandes [18]	0.0103	0.04	0.9921

In a second time, a potential value is applied on the top and bottom surfaces of the plate and results are presented in Table 7. No difference is observed between linear and quadratic electric potential approximations. The error is less than 1% for the ratios $a/e = 50$ and $a/e = 10$ and 4% is obtained in the limit plate domain $a/e = 5$.

6.2.2. Sandwich plate with intermediary electrical loads

This sandwich plate has the same geometric characteristics as the previous one with:

boundary conditions and loading: the rectangular plate is simply supported at its edges, and is subjected to an electric potential on the top and bottom faces and at the layer interfaces:

$$\phi(x_1, x_2, \pm e/2) = 100 \text{ V and } \phi(x_1, x_2, \pm 0.3e) = -100 \text{ V;}$$

material properties: PZT4/Epoxy/PZT4 with the following Epoxy properties

$C_{11} = C_{22} = 134.86$ GPa, $C_{12} = 5.156$ GPa, $C_{33} = 14.352$ GPa, $C_{23} = C_{13} = 7.133$ GPa, $C_{44} = C_{55} = 5.654$ GPa, $C_{66} = 64.852$ GPa, $\epsilon_{11} = \epsilon_{22} = .031$ nF/m, $\epsilon_{33} = .0266$ nF/m,

mesh used: $N = 4$ and one numerical layer in each physical layer;

results: $v_3(a/2, b/2, 0)$.

The maximum values for the transverse displacement are given for the two length to thickness ratios in Table 8

Table 8
Sandwich plate with intermediary electric potentials

	$a/e = 10$ v_3 (μm)	$a/e = 50$ v_3 (μm)
CL8NZ2	3.590	89.76
CL8NZ	3.600	90.00
CL8EZ	3.600	90.00
Fernandes [18]	3.625	89.84

in comparison with the three-dimensional reference solution from Fernandes [18]. Same results are obtained for all the electric potential approximations.

6.3. The simply supported square symmetric cross-ply piezoelectric laminated plate

The analytical solution of a simply supported square composite laminate has been presented by Ray et al. [19]. The surfaces are bonded with very thin piezoelectric layers under electrical and mechanical loads. A cross-ply laminate ($0^\circ/90^\circ/0^\circ$) made of three identical plies is used as the core of the structure. The plate has the following characteristics:

geometry: a square plate of length $a = 0.9$ m, of total thickness $e = 9.08$ mm while the core thickness is $e_c = 9$ mm and face one is $e_f = 0.04$ mm; a quarter of the plate is meshed;

boundary conditions and loading: the plate is simply supported at its edges, and is subject to doubly sinusoidal loads:

- a mechanical load $p_3(x_1, x_2) = P_0 \sin(\frac{\pi x_1}{a}) \sin(\frac{\pi x_2}{a})$, with $P_0 = 1$;
- a potential load $\phi(x_1, x_2, e/2) = \phi_0 \sin(\frac{\pi x_1}{a}) \sin(\frac{\pi x_2}{a})$ with $\phi_0 = 100$ and $\phi(x_1, x_2, \pm e/2) = 0$ V;

material properties:

- for the core:
 $E_1 = 25E_2^c$ $E_3 = E_2^c$ $G_{12} = G_{13} = 0.5E_2^c$ $G_{23} = 0.2E_2^c$
with $E_2^c = 6.9$ GPa, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$, $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 8.85 \times 10^{-12}$ F/m,
- for the piezoelectric faces: $E_1 = E_2 = E_3 = 2$ GPa, $\nu_{12} = \nu_{13} = \nu_{23} = 0.29$, $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0.1062 \times 10^{-9}$ F/m, $e_{31} = e_{32} = 0.0046$ C/m²,

mesh used: $N = 8$ and one numerical layer in each physical layer.

results: they are non-dimensionalized as

$$\begin{aligned} \tilde{v}_3 &= \frac{100E_2^c}{S^4 e_c} v_3 \left(\frac{L}{2}, \frac{L}{2}, 0 \right) (\tilde{T}_{11}, \tilde{T}_{22}, \tilde{T}_{12}) \\ &= \frac{1}{S^2} \left(T_{11} \left(\frac{L}{2}, \frac{L}{2}, \frac{\pm e_c}{2} \right), T_{22} \left(\frac{L}{2}, \frac{L}{2}, \frac{\pm e_c}{2} \right), T_{12} \left(0, 0, \frac{\pm e_c}{2} \right) \right) (\tilde{T}_{13}, \tilde{T}_{23}) \\ &= \frac{1}{S} \left(T_{13} \left(0, \frac{L}{2}, 0 \right), T_{23} \left(\frac{L}{2}, 0, 0 \right) \right) \text{ with } S = \frac{a}{e_c} \end{aligned}$$

This test was also studied by Cen et al. [14] and Sheikh et al. [13]. They proposed respectively a 4 node FE and a 9 node FE, based on FSDT theory with nodal electric dof. They showed that there is a mistake in the analytical calculation of the deflection values by Ray et al. [19] but the stress solutions are reliable.

As the applied loads depend on x_1 and x_2 , only CL8NZ and CL8NZ2 using nodal electric dof are used. In Table 9, transverse displacement and in-plane stresses are in good agreement with the other results, and the transverse shear stresses are more accurate with respect to the reference solutions. No difference is observed between the two electric potential approximations because the two piezoelectric skins are very thin.

6.4. A frequency test

The free vibration problem of a five-ply simply supported thin square laminated composite plate with surface bonded piezoelectric layers is now considered in order to validate the developed models in the evaluation of the first five frequencies. The laminate is made of three plies of Graphite/Epoxy ($0^\circ/90^\circ/0^\circ$) and two surfaces bonded PZT4 piezoelectric layers.

geometry: square plate of length a , total thickness e with ratio $\frac{a}{e} = 50$; each ply of the core has a thickness of $0.267e$, and the piezoelectric one is $0.1e$.

boundary conditions and loading: the square plate is simply supported at its edges. Two sets of electric boundary conditions are considered for the inner surfaces:

- a short circuited condition (S), with the potential forced to remain zero (grounded) at the outer and inner surfaces of the piezoelectric layers,
- an open circuited condition (O), where the electric potential remains free everywhere, excepted on the lower surfaces of the plate where it is forced to be zero;

Table 9
Deflection and stresses of smart laminated thin composite plates

Model	$\tilde{v}_3$	$\tilde{T}_{11}$	$\tilde{T}_{22}$	$\tilde{T}_{12}$	$\tilde{T}_{13}$	$\tilde{T}_{23}$
CL8NZ2	0.412	0.505/−0.519	0.159/−0.185	−0.020/0.021	0.382	0.089
CL8NZ	0.412	0.505/−0.519	0.159/−0.185	−0.020/0.021	0.382	0.089
FE (Cen) [14]	0.411	0.505/−0.519	0.159/−0.185	−0.020/0.021	0.378	0.080
FE (Sheikh) [13]	0.411	0.510/−0.514	−	−	−	−
Theory (Ray) [19]	0.447	0.504/−0.518	0.158/−0.184	−0.019/0.021	0.382	0.086

material properties:

- for the Graphite/Epoxy:

$E_1 = 132.38$ GPa, $E_3 = E_2 = 10.76$ GPa, $G_{23} = 3.61$ GPa, $G_{12} = G_{23} = 5.65$ GPa, $\nu_{12} = \nu_{13} = 0.24$, $\nu_{23} = 0.49$, $\epsilon_{11} = 3.5\epsilon_0$, $\epsilon_{22} = \epsilon_{33} = 3\epsilon_0$, $\epsilon_0 = 8.85 \times 10^{-12}$ F/m,

- for the piezoelectric:

$E_1 = E_2 = 81.3$ GPa, $E_3 = 64.5$ GPa, $G_{12} = 30.6$ GPa, $G_{13} = G_{23} = 25.6$ GPa, $\nu_{23} = \nu_{13} = 0.43$, $\nu_{12} = 0.33$, $d_{31} = d_{32} = -122 \times 10^{-12}$ C/N, $d_{33} = -285 \times 10^{-12}$ C/N, $\epsilon_{11} = \epsilon_{22} = 1475\epsilon_0$, $\epsilon_{33} = 1300\epsilon_0$.

In order to compare with the analytical results of Heyliger and Saravanos [20], all layers are assumed to have equal density: $\rho = 1$ kg/m³;

mesh: $N = 8$ for the whole plate and the core is modeled with one numerical layer in each physical layer. For the piezoelectric layers, one numerical layer is used for CL8NZ2 and one or two numerical layers for CL8NZ/CL8EZ;

results: the first five frequencies are non-dimensionalized using the following expression: $\tilde{f}_i = f_i \frac{a}{c} \sqrt{\rho}$.

The first five frequencies and the mode number (m_1, m_2) , are given in Table 10 for the two electric boundary conditions using one numerical layer in each physical layer. They are compared with results from the literature: a two-dimensional closed-form solution from Benjeddou and Deu [21], a finite element solution using FSDT model from Correia et al. [7] and a three-dimensional layerwise exact solution from Heyliger and Saravanos [20].

First of all, the difference between the open-circuit and the closed-circuit conditions is not very significant for CL8NZ2 and CL8NZ. Results obtained with CL8EZ are identical for the two electric boundary conditions due to the assumption of constant value on the elementary domain which imply $\phi_{,1} = \phi_{,2} = 0$. All the results are in

Table 10

First five frequencies $\tilde{f}_i$ (10^3 Hz (kg/m)^{1/2}) of a thin square simply supported plate with one numerical layer in each physical layer

(m_1, m_2)	(O)	(S)	(O)	(S)	(O) and (S)
	CL8NZ2		CL8NZ		CL8EZ
(1, 1)	246.967	246.940	246.912	246.815	246.462
(1, 2)	563.750	563.625	563.600	563.075	560.725
(2, 1)	701.000	700.825	700.900	700.050	698.300
(2, 2)	980.950	980.600	980.725	979.075	973.925
(1, 3)	1108.425	1107.975	1108.125	1105.95	1097.175
	2D closed-form solution [21]		2D FE solution Q9-FSDT [7]		
(1, 1)	246.068	246.067	245.349	206.304	
(1, 2)	559.621	559.615	558.988	519.444	
(2, 1)	693.606	693.601	694.196	663.336	
(2, 2)	967.155	967.141	962.017	907.636	
(1, 3)	1091.481	1091.458	1093.006	1020.102	
	3D layerwise solution [20]				
(1, 1)	245.942	245.941			

Table 11

First five frequencies $\tilde{f}_i$ (10^3 Hz (kg/m)^{1/2}) of a thin square simply supported plate with two numerical layers in each piezoelectric layer

(m_1, m_2)	CL8NZ		CL8EZ
	(O)	(C)	(O) and (C)
(1, 1)	246.952	246.857	246.502
(1, 2)	563.700	563.200	560.825
(2, 1)	700.975	700.150	698.375
(2, 2)	980.900	978.225	974.050
(1, 3)	1108.350	1106.200	1097.350

good agreement with the available reference solutions in Table 10. A discrepancy with the results given by Correia et al. [7] (FSDT 5P) is observed for the short-circuited conditions. They use only one electric potential dof by layer. This proves that a linear variation of the electric potential must be at least assumed in each layer.

In order to evaluate the accuracy of the linear variation approximation in the thickness direction, results obtained with two numerical layers in each piezoelectric layer are given in Table 11. It can be noticed that no significant improvement is observed in this table with respect to Table 10. Therefore, the linear variation approximation with one numerical layer in each piezoelectric layer is usable for the frequency analysis of hybrid sandwich plates.

6.5. Adaptive composite plate

As our future aim is to modelize piezoelectric actuators and sensors in the field of active control of vibration, the case where piezoelectric layers are not located on the entire top and bottom faces of the plate is now presented. For this purpose, CL8EZ is used and previous tests have proved that this FE has the same order of accuracy as CL8NZ and CL8NZ2 with a lower cost.

Therefore, a rectangular composite plate, see Fig. 8, with four bonded piezoelectric actuators presented by Correia et al. [7] is studied. The plate is made of six laminae with the lamination sequence $(90^\circ/0^\circ/90^\circ)_s$. The origin of the coordinate system is at the upper left corner of the plate

Fig. 8. The composite plate with four piezoelectric devices.

Table 12
Transverse displacement for the composite plate with actuators

Model	$\bar{v}_3 (\times 10^{-5})$
CL8EZ with θ_1 restrained	11.651
CL8EZ with θ_1 free	10.466
Correia [7] FSDT5P with θ_1 restrained	11.657
Correia [7] FSDT5P with θ_1 free	10.467
Theory (Correia) [7] without θ_1	11.878

on the laminate midplane. Each piezoelectric device is made of a pair of piezoelectric patches symmetrically bonded to the upper and lower faces of the plate and polarized in opposite directions. The dimensions of the i th patch are $a(i) \times b(i) \times t(i)$ and its geometric center is at $c_1(i)$ and $c_2(i)$.

geometry: the laminated plate has the following dimensions: $a = 6L$, $b = 12L$ with $L = 0.0254$ m. The thickness of each laminae is $0.02L$ and $0.01L$ for the actuator. The following geometric properties are used: $a(i) = 1.5L$ ($i = 1, 2, 3, 4$), $b(i) = 3L$ ($i = 1, 2, 3, 4$), $c_1(1) = c_1(2) = 1.5L$, $c_1(3) = c_1(4) = 4.5L$, $c_2(1) = c_2(3) = 3L$, $c_2(2) = c_2(4) = 9L$,

boundary conditions: The plate is clamped at $x_2 = 0$ and free at the other boundaries. The electric potential difference applied to each actuator is $\Delta\Phi = 300$ V;

material properties:

- for the composite:

$$E_1 = 97.97 \text{ GPa}, E_2 = E_3 = 7.9 \text{ GPa}, G_{12} = G_{13} = G_{23} = 5.5999 \text{ GPa}, \nu_{13} = \nu_{23} = \nu_{12} = 0.28, \rho = 1519.9 \text{ kg/m}^3,$$

- for the piezoelectric actuators:

$$E = 1.999 \times 10^9 \text{ GPa}, \nu = 0.29, \rho = 1799 \text{ kg/m}^3, e_{31} = e_{32} = 0.046 \text{ C/m}^2,$$

mesh: $N = 8$ with 2×2 elements for each actuator;

results: $\bar{v}_3 = Lv_3(\frac{a}{2}, b)$.

The transverse displacement obtained from the present element is compared to the finite element and the analytical solutions developed by Correia et al. [7], see Table 12. For the analytical solution, the rotation θ_1 is restrained. The finite element solutions obtained considering this assumption are in good agreement with the analytical results. When θ_1 is free, the present results are similar to those of Correia et al. [7].

7. Conclusion

A family of finite elements was presented for the calculation of multilayered plate structures with piezoelectric layers or patches. The formulation is based on one of the most efficient eight node finite elements with no transverse shear locking and high accuracy with coarse meshes. Furthermore, different approximations are evaluated for the electric potential: linear and quadratic variations in each layer with respect to the thickness; quadratic approximation in each finite element using its 8 nodes and con-

stant value on the elementary domain for the in-plane variation.

Several tests were analysed to give a clear comparison between the different electric potential approximations and the following remarks can be made:

- no relevant deviation is observed between CL8NZ and CL8EZ; there is no need to introduce an in-plane variation at the elementary level;
- monomorph test: the physical layer must be modeled with only one numerical layer for the quadratic approximation, while many more numerical layers are needed for the linear one to obtain the same accuracy. This is the only test where CL8NZ2 is more efficient than CL8NZ/CL8EZ because the piezoelectric layer is thick and the electric potential variation in the thickness direction has an important effect;
- sandwich cases: same order of accuracy is obtained with linear or quadratic variations for thin piezoelectric layers. Only one numerical layer for each physical layer is necessary for all cases;
- accurate results for displacement and electric potential are obtained using coarse meshes in the field of thin to semi-thick plates;
- the assumption of constant electric potential in each layer must be used carefully and the linear variation is more reliable;
- the less cost FE CL8EZ can be used for a large range of piezoelectric plates. Results obtained are of the same order of accuracy as the other FE with few dof for all tests.

Finally, CL8EZ can be chosen for studying the optimal location and shape of piezoelectric actuators and sensors for active control of plates. The 8 node FE is very useful in order to modelize curved boundaries and round shape piezoelectric patches. An adapted optimization method is currently studied to use the optimal location criteria developed in Bruant and Proslir [22].

Appendix A. The piezoelectric laminae and the behaviour law

A.1. The two-dimensional expressions

In this section, Latin indices $i, j, \dots$ take their values in the set $\{1, 2, 3\}$ while Greek indices $\alpha, \beta, \dots$ take their values in the set $\{1, 2\}$.

The following vectors have been introduced:

- the stress vector: $[T]^T = [T_{11} \ T_{22} \ T_{12} \ T_{23} \ T_{13}]$;
- the strain vector: $[S]^T = [S_{11} \ S_{22} \ S_{12} \ S_{23} \ S_{13}]$ and we have $S_{ij} = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ for $i \neq j$ and $S_{ii} = \frac{\partial u_i}{\partial x_i}$ otherwise;
- the electric field vector: $[E]^T = [E_1 \ E_2 \ E_3]$ with $E_i = -\frac{\partial \phi}{\partial x_i}$;
- the electric displacement vector: $[D]^T = [D_1 \ D_2 \ D_3]$.

Since only small strains are retained and neglecting the normal transverse stress, classical assumption for the plate domain provide: $T_{33} = 0$. Therefore, the bidimensional constitutive law can be deduced from the three-dimensional law and the following expressions are obtained:

$$\begin{aligned} [\bar{C}] &= \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \\ [\bar{e}] &= \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} \\ 0 & 0 & 0 & e_{24} & 0 \\ \bar{e}_{31} & \bar{e}_{32} & 0 & 0 & 0 \end{bmatrix} \quad [\bar{\epsilon}] = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix} \end{aligned} \quad (A.1)$$

with

$$\begin{aligned} \bar{C}_{\alpha\beta} &= C_{\alpha\beta} - C_{\beta 3} \frac{C_{\alpha 3}}{C_{33}} \quad \bar{e}_{3\alpha} = e_{3\alpha} - e_{33} \frac{C_{\alpha 3}}{C_{33}} \\ \bar{\epsilon}_{33} &= \epsilon_{33} + \frac{(e_{33})^2}{C_{33}} \end{aligned} \quad (A.2)$$

In this last expression, C_{ij} , e_{ij} and ϵ_{ii} are the three-dimensional coefficients.

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