On the reinforcement of uninorms and absorbing norms

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***Abstract*—Aggregation operators Reinforcement ... We propose a n-ary extension of absorbing norms, defined with the help of generative functions, and its relationship with additive generating functions of uninorms. In this paper, we also present new aggregation operators, namely the k-uninorms and k-absorbing norms. These operators are a generalization of usual uninorms and absorbing norms for which a set combination of inputs is introduced. Their main ability is to provide reinforcement for contradictory inputs, as nullnorms and as opposed to uninorms. On the other hand it still provides full reinforcement for agreeing inputs, as uninorms and as opposed to nullnorms. Numerous examples are given in order to illustrate the behavior of the proposed operators.**

finish the abstract : reinforcement (weak?) before

***Index Terms*—Absorbing norms, aggregation operator, nullnorms, reinforcement, uninorms**

I. Introduction

Deeper introduction ...

During the last decades, one has witnessed a tremendous growth in the use of aggregation functions theory and its applications. A number of aggregation functions are closely linked to the theory fuzzy sets. In particular, triangular (co)norms [15], [12], uninorms and nullnorms [27], [3], [8] are prototypical examples of aggregation functions used in practice. As opposed to usual aggregation operators such as the arithmetic mean, these functions allow more flexibility, and thus present a more interesting data-specific behavior than standard aggregation operators.

Another reason of using such operators is that they allow to deal with the inherent uncertainty of the data.

complete on uncertainty

We are interested in this paper in uninorms, as they generalize both triangular norms and triangular conorms thanks to their neutral element. We also inspect nullnorms, which are also operators combining triangular norms and triangular conorms. Both uninorms and nullnorms show an interesting structure on the unit interval [9], [14]. In particular, it can be shown that uninorms act as full reinforcement operators: the output is large when all the inputs are large, the output is low when all the inputs are low, and the output is moderate when the inputs are in-between. There is no such study, or property, exhibited for nullnorms. Furthermore, the property of full reinforcement is not always adapted to the attitude of the decision-aiding system. This paper addresses the aforementioned issues by proposing new classes of operators, namely the k-uninorms and the k-absorbing norms, that show more adaptability to the input than standard uninorms and nullnorms.

This paper is organized as follows. We first give some definitions and fundamentals about fuzzy binary connectives and their n-ary extensions in Section II. In Section III, we give a short tour on uninorms, nullnorms and absorbing norms, and propose function generated n-ary extensions of these operators.

Then we present some recent related works on this topic, and we propose the k-uninorms and k-absorbing norms, as well as their properties. We also give some detailed examples on prototypical examples, and their use in ranking for decision-aiding, in Section V. Finally, we draw a conclusion and give some perspectives in Section VI.

II. Preliminaries

In many domains, making the fusion of information, or aggregation of information, is an important task. Generally speaking, aggregating data corresponds to mapping several values into a single value that is representative of the input values. Often, aggregated values come from evaluation of whether an expert, or a machine. In this paper, we assume that, without loss of generality, input values belongs to the unit interval $I = [0, 1]$.[1]

More formally, an aggregation operator is defined as follows [12]

Definition An operator $\mathcal{A}$ is an aggregation operator if and only if it satisfies boundary conditions and monotonicity:

- $\mathcal{A}(0, \cdots, 0) = 0$ and $\mathcal{A}(1, \cdots, 1) = 1$
- $\forall n, x_1 \leq y_1, \cdots, x_n \leq y_n \Rightarrow \mathcal{A}(x_1, \cdots, x_n) \leq \mathcal{A}(y_1, \cdots, y_n)$

As additional properties, one can find symmetric, associative aggregation operators, see [16] for more details and other properties. Triangular norms, introduced by Menger [22], are a generalization of the triangle inequality in probabilistic metric

[1] if not, a simple normalization allows to obtain this property.

spaces. More recently they have been used as conjunctive operations in logic and intersections in lattices. Triangular norms (t-norms for short), together with their dual operators, triangular co-norms, which are disjunctive operators in logic, can be considered as a family of operators allowing to manage logical **and** and **or** connectives for multi-valued and fuzzy logic.

More precisely, a t-norm is a binary function $\top : [0,1]^2 \to [0,1]$ that is, for any x, y and z in the unit interval

- commutative, $\top(x,y) = \top(y,x)$
- monotonic, for $y \geq z$, $\top(x,y) \geq \top(x,z)$
- associative, $\top(x,\top(y,z)) = \top(\top(x,y),z)$
- having 1 as neutral element, $\top(1,x) = x$

It is interesting to note that the associativity of t-norms allows an easy extension to n-ary operators. Moreover, the representation theorem [20] allows to obtain generated t-norms thanks to additive or multiplicative generating functions [15]. A strictly decreasing function $t : [0,1] \to [0,\infty[$, with $t(1) = 0$, can be used to construct a t-norm

$$\top(x_1,\cdots,x_n) = t^{(-1)}\left(\sum_{i=1}^{n} t(x_i)\right), \tag{1}$$

where $t^{(-1)}$ is the pseudo-inverse function of t, defined by $t^{(-1)}(y) = \sup\left(x \in I | t(x) > y\right)$.

Example
- Let $t(x) = -\log(x)$, then $\top(x_1,\cdots,x_n) = \prod_{i=1}^{n} x_i$ is the product t-norm,
- Let $t(x) = 1 - x$, then $\top(x_1,\cdots,x_n) = \max\left(0, \sum_{i=1}^{n} x_i - (n-1)\right)$ is the Lukasiewicz t-norm.

Beside mathematical properties, t-norms also show an interesting behavioral property as they provide downward reinforcement [31]. More precisely, if all the input values are low, then the aggregation is even more low. Such aggregation is seen as pessimistic attitude toward decision.

The dual operator of t-norms (i.e. a commutative, monotonic and associative operator with neutral element 0) is called a triangular conorm $\perp$ (t-conorm for short). As for t-norms, it exists a strictly increasing function $s : [0,1] \to [0,\infty[$, with $s(0) = 0$, that can be used to construct a t-conorm:

$$\perp(x_1,\cdots,x_n) = s^{(-1)}\left(\sum_{i=1}^{n} s(x_i)\right), \tag{2}$$

where $s^{(-1)}$ is the pseudo-inverse function of s, defined by $s^{(-1)}(y) = \sup\left(x \in I | s(x) < y\right)$. Note that additive generators of a t-norm and its dual t-conorm are related by $s(x) = t(1-x)$.

Example
- Let $s(x) = -\log(1-x)$, then $\perp(x_1,\cdots,x_n) = 1 - \prod_{i=1}^{n}(1-x_i)$ is the sum product t-conorm,
- Let $s(x) = x$, then $\perp(x_1,\cdots,x_n) = \min\left(1, \sum_{i=1}^{n} x_i\right)$ is the bounded sum (Lukasiewicz t-conorm).

By duality of t-norms, t-conorms provide upward reinforcement : if all the inputs are large, then the result is even larger. Such aggregation is seen as optimistic attitude toward decision. More details on triangular norms are out of the scope of this paper, and the interested reader can refer to more complete surveys, see e.g. [15], for more informations.

III. Uninorms and nullnorms

A. Uninorms

Triangular norms and triangular conorms provide a downward and upward reinforcement, respectively, but not simultaneously. In order to deal with this problem, Yager and Rybalov [27] introduced an aggregation operator, the uninorm $\mathcal{U}$. This operator is a generalization of triangular norms and conorms. The neutral element e belonging to the unit interval allows to use a t-norm or t-conorm depending on the input values.

Definition An uninorm is a binary operator $\mathcal{U}$ which is commutative, associative, increasing with a neutral element e lying in the unit interval, and for which every $x \in [0,1]$, $\mathcal{U}(x,e) = x$ holds.

An interesting property of uninorms is that they allow a compensation between values that are separated by a specified neutral element. Consequently, uninorm operators are able to provide both downward and upward reinforcement, as well as compensation for in-between values. Naturally, the relationship of uninorms and triangular norms is important, and one can even write a triangular norm as a function of an uninorm

$$\top_{\mathcal{U}}(x,y) = \frac{\mathcal{U}(ex,ey)}{e} \tag{3}$$

Alternatively, the function

$$\perp_{\mathcal{U}}(x,y) = \frac{\mathcal{U}(e+(1-e)x, e+(1-e)y) - e}{1-e} \tag{4}$$

is a triangular conorm [10]. The behavior and the structure of an uninorm is strongly linked to triangular norms in $[0,e]^2$, while it is related to triangular conorms in $[e,1]^2$. On the other parts of the squared unit, $\mathcal{U}$ is bounded by the minimum and the maximum, i.e. for all $(x,y) \in [0,1]^2 \backslash ([0,e]^2 \cup [e,1]^2)$, $\mathcal{U}$ is a compensation operator. Since $\mathcal{U}$ is an associative function, we have $\mathcal{U}(0,1) \in \{0,1\}$. One calls a conjunctive uninorm an uninorm $\mathcal{U}$ satisfying $\mathcal{U}(0,1) = 0$, and a disjunctive uninorm $\mathcal{U}(0,1) = 1$. The functions defined in (5) and (6) provide a generic family of operators :

$$\mathcal{U}(x,y) = \begin{cases} e\top(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y) \in [0,e[^2 \\ e + (1-e)\perp\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2 \\ \min(x,y) & \text{otherwise} \end{cases} \tag{5}$$

$$\mathcal{U}(x,y) = \begin{cases} e\top(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y) \in [0,e]^2 \\ e + (1-e)\perp\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y) \in]e,1]^2 \\ \max(x,y) & \text{otherwise} \end{cases} \tag{6}$$

In (5), one can note that $\mathcal{U}(0,1) = 0$, implying that $\mathcal{U}$ is a conjunctive uninorm, while in (6), one have $\mathcal{U}(0,1) = 1$, making $\mathcal{U}$ a disjunctive uninorm. The properties of $\mathcal{U}$, $\top$ and $\perp$ allows to extend each uninorm to its n-ary operator [4] :

$$\mathcal{U}(x_1,\cdots,x_n) = \mathcal{U}\Big(\top^\star(min(x_1,e),\cdots,min(x_n,e)),\\ \perp^\star(max(x_1,e),\cdots,max(x_n,e)\Big),$$

where $\top^\star$, $\perp^\star$ are given by

$$\top^\star(x,y) = e\top(\frac{x}{e},\frac{y}{e})$$
$$\perp^\star(x,y) = e + (1-e)\perp(\frac{x-e}{1-e},\frac{y-e}{1-e})$$

Also, due to its definition, it is important to note that any uninorm $\mathcal{U}$ holds the compromise property (i.e. larger –or equal– than the minimum and lower –or equal– than the maximum) if the neutral element lies between the minimum and the maximum value. A interesting behavior of compensation for Archimedean uninorms [6] may occur for two values separated by the neutral element e, in particular, we have the following definition.

Definition An aggregation operator $\mathcal{U}$ is a continuous Archimedean uninorm if for any $(x_1,\cdots,x_n) \in \bigcup_{n\in\mathbb{N}}[0,1]^n$, $\{0,1\} \subset \{x_1,\cdots,x_n\}$, if and only if there exists a monotonic bijection $g : [0,1] \to [-\infty,\infty]$, with $g(e) = 0$, such that

$$\mathcal{U}(x_1,\cdots,x_n) = g^{-1}\Big(\sum_{i=1}^{n} g(x_i)\Big) \tag{7}$$

As defined by (7), $\mathcal{U}$ is a generated uninorm with additive generator g, with neutral element e. Note that if $g(1) = 0$, $\mathcal{U}$ is t-norm (see Eq. (1)), while if $g(0) = 0$, $\mathcal{U}$ is t-conorm (see Eq. (2)).

Uninorms provide full reinforcement, i.e. both downward and upward reinforcement. When inputs are mixed or contradictory (i.e. low and large values), the output is averaging the input. Consequently, an operator providing full reinforcement is optimistic for positive inputs, and pessimistic for negative inputs.

In this paper, we use the general additive function g defined as

$$g_\alpha(x) = \log\frac{x^\alpha}{1-x^\alpha}. \tag{8}$$

Note that using the value $\alpha = 1$ in (8) allows to retrieve the uninorm 3Π operator as defined in [31]:

$$\mathcal{U}(x_1,\cdots,x_n) = \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n x_i + \prod_{i=1}^n(1-x_i)}$$

The corresponding pseudo-inverse function g_α^{-1} of (8) is defined as

$$g_\alpha^{-1}(x) = \left(\frac{\exp(x)}{1+\exp(x)}\right)^{\frac{1}{\alpha}}. \tag{9}$$

Note that given a neutral element e of a generated uninorm, one can obtain the corresponding value of α as $\alpha = \dfrac{-\log(2)}{\log(e)}$. As can be observed, $\lim_{\alpha\to 0}(1/2)^{1/\alpha} = e = 0$, and $\lim_{\alpha\to+\infty}(1/2)^{1/\alpha} = e = 1$. Consequently, both t-norms and t-conorms can be obtained (by taking the limit). Conversely, having the function g, one can obtain the corresponding neutral element by noting that $e = g^{-1}(0)$.

Proposition III.1. *Let $\mathcal{U}_\alpha$ be defined as $\mathcal{U}_\alpha(x_1,\cdots,x_n) = g_\alpha^{-1}\Big(\sum_{i=1}^n g_\alpha(x_i)\Big)$. Then $\mathcal{U}_\alpha$ is decreasing with α.*

Proof. The general term of $\mathcal{U}_\alpha$ is given by

$$\mathcal{U}_\alpha = \left(\frac{\prod_{i=1}^n x_i^\alpha}{\prod_{i=1}^n x_i^\alpha + \prod_{i=1}^n 1 - x_i^\alpha}\right)^{1/\alpha}. \tag{10}$$

Taking the derivative of Eq. (10) with respect to α yields $\partial\mathcal{U}_\alpha/\partial\alpha \leq 0$, whatever α, concluding the proof.[2] □

For the sake of clarity and brevity, we use in this paper only three different values of α, 0.5, 1 and 2 that correspond to the neutral elements 0.25, 0.5 and 0.707, respectively. The corresponding iso-values of these uninorms are given in Figure 1, where pure white encodes the value 0, and blue the value 1. As can be observed, the values on the four bounds of the unit interval do not change, but this not the case for the diagonal $x \approx y$. In this case the lower e, the larger output for two low inputs, and the reverse for two large inputs.

B. Nullnorms

In the previous section, uninorms, which are ordinal sums of t-norms and t-conorms, have been presented. In this section, we devote our attention to a family of operators that are closely related to uninorms, the nullnorms [5]. Instead of relaxing the neutral element, the nullnorms relax the annihilator constraint from being one or zero to be any value in the unit interval. By duality to uninorms, nullnorms behave like a t-conorm when aggregated values are lower than the absorbing element, noted a, and behave like a t-norm when aggregated values are larger than a. In other cases, the nullnorms reduce to their absorbing element, a.

Definition A nullnorm $\mathcal{V}$ is a commutative, associative and increasing function, having a zero-element (absorbing element) $a \in [0,1]$, which satisfies $\mathcal{V}(x,0) = x$ for all $x \leq a$, and $\mathcal{V}(x,1) = x$ for all $x \geq a$.

We say that a is the annihilator of $\mathcal{V}$ (by monotonicity). For a given nullnorm $\mathcal{V}$ with annihilator element $a \in]0,1[$, the binary operator $T_\mathcal{V}$ defined by:

$$\top_\mathcal{V}(x,y) = \frac{\mathcal{V}(a+(1-a)x, a+(1-a)y) - a}{1-a} \tag{11}$$

is a t-norm, and for a nullnorm V with annihilator element $a \in]0,1[$, the binary operator $S_\mathcal{V}$ defined by:

$$\perp_\mathcal{V}(x,y) = \frac{\mathcal{V}(ax,ay)}{a} \tag{12}$$

is a t-conorm. Therefore, the structure of a nullnorm on $[0,a]^2$ is closely related to a t-conorm, and its structure on $[a,1]^2$ is closely related to a t-norm. When fixing $\top$ and $\perp$, a unique

[2]The complete derivation is omitted for sake of brevity.

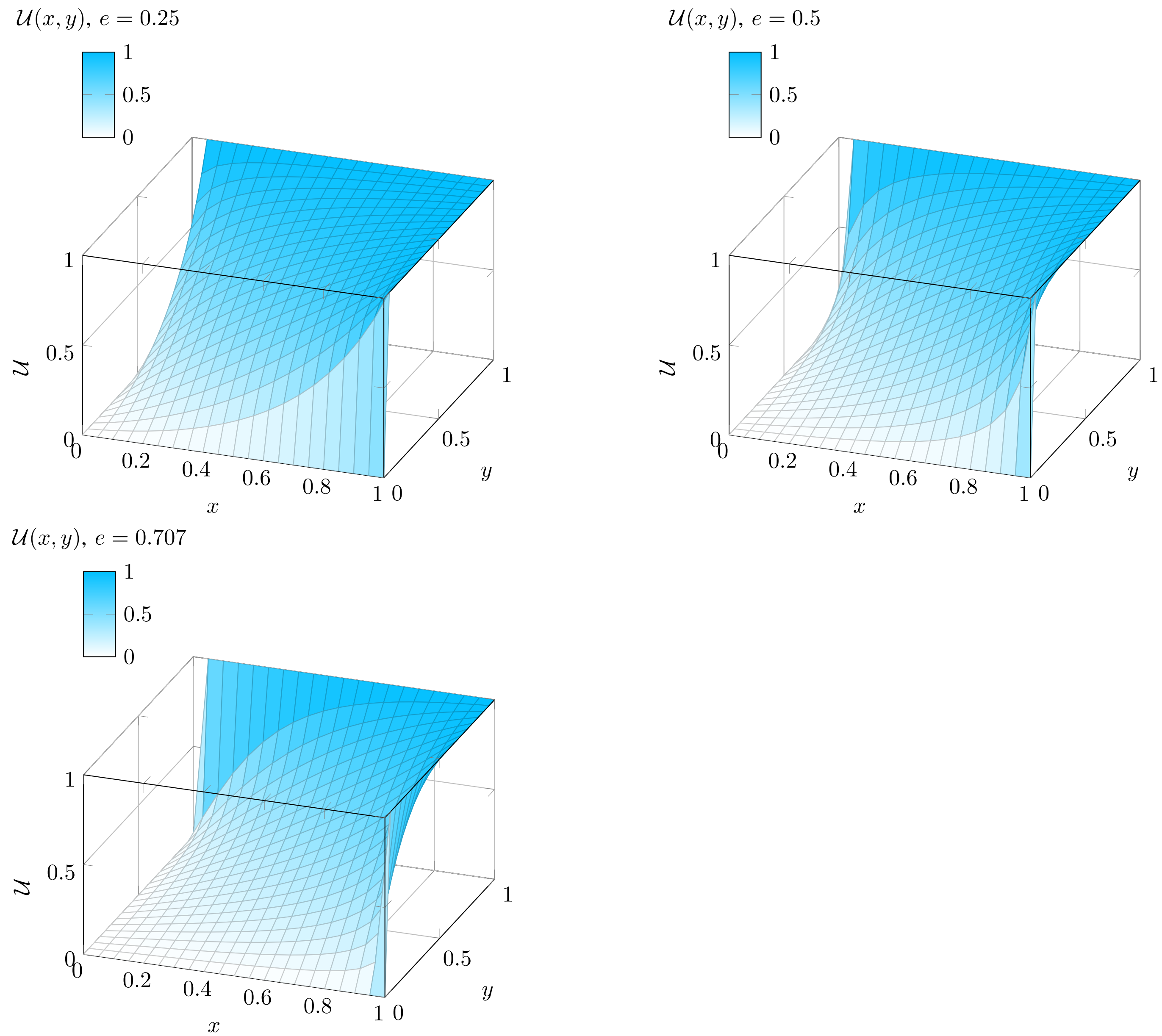

Figure 1. Values of generated continuous uninorms on the unit square, for three different values of α = 0.5, 1 and 2. These values are corresponding to the neutral elements e = 0.25, 0.5 and 0.707, respectively. The middle one ($e = 0.5$) coincides with the 3Π operator.

nullnorm satisfies Eq. (11) and Eq. (12) on the rest of the unit square. This is the nullnorm given by:

$$\mathcal{V}(x, y) = a \qquad \forall (x, y) \in [0, 1]^2 \backslash ([0, a]^2 \cup [a, 1]^2) \tag{13}$$

As for uninorms, one can show that a nullnorm can be written as

$$\mathcal{V}(x, y) = \begin{cases} a\bot(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2 \\ a + (1 - a)\top\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x, y) \in [a, 1]^2 \\ a & \text{otherwise} \end{cases} \tag{14}$$

One can also obtain nullnorms from generating functions.

Definition An aggregation operator $\mathcal{V}$ is a continuous nilpotent nullnorm with absorbing element a if and only if there exists an increasing bijection $q : [0, 1] \to [0, 1]$ such that

$$\mathcal{V}(x_1, \cdots, x_n) = q^{-1}\left(med\left(\sum_{i=1}^{n} q(x_i), \sum_{i=1}^{n} q(x_i) - (n - 1), q(a)\right)\right) \tag{15}$$

In practice, nullnorms are barely used, due to their properties. In particular, whenever there are two values, one above and one below the absorbing element, then the output is this absorbing element. Unfortunately, this is almost always the case for real data. A soften version of nullnorm, called absorbing norm, has been introduced in [25]. This relaxation particularly focus on the monotony of nullnorms, which is the reason why nullnorms can barely be used in practical applications.

Definition An absorbing norm V is a commutative and associative function, having a zero-element (absorbing element) $a \in [0,1]$, which satisfies $V(x,a) = a$ for all $x \in [0,1]$.

As for nullnorms, if we set a to 0, then $V(x,y)$ is a t-norm, and alternatively, setting a to 1 gives a t-conorm. Two important classes of absorbing norms are the conjunctive and disjunctive absorbing norms, defined by

$$V(x,y) = \begin{cases} a\perp(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a]^2 \\ a + (1-a)\top\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2 \\ M(x,y) & \text{otherwise} \end{cases} \tag{16}$$

where $M(x,y)$ can be either the minimum operator or the maximum operator. The most important consequence, compared to nullnorms, is that it is not constant and equal to the absorbing element in $[0,a] \times [a,1]$ and $[a,1] \times [0,a]$.

Proposition III.2. *Let us write the n-ary extension of absorbing norm using multiplicative generator as*

$$V(x_1, \cdots, x_n) = h^{-1}\left(\prod_{i=1}^{n} h(x_i)\right) \tag{17}$$

where the function $h : [0,1] \to]-\infty, \infty[$ is a strictly increasing and continuous mapping satisfying $h(0) = -\infty$, $h(a) = 0$ and $h(1) = +\infty$. $V(x_1, \cdots, x_n)$ is a an absorbing norm.

Proof. Commutativity is easily obtained by commutativity of the product. Let us consider the associative property.

$$\begin{aligned} V(x_1, V(x_2, \cdots, x_n)) &= h^{-1}(h(x_1)h(V(x_2, \cdots, x_n))) \\ &= h^{-1}\left(h(x_1)h(h^{-1}(\prod_{i=2}^{n} h(x_i)))\right) \\ &= h^{-1}\left(\prod_{i=1}^{n} h(x_i)\right) \\ &= h^{-1}\left(h(h^{-1}(\prod_{i=1}^{n-1} h(x_i)))h(x_n)\right) \\ &= V(V(x_1, \cdots, x_{n-1}), x_n) \end{aligned}$$

which concludes the proof. □

Note that if $h(0) = 0$, V is t-norm (see Eq. (1)), while if $h(1) = 0$, V is t-conorm (see Eq. (2)).

Proposition III.3. *The element a for which $h(a) = 0$ holds in (17) is an absorbing element of $V(x_1, \cdots, x_n)$.*

Proof. Let $a \in \{x_1, \cdots, x_n\}$, then we have $\prod_{i=1}^{n} h(x_i) = 0$. Therefore, we have $V(x_1, \cdots, x_n) = h^{-1}(0) = a$, which concludes the proof. □

Note that $V(x_1, \cdots, x_n)$ does not satisfy the boundary conditions on 0 and 1 of nullnorms. In particular, we have

$$V(x_1, \cdots, x_n, 0) = \begin{cases} 0 & \text{if for all i, } x_i > a \\ a & \text{if } a \in \{x_1, \cdots, x_n\} \\ 1 & \text{if } \exists i,\, x_i < a \end{cases} \tag{18}$$

and

$$V(x_1, \cdots, x_n, 1) = \begin{cases} 0 & \text{if } \exists i,\, x_i < a \\ a & \text{if } a \in \{x_1, \cdots, x_n\} \\ 1 & \text{if for all i, } x_i > a \end{cases} \tag{19}$$

Consequently, $V(x_1, \cdots, x_n)$, as defined by Eq. (17), is not a nullnorm, but an absorbing norm (see [1] and [13]). As described in [26], this operator can be seen as the negation of the fuzzy xor operator.

Proposition III.4. *The element e defined by $h^{-1}(1)$ is a neutral element of $V(x_1, \cdots, x_n)$.*

Proof. Let $h^{-1}(1) \in \{x_1, \cdots, x_n\}$, therefore $\exists k \in \{1, \cdots, n\}$ such that $x_k = e$. Then $V(x_1, \cdots, x_n) = h^{-1}\left(\prod_{i=1}^{n} h(x_i)\right) = h^{-1}\left(h(x_k)\prod_{i=1, i\neq k}^{n} h(x_i)\right)$.
Hence, $V(x_1, \cdots, x_n) = h^{-1}\left(\prod_{i=1, i\neq k}^{n} h(x_i)\right)$, concluding the proof. □

Nor nullnorms nor absorbing norms show any reinforcement property. In particular, when both elements are below the absorbing element, and therefore considered as low, the result is larger than the maximum value (due to the use of t-conorm) of the elements. Conversely, when both elements are above the absorbing element, the result is lower than the minimum. Consequently, such an operator is the dual of full reinforcement operators : it is optimistic for negative inputs, and pessimistic for positive inputs.

Proposition III.5. *Let g be an additive generative function of a representable uninorm. Then*

$$V(x_1, \cdots, x_n) = g^{-1}\left(\prod_{i=1}^{n} g(x_i)\right)$$

is an absorbing norm, where the neutral element of the uninorm is equal to the absorbing element of the absorbing norm.

Proof. The proofs of Propositions III.2, III.3 and III.4 gives the sufficient material to prove this proposition. □

In the sequel, we use the same generating function, defined by $g_\alpha(x) = h_\alpha(x) = \log \frac{x^\alpha}{1-x^\alpha}$ (see Eq. (8)) for both generated uninorms and generated absorbing norms. Let V_α be defined as $V_\alpha(x_1, \cdots, x_n) = g_\alpha^{-1}\left(\prod_{i=1}^{n} g_\alpha(x_i)\right)$.

Example Using $\alpha = 1$, we obtain

$$V_1(x_1, \cdots, x_n) = \frac{\exp\left(\prod_{i=1}^{n} \log \frac{x_i}{1-x_i}\right)}{1 + \exp\left(\prod_{i=1}^{n} \log \frac{x_i}{1-x_i}\right)}$$

Here again, for the sake of clarity and brevity, we use only three different values of α, 0.5, 1 and 2 that correspond

$\mathcal{U}_\alpha$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$
$\mathbf{v}_1$	0.998	0.990	0.952
$\mathbf{v}_2$	0.009	0.001	0.001
$\mathbf{v}_3$	0.926	0.6	0.181

Table I
VALUES OF GENERATED UNINORMS ON PROTOTYPICAL EXAMPLE VECTORS $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

V_α	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$
$\mathbf{v}_1$	0.999	0.965	0.743
$\mathbf{v}_2$	0.201	0	0
$\mathbf{v}_3$	0.173	0.774	0.999

Table II
VALUES OF GENERATED ABSORBING NORMS ON PROTOTYPICAL EXAMPLE VECTORS $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

to absorbing elements 0.25, 0.5 and 0.707, respectively. The corresponding iso-values of these absorbing norms are given in Figure 2, where pure white encodes the value 0, and blue the value 1.

As can be observed, the values on the four bounds of the unit interval do not change, whatever α. If we consider the diagonal $x \approx y$, one can see that it is decreasing till reaching the absorbing element a, and then increase towards 1. This operator gives a low output for contradictory inputs, and a large output for agreeing inputs, especially for elements close to the boundary. In general, increasing the value of the absorbing element tends to increase the output of the operator.

C. A numerical example

Let us consider three different vectors characterizing three different situations. Those three vectors will be used throughout all the paper in order to emphasize the behavior of the different aggregation functions that will be presented.

- three high values : $\mathbf{v}_1 = \{0.9, 0.8, 0.75\}$,
- three low values : $\mathbf{v}_2 = \{0.05, 0.1, 0.2\}$,
- mixed (contradictory) values : $\mathbf{v}_3 = \{0.4, 0.9, 0.2\}$.

Using generated uninorms on $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ gives the Table I. As can be expected, the output for $\mathbf{v}_1$ and $\mathbf{v}_2$ are respectively high and low, whatever the value of α. For these specific inputs, $\mathcal{U}_\alpha$ thus provides full reinforcement. Considering $\mathbf{v}_3$, the conclusion are not so clear, depending on the value of α. In particular, if $\alpha = 1$, corresponding to the 3Π-operator, the output value of 0.6 lies between the minimum and the maximum value, whereas it is not the case for $\alpha = 0.5$ and $\alpha = 2$. More precisely, using a small value for α only provides upward reinforcement $(0.926 \geq \max(\mathbf{v}_3))$, and a large value of α provides downward reinforcement $(0.181 \leq \min(\mathbf{v}_3))$. It can be explained by the fact that when α tends toward zero, then $\mathcal{U}_\alpha$ converges to a t-conorm. On the other hand, when α increases, $\mathcal{U}_\alpha$ converges to a t-norm. This behavior is interesting for a number of reasons, but particularly because it allows to use operators providing full reinforcement for unmixed inputs (e.g. $\mathbf{v}_1$ or $\mathbf{v}_2$), and whether downward (pessimistic attitude) or upward (optimistic attitude) reinforcement for mixed inputs ($\mathbf{v}_3$).

However, it does not differentiate well the amount of well satisfied criteria for mixed inputs. In particular, let us consider three criteria, and the following two inputs $\mathbf{a} = \{0.9, 0.8, 0.1\}$ and $\mathbf{b} = \{0.9, 0.3, 0.2\}$. Both $\mathbf{a}$ and $\mathbf{b}$ have contradictory inputs, but $\mathbf{a}$ clearly has two satisfied criteria, whereas $\mathbf{b}$ only one. This difference is not well represented by $\mathcal{U}_\alpha$, so that one needs a tool to cope with it. In particular, we are interested in providing operators that are able to evaluate to what extent an input presents mixed values, and how many high and low values, while keeping a full reinforcement behavior for clear inputs.

Using generated absorbing norms on $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ gives the Table II. The operator V_α is the absorbing norm generated by the function g_α, defined in (8). As can be seen, here again, the outputs obtained for $\mathbf{v}_1$ and $\mathbf{v}_2$ are high and low for any value of α, respectively. More precisely, as mentioned in the previous section, low values of α make V_alpha tends to a t-norm. Consequently, even if the output is larger than the maximum value of $\mathbf{v}_2$ $(0.201 > 0.2)$, as expected, it is a rather low value. On the opposite, when α increases, V_α converges to a t-conorm. In this case, we obtain an output which is lower than the minimum value of $\mathbf{v}_1$ $(0.743 < 0.75)$. Moreover, one can note that, for mixed inputs, the impact of α on V_α is the opposite of the impact of α on $\mathcal{U}_\alpha$. It can be noted that the output for $\mathbf{v}_2$ is low for the following reason: the first two elements (with any permutation) agree on being low, so that the intermediary output is large, being in contradiction with the third element, therefore resulting in a low output.

As a side remark, it is interesting to note that absorbing norms V_α are not increasing nor decreasing with respect to α, see e.g the values for $\mathbf{v}_1$ and $\mathbf{v}_3$.

D. Related works

Recently, in [19], the class of almost equitable uninorms has been proposed. These operators have been introduced to deal with contradictory inputs. In particular, they are uninorms respecting $U(x, N(x)) = e$, where $N(x)$ is the strong negation of x. In other terms, considering one value and its opposite is neutral in the aggregation. An interesting point is that if one considers representable uninorms with additive generator g, then using $N(x) = g^{-1}(-g(x))$ as strong negation provides an almost equitable uninorm.

In [28], Yager introduced the concept of noble reinforcement for aggregation operators. By noble reinforcement, the author means that only sufficiently large input should reinforce each output and produce a large output. This is in opposition with the behavior of t-conorms, where a large number of low values may result in a large output. As for full reinforcement operators, noble reinforcement operators are obtained thanks to a Takagi-Sugeno-Kang fuzzy system. The property of noble reinforcement is also extended so that reinforcement only occurs if a required number of large values is observed.

In [7], the property of mean reinforcement is presented. It is slightly related to full reinforcement, in that low scores

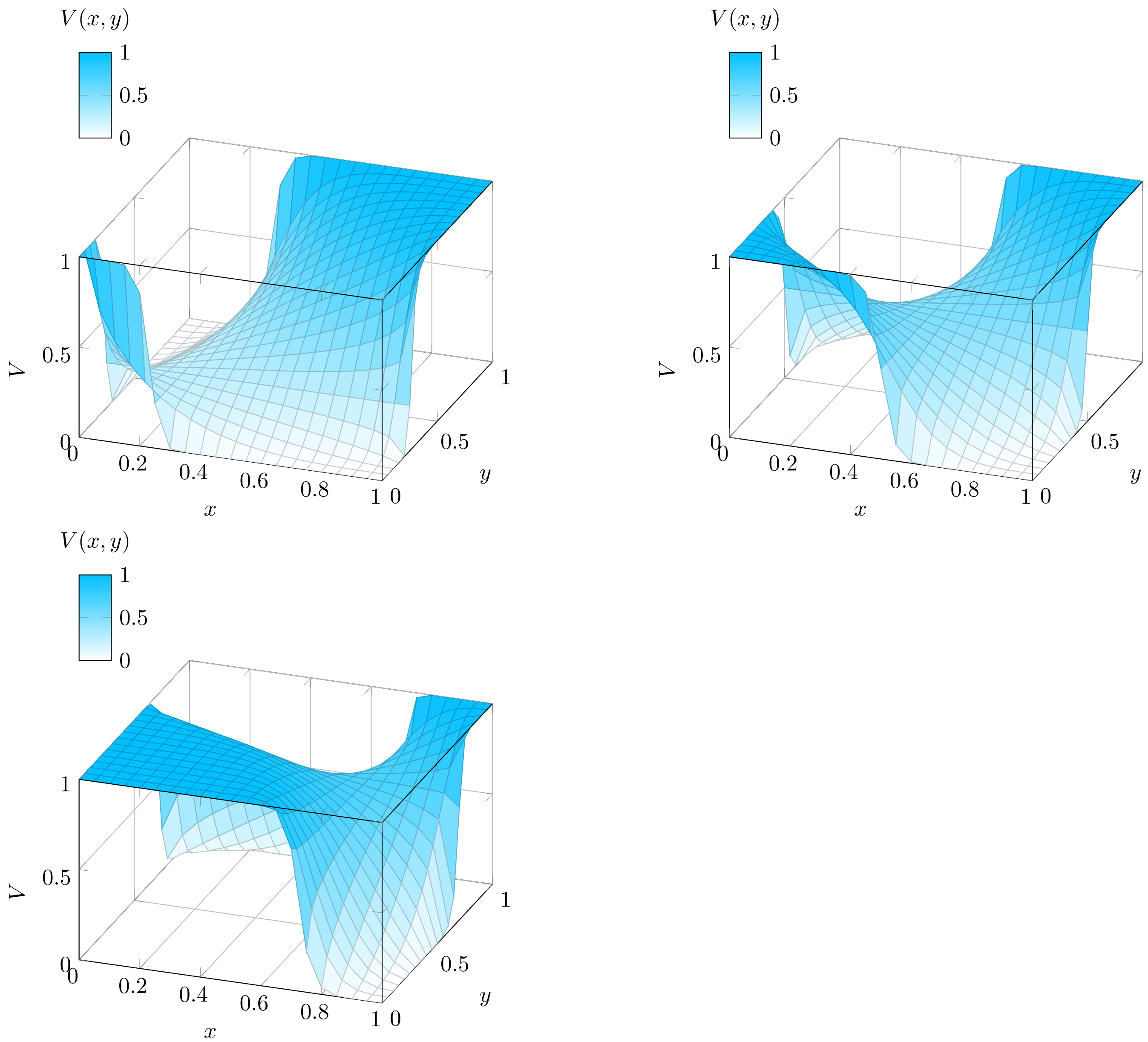

Figure 2. Values of generated continuous absorbing norms on the unit square, for three different values of α = 0.5, 1 and 2. These values are corresponding to the absorbing elements a = 0.25, 0.5 and 0.707, respectively.

produce low score, and large scores produce large scores. However, bounds are different. In case of low values, the output must be lower than the average, while it must be larger than the average for an input containing only large values. They propose a generalization of the 3Π operator with the help of symmetric sums, that is satisfying the proposed property.

In [23], the authors present the concept of fuzzy majority opinion. By relating the current problem to the quantification in fuzzy logic, they are using linguistic quantifiers for defining OWA operators [30]. Stating that similar values have close positions if they are sorted, they use an induced ordered weighted average (IOWA) operator defined as

$$IOWA() = \text{complete equation} \tag{20}$$

Talk about specificity measures [29]

In [21] and [18], the authors present k fuzzy connectives, aiming to assess to what extent several (k) values are high or low in unconstrained fuzzy sets. Their proposition is based on the set combination of triangular norms and triangular conorms. In particular, considering the powerset $\mathbb{P}$ of a set $\{1, \cdots, n\}$, and $\mathbb{P}(k)$ the subset of $\mathbb{P}$ such that $|\mathbb{P}(k)| = k$, the k fuzzy disjunction is defined by

$$\perp_k(x_1, \cdots, x_n) = \top_{\theta \in \mathbb{P}(k-1)} \perp_{j=1, j \notin \theta}^{n} x_j, \quad (21)$$

where we impose $1 \leq k \leq n$. Using additive generators of triangular norms and triangular conorms, one can write Eq. (21) as

$$\perp_k(x_1, \cdots, x_n) = t^{(-1)}\left(\sum_{\theta \in \mathbb{P}(k-1)} t\Big(s^{(-1)}\big(\sum_{j=1, j \notin \theta} s(x_j)\big)\Big)\right),$$

where t and s are additive generators of triangular norms (see Eq. (1)) and triangular conorms (see Eq. (2)), respectively. Naturally, one can use the same transformation for the dual operator, the k fuzzy conjunction

$$\top_k(x_1, \cdots, x_n) = s^{(-1)}\left(\sum_{\theta \in \mathbb{P}(k-1)} s\Big(t^{(-1)}\big(\sum_{j=1, j \notin \theta} t(x_j)\big)\Big)\right).$$

This operator is a step toward the objective of this paper, but it does not provide full reinforcement, thus showing the same drawback as triangular norms and triangular conorms.

IV. k-UNINORMS AND k-ABSORBING NORMS

A. k-uninorms

In order to provide both full reinforcement and the ability to discriminate the number of high (low) values, we propose to define the set combination of uninorms and absorbing norms. The k-uninorm is thus given by the absorbing norm of the uninorm, following the same principle as in Eq. (21)

More specifically, we define the k-uninorm as

$$U_k(x_1, \cdots, x_n) = V_{\theta \in \mathbb{P}(k-1)} \left(\mathcal{U}_{j=1, j \notin \theta}^{n}(x_j) \right).$$

Using additive generator and multiplicative generator of uninorms and absorbing norms, respectively, one can write

$$U_k(x_1, \cdots, x_n) = g^{-1}\left(\prod_{\theta \in \mathbb{P}(k-1)} g\left(g^{-1}\left(\sum_{j=1, j \notin \theta}^{n} g(x_j) \right)\right)\right),$$

which can be further simplified as

$$U_k(x_1, \cdots, x_n) = g^{-1}\left(\prod_{\theta \in \mathbb{P}(k-1)} \sum_{j=1, j \notin \theta}^{n} g(x_j) \right). \quad (22)$$

Proposition IV.1. *$U_k(x_1, \cdots, x_n)$, defined by (22), is an aggregation operator, i.e. it is a monotonic operator satisfying boundary conditions.*

Proof. Let us consider the boundary conditions. We consider the generic case where $1 \leq k \leq n$. With constant value $x_i = a$ for any i, we can write (22) as

$$U_k(a, \cdots, a) = g^{-1}\left(((n+1-k)g(a))^{\binom{n}{k-1}} \right)$$

Setting a to 0 gives $U_k(0, \cdots, 0) = 0$ due to the bound values of g. The same property gives $U_k(1, \cdots, 1) = 1$ for the other bound. Now we proof that $U_k(x_1, \cdots, x_n)$ is monotonic. By definition, g is monotonic and increasing, so that, for any θ, we have

$$\sum_{j=1, j \notin \theta}^{n} g(x_j) \leq \sum_{j=1, j \notin \theta}^{n} g(y_j),$$

for $x_1 \leq y_1, \cdots, x_n \leq y_n$. Finally, we obtain $U_k(x_1, \cdots, x_n) \leq U_k(y_1, \cdots, y_n)$, concluding the proof. ☐

U_k	$k=1$	$k=2$	$k=3$
$\mathbf{v}_1$	0.998	1	0.999
$\mathbf{v}_2$	0.009	0.002	0.201
$\mathbf{v}_3$	0.926	0.915	0.173

U_k	$k=1$	$k=2$	$k=3$
$\mathbf{v}_1$	0.990	1	0.965
$\mathbf{v}_2$	0.001	0	0
$\mathbf{v}_3$	0.6	0.068	0.774

U_k	$k=1$	$k=2$	$k=3$
$\mathbf{v}_1$	0.952	0.972	0.743
$\mathbf{v}_2$	0.001	0	0
$\mathbf{v}_3$	0.181	0.386	0.999

Table III
k-UNINORMS WITH DIFFERENT VALUES OF k AND α ($\alpha = 0.5, 1, 2$, FROM LEFT TO RIGHT).

Proposition IV.2. *Using $k = 1$ in equation (22) provides an uninorm as defined by (7).*

Proof. Replacing k by 1 in (22) gives

$$U_k(x_1, \cdots, x_n) = g^{-1}\left(\prod_{\theta \in \mathbb{P}(0)} \sum_{j=1, j \notin \theta}^{n} g(x_j) \right),$$

and therefore, we obtain

$$U_k(x_1, \cdots, x_n) = g^{-1}\left(\sum_{j=1}^{n} g(x_j) \right),$$

which is the definition of a generated uninorm, thus concluding the proof. ☐

Let us set $\alpha = 1$, so that the generated uninorm corresponds to the 3Π operator, and the generated absorbing norm is given by

$$V(x_1, \cdots, x_n) = \frac{\exp\left(\prod_{j=1}^{n} \log \frac{x_j}{1-x_j} \right)}{1 + \exp\left(\prod_{j=1}^{n} \log \frac{x_j}{1-x_j} \right)}.$$

In Table III, the output values of the k-uninorm operator is given, when applied to the same prototypical examples $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. The order k of the operator ranges from 1 to 3 (i.e. the length of the vector), and the parameter α takes the values 0.5, 1 and 2. As can be noted in this table, the columns for which $k = 1$ corresponds to uninorm operators, so that a full reinforcement can be observed. Conversely, the column for which $k = 3$ corresponds to absorbing norm operators, and same comments as in section III-C can be stated. Finally, the column for which $k = 2$ is an interesting intermediate step between uninorms and absorbing norms. The full reinforcement property is still observed, and is more prominent than for uninorms (e.g. for $\alpha = 2$, $1 > 0.990$ and $0 < 0.001$). However, for contradictory inputs ($\mathbf{v}_3$), depending on the value of α, the operator provides whether upward reinforcement ($\alpha = 0.5$), downward reinforcement ($\alpha = 1$) or compensation ($\alpha = 2$).

B. *k-absorbing norms*

One can define the dual operator of k-uninorms by considering the combination of absorbing norms and uninorms. In particular, we define the k-absorbing norm as

$$V_k(x_1, \cdots, x_n) = \mathcal{U}_{\theta \in \mathbb{P}(k-1)} \left(V_{j=1, j \notin \theta}^n (x_j) \right).$$

Again, using multiplicative generator and additive generator of absorbing norms and uninorms, respectively, one can write

$$V_k(\mathbf{x}_1, \cdots, \mathbf{x}_n) = h^{-1} \left(\sum_{A \in \mathcal{P}_{k-1}} h \left(h^{-1} \left(\prod_{j \in N \setminus A} h(x_j) \right) \right) \right), \tag{23}$$

which can be further simplified as

$$V_k(\mathbf{x}_1, \cdots, \mathbf{x}_n) = h^{-1} \left(\sum_{A \in \mathcal{P}_{k-1}} \prod_{j \in N \setminus A} h(x_j) \right). \tag{24}$$

Proposition IV.3. $V_k(x_1, \cdots, x_n)$*, defined by (22), is an aggregation operator, i.e. it is a monotonic operator satisfying boundary conditions.*

Proposition IV.4. *Using $k = 1$ in equation (24) provides an absorbing norm as defined by (17)*

Proof. There are no set (except the empty one) θ such that $\theta \in \mathbb{P}(0)$, so that $V_1(\mathbf{x}_1, \cdots, \mathbf{x}_n) = h^{-1} \left(\prod_{j=1}^n h(x_j) \right)$, which is the absorbing norm defined by (17). □

Proposition IV.5. *Using $k = n$ in equation (22) provides an absorbing norm as defined by (17).*

Proof. If $k = n$, then $U_n(x_1, \cdots, x_n) = g^{-1} \left(\prod_{\theta \in \mathbb{P}(n-1)} \sum_{j=1, j \notin \theta}^n g(x_j) \right)$, so that $U_n(x_1, \cdots, x_n) = g^{-1} \left(\prod_{j=1}^n g(x_j) \right)$. □

Proposition IV.6. *Using $k = n$ in equation (24) provides an uninorm as defined by (7).*

Proof. Let $k = n$, so that $V_n(\mathbf{x}_1, \cdots, \mathbf{x}_n) = h^{-1} \left(\sum_{\theta \in \mathbb{P}(n-1)} \prod_{j=1, j \notin \theta}^n h(x_j) \right)$, which is equal to $h^{-1} \left(\sum_{j=1}^n h(x_j) \right)$, giving the generated uninorm defined by Eq. (7). □

In Table IV, the output values of the k-absorbing norm operator is given, when applied to the same prototypical examples $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. The order k of the operator ranges from 1 to 3 (i.e. the length of the vector), and the parameter α takes the values 0.5, 1 and 2. Following from the propositions IV.4, IV.5 and IV.6, the column for which $k = 1$ and $k = 3$ have already be described in Table III: V_1 is an absorbing norm, and is equivalent to U_3, and V_3 is a generated uninorm. Consequently, we focus on the columns for which $k = 2$ in this analysis. Let us consider the case $\alpha = 1$. The output for agreeing inputs such as $\mathbf{v}_1$ and $\mathbf{v}_2$ is large, while it is low for

V_k	$k=1$	$k=2$	$k=3$
$\mathbf{v}_1$	0.999	0.999	0.998
$\mathbf{v}_2$	0.201	0.640	0.009
$\mathbf{v}_3$	0.173	0.490	0.926

V_k	$k=1$	$k=2$	$k=3$
$\mathbf{v}_1$	0.965	0.999	0.990
$\mathbf{v}_2$	0	0.999	0.001
$\mathbf{v}_3$	0.774	0.03	0.6

V_k	$k=1$	$k=2$	$k=3$
$\mathbf{v}_1$	0.743	0.890	0.952
$\mathbf{v}_2$	0	1	0.001
$\mathbf{v}_3$	0.999	0.385	0.181

Table IV
k-ABSORBING NORM WITH DIFFERENT VALUES OF k AND α ($\alpha = 0.5, 1, 2$, FROM LEFT TO RIGHT).

contradictory inputs, such as $\mathbf{v}_3$. The operator can be seen as a measure of agreement between the input values. Observing the values of $\mathbf{v}_1$ and $\mathbf{v}_2$, we see that the values of $\mathbf{v}_2$ are lower than the value of $\mathbf{v}_1$ are larger. It can be detected when increasing the value of α to 2: the measure of agreement of $\mathbf{v}_2$ is larger the one of $\mathbf{v}_1$ ($1 > 0.890$). Conversely, when α decreases, this behavior is reversed: the agreement within $\mathbf{v}_2$ decreases, while agreement within $\mathbf{v}_1$ remains constant. The value $V_2(\mathbf{v}_3$ for $\alpha = 1$ is low, meaning that there are no agreement between the input values. This is moderated by changing the value of α, either by increasing or decreasing it.

finish comments of Table IV

V. A CASE STUDY IN MULTI-CRITERIA DECISION MAKING

In this section, we consider a scenario of multi-criteria decision making. In particular, we have four different, and supposed independent, criteria C_i, $i = 1, \cdots, 4$ that may represent the degree of utility of an individual x to a context dependent task (e.g. hiring for a job, getting a bank credit, auctions, …). In our simple case study, we consider ten different individuals x_j, $j = 1, \cdots, 10$ that represent various prototypical characteristics. Complete values and characteristic of each individual are given in Table V. In this table, a $+$ symbol represents a large input, a $-$ symbol stands for a low input, and the symbol $\sim$ is used for moderated inputs.

We give in Table VI the corresponding outputs of k-uninorms and k-absorbing norms, for different values of α. In this table, we also give the three clusters obtained for each operator if we consider the following intervals

- output is considered as low if it belongs to $[0, 0.25]$
- output is considered as moderate if it belongs to $]0.25, 0.75[$
- output is considered as large if it belongs to $[0.75, 1]$

Each cluster is represented by a color: low, moderate and large. Interestingly, the 18 different possible operators all give a different partition of individuals, while presenting some similarities. In particular, for U_1, individuals $\{x_1, x_6, x_9\}$ and $\{x_2, x_8\}$ are often clustered together. This result was expected from the characteristics of these individuals (see Table V): the first cluster is considered as large and slightly large inputs, and the second one as low inputs. The operators U_2 and U_3 tend to cluster together $\{x_4, x_5\}$ and $\{x_7, x_8\}$. These two

Individual	C_1	C_2	C_3	C_4	Characteristic
x_1	0.8	0.9	0.95	0.75	$4+$
x_2	0.2	0.15	0.1	0.25	$4-$
x_3	0.85	0.7	0.3	0.2	$2+, 2-$
x_4	0.75	0.6	0.45	0.25	$1+, 3\sim$
x_5	0.55	0.5	0.5	0.45	$4\sim$
x_6	0.8	0.95	0.9	0.2	$3+, 1-$
x_7	0.05	0.25	0.25	0.8	$1+, 3-$
x_8	0.5	0.2	0.15	0.1	$1\sim, 3-$
x_9	0.5	0.9	0.8	0.75	$3+, 1\sim$
x_{10}	0.55	0.45	0.9	0.15	$1+, 2\sim, 1-$

Table V

DESCRIPTION OF EACH INDIVIDUAL IN TERMS OF CRITERIA C_i AND ITS CHARACTERISTIC.

clusters correspond to moderate inputs and slightly low inputs, respectively. U_4 (which is also V_1) clusters together $\{x_6\}$ and $\{x_7\}$, which show an odd number of well separated large (or low) values, therefore illustrating its property of negation of exclusive operator. Finally, V_2 and V_3 tends to provide the cluster $\{x_3, x_4, x_{10}\}$, which are moderated inputs having contradictory values.

In the next table, Table VII, the ranking of the output, for each of the 18 operators is given. As can be expected, x_1 is ranked first with 15 out of the 18 operators.

finish comments of Table VII

VI. Conclusion

We presented in this paper …

finish conclusion

A natural application of the proposed operators is consensus reaching for group decision making problems [24]. Among the potential perspectives, …on weighting operators, as in [18], in a context of similarity measurement [17], [11], or for recommender systems [2], or even

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Indiv.	U_1	U_2	U_3	U_4	V_2	V_3
$\alpha = 0.5$						
x_1	0.999	1.000	1.000	1.000	1.000	1.000
x_2	0.036	0.582	0.265	0.250	0.231	0.421
x_3	0.967	1.000	0.045	0.210	0.191	0.961
x_4	0.957	1.000	0.999	0.250	0.701	0.978
x_5	0.943	1.000	1.000	0.411	0.877	0.980
x_6	0.999	1.000	1.000	0.006	0.999	0.999
x_7	0.503	0.011	0.250	0.250	0.250	0.004
x_8	0.132	0.247	0.243	0.233	0.377	0.116
x_9	0.999	1.000	1.000	0.999	0.999	0.999
x_{10}	0.971	1.000	0.999	0.071	0.139	0.954
$\alpha = 1$						
x_1	0.999	1.000	1.000	0.999	0.999	0.999
x_2	0.001	1.000	1.000	0.996	0.001	0.999
x_3	0.586	0.807	0.500	0.848	0.437	0.042
x_4	0.551	0.523	0.500	0.524	0.438	0.216
x_5	0.500	0.500	0.500	0.500	0.500	0.489
x_6	0.994	1.000	0.500	0.000	0.000	0.989
x_7	0.022	1.000	0.990	0.007	0.999	0.633
x_8	0.004	1.000	1.000	0.500	0.005	0.999
x_9	0.990	1.000	1.000	0.500	0.965	0.999
x_{10}	0.613	0.340	0.500	0.538	0.495	0.020
$\alpha = 2$						
x_1	0.994	1.000	1.000	0.783	0.980	0.999
x_2	0.010	1.000	1.000	1.000	0.009	1.000
x_3	0.101	1.000	0.009	0.655	0.999	0.952
x_4	0.110	1.000	0.999	0.607	0.592	0.996
x_5	0.110	1.000	1.000	0.894	0.075	0.999
x_6	0.863	0.845	0.009	0.053	0.010	0.019
x_7	0.010	1.000	1.000	0.009	0.009	1.000
x_8	0.010	1.000	1.000	1.000	0.009	1.000
x_9	0.874	0.643	0.780	0.665	0.469	0.489
x_{10}	0.103	1.000	0.998	0.044	0.999	0.830

Table VI

OUTPUT OF k-UNINORMS AND k-ABSORBING NORMS, FOR DIFFERENT VALUES OF α. EACH OUTPUT IS COLORED ACCORDING TO ITS CLUSTER {HIGH, MODERATE, LOW}(RESPECTIVELY BLUE, GREEN AND RED, SEE TEXT FOR DETAILS).

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Operator	k	α	Ranking
U_k	1	0.5	$x_1 = x_6 = x_9 \succ x_{10} \succ x_3 \succ x_4 \succ x_5 \succ x_7 \succ x_8 \succ x_2$
U_k	2	0.5	$x_1 = x_3 = x_4 = x_5 = x_6 = x_9 = x_{10} \succ x_2 \succ x_7 \succ x_8$
U_k	3	0.5	$x_1 = x_5 = x_6 = x_9 \succ x_4 = x_{10} \succ x_2 \succ x_7 \succ x_8 \succ x_3$
U_k	4	0.5	$x_1 \succ x_9 \succ x_5 \succ x_2 = x_4 = x_7 \succ x_8 \succ x_3 \succ x_{10} \succ x_6$
V_k	2	0.5	$x_1 \succ x_6 = x_9 \succ x_5 \succ x_4 \succ x_8 \succ x_7 \succ x_2 \succ x_3 \succ x_{10}$
V_k	3	0.5	$x_1 \succ x_6 = x_9 \succ x_5 \succ x_4 \succ x_3 \succ x_{10} \succ x_2 \succ x_8 \succ x_7$
U_k	1	1	$x_1 \succ x_6 \succ x_9 \succ x_{10} \succ x_3 \succ x_4 \succ x_5 \succ x_7 \succ x_8 \succ x_2$
U_k	2	1	$x_1 = x_2 = x_6 = x_7 = x_8 = x_9 \succ x_3 \succ x_4 \succ x_5 \succ x_{10}$
U_k	3	1	$x_1 = x_2 = x_8 = x_9 \succ x_7 \succ x_3 = x_4 = x_5 = x_6 = x_{10}$
U_k	4	1	$x_1 \succ x_2 \succ x_3 \succ x_{10} \succ x_4 \succ x_5 = x_8 = x_9 \succ x_7 \succ x_6$
V_k	2	1	$x_1 = x_7 \succ x_9 \succ x_5 \succ x_{10} \succ x_4 \succ x_3 \succ x_8 \succ x_2 \succ x_6$
V_k	3	1	$x_1 = x_2 = x_8 = x_9 \succ x_6 \succ x_7 \succ x_5 \succ x_4 \succ x_3 \succ x_{10}$
U_k	1	2	$x_1 \succ x_9 \succ x_6 \succ x_4 = x_5 \succ x_{10} \succ x_3 \succ x_2 = x_7 = x_8$
U_k	2	2	$x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = x_8 = x_{10} \succ x_6 \succ x_9$
U_k	3	2	$x_1 = x_2 = x_5 = x_7 = x_8 \succ x_4 \succ x_{10} \succ x_9 \succ x_3 = x_6$
U_k	4	2	$x_2 = x_8 \succ x_5 \succ x_1 \succ x_9 \succ x_3 \succ x_4 \succ x_6 \succ x_{10} \succ x_7$
V_k	2	2	$x_3 = x_{10} \succ x_1 \succ x_4 \succ x_9 \succ x_5 \succ x_6 \succ x_2 = x_7 = x_8$
V_k	3	2	$x_2 = x_7 = x_8 \succ x_1 = x_5 \succ x_4 \succ x_3 \succ x_{10} \succ x_9 \succ x_6$

Table VII
RANKING OF THE 18 OPERATORS

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