Globally analytic p–adic representations of the pro–p Iwahori subgroup of GL(2) and base change, II: a Steinberg tensor product theorem

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Introduction

This is part II of a paper, the first part of which is [4]. In that article we considered the Iwasawa algebra of the pro–p Iwahori subgroup of $GL(2, L)$ for an unramified extension L of degree r of $\mathbb{Q}_p$ and gave a presentation of it by generators and relations, imitating [3]. A natural base change map then appears that, however, is well–defined only for the *globally analytic distributions* on the groups, seen as rigid–analytic spaces.

In §1.1 of [4], we stated that this should be related to a construction of base change for representations of these groups, similar to Steinberg's tensor product theorem [13] for algebraic groups over finite fields.

In this paper we give such a construction, and we show that it is compatible with the (p–adic) Langlands correspondence in the case of the principal series for $GL(2)$.

By the previous remark, we have to limit ourselves to globally analytic representations. These representations have been considered by Emerton in his exhaustive introduction (unfortunately unpublished) to p–adic representation theory [6]. See in particular sections 3.3, 5.1 in his paper; the restriction of scalars, central to our constructions, is considered in his section 2.3.

The first section of this paper contains preliminaries about rigid–analytic groups. The group associated to the pro–p Iwahori is (by Lazard's description) very simple, a product of copies of the rigid–analytic closed unit ball. In particular the algebras of functions we consider are all Tate algebras. We must, however, systematically consider restriction of scalars. Even for such simple spaces, this functor does not behave trivially, as was pointed out to me by Gaëtan Chenevier. See [1, 14]. However, this is the case for unramified extensions (§ 1.1.) It is then an easy matter to describe the natural functorial maps between Tate algebras (Proposition 1.4) and, dually, between (global) distribution algebras (§ 1.2). Nevertheless, the distribution algebra for a product is not a tensor product (even a completed tensor product.) This causes problems in the representation theory, which will be mentioned below; these "pathologies" are reviewed in the Appendix.

In section 2 of this paper, we review the properties of these representations, adding some complements to Emerton's results. In particular, we study tensor products of representations (Theorem 2.3).

In contrast with the category of locally analytic representations, we can work here with (p–adic) Banach spaces rather than with Fréchet spaces, or spaces of compact type [12, 6]. Indeed, the spaces $\mathcal{A}$ and $\mathcal{D}$ of globally analytic functions (resp. distributions) are Banach spaces. The unfortunate consequence is that they are not reflexive. In particular we cannot systematically use duality as in the admissible Banach theory [11] or the locally analytic theory [12]. A related problem is that the spaces $\mathcal{D}$ of distributions are not Nœtherian. See the remarks in § 2.3, as well as the Appendix.

In section 3, we take up the construction of the base change functor, i.e., the Steinberg tensor product. Once the requisite property of the tensor product has been established in section 2, this is totally natural. The main point is that a globally analytic representation will automatically extend, from the L–points of a rigid–analytic group G over L (we consider only very special groups, cf. section 2) to the F–points for any finite extension F of L. Although this is not explicit in [6], it follows from his definitions. The construction is given in §3.2.

Of course this is meaningful only if it is compatible with the expected Langlands correspondence. The end of section 3 is devoted to the proof of this fact for the principal series. We start with the pro–p Iwahori G of $GL(2, \mathbb{Q}_p)$. We must of course consider only the representations which have globally analytic vectors. This condition is specified in (3.4). In Theorem 3.6, we show that (under the same assumption as in [12]) the globally analytic representation of the pro–p Iwahori subgroup of $GL(2, \mathbb{Q}_p)$ is topologically irreducible.

In § 3.6, we extend these results to the pro–p Iwahori subgroup of $GL(2, L)$ where $L/\mathbb{Q}_p$ is unramified. Here we rely on the results of Orlik and Strauch [8]. We show that the formation of the Steinberg tensor product is compatible with Langlands functoriality (cf. Definition 3.8); the final result is

Theorem 3.8 which exhibits base change in this context.

These results concern only the pro–p Iwahori subgroups, not the full groups $GL(2,\mathbb{Q}_p)$, $GL(2,L)$. In § 3.5 we make some tentative remarks about the extension of base change to the full groups. Finally, the Appendix reviews some questions concerning the tensor products of distributions and the non-Noetherian character of these algebras.

While writing this paper I had the benefit of discussions or correspondence with Berthelot, Breuil, Chenevier, Raynaud and Schneider. I am very grateful to them, and especially to Peter Schneider who explained to me the facts reviewed in the Appendix. I also thank Ariane Mézard who read the manuscript and corrected several mistakes.

1 Restriction of scalars and base change maps for analytic functions and distributions on rigid–analytic unit ball groups

1.1

We consider an unramified extension L/L_0, of degree r, of p–adic fields (finite extensions of $\mathbb{Q}_p$). Let $X = B^1/L$ be the closed unit ball over L, a rigid–analytic space whose affinoid algebra is

$$\mathcal{T}_L^1 = L < x > \ .$$

There is a functor of restriction of scalars, which to $X = X/L$ associates a rigid–analytic space $Y = \mathrm{Res}_{L/L_0} X/L_0$.

Lemma 1.1. *(L/L_0 unramified).- Y is isomorphic to the r–th power of B^1/L_0.*

This is a special case of the more general results of Bertapelle [1]. Let (e_i) be a basis of $\mathcal{O}_L$ over $\mathcal{O}_{L_0}$, and let B be an affinoid L_0–algebra. Consider $f \in Hom_L(L < x >, B \bigotimes_{L_0} L)$, thus

$$f(x) = \sum \ b_i e_i \qquad (b_i \in B)\,.$$

We want to define canonically

$$g \in \mathrm{Hom}_{L_0}(L_0 < x_1, \ldots x_r >, B),\ \text{with}\ g(x_i) = b_i\,.$$

(Thus $Y \cong B^r/L_0$ is canonical, given the choice of the basis (e_i).) This is possible if, and only if, $\|b_i\|_{\text{Sup}} \le 1$ assuming $\|b\|_{\text{Sup}} \le 1$ where the sup norms are relative to the affinoid algebras B and $B \bigotimes\limits_{L_0} L$.

Assume first that B is a finite field extension of L_0. Then $B \bigotimes\limits_{L_0} L$ is a product of finite, unramified extensions B_α of L, and the integers $\mathcal{O}(B \bigotimes\limits_{L_0} L) = \prod\limits_\alpha \mathcal{O}(B_\alpha)$ satisfy, the extensions being unramified, $\mathcal{O}(B \bigotimes\limits_{L_0} L) = \mathcal{O}_B \bigotimes\limits_{\mathcal{O}_{L_0}} \mathcal{O}_L = \bigoplus \mathcal{O}_B e_i$. To say that $\|b\|_{\text{Sup}} \le 1$ for $b \in B \otimes L$ is to say that $b_\alpha \in \mathcal{O}(B_\alpha)$, or $b \in \mathcal{O}(B \bigotimes\limits_{L_0} L)$.

This implies that $\|b_i\| \le 1$.

Now let B be a general affinoid algebra over L_0, and $B' = B \bigotimes\limits_{L_0} L$. If $b = \sum b_i e_i \in B'$ $(b_i \in B)$, the computation in [1, p. 444] shows that

$$\|b\|_{\text{Sup}} = \operatorname*{Sup}_{y \in \operatorname{Max} B} \ \operatorname*{Max}_{\substack{x \in \operatorname{Max} B' \\ x | y}} \left\| \left(\sum b_i e_i \right)(x) \right\|_{\text{Sup}} .$$

However, y corresponds to a finite extension K_0 of L_0, x to a finite extension K of L contained in $L \bigotimes\limits_{K_0} L_0$ so unramified over L. The previous result implies that $\| \sum b_i e_i(x) \|_{\text{Sup}} = \text{Sup} \|b_i(x)\|$. Thus $\|b_i\|_{\text{Sup}} \le 1$ if $\|b\|_{\le 1}$. We note that we have in fact:

Lemma 1.2. *The isomorphism $Y \xrightarrow{\simeq} (B^1/L_0)^r$ is canonically defined by the choice of the basis (e_i).*

In fact, the function g (for instance if $B = K_0$ is a field extension of L_0) is defined by

$$g(x_1, \dots x_r) = f(\Sigma e_i x_i) . \tag{1.1}$$

$(|x_i| \le 1)$. The e_i being integral, it is easy to check that for $f \in \mathcal{T}_L^1$, the infinite series in the right is convergent.

Since restriction of scalars is compatible with direct products [1, Prop. 1.8] we have likewise

$$\operatorname{Res}_{L/L_0} (B^1/L)^d = (B^1/L_0)^{dr}$$

the isomorphism being canonical once we have fixed the basis (e_i).

1.2

We now consider a rigid–analytic group G_L over L, isomorphic as a rigid–analytic space to $(B^1/L)^d$. (In particular $G_L(L)$ is dense in G_L for the Zariski topology.) Let $\mathcal{A}(G_L) \cong \mathcal{T}_L^d$ be the space of analytic functions on G_L. The multiplication in G_L is defined by a morphism

$$m^* : \mathcal{A}(G_L) \longrightarrow \mathcal{A}(G_L)\widehat{\otimes}\mathcal{A}(G_L)$$

(completed tensor product). In this case, the product is given, in co–ordinates, by integral functions, [2, Cor. 5.1.3.5] so

$$m^* : \mathcal{A}^0(G_L) \longrightarrow \mathcal{A}^0(G_L)\widehat{\otimes}\mathcal{A}^0(G_L)\,.$$

Then m^* defines naturally a map

$$\text{Res } m^* : \mathcal{A}^0(\text{Res } G_L) \longrightarrow \mathcal{A}^0(\text{Res } G_L)\widehat{\otimes}\mathcal{A}^0(\text{Res } G_L)\,,$$

Res being the restriction of scalars of G_L, a group over L_0.

Assume now that the group G_L is actually defined over L_0, i.e., is obtained by *extension* of scalars from L_0. Then $\mathcal{A}(G_L) = \mathcal{A}(G_{L_0}) \otimes L$. The map m^* is obtained by extension of scalars from

$$m_0^* : \mathcal{A}(G_{L_0}) \longrightarrow \mathcal{A}(G_{L_0})\widehat{\otimes}\mathcal{A}(G_{L_0})\,.$$

The integrality property for G_L and the property for G_{L_0} are equivalent.

Now the previous construction associates to $f \in \mathcal{A}(G_L)$ (with L–coefficients, i.e. in $\mathcal{T}^n(L)$) a function g in $\mathcal{A}(\text{Res}G_L) \otimes L$ (the function g defined by (1.1) will have coefficients in L). In particular we get a map $\mathcal{A}(G_{L_0}) \to \mathcal{A}(\text{Res } G_L) \otimes L$ by composition with the previous "tautological" map $\mathcal{A}(G_{L_0}) \to \mathcal{A}(G_L)$.

Definition 1.3. *This map b_1: $\mathcal{A}(G_{L_0}) \to \mathcal{A}(\text{Res } G_L) \otimes L$ is the holomorphic base change map.*

This map commutes with the comultiplications m_0^* and Res m^*: it is obvious if we consider m_0^* and m^*, and for m^* and Res m^* it follows from the formal properties of restriction of scalars. Furthermore b_1 sends $\mathcal{A}^0(G_{L_0})$ to $\mathcal{A}^0(\text{Res } G_L \otimes L)$.

The unramified extension L/L_0 is Galois. Thus the Galois group $\Sigma = \text{Gal}(L/L_0)$ acts naturally on G_L (by σ–linear automorphisms of the Tate algebra) and acts on Res G_L by L_0–automorphisms.

Definition 1.4. *The map* $b : \mathcal{A}(G_{L_o}) \to \mathcal{A}(\mathrm{Res}\ G_L) \otimes L$ *is defined by*

$$b(f) = \prod_{\sigma \in \Sigma} b_1(f)^\sigma \,.$$

Since b_1 commutes with the comultiplication, the same is true for the product $\Pi\ b_1^\sigma$. We also note the following: Assume we *extend* scalars from L_0 to L for the L_0–groups. Then

$$\mathrm{Res}\ G_L \otimes_{L_0} L$$

is naturally isomorphic to $\prod\limits_\sigma G_L$. Indeed, if B is an L–algebra (in particular an affinoid algebra), $B \otimes_{L_0} L \cong \bigoplus\limits_\sigma B_\sigma$ where $B_\sigma = \{\beta \in B \otimes L : \lambda_1 \beta = \lambda_2^\sigma \beta$ where $\lambda \in L$ and λ_1 is the action of λ on $B \otimes L$ by the first component, λ_2 by the second component. Now, $\mathcal{A}$ denoting a Tate algebra:

$$\begin{aligned}
\mathrm{Hom}_L \quad & (\mathcal{A}(\mathrm{Res} G_L) \otimes_{L_0} L, B) \qquad (B/L) \\
& = \mathrm{Hom}_{L_0}(\mathcal{A}(\mathrm{Res}\ G_L), B_0) \\
& \quad (B_0 \text{ being equal to } B/L_0) \\
& = \mathrm{Hom}_L(\mathcal{A}(G_L), B_0 \otimes L) \\
& = \bigoplus_\sigma \mathrm{Hom}_L(\mathcal{A}(G_L), B_\sigma)\,.
\end{aligned}$$

In particular, after extension of scalars to L, $\mathcal{A}(\mathrm{Res}\ G_L) \otimes L \cong \widehat{\bigotimes_\sigma} \mathcal{A}(G_L)$. The map b is then a tensor product: b_1 sends $\mathcal{A}(G_{L_0})$ to the functions on G_L that are L–holomorphic (given by power series $\Sigma a_m \underline{x}^m$, $\underline{x} = (x_1, \ldots x_d)$ being the variable) while the component associated to σ sends a power series in $\mathcal{A}(G_0)$ to $\Sigma a_m \sigma(\underline{x})^m$.

We now agree to consider all Tate algebras as having coefficients in L, and we denote them by $\mathcal{A}_L$.

Summarizing, we now have the following result:

Proposition 1.5. *(i) There exists a natural map* b_1: $\mathcal{A}_L(G_{L_0}) \to \mathcal{A}_L(\mathrm{Res}\ G_L)$. *It commutes with the comultiplications.*

(ii) There exists a natural map $b = \prod\limits_{\sigma \in \Sigma} b_1^\sigma : \mathcal{A}_L(G_{L_0}) \to \mathcal{A}_L(\mathrm{Res}\ G_L)$. *It commutes with the comultiplication.*

(iii) In the isomorphism $\mathcal{A}_L(\mathrm{Res}\ G_L) \cong \widehat{\bigotimes_\sigma} \mathcal{A}(G_L)$ $(\mathcal{A}(G_L) = \mathcal{A}_L(G_L))$, $b = \bigotimes\limits_\sigma b_1^\sigma$.

(iv) The maps b_1 and b send the unit balls $\mathcal{A}^0_L(G_{L_0})$ to $\mathcal{A}^0_L(\text{Res } G_L)$. (The norm being the sup norm of coefficients).

We now consider the spaces of (L–valued) global distributions on G_{L_o} and Res G_L). We denote them by $\mathcal{D}_L(G_{L_0})$, $\mathcal{D}_L(\text{Res } G_L)$. These are the Banach spaces dual to the Banach spaces of analytic functions (for the sup norms). We obtain, dually, a map

$$b_1^* : \mathcal{D}_L(\text{Res } G_L) \to \mathcal{D}_L(G_{L_0})$$

and also

$$b^* : \mathcal{D}_L(\text{Res } G_L) \to \mathcal{D}_L(G_{L_0}) .$$

These are homomorphisms, for the convolution of distributions. Using (iii) in the Proposition, we can write

$$\bigotimes_\sigma \mathcal{D}_L(G_L) \subset \mathcal{D}_L(\text{Res } G_L)$$

and b^* is then, on this subspace, given by

$$\bigotimes_\sigma T_\sigma \mapsto \mathop{*}_\sigma T_\sigma$$

(where $T_\sigma \in \mathcal{D}_L(G_{L_0})$ is σ–holomorphic). However, $\widehat{\bigotimes_\sigma} \mathcal{D}_L(G_L)$ is **not** equal to $\mathcal{D}_L(\text{Res } G_L)$. Since, after extension of scalars, our groups become products, this can be seen as follows.

We may forget for a moment the restriction of scalars, and consider two groups G, H isomorphic (as rigid–analytic spaces) to $(B^1)^d$, $(B^1)^{d'}$ over L. The spaces of analytic functions are $\mathcal{T}^d(L)$, $\mathcal{T}^{d'}(L)$, with the sup norm. The dual $\mathcal{D}_L(G)$ of the space of functions

$$f(x) = \sum_n a_n \, \underline{x}^n , \qquad a_n \to 0$$

($n \in \mathbb{N}^d$, $\underline{x} = (x_1, \dots x_d)$, $|x_i| \leq 1$) is the space of distributions

$$T = \sum_n c_n \; \delta_n \qquad (|c_n| \leq C)$$

where $\delta_n(f) = a_n = n \,! \, \frac{\partial^n f}{\partial \underline{x}^n}(0)$. It is a Banach space, the norm being $\sup |c_n|$. The same description applies to a distribution S on H, and a distribution on

$G \times H$. However, these Banach spaces are ℓ^∞ spaces in the indexes, and for three (countable) sets X, Y, $X \times Y$, it is not true that

$$\ell^\infty(X) \mathbin{\widehat{\otimes}} \ell^\infty(Y) = \ell^\infty(X \times Y)\,.$$

In order to form tensor products, we must consider the unit balls in $\mathcal{D}_L(G)$, $\mathcal{D}_L(H)$ (with their weak topology) and apply a result of Lazard. This was explained to me by Peter Schneider; we will return to it at the end.

2 Globally analytic representations

2.1

In this section we review some basic properties of globally analytic representations of a rigid–analytic group on a Banach space, mostly following Emerton [6]. We assume given L and G/L as in § 1.2. We denote by $\mathcal{A} \cong \mathcal{T}_d(L)$ the space of globally analytic functions on G. We will often write G for $G(L)$ if this does not lead to confusion ; $G(L)$ is dense in G for the Zariski topology.

2.2

Let V be a Banach space over a field K containing L. We assume again K finite over $\mathbb{Q}_p$. If $g \mapsto \pi(g)$ is a representation of G on V, we say that π (or V) is a globally analytic representation if the map

$$g \mapsto g \cdot v = \pi(g)v$$

is (globally) analytic on G for all $v \in V$. Thus, in coordinates $(x_1, \ldots x_n)$:

$$g \cdot v = \sum_m \underline{x}^m v_m$$

where $v_m \in V$ and $\|v_m\| \to 0$.

Here $m = (m_1, \ldots, m_d)$ and $\underline{x}^m = x_1^{m_1} \cdots x_d^{m_d}$, $m_i \in \mathbb{N}$. Such a representation is automatically continuous, and even differentiable. We will simply use the term "analytic" for "globally analytic". Note that it is relative to the L–structure on V.

In this situation V is endowed with two natural norms, the given norm and

$$\|v\|_\omega = \operatorname*{Sup}_m \|v_m\|\,.$$

The second norm is the norm of the map $g \mapsto gv$ in the Banach space $\mathcal{A}(G,V) = \mathcal{A}(G)\hat{\otimes}V$ (for this isomorphism cf. e.g. [6, § 2.1]). The map $(V, \| \ \|_\omega) \to (V, \| \ \|)$ is bijective and obviously continuous. Since V, with the norm $\| \ \|_\omega$, is complete [6, 3.3.1, 3.3.3] it is bicontinuous by Banach's isomorphism theorem [9, Cor. 8.7].

We recall the proof of the completeness of $(V, \| \ \|_\omega)$, as we will require similar arguments. Thus let $(v^\alpha)_\alpha$ be a Cauchy sequence in V for $\| \ \|_\omega$. For each α, $(v^\alpha_m)_{m\in M}$ is an element of $C^0(M,V)$ where $M = \mathbb{N}^d$ is the set of exponents. Since this space is complete, $(v^\alpha_m)_m \mapsto (v_m)$ in $C^0(M,V)$ for an element $v_m \in C^0(M,V)$. In particular $v^\alpha = v^\alpha_0 \to v_0 \in V$. Now $gv = \lim\limits_\alpha gv^\alpha_o$ $(g \in G)$, so $gv = \lim\limits_\alpha(\sum\limits_m \underline{x}^m v^\alpha_m)$. Since

$$\Big\| \sum_m \underline{x}^m (v^\alpha_m - v_m) \Big\| \leq \operatorname*{Sup}_m \Big\| v^\alpha_m - v_m \Big\| \longrightarrow 0 \qquad (\alpha \to \infty)$$

we see that $gv = \sum \underline{x}^m v_m$, which implies that $\|v - v^\alpha\|_\omega \to 0$.

Corollary 2.1. *There exists a constant C_V (depending on V) such that $\|v\|_\omega \leq C_V \|v\|$ $(v \in V)$.*

In particular $\|gv\| \leq C_V \|v\|$ $(g \in G)$.

In fact the original norm can be replaced by an equivalent norm such that $\|gv\| = \|v\|$: see Emerton [6, §6.5].

Lemma 2.2. *Let $(V, \| \ \|)$ be a continuous Banach representation of G, and let $W \subset V$ be a subspace comprised of analytic vectors. Assume that $\|w\|_\omega \leq C\|w\|$ $(C > 0)$ for $w \in W$. Then any vector of $\bar{W} \subset V$ (the closure for the topology of V) is analytic.*

Proof.— Consider a sequence $(w^\alpha)_\alpha$ of vectors in W, such that $\|w^\alpha - v\| \to 0$ $(v \in V)$. Then w^α is a Cauchy sequence for $\| \ \|$, so also for $\| \ \|_\omega$. If

$$g \cdot w^\alpha = \sum_m \underline{x}^m w^\alpha_m \,,$$

the sequence $(w^\alpha_m)_{m\in M}$ has a limit (v_m) in $C^0(M,V)$. In particular $v_0 = v$. Again

$$g\, w^\alpha = \sum_m \underline{x}^m w^\alpha_m \longrightarrow gv \qquad (\alpha \to \infty$$

and $\| \sum_m \underline{x}^m (w_m^\alpha - v_m)\| \leq \operatorname{Sup}_m \|w_m^\alpha - v_m\| \to 0$ $(\alpha \to \infty)$ which implies that $gv = \sum_m \underline{x}^m v_m$.

Consider now two rigid analytic groups G, H verifying our assumptions. Let V, W be analytic representation of G, H on Banach spaces. We assume the norms invariant, using Emerton's result. Then $G \times H$ acts on the algebraic tensor product $V \otimes W$. By [9, Prop. 2.1.7.5] this action extends to $V \hat{\otimes} W$, with $\|(g,h)u\| = \|u\|$ $(u \in V \hat{\otimes} W)$.

Now $V \otimes W$ is dense in $V \hat{\otimes} W$, and is comprised of analytic vectors : if $v \in V$, $w \in W$ and

$$gv = \sum_m \underline{x}^m v_m \,, \quad hw = \sum_p \underline{y}^p w_p$$

$(g \in G,\ h \in H)$ then

$$(g,h)(v \otimes w) = \sum_{m,p} \underline{x}^m \underline{y}^p v_m \otimes w_p \,.$$

Since $\|v_m \otimes w_p\| = \|v_m\| \, \|w_p\|$ (Schneider [9, Prop.17.4]), this yields an analytic expansion.

Now endow $V \otimes W$ with its analytic norm $\| \, \|_\omega$, for the action of $G \times H$. We have

$$\begin{aligned} \|v \otimes w\|_\omega &= \operatorname*{Max}_{m,p} \|v_m \otimes w_p\| \\ &= \operatorname{Max}\|v_m\| \operatorname{Max}\|w_p\| \\ &= \|v\|_\omega \, \|w\|_\omega. \end{aligned}$$

Now consider any vector $u \in V \otimes W$. The tensor product norm is defined by

$$\|u\| = \inf \operatorname*{Max}_i \|v_i\| \, \|w_i\|$$

over the decompositions $u = \sum v_i \otimes w_i$. Choose $\varepsilon > 0$, and a decomposition such that

$$\begin{aligned} & \|u\| && \geq \operatorname{Max}\|v_i\| \, \|w_i\| - \varepsilon. \\ \text{Then} \quad & \|u\|_\omega && \leq \operatorname*{Max}_i \|v_i \otimes w_i\|_\omega \\ & && \leq C_V C_W \operatorname*{Max}_i \|v_i \otimes w_i\| \leq C_V C_W(\|u\| + \varepsilon). \end{aligned}$$

Thus $\|u\|_\omega \leq C_V C_W \|u\|$, and $V \otimes W \subset V \hat{\otimes} W$ verifies the assumption of the Lemma. This implies:

Theorem 2.3. *If V, W are (globally) analytic representations of G, H, $V\hat{\otimes}W$ is a globally analytic representation of $G \times H$.*

(For a similar result, but for locally analytic representations, see Emerton [6, 3.6.18]).

We also note the following property. Let $\mathfrak{g}$ be the Lie algebra of G (over $\mathbb{Q}_p$).

Proposition 2.4. *If V is a globally analytic representation of G and $W \subset V$ is a closed subspace, W is G–invariant if and only if W is invariant by the enveloping algebra $U(\mathfrak{g})$.*

(Recall from [12] that the Lie algebra, or $U(\mathfrak{g})$, acts on a space of analytic vectors). If W is G–invariant, it contains the derivatives $Xw = \lim\limits_{t\to 0} \left(\frac{e^{tX}-1}{t}\right) w$ of its vectors by elements $X \in \mathfrak{g}$. Conversely, if

$$gw = \sum_m \underline{x}^m v_m$$

then $v_m = \frac{1}{m!}\frac{d^m}{d\underline{x}^m}\Big|_0 (gv)$, the derivative being computed with respect to the variables $\underline{x}$. However the enveloping algebra (acting via $uf = (u * f)(0)$ for $u \in U(\mathfrak{g})$, f an analytic function on G) also spans the space of derivatives at 0. If W is invariant by $U(\mathfrak{g})$, the coefficient v_m belong to W and therefore $gw \in W$.

By contrast with the case of complex unitary representations, we do not know if $V\hat{\otimes}W$ is (topologically) irreducible if V, W are topologically irreducible. The only, obvious, property is that $V\hat{\otimes}W$ is topologically cyclic (i.e., the closed subspace generated by a suitable vector is equal to $V\hat{\otimes}W$ if V and W are - in particular if they are irreducible.) Indeed, if v spans V and w spans W, $v \otimes w$ spans $V\hat{\otimes}W$.

2.3

Finally, we also recall from Emerton's paper that there is a duality theory for globally analytic representations, similar to the duality for locally analytic (or Banach admissible) representations. If V is a globally analytic representation, the distribution algebra $\mathcal{D}_K(G)$ acts on the dual V'. There is a duality between closed submodules of $(\mathcal{A}(G) \otimes K)^n$ and quotients of $\mathcal{D}_K(G)^n$. See [6, Theorem 5.1.15]. We will not be able to use this, however. There are two obstacles: the algebra $\mathcal{D}_K(G)$ is not Nœtherian; furthermore, as noticed at the end of § 1, it does not behave well with respect to the product of groups.

Let us define an *admissible globally analytic* representation as a globally analytic Banach representation that is a closed submodule of $(\mathcal{A}(G) \otimes K)^n$. Recall also from [10, 6] that there is a category of admissible (continuous) Banach representations and of admissible locally analytic Banach representations on spaces of compact type [11]. In general, an admissible globally analytic representation is not an admissible locally analytic representation (an infinite–dimensional Banach space is not of compact type) and is not an admissible Banach representation. Indeed, if E is such a representation and E^0 is its unit ball (for a given G–invariant norm), and if ϖ is a uniformising parameter of K, it is known that $E^0/\varpi E^0 = \bar{E}$ is a smooth admissible representation of G over the finite residue field k of K [10],[6, 6.5.7]. However, $\mathcal{A}(G)$ does not have that property.

For instance, if G is the additive unit ball, so $V = \mathcal{A}(G) \otimes K = \mathcal{T}_1(K)$, its unit ball is translation–invariant and the subgroup $\varpi_L \mathcal{O}_L$ of $G(L) = \mathcal{O}_L$ acts trivially on $\bar{V} = k[x]$, so this representation is not admissible.

Assume however that E is an admissible Banach representation. Then E is a closed subspace of $\mathcal{C}(G, K)^n$ for some n [10],[6, § 6]. Let $V = E^{an}$ be the space of globally analytic vectors. Emerton's results (see the proof recalled before Cor. 2.1) show that V is complete for the norm $\| \ \|_\omega$. It is an analytic Banach representation [6, Cor. 3.3.6].

Assume $V = E^{an}$ is dense in E. Since $\mathcal{C}(G, K)^{an}$ is equal to $\mathcal{A}(G) \otimes K$, V is sent to $(\mathcal{A}(G) \otimes K)^n$. Let $j = (j_i)_{i=1,\dots n}$ be the closed embedding $E \to \mathcal{C}(G, K)^n$. By Banach's theorem $\|v\| \geq C \operatorname*{Sup}_i \|j_i(v)\|$ for $v \in E$, C being a > 0 constant. This implies that $\|v\|_\omega \geq C \operatorname*{Sup}_i \|j_i(v)\|_\omega$ for $v \in V$. The canonical norm $\| \ \|_\omega$ on $\mathcal{A}(G)$ is the usual norm – the sup norm on coefficients. (See Proposition 2.7 below.) Thus V is a closed subspace of $(\mathcal{A}(G) \otimes K)^n$. Conversely, if V is such a subspace, we can consider its closure $E \subset \mathcal{C}(G), K)^n$. It is an admissible Banach representation in which V is dense. Clearly $V \subset E^{an}$, but it does not seem to follow that V is equal to E^{an}. To summarise:

Proposition 2.5. *Any any admissible globally analytic representation is a dense subspace of an admissible Banach representation. If E is an admissible Banach representation, E^{an} is an admissible globally analytic representation.*

The admissible analytic representations have further interesting properties. Recall that in general, if V is an analytic representation, there is an action of $\mathcal{D}(G) \otimes K$ on the continuous dual V' [6, 5.1.8]. If V is admissible, we can say more.

Assume $T \in \mathcal{D}(G)$ (we forget the extension of scalars for simplicity of notation.) If $f \in \mathcal{A}(G)$, we can define a function $T * f$ by

$$T * f(x) = \int T(z) f(z^{-1}x) dz \tag{2.1}$$

in functional notation, i.e. T applied to the function of z, $z \mapsto z^{-1}x$. Since $f(z^{-1}x)$ is in the Tate algebra of $G \times G$, this is well–defined and, moreover, defines a function in $\mathcal{A}(G)$. Thus $\mathcal{D}(G)$ acts by convolution on $\mathcal{A}(G)$, and this is compatible with the convolution product.

Assume now that $V \subset \mathcal{A}(G)$ is a closed invariant subspace. Then V is invariant by the differential operators $\frac{1}{m!}\frac{d^m}{dx^m}$. If $f \in \mathcal{A}(G)$ and

$$T = \sum_m c_m \frac{1}{m!} \frac{d^m}{dx^m}\Big|_0$$

$\in \mathcal{D}(G)$ (with c_m bounded), $T * f$ is the limit in $\mathcal{A}(G)$ of $T_X f$,

$$T_X f = \sum_{|m| \leq X} c_m \frac{1}{m!} \frac{d^m}{dx^m} \; f$$

as can be seen by expanding the function $f(z^{-1}x)$ in (2.1) in the Tate algebra of $G \times G$. Therefore V is invariant by $\mathcal{D}(G)$. The same extends to an embedding $V \to \mathcal{A}(G)^n$. Thus:

Proposition 2.6. *If V is an admissible globally analytic representation, the distribution algebra $\mathcal{D}(G)$ acts naturally on V. The action is continuous if $\mathcal{D}(G)$ is equipped with its weak dual topology.*

The continuity follows from the precious argument. It implies in particular that the action is intrinsic.

We recall that for locally analytic representations this construction is due to Schneider and Teitelbaum [11, § 3]. However their proof relies on an isomorphism

$$\mathcal{L}(\mathcal{D}_{loc}(G), V) \cong \mathcal{A}_{loc}(G, V)$$

([11, Thm. 2.2]; here $\mathcal{A}_{loc}(G)$ is the space of locally analytic functions and $\mathcal{D}_{loc}(G)$ its dual space, and V is a suitable topological space. The analogue is not true in our context. Indeed

$$\mathcal{A}(G,V)=\mathcal{A}(G)\hat{\otimes}V\cong C_0(M,V)$$

where M is our set of exponents, while $\mathcal{D}(G)\cong\ell^\infty(M,L)$. Since $\ell^\infty(\mathbb{N})'$ is distinct from $C_0(\mathbb{N})$, we see a fortiori that these spaces are not isomorphic.

Because the comultiplication is given by integral series, we also have:

Proposition 2.7. *Consider the admissible representation $V=\mathcal{A}(G)\otimes K$ of G, with its usual norm (sup of the coefficients.) Then*

(i) V is a unitary representation.

(ii) On V, $\| \ \|_\omega = \| \ \|$.

(iii) For $T\in\mathcal{D}^0=(V')^0$ and $f\in\mathcal{A}^0$, the function $g\mapsto <T,gf>$ is in $\mathcal{A}^0(G,K)$.

These facts easily follow from the property of the coproduct. Since an admissible analytic representation embeds as a closed subspace of $(\mathcal{A}(G)\otimes K)^n$, it follows that:

Corollary 2.8. *Properties (i-iii) of Proposition 2.7 are true for an admissible analytic representation.*

3 Unramified base change : the pro–p Iwahori for $GL(2)$

3.1

The content of this section is twofold: we first describe a functor producing, for an unramified extension L/L_0 and a globally analytic representation of $G(L_0)$ (the assumptions are those of § 1), a representation of $G(L)$ of the same kind. In fact, as in § 1 for distribution algebras, there are two such functors. The first produces a "holomorphic" extension to $G(L)$. The second ("full base change") is the one that should be related to Langlands functionality. It is the "Steinberg tensor product" described at the end of section 1.1 of [4].

We then show that for $GL(2)$ and principal series representations of the pro–p Iwahori subgroup, this is compatible with base change for the principal series described by Schneider–Teitelbaum and Orlik–Strauch [11, 8]. In particular we show that certain globally analytic tensor products are irreducible.

3.2

Let L/L_0 denote an unramified extension of p–adic groups and G a rigid analytic group over L_0 verifying the conditions of § 1. We fix a p–adic field K (finite over $\mathbb{Q}_p$) and an injection $L : L \subset K$. If $\sigma \in \mathrm{Gal}(L/L_0)$, we then have the injection $L \circ \sigma : L \to K$.

Let V denote a (globally) analytic representation of $G(L_0)$ on a K–Banach space.

Proposition 3.1. *(i) V extends naturally to an analytic representation of $G(L)$.*

(ii) If V is admissible, the corresponding representation of $G(L)$ is admissible.

The group $G(L_0)$ acts on V by

$$g \cdot v = \sum_m \underline{x}^m v_m \tag{3.1}$$

with the notations of § 2, and $v_m \to 0$. If $g \in G(L)$, the same expansion (with $\underline{x} = (x_1, \dots x_n) \in \mathcal{O}_L^n$) is convergent, and we define $g \cdot v$ by (3.1).We must check that this defines a group representation of $G(L)$. The map

$$\begin{array}{l} (g,h) \mapsto gh.v = F(g,h) \\ G(L) \times G(L) \longrightarrow V \end{array}$$

is the composition of the map $(g,h) \mapsto gh$, analytic in the two variables, and of an analytic map $G(L) \to V$. It is analytic in the two variables.

On the other hand we have for g, $h \in G(L_0)$:

$$g(hv) = g\, F(1,h)\,. \tag{3.2}$$

Write $\underline{x}$ for the co-ordinates of g and $\underline{y}$ for the co-ordinates of h. Then

$$F(1,h) = hv = \sum_m \underline{x}^m v_m\,.$$

On the other hand, for any v_m,

$$gv_m = \sum_p \underline{y}^p v_{m,p}$$

with $v_{m,p} \to 0$ ($|p| \to \infty$).

Since $\|v_m\| \leq C_V \|v\|$ for any m and $v \in V$,

$gF(1,h) = \sum\limits_{m,p} \underline{x}^m \underline{y}^p v_{mp}$,

the double sum being convergent: if $|m| + |p| \to \infty$, either $m \to \infty$ and $\|v_{m,p}\| \leq C_V \|v_m\| \to 0$ or m is bounded and, again, $\|v_{m,p}\| \to 0$. Thus the function $gF(1,h) : G(L_0) \times G(L_0) \to V$ is a Tate series (with coefficients in V) in the two variables, and extends to an analytic function $G(L) \times G(L) \to V$. Since $F(g,h) = gF(1,h)$ for $g \in G(L_0)$, these two analytic functions coincide: indeed $G(L_0)$ is Zariski–dense in G, and the result follows (for instance, evaluate the two functions against a continuous linear form $\lambda \in V'$).

This proves (i). Assume now V is a closed subspace of $\mathcal{A}(G_{L_0}, K)$. Note that the same argument applies to $\mathcal{A}(G_{L_0}, K)$, an analytic representation of $G(L_0)$. But $\mathcal{A}(G_{L_0}, K) = \mathcal{A}(G_L, K)$ and now V (as a representation of $G(L)$ is a closed subspace of $\mathcal{A}(G_L, K)$.

We will call the extension of Proposition 3.1 the holomorphic base change of V. Its coefficients are L–analytic (for the given embedding $L \to K$): it is L–analytic in the sense of Emerton [6].

If $\sigma \in \mathrm{Gal}(L/L_0)$ we write V^σ for the representation of $G(L)$ associated to $\iota \circ \sigma$. It is L–analytic for $\iota \circ \sigma$.

Definition 3.2. *The full base change of V is the globally analytic representation of* $\mathrm{Res}_{L/L_0} G(L)$ *on* $W = \widehat{\bigotimes_\sigma} V^\sigma$.

It is analytic for $\mathrm{Res}_{L/L_0} G(L)$ by the results of § 1. (Note that L/L_0 being unramified, $\mathrm{Res}_{L/L_0} G(L)$ is again a group of the same type.) The fact that the completed tensor product is globally analytic follows from § 2.

When V is the restriction to $G(L)$ — the L–points of a rigid–analytic group deduced from a suitable integral structure of a reductive group $\mathcal{G}/L$ — of a representation (still denoted by V) of $\mathcal{G}(L)$, we conjecture that this will be compatible, in some sense, with Langlands base change (still conjectural) for p–adic Banach representation of $\mathcal{G}(L)$. Of course the relation between admissible Banach representations and globally analytic Banach representations (for $G(L)$) is not one–to–one, cf. Proposition 2.5. It would be interesting to determine which Banach spaces E give rise to a given V, for instance if V is irreducible. Furthermore, even in the case of irreducible principal series V for $\mathcal{G}(L)$, the restriction to $G(L)$ is not irreducible. The full base change of Definition 3.2 then describes only certain of its submodules.

This will be clear for the principal series.

3.3

We now consider the case of the principal series for $GL(2)$. For simplicity we assume $L_0 = \mathbb{Q}_p$. We assume $p > 2$, so G is a product of 1–dimensional balls. Thus $\mathcal{G}(L_0) = GL(2, \mathbb{Q}_p)$ while G is the rigid–analytic group studied in the first part of this paper. The principal series is then described by Schneider and Teitelbaum [11]. (They define the Iwahori subgroup by matrices that are lower triangular $\mod p$ while in [4] we consider upper–triangular matrices. We will follow their choices.)

Let $B = \left\{ g \in GL(2, \mathbb{Z}_p) : g \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} [p] \right\}$, so our group $G = G(\mathbb{Q}_p)$ is a subgroup of B. Let $P_0 \supset T_0$ be the set of upper triangular (resp. diagonal) matrices in B. Let $\chi : T_0 \to K^\times$ be a locally analytic character, such that

$$\chi \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} = \exp(c(\chi) log(t))$$

for $t \in T_0 = (\mathbb{Z}_p^\times)^2$ when t is sufficiently close to 1. Thus $c(\chi) \in K$.

We consider first, as they do, the locally analytic induced representation of B

$$J_{loc} = \mathrm{ind}_{P_0}^B(\chi) = \{f \in \mathcal{A}_{loc}(B, K) : f(gb) \equiv \chi(b^{-1}) f(g)\}$$

$(b \in P_0)$, where χ is naturally extended to P_0. We have

$$B = UP_0\,, \quad U = \left\{ \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix}, \quad z \in \mathbb{Z}_p \right\}. \tag{3.3}$$

Note that since χ is fixed, the restriction of the functions of J_{loc} to $G \subset B$ is injective. With

$$Q_0 = P_0 \cap G = \left\{ \begin{pmatrix} s & x \\ 0 & t \end{pmatrix} : s, t \equiv 1\,,\ x \equiv 0\ [p] \right\}$$

we see that the space of J_{loc} is

$$I_{loc} = \{f \in \mathcal{A}_{loc}(G, K) : f(gb) \equiv \chi(b^{-1}) f(g)\}$$

$(b \in Q_0)$. With (3.3) replaced by $G = UQ_0$, we see that $I_{loc} \cong \mathcal{A}_{loc}(\mathbb{Z}_p, K)$ where $\mathbb{Z}_p$ is seen as the rigid analytic (additive) group $B^1(\mathbb{Z}_p)$. The group G acts by left translations, thus by $f(g) \mapsto f(h^{-1}g)$. We now have [11, Lemma 5.2]:

Lemma 3.3. *For $y \in \mathbb{Z}_p$, $x \in p\mathbb{Z}_p$, s, $t \equiv 1$ $[p]$:*

$$\begin{aligned}
(i) \quad & \begin{pmatrix} 1 & \\ y & 1 \end{pmatrix} f(z) &=& f(z-y) \\
(ii) \quad & \begin{pmatrix} s & \\ & t \end{pmatrix} f(z) &=& f(st^{-1}z)\chi(s,t) \\
(iii) \quad & \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} f(z) &=& f\Big(\tfrac{z}{1-xz}\Big)\chi((1-xz)^{-1},\ 1-xz)\,.
\end{aligned}$$

We now seek conditions such that $\mathcal{A}(B^1, K)$, where B^1 is the unit ball in the z–variable, is a globally analytic representation. We simply denote this space by $\mathcal{A}$; we will similarly drop the subscript K in this section.

Lemma 3.4. *It suffices to check analyticity separately for the 1–parameter (rigid–analytic) subgroups of which G is the product.*

Changing notation, denote by x, y, u, w the variables in $\mathbb{Z}_p$ deduced from the natural variables. (So x is $p^{-1}x'$ where x' is the coordinate in (iii)). Assume for instance $yf = \sum_o^\infty y^m f_m$, $\|f_m\| \to 0$ for any f, where $\| \ \|$ is the natural norm on $\mathcal{A}$. Then, with obvious notation:

$$\begin{aligned}
xyf &= x\sum_o^\infty y^m f_m \\
&= \sum_{m=0}^\infty y^m \sum_{p=0}^\infty x^p f_{m,p}
\end{aligned}$$

where, for each m, $f_{m,p} \to 0$ with p.

However, the norm on $\mathcal{A}$ is equivalent to the norm $\| \ \|_{\omega,x}$ deduced from the action of the (rigid–analytic) x–group. Thus we can assume that $\|f_{m,p}\| \le C\|f_m\|$ (for this new norm). Then $\|f_{m,p}\| \to 0$ when $|m| + |p| \to \infty$. The same argument applies to any number of variables.

For $f \in \mathcal{A}$ and $z \in \mathbb{Z}_p$, $f \mapsto f(z)$ is a continuous linear form. For $s = t$, (ii) yields:

$$\begin{pmatrix} s & \\ & s \end{pmatrix} f(z) = f(z)\chi(s,s)\,.$$

If the action is analytic, we see that $\chi(s,s)$ must be an analytic function of s for $s \equiv 1$ $[p]$. Now $\chi(s,t) = \chi(t,t)\chi(st^{-1},1)$. We may then consider

$$\begin{pmatrix} s & \\ & 1 \end{pmatrix} f(z) = f(sz)\chi(s,1)\,.$$

Taking $f = 1$, we see that $\chi(s,1)$ must be analytic. Moreover, if

$$f(z) = \sum_{m \geq 0} a_m z^m$$

and $s = 1 + w$ $(p|w)$ then

$$f(sz) = \sum_n w^n \sum_{m \geq n} a_m \binom{m}{p} f^n = \sum_n w^n f_n(z)$$

yields an analytic expansion, in $\mathcal{A}$, of $f(sz)$. .

The condition on the analyticity of $\chi(s,t)$ is as follows. Write $\chi = (\alpha, \beta)$ with

$$\alpha(1+pu) = e^{a \log(1+pu)},\ \beta(1+pu) = e^{b \log(1+pu)}$$

$(a, b \in K)$ for $u \in \mathbb{Z}_p$ close to 0. The exponential is analytic (in K) in the domain $v_p(z) > \frac{e}{p-1}$ where $e = e(K)$; v_p is always the normalized valuation, $v_p(p) = 1$. Now

$$v_p(a\ \log(1+pu)) = v_p(a) + 1 + v_p(u)$$

since $p > 2$, so we must have $v_p(a) + 1 > \frac{e}{p-1}$, i.e.:

$$\begin{aligned} v_p(a), v_p(b) \quad &> \frac{e}{p-1} - 1 \\ &= \frac{-p}{p-1} \qquad \text{if } K \text{ is unramified.} \end{aligned} \tag{3.4}$$

Henceforth we assume that α, β verify these conditions. (" α, β are analytic" for short.) Now the action of $\begin{pmatrix} s & \\ & t \end{pmatrix}$ is a twist of the action associated to $\chi = 1$ by an analytic character. Thus (i), (ii) yield analytic actions.

Now $\alpha(1+v)$ belongs to the Tate algebra on the ball $|v| \leq p^{-1}$, so $\alpha(1-xz)$ belongs to the Tate algebra of two variables on $B^1 = B(1) \times B(p^{-1})$. In particular it has a convergent expression

$$\sum_{m \geq 0} x^m\ \alpha_m(z)\,, \qquad \alpha_m \in \mathcal{A}$$

on this domain, convergent (for $|x| \leq p^{-1}$) as a series in $\mathcal{A}$. Now for $|v| < 1$

$$(1-v)^{-m} = \sum_{q=0}^{\infty} \binom{m+q-1}{q} v^q\,, \text{ so}$$

for $f = \sum_0^\infty a_m z^m$,

$$\begin{aligned} f\Big(\frac{z}{1-xz}\Big) &= \sum_0^\infty a_m \sum_{q=0}^\infty \binom{m+q-1}{q} x^q z^q \\ &= \sum_{q=0}^\infty \; x^q \; \sum_{m=0}^\infty \binom{m+q-1}{q} a_m z^q\,. \end{aligned}$$

We have to remember that $x \in p\mathbb{Z}_p$, so the analyticity of the action (iii) must be seen in the variable $\xi = \frac{x}{p} \in B^1$. The expression now becomes

$$\begin{aligned} &\sum_{q=0}^\infty \xi^q \Big(p^q \sum_{m=0}^\infty \binom{m+q-1}{q} a_m z^q \\ &= \sum_{q=0}^\infty \xi^q f_q(z) \end{aligned}$$

with obviously $\|f_q\|_{\mathcal{A}} \leq p^{-q}\|f\|_{\mathcal{A}}$. Thus the action (iii) is analytic.

Let $\mathcal{A}_{\text{loc}} \supset \mathcal{A}$ be the space of locally analytic functions. The representation of G on $\mathcal{A}_{\text{loc}}$ is studied by Schneider and Teitelbaum in [11]. Let $\mathcal{D}_{\text{loc}}$ be the space of distributions on $U = \left\{ \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} \right\}$ in their sense, i.e. the topological dual of $\mathcal{A}_{\text{loc}}$. We recall that $\mathcal{A}_{\text{loc}} = \varinjlim_n \mathcal{A}(n)$ where $\mathcal{A}(n)$ is the space of functions globally analytic on each ball of radius p^{-n}. Thus $\mathcal{A} = \mathcal{A}(0)$. The transition maps are injective and compact, with dense image. Dually we have $\mathcal{D}_{\text{loc}} = \varprojlim \mathcal{D}(n)$. This is a projective limit of Banach spaces, the projection maps being compact with dense image; $\mathcal{D} = \mathcal{D}(0)$.

Similarly for the rigid–analytic group G, we have $\mathcal{A}_{\text{loc}}(G)$, $\mathcal{D}_{\text{loc}}(G)$ with similar properties. Consider the maps

$$\begin{aligned} &r : \mathcal{D}_{\text{loc}} \longrightarrow \mathcal{D} = \mathcal{A}' \qquad \text{(continuous dual of } \mathcal{A}) \\ &R : \mathcal{D}_{\text{loc}}(G) \longrightarrow \mathcal{D}(G)\,. \end{aligned}$$

We have natural actions of $\mathcal{D}_{\text{loc}}(G)$ on $\mathcal{D}_{\text{loc}}$ and of $\mathcal{D}(G)$ on $\mathcal{D}$ (see 2.3), which we denote by the convolution sign.

Lemma 3.5. *For $T \in \mathcal{D}_{\text{loc}}(G)$, $F \in \mathcal{D}_{\text{loc}}$, $r(T * F) = R(T) * r(F)$.*

The maps r and R are continuous. The map $(t, f) \mapsto t * f$ $(t \in \mathcal{D}(G), f \in \mathcal{D})$ is continuous in t; similarly $(T, F) \mapsto T * F$ $(T \in \mathcal{D}_{\text{loc}}(G), F \in \mathcal{D}_{\text{loc}})$ is continuous [11]. Furthermore the finite group algebra $K[G]$ is dense in $\mathcal{D}_{\text{loc}}(G)$. It suffices then to check the formula for a single Dirac measure $T = \delta_g$, where it is obvious. We can now deduce from the results of Schneider–Teitelbaum:

Theorem 3.6. *If $b - a \notin \mathbb{N}$, the globally analytic representation of G on $\mathcal{A}$ is topologically irreducible and admissible.*

Consider the G–map $r : \mathcal{D}_{\text{loc}} \to \mathcal{D}$, and let $X \subset \mathcal{D}$ be a closed submodule (for the action of G). Then $r^{-1}X \subset \mathcal{D}_{\text{loc}}$ is a closed submodule, invariant by $\mathcal{D}_{\text{loc}}(G)$. In [11], Schneider and Teitelbaum consider in fact the action of $\mathcal{D}_{\text{loc}}(B)$. By [11, Theorem 5.4], $\mathcal{D}_{\text{loc}}$ is (algebraically) irreducible under $\mathcal{D}_{\text{loc}}(B)$. However a glance at their proof shows that it remains irreducible under $\mathcal{D}_{\text{loc}}(G)$: the proof involves only the action of the Lie algebra, except for the argument at the bottom of p. 460. Here it must be checked that a submodule V of $\mathcal{D}_{\text{loc}}$, under the action of B, is generated by distributions T with Amice transform having only zeroes in the set of elements of the form $\zeta - 1$ where ζ is a root of unity (in $\mathbb{C}_p$) of p^n-order. The argument relies on the action of T_0; however it is easily seen that the action of the group of elements congruent to 1 mod p (contained in G) is sufficient.

Thus $r^{-1}X$ is null or equal to $\mathcal{D}_{\text{loc}}$. Since X is closed and $r : \mathcal{D}_{\text{loc}} \to \mathcal{D}$ has dense image, we deduce that X is equal to $\{0\}$ or $\mathcal{D}$. However $\mathcal{D}$ is the Banach dual of $\mathcal{A}$. If $Y \subset \mathcal{A}$ is a closed subspace, Y' is a quotient of $\mathcal{D} = \mathcal{A}'$ by the Hahn–Banach theorem, thus $Y' = \{0\}$ or $\mathcal{D}$, which implies that $Y = \{0\}$ or $\mathcal{A}$.

Finally, the representation on $\mathcal{A}$ is admissible: indeed, $\mathcal{A}$ is the subspace of $\mathcal{A}(G)$ defined by the conditions $f(gb) \equiv \chi(b^{-1})f(g)$ (f is then analytic on G since χ is so) and this is a closed subspace.

3.4

Let now L be an unramified extension of $\mathbb{Q}_p$, of degree r. All the arguments of the previous section extend to the group $G(L)$. Here $\chi = (\alpha, \beta)$ must be a couple of characters of $\mathcal{O}_L^\times$. The conditions on the logarithmic parameters $a, b \in K$ remain the same since L is unramified. Note that the representation on $\mathcal{A}(B^1, K)$ (where now B^1 is seen as an L–analytic space) given by the formulas of Lemma 3.3 is L–analytic. Its restriction to $G(\mathbb{Q}_p)$ is simply the previous representation. Indeed, the representation of $G(L)$ is obtained from the representation of $G(\mathbb{Q}_p)$ by the procedure described in Proposition 3.1.

Denote by $I_{\mathbb{Q}_p}(\chi)$ and $I_L(\chi)$, respectively, the two *globally analytic* representations. (Since χ is defined by its parameters (a, b) we agree to identify the characters for the two fields.) Then we have:

Proposition 3.7. *If $b - a \notin \mathbb{N}$, $I_L(\chi)$ is irreducible; it is the holomorphic base change of $I_{\mathbb{Q}_p}(\chi)$ (for the given embedding $L \subset K$).*
(The irreducibility clearly follows from the irreducibility under $G(\mathbb{Q}_p)$.)

We now compare the base change functor we have constructed with the natural consequences of a (conjectural) Langlands functoriality for p–adic representations. We refer the reader to the Introduction to [3] for more motivation. The principal series representation of $G(\mathbb{Q}_p)$ is one of two summands (under $G(\mathbb{Q}_p)$) of an irreducible representation π of $\mathcal{G}(\mathbb{Q}_p) = GL(2, \mathbb{Q}_p)$ [11, § 5], the principal series associated to the representation of the Galois group

$$\sigma \mapsto \alpha(\sigma) \oplus \beta(\sigma)$$

($\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p)$). Here we have assumed α, β extended to $\mathbb{Q}_p^\times$, thus giving characters of the Weil group $W_{\mathbb{Q}_p}$, and $\alpha(p)), \beta(p)$ units so the representation of $W_{\mathbb{Q}_p}$ actually extends to the Galois group.

In conformity with the general formalism, the base change π_L of π should be associated to the couple of characters $(\alpha \circ N_{L/\mathbb{Q}_p},\ \beta \circ N_{L/\mathbb{Q}_p})$. Thus, instead of

$$\alpha(1 + pu) = e^{a \log(1+pu)} \qquad (u \in \mathcal{O}_L)$$

we should consider

$$\begin{aligned} \alpha_L(1 + pu) &= \prod_\sigma \alpha(1 + p\sigma(u)) \\ &= e^{a\Sigma \log(1+p\sigma(u))}. \end{aligned}$$

In other terms, we should induce the character given an $1 + p\mathcal{O}_L$ by $\prod_\sigma \alpha^\sigma$, and similarly for β. Furthermore, the representation should not be realized on a space of (L–holomorphic) functions on $U(L)$, but on a space of $\mathbb{Q}_p$–holomorphic functions on $\mathrm{Res}_{L/\mathbb{Q}_p} U(\mathbb{Q}_p)$.

Therefore we must consider the full base change of $I_{\mathbb{Q}_p}(\chi)$ to $\mathrm{Res}_{L/\mathbb{Q}_p} G(\mathbb{Q}_p)$.

Definition 3.8. *The (Langlands) base change of* $I_{\mathbb{Q}_p}(\chi)$ *is the representation of* $\mathrm{Res}_{L/\mathbb{Q}_p}G(\mathbb{Q}_p)$ *on*

$$\widehat{\bigotimes_{\sigma}}(I_L(\chi))^{\sigma} := I(\chi \circ N)\,.$$

Note that each factor is irreducible for $b - a \notin \mathbb{N}$ and admits the same description as in Proposition 3.7; the embedding $\iota : L \to K$ must be replaced by $\iota \circ \sigma$. Each factor is holomorphic for this embedding.

Now the space of the representation $I(\chi \circ N)$ is $\widehat{\bigotimes_{\sigma}}\mathcal{A}(U, K) = \mathcal{A}(\mathrm{Res}_{L/\mathbb{Q}_p}U, K)$. The representation is now a space of globally analytic vectors (by Proposition 3.7) in the locally analytic representation $I_{la}(\chi \circ N)$ of $\mathrm{Res}_{L/\mathbb{Q}_p}G$. The representation of the full Iwahori subgroup on I_{la} and its dual, the "locally analytic" distributions on $U(\mathcal{O}_L)$, has been studied by Orlik and Strauch. They prove the analogue of Theorem 5.4 in [10]: assume that we are given characters $(\underline{\alpha}, \underline{\beta})$ of $L^{\times}$ with values in K. They give us a family of logarithmic parameters (a_σ, b_σ) for the embeddings $L \to K$. Then the space of distributions, dual to $I_{la}(\chi \circ N)$, is irreducible under $\mathcal{D}(\mathrm{Res}_{L/\mathbb{Q}_p}G, K)$ if the Verma module for $U(\mathfrak{g} \otimes L) \equiv \bigotimes_{\sigma} U(\mathfrak{g})$ induced from the *dual* of the linear form

$$(\mathbf{a}, \mathbf{b}) : (X, Y) \in \mathfrak{t} \otimes L \mapsto \sum (a_\sigma X_\sigma + b_\sigma Y_\sigma)$$

is irreducible. Here $\mathfrak{t}$ is the Lie algera of the diagonal torus. See [8, Theorem 3.4.12]. Note that their result is again stated for the full Iwahori subgroup, but that, their argument being differential, it extends to G.

In our case the highest weights are parallel, given by (a, b). The highest weight to be considered is then $(-a, -b)$. The Verma module will be *reducible* if $(X, -X) \mapsto (b - a)X$, where $(X, -X)$ is in the Lie algebra of the diagonal torus in $SL(2)$, is the highest weight of a finite–dimensional representation: that is, if $b - a \in \mathbb{N}$. Otherwise, all the factors are irreducible, and the corresponding algebraic tensor product, under $U(\mathfrak{g} \otimes L)$, is irreducible. Finally, we then have:

Theorem 3.9. *Assume* $b - a \notin \mathbb{N}$. *Then the tensor product* $\widehat{\bigotimes_{\sigma}} I_L(\chi)^{\sigma}$ *is irreducible, and is the representation of* $G(L)$ *on the space of globally analytic vectors in the representation of* $G(L)$ *induced from* $\chi \circ N$.

3.5

It is not our intention here to explore the situation if one considers the full groups $\mathcal{G}(\mathbb{Q}_p)$, $\mathcal{G}(L)$. The Bruhat decomposition yields a decomposition of the induced representation π of $\mathcal{G}(\mathbb{Q}_p)$ (say, on locally analytic functions), under the Iwahori subgroup, as a sum of two subspaces, one (the representation we have considered), π_w, associated to the big orbit, and another, say

$$\pi = \pi_w \oplus \pi_1 .$$

The corresponding irreducibility results for π_1, under the Iwahori subgroup, are proved by Schneider–Teitelbaum, and by Orlik and Strauch in the general case. We expect that the analogue of Theorem 3.9 remains true for the other summand (with now the condition $a - b \notin \mathbb{N}$). Now the full completed tensor product will be

$$\widehat{\bigotimes_\sigma}(I(w)^\sigma \oplus I(1)^\sigma)\,, \tag{3.5}$$

where I denotes the globally analytic representation. On the other hand, the representation of $G(L) = (\mathrm{Res}_{L/\mathbb{Q}_p} G)(\mathbb{Q}_p)$ is

$$\widehat{\bigotimes_\sigma} I(w)^\sigma \oplus \widehat{\bigotimes_\sigma} I(1)^\sigma\,. \tag{3.6}$$

(The Weyl group of $\mathcal{G}(L)$ still has two elements). Thus the mixed terms in (3.5) should be deleted. Note that this is compatible with the properties of base change for irreducible admissible representations of $\mathcal{G}(\mathbb{Q}_p)$, $\mathcal{G}(L)$ over $\mathbb{C}$, where the base change is expressed by an identity of traces, vs. twisted traces. If one considers the trace of a representation of the form (3.5), twisted by the action of the Galois group $<\sigma>$, only the terms in (3.6) contribute. (We are aware that there is no character theory for p–adic representations. . .)

A Appendix: “Pathologies”

We review some properties of the distribution algebras on our groups, relative to tensor products or the Nœtherian property. They introduce some difficulties explained in the main text.

A.1 $\mathcal{D}(X \times Y) \not\equiv \mathcal{D}(X)\widehat{\otimes}\mathcal{D}(Y)$

Here X, Y are rigid–analytic spaces isomorphic to products of unit balls, over a p–adic field L. Then $\mathcal{A}(X \times Y)$ is naturally isomorphic to $\mathcal{A}(X)\widehat{\otimes}\mathcal{A}(Y)$. Moreover, each space of analytic functions is, as a Banach space, isomorphic to $C_0(M)$ where M is the set of exponents: $M = \mathbb{N}^d$, $d = \dim(X)$. Here $C_0(M) = C_0(M, L)$. We have, for two countable sets M, N.

$$C_0 M \times N) \cong C_0(M)\widehat{\otimes}C_0(N)$$

(Schneider [9, p. 112]) which yields the requisite (well–known) isomorphism for the Tate algebras. We can assume that our sets M, N are equal to $\mathbb{N}$. We now have the following result, certainly well–known:

Proposition A.1. *The natural map*

$$\ell^\infty(\mathbb{N})\widehat{\otimes}\ell^\infty(\mathbb{N}) \longrightarrow \ell^\infty(\mathbb{N} \times \mathbb{N})$$

is injective, but is not an isomorphism.

Here $\ell^\infty(\mathbb{N}) = \ell^\infty(\mathbb{N}, L)$ is the Banach space of bounded sequences, the dual of $C_0(\mathbb{N})$. We will denote by V' the dual of a Banach space V, with its strong topology (the topology as a Banach space.) We denote by $\mathcal{L}(V, W)$ the space of continuous linear maps $V \to W$, again with the topology of the norm. Now we have [2, 2.1.7.2]

$$\mathcal{L}(V\widehat{\otimes}W, X) \xrightarrow{\approx} \mathcal{L}(V, \mathcal{L}(W, X))$$

(isometric isomorphism) for three Banach spaces, the map being the natural one, so

$$(C_0\widehat{\otimes}C_0)' = \mathcal{L}(C_0\widehat{\otimes}C_0, L) \cong \mathcal{L}(C_0, \ell^\infty)\,.$$

Since $(C_0\widehat{\otimes}C_0)' = \ell^\infty(\mathbb{N} \times \mathbb{N})$, it suffices to check:

Proposition A.2. *The natural map*

$$\ell^\infty\widehat{\otimes}\ell^\infty \longrightarrow \mathcal{L}(C_0, \ell^\infty)$$

obtained by completion from

$$\ell^\infty\otimes\ell^\infty \longrightarrow \mathcal{L}(C_0, \ell^\infty)$$

is injective, but is not an isomorphism.

Here $\mathcal{L}(C_0, \ell^\infty)$ is provided with the strong (= Banach) topology. Schneider [9, Proposition 18.2] shows that this map is an isomorphism onto its image, whence the injectivity in Proposition A1. He also shows that its image is the space

$$\mathcal{CC}(C_0, \ell^\infty)$$

of completely continuous operators. Thus it suffices to show

$$\mathcal{CC}(C_0, \ell^\infty) \neq \mathcal{L}(C_0, \ell^\infty)\,. \tag{A.1}$$

Now $F : C_0 \to \ell^\infty$ is in $\mathcal{CC}$ if, and only if, $F(B)$ is compactoid for any bounded set $B \subset C_0(X)$. We can simply consider the unit ball.

Recall also that $\Omega \subset V$ (a Banach space) is compactoid [9, p. 71] if $\forall r > 0$ $\exists (v_i)_{i=1,\dots N}$, $v_i \in V$, such that

$$\Omega \subset B_V(r) + \sum_1^N \mathcal{O}_L v_i\,.$$

We consider the identity map $F : C_0 \to \ell^\infty$ and show that it is not completely continuous. If it were, we would have for $f \in C_0$:

$$\|f\|_\infty \leq 1 \Longrightarrow \forall r \quad \|f - \sum_1^{N(r)} z_i f_i\|_\infty \leq r$$

for some functions $f_i \in \ell^\infty$, and integers z_i, depending on r (but not on f).

For simplicity assume $L = \mathbb{Q}_p$. Fix $r = p^{-n}$, $n \geq 0$. The f_i take values in $p^{-M}\mathbb{Z}_p$, $M \geq 0$. The function f defines $\bar{f} : \mathbb{N} \to \mathbb{Z}_p/p^n\mathbb{Z}_p$, and

$$\bar{f}(x) = \sum_1^{N(r)} \bar{z}_i \bar{f}_i(x)\,, \tag{A.2}$$

a linear combination of functions $\mathbb{N} \to p^{-M}\mathbb{Z}_p/p^n\mathbb{Z}_p$. Since f is arbitrary in $B(1) \subset C_0$, $\bar{f}$ can be an arbitrary function with finite support $\mathbb{N} \to \mathbb{Z}_p/p^n\mathbb{Z}_p$. However, when (z_i) varies in $\mathbb{Z}_p^{N(r)}$, the set of functions on the right–hand side of (A2) with values in $p^{-M}\mathbb{Z}_p/p^n\mathbb{Z}_p$ is finite, and this is a contradiction.

A.2 $\mathcal{D}(G)$ *(as a convolution algebra) is not Nœtherian for left- or right- ideals.*

Here the rigid–analytic group is assumed to verify the assumptions in section 1. We start with $B(1)$ (rigid–analytic ball) seen as an additive group. Then

$$\begin{aligned} \mathcal{A}(G) &= \Big\{ \sum_0^\infty a_n x^n,\ a_n \to 0 \Big\}, \\ a_n &= \frac{1}{n!}\Big(\frac{d^n}{dx^n} f\Big)(0)\,. \end{aligned}$$

The algebra $\mathcal{D}(G)$ is isomorphic to ℓ^∞ by taking the basis dual to the x^n; we can write a distribution $T \in \mathcal{D}(G)$ as

$$T = \sum_0^\infty c_n \frac{1}{n!}\Big(\frac{d^n}{dx^n}\Big)_0, \quad (c_n) \in \ell^\infty$$

and then $\mathcal{D}$ is naturally isomorphic to an algebra of divided power series:

$$T = \sum_0^\infty \frac{c_n}{n!} t^n, \quad t = \Big(\frac{d}{dx}\Big)_0, \quad (c_n) \in \ell^\infty\,.$$

As pointed out by Berthelot, this algebra is not Nœtherian[1]. Since it is a Banach algebra for convolution (which becomes here the product of the series), it suffices to check that there is an ideal which is not closed [2, 3.7.2.2]. In fact:

Lemma A.3 (Berthelot)**.** *The ideal $(t) \in \mathcal{D}$ is not closed.*

Indeed if $T \in (t)$,

$$\begin{aligned} T = TS &= t\Big(\sum_0^\infty \frac{c_n}{n!} t^n\Big) \\ &= \sum_1^\infty t^n \frac{d_n}{n!} \end{aligned}$$

with $d_n = nc_{n-1}$, so

$$T \in (t) \Leftrightarrow d_0 = 0 \text{ and } \Big|\frac{d_n}{n}\Big| \le C\,.$$

[1] Berthelot considered the subalgebra given by $(c_n) \in C_0$, but this makes no difference.

But this subspace of $\mathcal{D}$ is clearly not closed. For instance, if $T = (d_n)$ has support on $\{n = p^{2n}\}$ ($r \geq 0$ even) with

$$\begin{aligned} d_{p^r} &= p^{r/2} \text{ (so } T \in \ell^\infty) \\ \left|\frac{d_{p^r}}{p^r}\right| &= |p^{-r/2}| \longrightarrow \infty \text{ (so } T \notin (t)) \,, \end{aligned}$$

T is the limit in $\mathcal{D}$ of the truncated series T^α with $d^\alpha_{p^r} = p^{r/2}$ ($r \leq \alpha$), $d^\alpha_{p^r} = 0$ ($r > \alpha$) which obviously belong to the ideal.

Consider now a rigid–analytic group G, isomorphic to a product $B(1)^d$ over $\mathbb{Q}_p$ as a rigid–analytic space, the coproduct being then given by Tate series with integral coefficients. We further assume (as is the case in this paper) that the factors are (additive) analytic subgroups, and their distribution algebras are therefore as in the previous proof.

Proposition A.4. *Under these assumptions $\mathcal{D}(G)$ is not Nœtherian (for left- or right- ideals).*

Indeed we have, as in the commutative case:

Lemma A.5 (Schneider–Teitelbaum)**.** *If $\mathcal{D}(G)$ is (left) Nœtherian, any (left) ideal is closed.*

See Proposition 2.1 in [12]. For completeness we provide a proof. In the commutative case this is [2, 3.7.2.2]. A glance at their proof whose that it suffices to prove Nakayama's lemma for the ideal $A^\vee = \{a \in A : \|a\| < 1\}$ in A^0, where we have written $A = \mathcal{D}(G)$. We may assume that the (Banach) norm on A is submultiplicative [2, 1.2.1.2]. Since the argument in [2, 1.2.3.6] for Nakayama's lemma uses determinants, we rephrase it (using moreover $A^\vee$ rather than the set $\check{A}$ of topologically nilpotent elements):

Lemma A.6. *Let M be an A–module, and N a submodule of M such that there exist $x_1, \dots x_n \in M$ with $M \subset N + \sum_1^n A^\vee x_i$. Then $N = M$.*

As in [2], loc. cit., we can write

$$\underline{x} = \underline{y} + C\underline{x}$$

where $\underline{x}$ is a column vector in M^n with coordinates x_i, and $\underline{y}$ has coordinates in N, and the matrix $C \in M_n(A)$ has entries in $A^\vee$. Thus $\underline{y} = (1 - C)\underline{x}$. The matrix $1 - C$ is invertible: if $M_n(A)$ is endowed with the operator norm, this norm is submultiplicative, and $\|C\| < 1$. This implies that $\underline{x} = (1-C)^{-1}\underline{y} \in N^n$. (It is not clear to us that the argument applies for $\check{A}$.)

We now return to the proof of Proposition A4. Assume that (as a rigid analytic space)

$$G = G_1 \times \cdots \times G_d$$

where each G_i is a rigid–analytic group isomorphic to the additive unit ball over $\mathbb{Q}_p$.

In particular we have a bijection

$$\begin{array}{rcl} \mathbb{Z}_p^d = G_1(\mathbb{Q}_p) \times \cdots \times G_d(\mathbb{Q}_p) & \longrightarrow & G(\mathbb{Q}_p) \\ (g_1, g_2, \ldots, g_d) & \longmapsto & g_1 \cdots g_d \,. \end{array}$$

The Tate algebra $\mathcal{A}_G$ is isomorphic with $\widehat{\bigotimes_{i=1,\ldots d}} \mathcal{A}_{G_i}$ where each $\mathcal{A}_{G_i}$ is the Tate algebra in one variable. Evaluated on the points of $G(\mathbb{Q}_p)$, this yields the map $f \mapsto f(g_1, \ldots, g_d)$ $(f \in \mathcal{A}_G)$.

Each injection $j_i : G_d \longrightarrow G$ is an homomorphism, and the restriction $\mathcal{A}_G \longrightarrow \mathcal{A}_{G_i}$ is therefore compatible with the coproduct. Dually, we get

$$(j_i)_* : \mathcal{D}_{G_i} \longrightarrow \mathcal{D}_G \,,$$

compatible with convolution. If we denote by x_i the local variable on G_i, a function $f \in \mathcal{A}_G$ being then in the Tate algebra in the x_i, an element of $\mathcal{D}_G$ can be written

$$T = \sum_n c_n \frac{1}{n!} \partial_1^{n_1} \cdots \partial_d^{n_d} := \sum_n c_n \delta_n \tag{A.3}$$

with $n = (n_1, \ldots n_d) \in \mathbb{N}^d$, $n! = \Pi(n_i)!$,

$$\partial_i f(x_1, \ldots, x_d) = \frac{d}{dx_i} f(x_1, \ldots, x_d)(0), \text{ and } (c_n) \in \ell^\infty(\mathbb{N}^d).$$

Let us write $\mathcal{D}_1$ for the subalgebra $(j_1)_*(\mathcal{D}_{G_1})$, given by $c_n = 0$ if $n_i > 0$ for some $i \geq 2$. This is clearly a closed subalgebra. The element $\partial_1 = \delta_1$ - abuse of notation for $\delta_{(1,0,\ldots,0)}$ - is equal to $(j_1)_*((\frac{d}{dx})_0)$. Moreover, for the convolution product in G, we have

$$\delta_1 * \delta_n = \frac{1}{n!} \frac{d}{dx_1} \Big(\frac{d}{dm}\Big)^{n_1} \cdots \Big(\frac{d}{dx_d}\Big)^{n_d}$$

(evaluated at $(0,\ldots,0)) = n_1\frac{1}{(n_1+1)!n_2!\cdots n_d}(\frac{d}{dx_1})^{n_1+1}\cdots(\frac{d}{dx_d})^{n_d} = n_1\delta_{n'}$ where $n' = (n_1+1,\ldots,n_d)$.

We will show that the left ideal $\delta_1\mathcal{D}(G)$ is not closed in $\mathcal{D}(G)$. Indeed, if it were, its intersection with the closed subalgebra $\mathcal{D}_1$ would be so. But the intersection is isomorphic (as an algebra) with the algebra

$$\mathcal{D}_{G_1} = \Big\{\sum_{m\geq 0} c_m\delta_m\Big\} = \Big\{\sum_{m\geq 0} c_m\frac{1}{m!}\Big(\frac{d}{dx}\Big)^m\Big\}$$

in one variable, and the previous computation shows that the intersection is the ideal $(\frac{d}{dx}) \subset \mathcal{D}_{G_1}$ considered in the first part of the proof. Therefore $\delta_1\mathcal{D}(G)$ is not closed; this completes the proof of Proposition A6.

A.3

Finally, we note that there is a possible substitute for the consideration of $\mathcal{D}(G)$, which could obviate the problems we encountered. (This was pointed out by Schneider.)[2]

This algebra was already introduced by Lazard [7, III.3.3.3] who calls in $Ala(G)$. To use Lazard's results, we keep the assumptions of § A2 and assume moreover that the factor $G_i(\mathbb{Q}_p) \cong \mathbb{Z}_p$ form a Lazard basis – a "base ordonnée" in the sense of [7, III.2.2.4]. The Iwasawa algebra of G (with integral coefficients) is then given by the series

$$\sum_n a_n z_1^{n_1}\cdots z_d^{n_d} = \sum a_n z^n$$

with $a_n \in \mathbb{Z}_p$ and $z_i = \delta_1 - \delta_0$ in $G_i(\mathbb{Q}_p)$. Lazard defines the algebra $Sat\ Al(G)$, where $Al(G) = \Lambda_G$ is the Iwasawa algebra, and $Sat\ Al(G)$ is given by $val(a_n) \geq -|n|$. (Thus $SatAl(G)$ is contained in the completion of $\Lambda_G\otimes\mathbb{Q}_p$).

Recall that $\Lambda_G \subset \mathcal{D}_G$.

Lazard defines $Ala(G) \subset Sat\ Al(G)$ as the compact $\mathbb{Z}_p$–module generated by the $\dfrac{z^n}{n!}$, with the usual notation. If we recall that the basis elements z^n are dual to the Mahler basis $\dbinom{x}{n}$ of Λ, we see that this is the unit ball in the Banach dual of the functions $\sum_n c_n n!\dbinom{x}{n}$ $(c_n \to 0)$ in $\mathcal{C}(G)$, with the weak

[2] In this section we will skip details, as we only want to outline an alternative approach.

topology. By Amice's theorem [5, I.4] this space of functions is $\mathcal{A}(G)$. Thus $Ala(G) \subset \mathcal{D}_G$, and $Ala(G)$ is the unit ball in $\mathcal{D}_G$, with the weak topology. Note that both the $n!\binom{x}{n}$ and the x^n $(x = (x_1, \dots x_d))$ are orthonormal bases of $\mathcal{A}_G$: see Colmez [5, I.4.3]. Thus this coincides with the description of $\mathcal{D}_G$ as an ℓ^∞ space on the set of powers n.

Now if we consider these integral spaces (isomorphic to $\mathbb{Z}_p^N$, N the set of indices, with the product topology), we obtain indeed, for a product $G \times H$, an isomorphism $Ala(G \times H) \cong Ala(G)\widehat{\bigotimes_{\mathbb{Z}_p}} Ala(H)$. This is stated (but not proven) by Lazard [7, III.3.]. The completed tensor product (*over* $\mathbb{Z}_p$) is defined in [7, I.3.2.6, I.3.2.9].

Recall that we have assumed that the coproduct for G (and H, a group of the same type) was given by integral Tate expansions. Then $\mathcal{D}_G(\mathbb{Z}_p)$, the distributions with integral coefficients, is the unit ball in $\mathcal{D}_G$, and its weak topology is as we saw, the product topology. The tensor product $\mathcal{D}_G(\mathbb{Z}_p) \times \mathcal{D}_H(\mathbb{Z}_p) \to \mathcal{D}_{G\times H}(\mathbb{Z}_p)$ is simply given, M being the set of exponents for H, by

$$(z_n, w_m) \longmapsto (z_n w_m)_{n,m}$$

where $N \times M$ is the set of exponents for $G \times H$. It is easy to see that it yields, as asserted by Lazard, an isomorphism $\mathbb{Z}_p^N \widehat{\otimes} \mathbb{Z}_p^M \to \mathbb{Z}_p^{N\times M}$. In particular, under our standing assumptions on the groups (in particular, the integrality conditions), we have:

Proposition A.7. *Let $\mathcal{D}_G^0$, $\mathcal{D}_H^0$ denote the unit balls of $\mathcal{D}_G$, $\mathcal{D}_H$ with their weak topology. Then*

$$\mathcal{D}_{G\times H}^0 \cong \widehat{\mathcal{D}_G^0 \bigotimes_{\mathbb{Z}_p} \mathcal{D}_H^0}\,.$$

Now assume V, W are globally analytic Banach representations of G, H. The algebra $\mathcal{D}_G$ acts on V', by

$$< Tv', v >= \int < v', gv > dT(g)$$

where we have used the integral notation for T.

Now write V_0' for the unit ball in V', and V_0 for the unit ball in V. We may assume, as in § 2.1, that the action of G on V preserves the norm. Then if $v \in V_0$ and $v' \in V_0'$ $< v', gv >$ is a function f on G, given by a Tate series, and such that $\operatorname*{Sup}_{g \in G} |f(g)| \leq 1$.

If moreover V is admissible, the Tate norm $\|f\|$ is ≤ 1 by Corollary 2.8. Thus Tv' belongs to V_0'. We obtain an action of $\mathcal{D}_G^0$ on the unit ball of V', compatible with the weak topologies.

If now we consider another group H, an admissible globally analytic representation W of H, and the product $G \times H$, we see that $\mathcal{D}_G^0 \widehat{\otimes} \mathcal{D}_H^0$ acts on $V_0' \widehat{\otimes} W_0'$. Replacing the distribution algebras by their unit balls, we are very close to the original construction of Schneider–Teitelbaum [10] for Banach representations.

Since $\mathcal{D}_G$ is not Nœtherian, however, it easily follows that $\mathcal{D}_G^0$ is not Nœtherian.

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