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GLOBAL SOLUTIONS FOR A HYPERBOLIC-PARABOLIC SYSTEM OF CHEMOTAXIS

RAFAEL GRANERO-BELINCHÓN

Abstract. We study a hyperbolic-parabolic model of chemotaxis in dimensions one and two. In particular, we prove the global existence of classical solutions in certain dissipation regimes.

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1. Introduction

In this note we study the following system of partial differential equations

\[
\begin{align*}
\partial_t u &= -\Lambda^\alpha u + \nabla \cdot (uq), \quad \text{for } x \in \mathbb{T}^d, \quad t \geq 0, \\
\partial_t q &= \nabla f(u), \quad \text{for } x \in \mathbb{T}^d, \quad t \geq 0,
\end{align*}
\]

where \( u \) is a non-negative scalar function, \( q \) is a vector in \( \mathbb{R}^d \), \( \mathbb{T}^d \) denotes the domain \( [-\pi, \pi]^d \) with periodic boundary conditions, \( d = 1, 2 \) is the dimension, \( f(u) = u^2/2 \), \( 0 < \alpha \leq 2 \) and \( (-\Delta)^{\alpha/2} = \Lambda^\alpha \) is the fractional Laplacian.

This system was proposed by Othmers & Stevens [21] based on biological considerations as a model of tumor angiogenesis. In particular, in the previous system, \( u \) is the density of vascular endothelial cells and \( q = \nabla \log(v) \) where \( v \) is the concentration of the signal protein known as vascular endothelial growth factor (VEGF) (see Bellomo, Li, & Maini [1] for more details on tumor modelling). Similar hyperbolic-dissipative systems arise also in the study of compressible viscous fluids or magnetohydrodynamics (see S. Kawashima [8] and the references therein).
Equation (1) appears as a singular limit of the following Keller-Segel model of aggregation of the slime mold Dictyostelium discoideum [9] (see also Patlak [20])

\[
\begin{align*}
\frac{\partial t u}{\partial t} &= -\Lambda^{\alpha} u - \chi \nabla \cdot (u \nabla G(v)), \\
\frac{\partial t v}{\partial t} &= \nu \Delta v + (f(u) + \lambda) v,
\end{align*}
\]

when \( G(v) = \log(v) \) and the diffusion of the chemical is negligible, i.e. \( \nu \to 0 \).

Similar equations arising in different context are the Majda-Biello model of Rossby waves [18] or the magnetohydrodynamic-Burgers system proposed by Fleischer & Diamond [3].

Most of the results for (1) corresponds to the case where \( d = 1 \). Then, when the diffusion is local i.e. \( \alpha = 2 \), (1) has been studied by many different research groups. In particular, Fan & Zhao [2], Li & Zhao [13], Mei, Peng & Wang [19], Li, Pan & Zhao [12], Jun, Jixiong, Huijiang & Changjiang [7] Li & Wang [16] and Zhang & Zhu [25] studied the system (1) when \( \alpha = 2 \) and \( f(u) = u \) under different boundary conditions (see also the works by Jin, Li & Wang [6], Li, Li & Wang [14], Wang & Hillen [22] and Wang, Xiang & Yu [23]). The case with general \( f(u) \) was studied by Zhang, Tan & Sun [26] and Li & Wang [17].

Equation (1) in several dimensions has been studied by Li, Li & Zhao [11], Hau [5] and Li, Pan & Zhao [15]. There, among other results, the global existence for small initial data in \( H^{s}, s > 2 \) is proved.

To the best of our knowledge, the only result when the diffusion is non-local, i.e. \( 0 < \alpha < 2 \), is [4]. In that paper we obtained appropriate lower bounds for the fractional Fisher information and, among other results, we proved the global existence of weak solution for \( f(u) = u^r/r \) and \( 1 \leq r \leq 2 \).

In this note, we address the existence of classical solutions in the case \( 0 < \alpha \leq 2 \). This is a challenging issue due to the hyperbolic character of the equation for \( q \). In particular, \( u \) verifies a transport equation where the velocity \( q \) is one derivative more singular than \( u \) (so \( \nabla \cdot (uq) \) is two derivatives less regular than \( u \)).

2. STATEMENT OF THE RESULTS

For the sake of clarity, let us state some notation: we define the mean as

\[
\langle g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(x) dx.
\]

Also, from this point onwards, we write \( H^s \) for the \( L^2 \)-based Sobolev space of order \( s \) endowed with the norm

\[
\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \|\Lambda^s u\|_{L^2}^2, \quad \|u\|_{\dot{H}^s} = \|\Lambda^s u\|_{L^2}.
\]

For \( \beta \geq 0 \), we consider the following energies \( E_\beta \) and dissipations \( D_\beta \),

\[
E_\beta(t) = \|u\|_{H^\beta}^2 + \|q\|_{H^\beta}^2, \quad D_\beta(t) = \|u\|_{H^{\beta+\alpha/2}}^2, \quad D_\beta(t) = \|u\|_{H^{\beta+\alpha/2}}^2.
\]

Recall that the lower order norms verify the following energy balance [4]

\[
\frac{1}{2} (\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2) + \int_0^t \|u(s)\|_{H^{\alpha/2}}^2 ds = \frac{1}{2} (\|u_0\|_{L^2}^2 + \|q_0\|_{L^2}^2).
\]
2.1. On the scaling invariance. Notice that the equations (1)-(2) verify the following scaling symmetry: for every $\lambda > 0$

$$u_\lambda(x, t) = \lambda^{\alpha - 1} u(\lambda x, \lambda^\alpha t), \quad q_\lambda(x, t) = \lambda^{\alpha - 1} q(\lambda x, \lambda^\alpha t).$$

This scaling serves as a zoom in towards the small scales. We also know that

$$\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2$$

is a conserved quantity. Then, in the one dimensional case, the $L^2$ norms of $u$ and $q$ are invariant under the scaling of the equations when $\alpha = 1.5$. That makes $\alpha = 1.5$ the critical exponent for the global estimates known. Equivalently, if we define the rescaled (according to the scaling of the strongest conserved quantity $\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2$) functions

$$u_\gamma(x, t) = \gamma^{0.5} u(\gamma x, \gamma^\alpha t), \quad q_\gamma(x, t) = \gamma^{0.5} q(\gamma x, \gamma^\alpha t).$$

we have that $u_\gamma$ and $q_\gamma$ solve

$$\partial_t u_\gamma = -\Lambda^\alpha u_\gamma + \gamma^{\alpha - 1.5} \partial_x (u_\gamma q_\gamma),$$
$$\partial_t q_\gamma = \gamma^{\alpha - 1.5} u_\gamma \partial_x u_\gamma.$$

Larger values of $\alpha$ form the subcritical regime where the diffusion dominates the drift in small scales. Smaller values of $\alpha$ form the supercritical regime where the drift might be dominant at small scales.

Similarly, the two dimensional case has critical exponent $\alpha = 2$.

Remark 1. Notice that the equations (1)-(2) where $f(u) = u$ have a different scaling symmetry but the same critical exponent $\alpha = 1.5$. In this case, the scaling symmetry is given by

$$u_\lambda(x, t) = \lambda^{2\alpha - 2} u(\lambda x, \lambda^\alpha t), \quad q_\lambda(x, t) = \lambda^{\alpha - 1} q(\lambda x, \lambda^\alpha t),$$

while the conserved quantity is $\|u(t)\|_{L^1} + \|q(t)\|_{L^2}^2/2$. Thus, if we define the rescaled (according to the scaling of the conserved quantity) functions

$$u_\gamma(x, t) = \gamma u(\gamma x, \gamma^\alpha t), \quad q_\gamma(x, t) = \gamma^{0.5} q(\gamma x, \gamma^\alpha t).$$

we have that $u_\gamma$ and $q_\gamma$ solve

$$\partial_t u_\gamma = -\Lambda^\alpha u_\gamma + \gamma^{\alpha - 1.5} \partial_x (u_\gamma q_\gamma),$$
$$\partial_t q_\gamma = \gamma^{\alpha - 1.5} u_\gamma \partial_x u_\gamma.$$

A global existence result when $\alpha$ is the range $1.5 \leq \alpha < 2$ for the problem where $f(u) = u$ is left for future research.

2.2. Results in the one-dimensional case $d = 1$. One of our main result is

Theorem 1. Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(T) \times H^2(T)$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. Assume that $\alpha \geq 1.5$. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(T)) \cap L^2(0, T; H^{2+\alpha/2}(T)), q \in L^\infty(0, T; H^2(T)).$$

Furthermore, the solution is uniformly bounded in

$$(u, q) \in C([0, \infty), H^1(T) \times C([0, \infty), H^1(T)).$$
In the case where the strength of the diffusion, $\alpha$, is even weaker, we have the following global existence result for small data:

**Theorem 2.** Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. There exists $C_\alpha$ such that if $1.5 > \alpha > 1$ and

$$\|u_0\|_{H^{\alpha/2}}^2 + \|q_0\|_{H^{\alpha/2}}^2 \leq C_\alpha$$

then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(\mathbb{T})) \cap L^2(0, T; H^{2+\alpha/2}(\mathbb{T})), q \in L^\infty(0, T; H^2(\mathbb{T})).$$

Furthermore, the solution verifies

$$\|u(t)\|_{H^{\alpha/2}}^2 + \|q(t)\|_{H^{\alpha/2}}^2 \leq \|u_0\|_{H^{\alpha/2}}^2 + \|q_0\|_{H^{\alpha/2}}^2.$$  

**Corollary 1.** Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. Assume that $1 \geq \alpha \geq 0.5$

and

$$\|u_0\|_{H^1}^2 + \|q_0\|_{H^1}^2 < \frac{4}{9C_S^2}$$

where $C_S$ is defined in (7). Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(\mathbb{T})) \cap L^2(0, T; H^{2+\alpha/2}(\mathbb{T})), q \in L^\infty(0, T; H^2(\mathbb{T})).$$

Furthermore, the solution verifies

$$\|u(t)\|_{H^1}^2 + \|q(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + \|q_0\|_{H^1}^2.$$  

2.3. **Results in the two-dimensional case** $d = 2$. In two dimensions the global existence read

**Theorem 3.** Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}^2) \times H^2(\mathbb{T}^2)$ be the initial data such that $0 \leq u_0$ and $\text{curl} q_0 = 0$. Assume that $\alpha = 2$. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^3(\mathbb{T}^2)), q \in L^\infty(0, T; H^2(\mathbb{T}^2)).$$

Furthermore, the solution is uniformly bounded in

$$(u, q) \in C([0, \infty), H^1(\mathbb{T}^2)) \times C([0, \infty), H^1(\mathbb{T}^2)).$$

**Corollary 2.** Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}^2) \times H^2(\mathbb{T}^2)$ be the initial data such that $0 \leq u_0$ and $\text{curl} q_0 = 0$. Assume that $2 > \alpha \geq 1$ and

$$\|u_0\|_{H^1}^2 + \|q_0\|_{H^1}^2 < C$$

where $C$ is a universal constant. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^{2+\alpha/2}(\mathbb{T}^2)), q \in L^\infty(0, T; H^2(\mathbb{T}^2)).$$

Furthermore, the solution verifies

$$\|u(t)\|_{H^1}^2 + \|\nabla \cdot q(t)\|_{L^2}^2 \leq \|u_0\|_{H^1}^2 + \|\nabla \cdot q_0\|_{L^2}^2.$$
2.4. Discussion. Due to the hyperbolic character of the equation for \( q \), prior available global existence of classical solution for equation (1) impose several assumptions. Namely,

- either \( d = 1 \) and \( \alpha = 2 \) [26, 17],
- or \( d = 2, 3 \), \( \alpha = 2 \) and the initial data verifies some smallness condition on strong Sobolev spaces \( H^s, \ s \geq 2 \) [24, 27].

Our results removed some of the previous conditions. On the one hand, we prove global existence for arbitrary data in the cases \( d = 1 \) and \( \alpha \geq 1 \) and \( d = 2 \) and \( \alpha = 2 \). On the other hand, in the cases where we have to impose size restrictions on the initial data, the Sobolev spaces are bigger than \( H^2 \) (thus, the norm is weaker). Finally, let us emphasize that our results can be adapted to the case where the spatial domain is \( \mathbb{R}^d \).

A question that remains open is the trend to equilibrium. From (5) is clear that the solution \( (u(t), q(t)) \) tends to the homogeneous state, namely \((\langle u_0 \rangle, 0)\). However, the rate of this convergence is not clear.

3. Proof of Theorem 1

**Step 1; \( H^1 \) estimate:** Testing the first equation in (1) against \( \Lambda^2 u \), integrating by parts and using the equation for \( q \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2_{\dot{H}^1} + \| \partial_x u \|^2_{\dot{H}^{\alpha/2}} = - \int_T \partial_x (uq) \partial_x^2 u dx = \frac{1}{2} \int_T \partial_x q (\partial_x u)^2 dx - \int_T \partial_x q u \partial_x^2 u dx = \frac{1}{2} \int_T \partial_x q (\partial_x u)^2 dx - \int_T \partial_x q (\partial_t \partial_x q - (\partial_x u)^2) dx,
\]

so

\[
\frac{1}{2} \frac{d}{dt} (\| u \|^2_{\dot{H}^1} + \| q \|^2_{\dot{H}^1}) + \| u \|^2_{\dot{H}^{1+\alpha/2}} = \frac{3}{2} \int_T \partial_x q (\partial_x u)^2 dx.
\]

Denoting

\[ I = \frac{3}{2} \int_T \partial_x q (\partial_x u)^2 dx, \]

and using Sobolev embedding and interpolation, we have that

\[
I \leq \frac{3}{2} \| q \|_{\dot{H}^1} \| \partial_x u \|^2_{L^4} \leq \frac{3}{2} C_S \| q \|_{\dot{H}^1} \| \partial_x u \|^2_{\dot{H}^{0.25}},
\]

where \( C_S \) is the constant appearing in the embedding

\[
\| g \|_{L^4} \leq C_S \| g \|_{\dot{H}^{0.25}}.
\]

Using the interpolation

\[
H^{1+\alpha/2} \subset H^{1.25} \subset H^{\alpha/2},
\]

and Poincaré inequality (if \( \alpha > 1.5 \)) we conclude

\[
I \leq c \| q \|_{\dot{H}^1} \| \Lambda^{\alpha/2} u \|_{L^2} \| u \|_{\dot{H}^{1+\alpha/2}},
\]

Using (4), we have that

\[
\frac{d}{dt} E_1 + D_1 \leq c \| u \|^2_{\dot{H}^{\alpha/2}} E_1.
\]
Using Gronwall’s inequality and the estimate (5), we have that
\[ \sup_{0 \leq t < \infty} E_1(t) \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}). \]

\[ \int_0^T D_1(s)ds \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}, T), \forall 0 < T < \infty. \]

**Step 2; \( H^2 \) estimate:** Now we prove that the solutions satisfying the previous bounds for \( E_1 \) and \( D_1 \) also satisfy the corresponding estimate in \( H^2 \). We test the equation for \( u \) against \( \Lambda^4 u \). We have that
\[ \frac{1}{2} \frac{d}{dt} \|u\|_{H^2}^2 + \|u\|_{H^{2+\alpha/2}}^2 = -\int_T \partial_x^2(uq)\partial_x^2udx \]
\[ = \int_T \partial_x q \frac{5}{2}(\partial_x^2 u)^2dx - \int_T \partial_x^2 q(\partial_x \partial_x^2 q - 5\partial_x u\partial_x^2 u)dx, \]
so
\[ \frac{1}{2} \frac{d}{dt}(\|u\|_{H^2}^2 + \|q\|_{H^2}^2) + \|u\|_{H^{2+\alpha/2}}^2 = \frac{5}{2} \int_T \partial_x q(\partial_x^2 u)^2dx + 5 \int_T \partial_x^2 q\partial_x^2 u\partial_x udx. \]
We define
\[ J_1 = \frac{5}{2} \int_T \partial_x q(\partial_x^2 u)^2dx, \quad J_2 = 5 \int_T \partial_x^2 q\partial_x^2 u\partial_x udx. \]
Then, we have that
\[ J_1 \leq c\|\partial_x q\|_{L^\infty} \|\partial_x^2 u\|_{L^2}^2 \leq c\|\partial_x^2 q\|_{L^2} \|u\|_{H^{1+\alpha/2}}^{0.5} \|u\|_{H^{2+\alpha/2}}^{2-\alpha}, \]
so, using Young’s inequality,
\[ J_1 \leq c\|\partial_x^2 q\|_{L^2} \|u\|_{H^{1+\alpha/2}}^{2} + \frac{1}{4} \|u\|_{H^{2+\alpha/2}}^{2}. \]
Similarly, using Poincaré inequality and \( \alpha \geq 0.5 \),
\[ J_2 \leq c\|\partial_x u\|_{L^4} \|\partial_x^2 u\|_{L^4} \|\partial_x^2 q\|_{L^2} \leq c\|u\|_{H^{1+\alpha/2}} \|u\|_{H^{2+\alpha/2}} \|\partial_x^2 q\|_{L^2}, \]
and
\[ J_2 \leq c\|u\|_{H^{1+\alpha/2}}^{2} \|\partial_x^2 q\|_{L^2}^{2} + \frac{1}{4} \|u\|_{H^{2+\alpha/2}}^{2}. \]
Finally,
\[ \frac{d}{dt} E_2(t) + D_2(t) \leq c\|u\|_{H^{1+\alpha/2}}^{2}(E_2(t) + 1) \]
and we conclude using Gronwall’s inequality.

4. Proof of Theorem 2

**Step 1; \( H^{\alpha/2} \) estimate:** Testing the first equation in (1) against \( \Lambda^\alpha u \), we obtain
\[ \frac{1}{2} \frac{d}{dt} \|u\|_{H^{\alpha/2}}^2 + \|u\|_{H^\infty}^2 = \int_T \partial_x(uq)\Lambda^\alpha udx \]
\[ = -\int_T \Lambda^\alpha(u)\partial_x udx \]
\[ = -\int_T (\Lambda^\alpha(u) - u\Lambda^\alpha q) \partial_x udx - \int_T \Lambda^\alpha qu\partial_x udx, \]
so
\[ \frac{1}{2} \frac{d}{dt} (\|u\|_{H^{\alpha/2}}^2 + \|q\|_{H^{\alpha/2}}^2) + \|u\|_{H^\infty}^2 \leq -\int_T [\Lambda^\alpha, u]qu\partial_x udx. \]
We define
\[ K = - \int_T [\Lambda^\alpha, u] q \partial_x u \, dx. \]

Using the classical Kenig-Ponce-Vega commutator estimate \cite{10} and Sobolev embedding, we have that
\[
\| [\Lambda^\alpha, u] q \|_{L^2} \leq \epsilon \left( \| \partial_x u \|_{L^{2+\epsilon}} \| \Lambda^{\alpha - 1} q \|_{L^2} + \| \Lambda^\alpha u \|_{L^\infty} \right) + \epsilon \left( \| u \|_{\dot{H}^{1+\epsilon}} \left\| \Lambda^{\alpha - 1} q \right\|_{\dot{H}^{\frac{1}{2} - \frac{\epsilon}{4\alpha}}} + \| u \|_{\dot{H}^{\alpha}} \| q \|_{\dot{H}^{\frac{\alpha}{2}}} \right).
\]

Thus, taking \( \epsilon \) such that
\[ 1 + \frac{\epsilon}{4 + 2\epsilon} = \alpha, \quad \text{i.e.} \quad \epsilon = \frac{4\alpha - 4}{3 - 2\alpha}, \]
Equation (8) reads
\[
\| [\Lambda^\alpha, u] q \|_{L^2} \leq C_{\delta} \left( \| u \|_{\dot{H}^{1+\epsilon}} \left\| \Lambda^{\alpha - 1} q \right\|_{\dot{H}^{\frac{1}{2} - \frac{\epsilon}{4\alpha}}} + \| u \|_{\dot{H}^{\alpha}} \| q \|_{\dot{H}^{\frac{\alpha}{2}}} \right)
\]
Using (9) and Poincaré inequality, we have that
\[
K \leq c \| u \|_{\dot{H}^{1+\epsilon}} \| q \|_{\dot{H}^{\frac{1}{2}}}.
\]
Then, we have that
\[
d \frac{d}{dt} E_{\alpha} + D_{\alpha} \leq c \sqrt{E_{\alpha}} D_{\alpha}.
\]
Thus, due to the smallness restriction on the initial data, we obtain
\[
E_{\alpha}(t) + \delta \int_0^t D_{\alpha}(s) \, ds \leq E_{\alpha}(0)
\]
for \( 0 < \delta \) small enough.

**Step 2: \( H^1 \) estimate:** Our starting point is (6). Then we use the interpolation
\[
\left\| g \right\|_{\dot{H}^{0.25}}^2 \leq c \| g \|_{L^2} \left\| g \right\|_{\dot{H}^{0.5}},
\]
to obtain
\[
I \leq c \left\| q \right\|_{\dot{H}^1} \| u \|_{\dot{H}^{1.5}} \| u \|_{\dot{H}^{1.5}} \leq c E_1 D_{\frac{1}{2}} + \frac{D_1}{2}.
\]
Collecting all the estimates, we have that
\[
\frac{d}{dt} E_1 + D_1 \leq c E_1 D_{\frac{1}{2}},
\]
and we conclude using Gronwall’s inequality. The \( H^2 \) estimates follows as in the proof of Theorem 1.

5. **Proof of Corollary 1**

Using \( \alpha \geq 0.5 \) and the estimate (6), we have that
\[
I \leq \frac{3}{2} C_S \| q \|_{\dot{H}^{1+\alpha/2}} \| u \|_{\dot{H}^{1+\alpha/2}} \leq \frac{3}{2} C_S \sqrt{E_1 D_1}.
\]
Thus,
\[
\frac{1}{2} \frac{d}{dt} E_1 + D_1 \leq \frac{3}{2} C_S \sqrt{E_1 D_1}.
\]
Thus, due to the smallness restriction on the initial data, we obtain
\[
E_1(t) + \delta \int_0^t D_1(s) \, ds \leq E_1(0)
\]
for \(0 < \delta\) small enough. Equipped with this estimates, we can repeat the argument as in Step 2 in Theorem 1.

6. Proof of Theorem 3

Recall that the condition
\[
\text{curl } q_0 = 0
\]
propagates in time, \textit{i.e.}
\[
\text{curl } q(t) = \text{curl } q_0 + \frac{1}{2} \int_0^t \text{curl} \nabla u^2 \, ds = 0.
\]
As a consequence, every coordinate of \(q\) satisfy
\[
(q_i(t)) = 0,
\]
and the Poincaré-type inequality
\[
\|q\|_{L^2} \leq c \|\nabla \cdot q\|_{L^2}.
\]
Notice that in two dimensions we also have the energy balance (5). We test equation (1) against \(\Lambda^2 u\) and use the equation for \(q\). We obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) = -\|u\|_{H^2}^2 - \int_{T^2} \nabla u \cdot q \Delta u \, dx + \int_{T^2} |\nabla u|^2 \nabla \cdot q \, dx.
\]
Using Hölder inequality, Sobolev embedding and interpolation, we have that
\[
\frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + 2\|u\|_{H^2}^2 \leq c \left( \|u\|_{H^{1.5}} \|q\|_{L^1} \|u\|_{H^2} + \|u\|_{H^{1.5}}^2 \|\nabla \cdot q\|_{L^2} \right)
\leq c\|u\|_{H^{1.5}} \|q\|_{L^1} \|u\|_{H^2}^2 + \|u\|_{H^{1.5}} \|\nabla \cdot q\|_{L^2}^2.
\]
Using the Hodge decomposition estimate together with the irrotationality of \(q\) and (10), we have that
\[
\|q\|_{H^1} \leq c(\|q\|_{L^2} + \|\nabla \cdot q\|_{L^2}) \leq c \|\nabla \cdot q\|_{L^2}.
\]
Due to (5), we obtain that
\[
\frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + \|u\|_{H^2}^2 \leq c\|u\|_{H^2}^2 \|\nabla \cdot q\|_{L^2}^2.
\]
Using Gronwall’s inequality and (5), we obtain
\[
E_1 \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}),
\]
\[
\int_0^T D_1(s) \, ds \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}, T), \forall 0 < T < \infty.
\]
To obtain the \(H^2\) estimates, we test against \(\Lambda^4 u\). Then, using the previous \(H^1\) uniform bound and
\[
\|q\|_{L^\infty} \leq c\|q\|_{L^2} \|q\|_{H^2} \leq c\|q\|_{L^2} \|\Delta q\|_{L^2},
\]
we have that
\[
\frac{1}{2} \frac{d}{dt} \|\Delta u\|_{L^2}^2 + \|u\|_{H^3}^2 = -\int_{T^d} \nabla \Delta u \nabla (\nabla u \cdot q) \, dx
\]
\[
- \int_{T^d} u \nabla \Delta u \cdot \nabla \cdot q \, dx - \int_{T^d} \nabla u \cdot \nabla \Delta u \nabla \cdot q \, dx.
\]
Due to the irrotationality of $q$ and the identity
\[ \nabla \nabla \cdot q - \Delta q = \text{curl} \, (\text{curl} \, q), \]
we have
\[ \partial_t \Delta q = \partial_t \nabla (\nabla \cdot q) = \nabla |\nabla u|^2 + u \nabla \Delta u + \nabla u \Delta u. \]
Applying Sobolev embedding and interpolation, we obtain that (12) can be estimated as
\[ \frac{d}{dt} \left( \|\Delta u\|_{L^2}^2 + \|\Delta q\|_{L^2}^2 \right) + \|u\|_{H^3}^2 \leq c \|\Delta q\|_{L^2}^2, \]
so,
\[ \frac{d}{dt} E_2 + D_2 \leq c E_2, \]
and we conclude using Gronwall’s inequality.

7. Proof of Corollary 2

We test the equation (1) against $\Lambda^2 u$. We obtain that
\[ \frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) = -\|u\|_{H^{1+\frac{\alpha}{2}}}^2 + \int_{T^2} \nabla (\nabla u \cdot q) \nabla u dx + \int_{T^2} |\nabla u|^2 \nabla \cdot q dx. \]
After a short computation, using Hölder estimates, Sobolev embedding and interpolation, we obtain that
\[ \frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + \|u\|_{H^{1+\frac{\alpha}{2}}}^2 \leq c \|u\|_{H^{1.5}}^2 \|q\|_{H^1}. \]
Using (11) and $\alpha \geq 0$, we obtain
\[ \frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + D_1 \leq c D_1 \|\nabla \cdot q\|_{L^2}^2. \]
We conclude the result with the previous ideas.

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References


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