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# A new asymmetric correlation inequality for Gaussian measure

Kunal Dutta <sup>\*†</sup>      Arijit Ghosh <sup>\*‡</sup>      Nabil H. Mustafa <sup>§</sup>

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## Abstract

The Khatri-Šidák lemma says that for any Gaussian measure  $\mu \in \mathbb{R}^n$ , given a convex set  $K$  and a slab  $L$ , both symmetric about the origin, one has  $\mu(K \cap L) \geq \mu(K) \cdot \mu(L)$ . We state and prove a new asymmetric version of the Khatri-Šidák lemma when  $K$  is a symmetric convex body and  $L$  is a slab (not necessarily symmetric about the barycenter of  $K$ ). Our result also extends that of Szarek and Werner (1999), in a special case.

**Keywords.** Convex bodies, Gaussian measure, correlation inequalities, logarithmically concave functions

## 1 Introduction

Let  $N^n(a, \sigma^2)$ ,  $a \in \mathbb{R}^n$  to indicate the  $n$ -dimensional Gaussian distribution centered at  $a$ , with variance  $\sigma^2$ . Let  $\gamma_n$  denote the corresponding measure, i.e. the measure on  $\mathbb{R}^n$  having density  $\frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{\|x\|^2}{2}\right)$ . The Gaussian measure of a Borel set  $B \subset \mathbb{R}^n$  is the probability that a vector in  $\mathbb{R}^n$  drawn according to the distribution  $N^n(0, 1)$ , lies inside  $B$ .

A central problem in convex geometry and probability theory, which “has been the subject of intense efforts of many probabilists over the last thirty years” [Lat02] is the *Gaussian correlation inequality* (GCI), which states that the Gaussian measures of any two symmetric, convex bodies in  $\mathbb{R}^n$  are positively correlated, that is, given two symmetric, convex sets  $K, L \subset \mathbb{R}^n$ ,

$$\gamma_n(K \cap L) \geq \gamma_n(K) \cdot \gamma_n(L). \quad (1)$$

This was first proposed by Das Gupta *et al.* [DGEO<sup>+</sup>72] in 1972, though special cases had been asked as open problems as early as 1955 (see e.g. [DGEO<sup>+</sup>72], [SSZ98], for more details and references). After many partial results by several authors, e.g. [Pit77, Bor81, SSZ98, Har99, Sha03], in a very

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recent breakthrough [Roy14], Royen has proved the Gaussian correlation inequality (see also Latała and Matlak’s recent draft [LM15] for a discussion on Royen’s proof [Roy14]).

An important special case of the inequality, when one of the bodies is a symmetric slab (i.e., the set  $\{x \in \mathbb{R}^n : |\langle x, \hat{u} \rangle| \leq b\}$  for some unit vector  $\hat{u} \in \mathbb{R}^n$  and  $b \in \mathbb{R}^+$ ), was proved independently by Khatri [Kha67], and Šidák [Sid68]:

**Lemma 1 (Khatri-Šidák (1967-68))** *Let  $K \subset \mathbb{R}^n$  be a symmetric convex body, and  $L \subset \mathbb{R}^n$  be a symmetric slab. Then*

$$\gamma_n(K \cap L) \geq \gamma_n(K) \cdot \gamma_n(L).$$

A related, but somewhat different extension of the Khatri-Šidák lemma was proved by Szarek and Werner [SW99], who proved that positive correlation holds between a convex (not necessarily symmetric) set  $K$  and a slab  $L$ , as long as the barycenters (with respect to the Gaussian density) of  $K$  and  $L$  coincide, i.e. they have the same projection on the normal vector  $\hat{u}$  of  $L$ .

The Khatri-Šidák lemma, in particular, has proved to be extremely useful in a number of applications in probability and statistics (where it originally arose), as well as convex geometry - where it gives a lower bound on the Gaussian measure of symmetric convex sets (since any such set is the intersection of symmetric slabs), as well as small ball probabilities. A recent algorithmic application was given by Rothvoss [Rot14], who used the lemma to obtain an efficient algorithm for low-discrepancy colorings for finite set-systems.

## Our contribution

In this paper we give a new asymmetric correlation inequality for the Gaussian measure. To do this, we look at the relation of Brownian motion inside a convex set with the Gaussian measure of that set, in a non-symmetric setting. There are several results (see e.g. [Mar]), which show that the expected time for a Brownian motion starting at  $x_0$  to exit from a set (for the first time) gives “some sort of measure of that set” [Mar]. Our result is another step along this direction.

Specifically, by the Bachelier formula, the spatial distribution of a Brownian motion starting at  $x_0$ , at time  $t$ , is the normal distribution  $N^n(x_0, t)$ . Suppose now that we want to consider the Gaussian measure from a point  $x_0$  other than the origin, then by the previous intuition, this corresponds to a Brownian motion with starting point  $x_0$ , and the variance of the Gaussian measure should correspond to the exit time of the corresponding Brownian motion. Applying this to the context of the Gaussian correlation problem, suggests the formulation of Theorem 2.

**Theorem 2** *Let  $K$  be a convex body, symmetric about  $x_0 \in \mathbb{R}^n$ . Let  $L$  be a slab symmetric about the origin, i.e.*

$$L := \{x \in \mathbb{R}^n : |\langle x, \hat{u} \rangle| \leq b\},$$

where  $b \in \mathbb{R}^+$ . If  $|\langle x_0, \hat{u} \rangle| < b$ , then

$$\gamma'(K \cap L) \geq \gamma'(K)\gamma'(L),$$

where  $\gamma'$  is the Gaussian measure centered at  $x_0$  with variance

$$\sigma^2 = 1 - \frac{\langle x_0, \hat{u} \rangle^2}{b^2},$$

and covariance matrix  $\sigma^2 I$ .

Note that for small enough variance, one could have positive correlation for *any* class of convex bodies containing the origin. The contribution of the above theorem is therefore, determining the unique variance (with the help of the intuition discussed previously) which makes positive correlation hold for the case of symmetric convex bodies and asymmetric slabs.

Theorem 2 also extends a special case of the result of Szarek and Werner [SW99]. In the Szarek-Werner theorem, the barycenters of the body  $K$  and the slab  $L$  (with respect to  $\gamma$ ) need to coincide for positive correlation to hold, i.e. the hyperplane containing the barycenter of  $L$  and perpendicular to its normal vector  $\hat{u}$ , should contain the barycenter of  $K$  for positive correlation to hold. However in our case, the barycenter of  $L$  lies on  $\langle x, \hat{u} \rangle = 0$ , whereas  $K$ 's barycenter lies on the hyperplane  $\langle x, \hat{u} \rangle = \langle x_0, \hat{u} \rangle$ , where the RHS could be non-zero in general. Under the special case that  $K$  is symmetric about  $x_0$ , Theorem 2 shows that positive correlation is possible even in this case, *provided* we decrease the variance, i.e. make the Gaussian more concentrated. The factor by which the variance decreases, depends only on the distance between the barycenters of  $L$  and  $K$ , projected on the the normal to  $L$ . Thus, Theorem 2 also provides a stability result for Gaussian measure, showing how the Gaussian measure (with unit variance) decays as one of the convex bodies is translated along an axis.

We provide some preliminary background in Section 2. In Section 3, we give the proof of Theorem 2. We end with some remarks and open questions.

## 2 Preliminaries

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is *logarithmically concave*, or log-concave, if its domain is a convex set, and for all  $0 \leq \lambda \leq 1$ , and all  $x, y \in \mathbb{R}^n$ , it satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \geq (f(x))^\lambda (f(y))^{1-\lambda}.$$

A deep result for log-concave measures was proved by Prékopa and Leindler:

**Theorem 3 (Prékopa-Leindler [Pre73])** *Let  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  be log-concave on  $\mathbb{R}^{n+m}$ . Then the function*

$$h(x) = \int_{\mathbb{R}^m} f(x, y) dy$$

*is log-concave on  $\mathbb{R}^n$ .*

The following proposition follows easily from the definitions of log-concavity and symmetry.

**Proposition 4** *Let  $h : \mathbb{R}^m \rightarrow \mathbb{R}$  be symmetric and log-concave on  $\mathbb{R}^m$ . Then the level set*

$$I_h(s) := \{x \in \mathbb{R}^m : h(x) \geq s\},$$

*where  $s \in \mathbb{R}_+$ , is a symmetric convex set in  $\mathbb{R}^m$ .*

## 3 Gaussian correlation: asymmetric case

Our approach is based on an approach described in e.g. Giannopoulos [Gia97] and Schechtmann-Schlumprecht-Zinn [SSZ98] for proving the Khatri-Šidák lemma. The idea there is to use the Prékopa-Leindler theorem to go from the  $n$ -dimensional to the one-dimensional case, where the Khatri-Šidák result follows trivially. In our case, the one-dimensional situation is not obvious, and

we need to prove positive correlation. The  $n$ -dimensional case then follows via the Prékopa-Leindler theorem.

Before describing the proof of Theorem 2, we shall need a few preparatory lemmas. In the following,  $\text{sgn}(\cdot)$ <sup>1</sup> is the sign function.

**Proposition 5** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a  $k$ -differentiable function, and  $a, b \in \mathbb{R}$ , be such that the first  $(k - 1)$ -order derivatives of  $f$  vanish at  $x = b$ , but the  $k$ -th order derivative does not, that is,  $f_0(b) = f_1(b) = \dots = f_{k-1}(b) = 0 \neq f_k(b)$ , where  $f_i(c) = \frac{d^i f(x)}{dx^i}|_{x=c}$ ;  $f(x) := f_0(x)$ . Then,*

(i) *if  $a < b$ , and if for all  $x, y \in (a, b]$  we have  $\text{sgn}(f_k(x)) = \text{sgn}(f_k(y))$ , then for all  $x \in (a, b)$ , for  $i = 0, 1, \dots, k$ :*

$$\text{sgn}(f_i(x)) = (-1)^{k-i} \text{sgn}(f_k(b)).$$

(ii) *if  $a > b$ , and if for all  $x, y \in [b, a)$  we have  $\text{sgn}(f_k(x)) = \text{sgn}(f_k(y))$ , then for all  $x \in [b, a)$ ,  $i = 0, \dots, k$ .*

$$\text{sgn}(f_i(x)) = f_k(b)$$

*Proof* For  $a < b$ , it is easy to see that for  $i \in \{0, 1, \dots, k - 1\}$ , if for all  $x, y \in (a, b)$  we have  $\text{sgn}(f_{i+1}(x)) = \text{sgn}(f_{i+1}(y))$ , then

$$\begin{aligned} \text{sgn}(f_i(x)) &= \text{sgn}(f_i(y)), \quad \forall x, y \in (a, b), \quad \text{and} \\ \text{sgn}(f_i(x)) &= (-1)\text{sgn}(f_{i+1}(x)), \quad \forall x \in (a, b), \end{aligned}$$

since  $f_i(x)$  must be a monotone function and  $f_i(b) = 0$ . Since  $f_k(x)$  does not change sign in  $(a, b]$ , the previous statement indeed applies for  $i = k - 1$ . By reverse induction on  $i$ , the first part of the proposition holds.

For  $a > b$ , let  $i = k - 1$ . If  $\text{sgn}(f_k(b)) = 1$ , then since  $f_{k-1}(b) = 0$  and  $f_k(x) > 0$  for all  $x \in [b, a)$ , we get  $f_{k-1}(x) > f_{k-1}(b) = 0$ , for all  $x \in (b, a)$ , i.e.  $\text{sgn}(f_{k-1}(x)) = \text{sgn}(f_k(b))$ . The case is similar when  $\text{sgn}(f_k(b)) = -1$ . Again by reverse induction on  $i$ , the second statement of the proposition now follows for all  $i = 0, 1, \dots, k - 1$ .  $\square$

Next, let  $f : [0, 1) \rightarrow [0, 1)$  be given by

$$f(z) := \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\frac{1+z}{1-z}}}^{\sqrt{\frac{1-z}{1+z}}} \exp(-y^2/2) dy, \quad (2)$$

and define

$$f_1(z) := \frac{df(z)}{dz} \quad (3)$$

**Proposition 6**  $f_1(z) \leq 0$ , for all  $z \in [0, 1)$ .

<sup>1</sup>The sign function  $\text{sgn} : \mathbb{R} \setminus \{0\} \rightarrow \{-1, +1\}$  is given by  $\text{sgn}(x) := \frac{x}{|x|}$ .

*Proof* Consider the derivative of  $f$  with respect to  $z$ :

$$f_1(z) = -\frac{1}{\sqrt{2\pi(1-z^2)}} \left( \frac{\exp\left(-\frac{1-z}{2(1+z)}\right)}{1+z} - \frac{\exp\left(-\frac{1+z}{2(1-z)}\right)}{1-z} \right).$$

At  $z = 0$ ,  $f_1(z) = 0$ . We shall show, that as  $z$  goes from zero to 1,  $f_1(z)$  remains non-positive. To see this, apply the substitution  $y = \frac{1-z}{1+z}$ . Then  $z = \frac{1-y}{1+y}$ , and when  $z \in [0, 1)$ ,  $y \in (0, 1]$ . We get

$$\begin{aligned} f_1(z) &= \frac{-(1+y)}{2\sqrt{2\pi y}} \left[ \frac{(1+y)}{2\exp(y/2)} - \frac{(1+y)}{2y \cdot \exp(1/(2y))} \right] \\ &= \frac{-(1+y)^2}{4\sqrt{2\pi y}} \left[ \frac{1}{\exp(y/2)} - \frac{1}{y \cdot \exp(1/(2y))} \right] \end{aligned}$$

To show that the above expression is non-positive when  $y \in (0, 1]$ , it suffices to compare the denominators of the terms in the expression inside the square brackets. In particular, let

$$g(y) = y \cdot \exp\left(\frac{1}{2y}\right) - \exp\left(\frac{y}{2}\right),$$

then we want to show that for  $y \in (0, 1]$ ,  $g(y) \geq 0$ . Let us compute the derivatives of  $g$ :

$$\begin{aligned} g(y) &= y \cdot \exp\left(\frac{1}{2y}\right) - \exp\left(\frac{y}{2}\right) \\ g_1(y) &= \left(1 - \frac{1}{2y}\right) \exp\left(\frac{1}{2y}\right) - \frac{1}{2} \cdot \exp\left(\frac{y}{2}\right) \\ g_2(y) &= \frac{1}{4y^3} \cdot \exp\left(\frac{1}{2y}\right) - \frac{1}{4} \cdot \exp\left(\frac{y}{2}\right) \\ g_3(y) &= -\left(\frac{1}{12y^4} + \frac{1}{8y^5}\right) \cdot \exp\left(\frac{1}{2y}\right) - \frac{1}{8} \cdot \exp\left(\frac{y}{2}\right) \end{aligned}$$

Observe that  $y = 1$  is a point of inflection for the function  $g(y)$ , i.e.  $g(1) = g_1(1) = g_2(1) = 0 \neq g_3(1)$ , and  $g_3(y) \leq 0$  for all  $y \in (0, 1]$ . Therefore,  $g(y)$  satisfies the conditions for Proposition 5 to apply, with  $k = 3$ ,  $a = 0$ ,  $b = 1$  and  $\text{sgn}(g_3(1)) = -1$ . Hence, the sign of  $g(y)$  is  $(-1)^3 \text{sgn}(g_3(1))$  for  $y \in (0, 1]$ , i.e.  $g(y) \geq 0$  for  $y$  in the given range, which completes the proof.  $\square$

The following claim now discusses the one-dimensional situation:

**Claim 7** *Let  $K$  be a real interval, symmetric about  $a$  where  $|a| \in [0, 1)$ , and let  $L := [-1, 1]$ . Then,*

$$\gamma'(K \cap L) \geq \gamma'(K)\gamma'(L),$$

where  $\gamma' \sim N(a, 1 - a^2)$ .

*Proof* If  $K \subseteq L$  or  $L \subseteq K$ , then we are done (as in [Gia97]), because then  $\gamma'(K \cap L) = \min\{\gamma'(K), \gamma'(L)\}$  and  $\gamma'$  is a probability measure. However, in general,  $K \not\subseteq L$ , and  $L \not\subseteq K$ . Since  $K = (K \cap L) \sqcup (K \setminus L)$ , the lemma holds if and only if

$$\gamma'(K)(1 - \gamma'(L)) \geq \gamma'(K \setminus L).$$

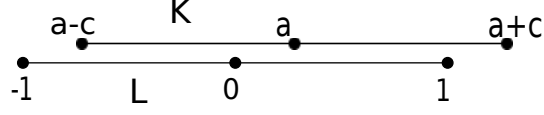


Figure 1:  $K, L$  (separated)

Let  $K := [a - c, a + c]$ . We shall analyse the case when  $a \geq 0$ . The complementary case follows analogously. As  $K = [a - c, a + c]$ , we have  $-1 \leq a - c \leq 1$ . Otherwise either  $K \subseteq L$  or  $L \subseteq K$ .

Let

$$\begin{aligned} P_1 &:= \gamma'(K) = \int_{a-c}^{a+c} d\gamma', \\ P_2 &:= 1 - \gamma'(L) = 1 - \int_{-1}^1 d\gamma', \\ Q &:= \gamma'(K \setminus L) = \int_1^{a+c} d\gamma'. \end{aligned}$$

We want to show that for all  $c \geq 0$  and  $a \in [0, 1)$ , we have  $P_1 P_2 \geq Q$ . Consider  $P_1 = P_1(a, c)$ ,  $P_2 = P_2(a, c)$ , and  $Q = Q(a, c)$ . At  $a = 0$ , we have  $P_1 P_2 \geq Q$ , from the fact that  $K$  and  $L$  are both intervals, and  $\gamma'$  is a probability measure. As  $a \rightarrow 1$ , we have

$$\lim_{a \rightarrow 1} \left( \frac{P_1}{2} - Q \right) = \lim_{a \rightarrow 1} \left( \int_a^{a+c} d\gamma' - \int_1^{a+c} d\gamma' \right) = 0,$$

while

$$\lim_{a \rightarrow 1} P_2 = \lim_{a \rightarrow 1} (1 - \gamma'(L)) = 1 - \int_{-\infty}^0 d\gamma = \frac{1}{2}.$$

Let

$$T := T(a, c) = P_1 P_2 - Q.$$

From the preceding discussion,  $T(0, c) \geq 0$ , and further since  $P_1, P_2$  and  $Q$  are bounded for all  $a \in [0, 1)$ , we have

$$\lim_{\varepsilon \rightarrow 1^-} T(\varepsilon, c) = \left( \lim_{\varepsilon \rightarrow 1^-} P_1(\varepsilon, c) \right) \left( \lim_{\varepsilon \rightarrow 1^-} P_2(\varepsilon, c) \right) - \left( \lim_{\varepsilon \rightarrow 1^-} Q(\varepsilon, c) \right) = 0.$$

Hence it is clear that if  $\frac{\partial T}{\partial a} \leq 0$  for all  $a \in [0, 1)$ , then we would be done. Let us therefore look at  $\frac{\partial T}{\partial a}$ . We have,

$$\frac{\partial T}{\partial a} = \frac{\partial P_1}{\partial a} P_2 + P_1 \frac{\partial P_2}{\partial a} - \frac{\partial Q}{\partial a}. \quad (4)$$

Applying the transform  $y = \frac{x-a}{\sigma}$  to the definitions of  $P_1, P_2, Q$ , with  $\sigma^2 = 1 - a^2$ , we get:

$$\begin{aligned} P_1 &= \frac{1}{\sqrt{2\pi}} \int_{-c/\sigma}^{c/\sigma} \exp(-y^2/2) dy, \\ P_2 &= 1 - \frac{1}{\sqrt{2\pi}} \int_{-(1+a)/\sigma}^{(1-a)/\sigma} \exp(-y^2/2) dy, \\ Q &= \frac{1}{\sqrt{2\pi}} \int_{(1-a)/\sigma}^{c/\sigma} \exp(-y^2/2) dy. \end{aligned}$$

By Proposition 6, we have that  $P_2$  increases as  $a$  goes from zero to one, since  $\frac{\partial P_2}{\partial a} = -f_1(a)$ . See the definitions of  $f$  and  $f_1$  from Equations (2) and (3).

Now,

$$\begin{aligned}\frac{\partial P_1}{\partial a} P_2 &= P_2 \cdot \frac{2}{\sqrt{2\pi}} \cdot \exp\left(\frac{-c^2}{2(1-a^2)}\right) \cdot \frac{ca}{(1-a^2)^{3/2}}, \\ \frac{\partial P_2}{\partial a} P_1 &= -P_1 f_1(a) \\ &= \frac{P_1}{\sqrt{2\pi(1-a^2)}} \left[ \exp\left(\frac{-(1-a)}{2(1+a)}\right) \cdot \frac{1}{(1+a)} - \exp\left(\frac{-(1+a)}{2(1-a)}\right) \cdot \frac{1}{(1-a)} \right], \\ \frac{\partial Q}{\partial a} &= \frac{1}{\sqrt{2\pi(1-a^2)}} \left[ \frac{ca}{(1-a^2)} \cdot \exp\left(\frac{-c^2}{2(1-a^2)}\right) + \exp\left(\frac{-(1-a)}{2(1+a)}\right) \cdot \frac{1}{1+a} \right].\end{aligned}$$

Substituting in equation (4) and collecting the terms having the same exponent, we get

$$\begin{aligned}\frac{\partial T}{\partial a} &= (2P_2 - 1) \cdot \exp\left(\frac{-c^2}{2(1-a^2)}\right) \cdot \frac{ca}{\sqrt{2\pi}(1-a^2)^{3/2}} \\ &\quad - (1 - P_1) \cdot \exp\left(\frac{-(1-a)}{2(1+a)}\right) \cdot \frac{1}{\sqrt{2\pi}(1+a)(1-a^2)^{1/2}} \\ &\quad - P_1 \cdot \exp\left(\frac{-(1+a)}{2(1-a)}\right) \cdot \frac{1}{\sqrt{2\pi}(1-a)(1-a^2)^{1/2}}.\end{aligned}$$

This is the sum of three terms, each of which are negative, as follows: The second and third terms are clearly negative. The first term is negative since

$$\begin{aligned}P_2(a=0) &= 1 - \gamma([-1, 1]) < 1/2, \\ \lim_{a \rightarrow 1^-} P_2 &= 1/2, \text{ and} \\ \frac{\partial P_2}{\partial a} &= -f_1(a) \geq 0, \quad \forall a \in [0, 1),\end{aligned}$$

and hence  $P_2(a) \leq 1/2$  for all  $a \in [0, 1)$ . So  $2P_2 - 1 \leq 0$  for  $a \in [0, 1)$ .  $\square$

Now we can give the proof of Theorem 2.

*Proof of Theorem 2* By the rotational symmetry of the Gaussian measure, we can assume  $\hat{u}$  to be the coordinate axis  $\hat{e}_1$ . Let  $a := \langle x_0, \hat{e}_1 \rangle$ , and  $\sigma := \sqrt{1-a^2}$ . Given Claim 7, the proof of the theorem follows by using the idea described earlier. We shall assume the slab to be a unit slab, i.e.  $b = 1$ . The general case follows by a standard scaling argument. Let  $h(x)$  be the marginal of  $\mathbf{1}_K$  according to the distribution  $\gamma'$ , along the direction  $\hat{u}$ , that is,

$$\forall x \in \mathbb{R} : h(x) = \frac{1}{(2\pi)^{(n-1)/2} \sigma} \int_{y \in \mathbb{R}^n : \langle y, \hat{u} \rangle = x} \mathbf{1}_K \cdot \exp\left(-\frac{\|y - x_0 - (x-a)\hat{u}\|_2^2}{2\sigma^2}\right) dy.$$

Hence,

$$\begin{aligned}\gamma'(K) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{x \in \mathbb{R}} h(x) \exp\left(-\frac{\|x-a\|_2^2}{2\sigma^2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{y \in \mathbb{R}} h(a + \sigma y) \exp\left(-\frac{\|y\|_2^2}{2}\right) dy \\ &= \int_{\mathbb{R}} h(a + y\sigma) d\gamma,\end{aligned}$$



and

$$\begin{aligned}\gamma'(K \cap L) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-1}^1 h(x) \exp\left(-\frac{\|x-a\|_2^2}{2\sigma^2}\right) dx \\ &= \int_{-(1+a)/\sigma}^{(1-a)/\sigma} h(a + \sigma y) d\gamma.\end{aligned}$$

Let

$$I_h(s) := \{x \in \mathbb{R} : h(a + \sigma y) \geq s\}.$$

By the Prékopa-Leindler inequality and the symmetry of  $K$  about  $x_0$ , we have that  $h(a + \sigma y)$  is a log-concave function of  $y$ , symmetric about  $a$ . Therefore, for all  $s \geq 0$ ,  $I_h(s)$  is an interval, symmetric about  $a$ . We have

$$\begin{aligned}\gamma'(K \cap L) &= \int_{s=0}^{\infty} \gamma'(I_h(s) \cap [-1, 1]) ds \\ &\geq \int_{s=0}^{\infty} \gamma'(I_h(s)) \cdot \gamma'([-1, 1]) ds \\ &= \left( \int_{s=0}^{\infty} \gamma'(I_h(s)) ds \right) \cdot \gamma'([-1, 1]) \\ &= \left( \int_{\mathbb{R}} h(x) d\gamma \right) \cdot \gamma'([-1, 1]) \\ &= \gamma'(K) \gamma'(L).\end{aligned}$$

where the inequality in the second step follows by Claim 7. □

## 4 Conclusion

The Theorem (2) is tight, as can be seen when  $a \rightarrow 1$ . It would be interesting to study other structural properties of the Gaussian measure of convex bodies, and the relation to algorithmic applications. We conjecture that Theorem 2 is part of a more general phenomenon:

**Conjecture 8** *Let  $K, L \in \mathbb{R}^n$  be convex bodies, with  $K$  symmetric about the origin, and  $L$  symmetric about  $x_0 \in \mathbb{R}^n$ . Then the following holds:*

$$\gamma'(K \cap L) \geq \gamma'(K) \gamma'(L),$$

where  $\gamma' \sim N(x_0, \tau)$ , with  $\tau$  being the expected exit time of a standard Brownian motion starting at  $x_0$ , from the region  $L$ .

Another interesting question that arises is, whether the symmetricity condition in our theorem can be relaxed to get a generalization of the result of Szarek and Werner.

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