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Abstract

The purpose of this paper is to prove an invariance theorem for m-components strongly coupled reaction-diffusion systems with non-homogenous boundary conditions. A characteristics of these systems have been defined and a new balance law has been constructed. Global existence in time of solutions is deduced without conditions on the nonlinearities growth.
AMS Classification: 35K57, 35K45
Keywords: Reaction diffusion systems, Invariant regions, Global existence.

1 INTRODUCTION

We consider the reaction-diffusion system

$$\frac{\partial u_i(t,x)}{\partial t} - \sum_{j=1}^{m} a_{ij}\Delta u_j(t,x) = f_i(t,x,u), \qquad \text{in } \mathbb{R}^+ \times \Omega, \tag{1}$$

with the boundary conditions

$$\frac{\partial u_i(t,x)}{\partial \eta} = g_i(x,u) \text{ on } \Gamma^1 \text{ and } \quad u_i(t,x) = \theta_i(x) \quad \text{on } \Gamma^2. \tag{2}$$

and the initial data

$$u_i(0,x) = u_i^0(x), \quad \text{in } \Omega, \tag{3}$$

where Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma^1 \cup \Gamma^2$ is a disjoint union with $\Gamma^1 = \prod_{i=1}^{m} \Gamma_i^1$, $\Gamma^2 = \prod_{i=1}^{m} \Gamma_i^2$

and $u=(u_1,...,u_m)$. The diffusion matrix $A=(a_{ij})_{1\leq i,j\leq m}$ with real entries, is supposed to be diagonalizable with positive eigenvalues: $0<\lambda_1<\lambda_2<...<\lambda_l$, each λ_i is of multiplicity $\mu_i\geq 1$. The reaction term $f=(f_1,...,f_m)$ is a continuous function from $\mathbb{R}^+\times\Omega\times\mathbb{R}^m$ into $\mathbb{R}^m$, the functions $g=(g_1,...,g_m)$ and $\theta=(\theta_1,...,\theta_m)$ on the boundary are continuous from $\partial\Omega\times\mathbb{R}^m$ into $\mathbb{R}^m$ and $\dfrac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$.
Systems of this form can be interpreted as models in several chemical, biological and dynamics of populations models where the quantity u_i stands for the chemical or molecular concentration and population density of the i-th specie respectively(see [5], [3], [7], [13] and [22]). The technique based on the bounded invariant regions is among those used to solve the problem of global existence in time of strong solutions for systems such as these and their quasilinear generalizations and accordingly it has received much attention.
There exists a wide bibliography on invariant sets for nonlinear parabolic and elliptic systems. The reader is referred to the expository articles [1], [2], [11], [12], [20], [23], [25], [10], [8], [9] and references there).
Whereas all authors mentioned thus far have considered the balance law condition only for tow components systems, here we define a characteristics and introduce a new balance law for the general system (1). Then we prove an invariance theorem which gives global existence of solutions when the constructed invariant sets are bounded.
This work was motivated by the interesting results of [12] who studied the two component strongly coupled system on the form (1) satisfying the following balance law conditions

$$\begin{gathered} -\lambda_2\nu(x)+(\lambda_1-a_{22})\left|f_1(t,x,u)\right|\leq a_{12}\left|f_2(t,x,u)\right| \\ \leq(\lambda_2-a_{22})\left|f_1(t,x,u)\right|+\lambda_1\nu(x), \\ \text{and} \\ -\xi(x)+(\lambda_1-a_{22})\left|g_1(x,u)\right|\leq a_{12}\left|g_2(x,u)\right| \\ \leq(\lambda_2-a_{22})\left|g_1(x,u)\right|+\xi(x), \end{gathered} \tag{4}$$

on $\mathbb{R}^+\times\Omega\times\mathbb{R}^2$ and $\partial\Omega\times\mathbb{R}^2$ respectively, where $\nu(x)$ and $\xi(x)$ are measurable functions on Ω and $\partial\Omega$ respectively. He proved global existence of solutions via bounded invariant regions technique under some conditions on the diffusion matrix A (with positive distinct eigenvalues and

$a_{12} > 0$) and the following dissipative conditions

$$\begin{aligned} u_1 f_1(t,x,u) \leq 0 \text{ and } u_1 f_2(t,x,u) \geq 0, \\ u_1 g_1(x,u) \leq 0 \text{ and } u_1 g_2(x,u) \geq 0, \end{aligned} \tag{5}$$

on $\mathbb{R}^+ \times \Omega \times \mathbb{R}^2$ and $\partial\Omega \times \mathbb{R}^2$ respectively. As [12] when $m = 2$, by these results we generalize the works of [8] and [9] to more general diffusion matrices by eliminating the need for the difficult-to-establish bound on the diffusion terms. Also we eliminate for the reaction terms that need to satisfy a polynomial growth condition and a strict balance law. Following [8], [9] and [12] we suppose that $u_p f_k$, $k = 1, ..., m$ don't change sign for some fixed $p = 1, ..., m$ not on the hole space $\mathbb{R}^n$ but for $|u_p|$ sufficiently large. Simultaneously permuting unknowns, reactions and rows-columns of the diffusion matrix, if necessary, we may assume without lost of generality $p = 1$.

The paper is organized as follows. After clarifying the notations and stating the various definitions in the next section, we present the results and proofs in the third section where we begin by treating the homogenous case in the first subsection. In the second one we treat the non-homogenous case. The last section will be consecrated to some applications for the two and three components reaction diffusion systems.

2 Definitions and notations

In this section we present some necessary notations and definitions which are needed in our subsequent sections. For technical reasons due to the difficult calculus of the explicit eigenvalues, namely when $m \geq 3$, we suppose the diffusion matrix A to be symmetric or at last Symmetrizable, then there exists a diagonal matrix S with positive entries (called Symmetrizer), such that the matrix $S.A$ is symmetric and the system can be transformed by the change variable: $v = S^{\frac{1}{2}} u$ to an equivalent system with symmetric matrix $S^{\frac{1}{2}} A S^{-\frac{1}{2}}$. Note that, from the Cauchy interlace Theorem for $m \times m$ symmetrizable matrices, the $(m-1)$ order principal minors have the same sign. As examples of such matrices we cite sign symmetric (i.e. $a_{ij}.a_{ji} > 0$ and $a_{ij} = 0 \Rightarrow a_{ji} = 0,\ 0 \leq i, j \leq m$) Tridiagonal, Pentadiagonal, Heptadiagonal and particularly, Symmetric matrices. The case $m = 2$ does not require the symmetrizability of the diffusion matrix and will be treated alone as a remark at the end of the next section.

Since the diffusion matrix is supposed to be diagonalizable, then it possess a full set of eigenvectors. If we denote by $x_i = \left(x_{i1}, \cdots, x_{im}\right)^t$ an

eigenvector of the matrix A^t (left eigenvector of A) associated with its eigenvalue λ_i and X the $m \times m$ matrix formed with the rows x_i, the dissipativity condition on the reactions (i.e. $u_1 f_k$ don't change sign, $k = 1, ..., m$) requires $x_{i1} \neq 0,\ i = 1, ..., m$ (i.e. the first column of the matrix X does not contain zeros). This condition is satisfied by those eigenvectors associated to the simple eigenvalues λ_i since the rank of the matrix $A^t - \lambda_i I$ which is also symmetrizable, is equal to $m-1$ and then, all its principal minors of order $m-1$ have the same sign and consequently non null, particularly those obtained by deleting the first row-column pair.
When the eigenvalue λ_i is of multiplicity μ_i, the rank of the matrix $A^t - \lambda_i I_m$ is equal to $m - \mu_i$, then at last one of its principal minor of order $m - \mu_i$ is non null. Let us choose μ_i integers $1 \leq i_1 < ... < i_{\mu_i} \leq m$ such that $\Delta_{\mu_i} \neq 0$, where

$$\Delta_{\mu_i} = \det\left((a_{i_r,\ i_s})_{r,s\neq 1,...,\mu_i} - \lambda_i I_{m-\mu_i} \right), \tag{6}$$

and where $I_{m-\mu_i}$ denotes the unit diagonal matrix of order $m - \mu_i$. Let $V_i = \left\{x_{i_1},\ x_{i_2}, ..., x_{i_{\mu_i}}\right\}$ be the eigenspace associated to the multiple eigenvalue λ_i, then for each $j = 1, ..., \mu_i$, λ_i is a simple eigenvalue of the ν_i-order submatrix $A^{(i_j)}$ obtained from A^t by deleting the $(\mu_i - 1)$ row-column pairs $i_1, ..., i_{j-1}, i_{j+1}, ..., i_{\mu_i}$

$$A^{(i_j)} = \begin{pmatrix} a_{l_1 l_1} & a_{l_2 l_1} & \cdots & a_{l_{\nu_i} l_1} \\ a_{l_1 l_2} & a_{l_2 l_2} & \cdots & a_{l_{\nu_i} l_2} \\ \vdots & \cdots & \ddots & \vdots \\ a_{l_1 l_{\nu_i}} & a_{l_2 l_{\nu_i}} & \cdots & a_{l_{\nu_i} l_{\nu_i}} \end{pmatrix}, \tag{7}$$

where $\nu_i = m - \mu_i + 1$ and $l_1 < l_2 < ... < l_{r-1} < l_r =: i_j < l_{r+1} < ... < l_{(m-\mu_i+1)}$ are different from $i_1, ..., i_{j-1}, i_{j+1}, ..., i_{\mu_i}$. Since the first component of the jth eigenvector x_{i_j} associated to λ_i is supposed to be non null, then

$$\Delta_{i_j,1} = \det\left((a_{l_s l_r})_{s,\ r\neq 1} - \lambda_i I_{m-\mu_i} \right)$$

$$=: \begin{vmatrix} a_{l_2 l_2} - \lambda_i & \cdots & a_{l_{\nu_i} l_2} \\ \cdots & \ddots & \vdots \\ a_{l_2 l_{\nu_i}} & \cdots & a_{l_{\nu_i} l_{\nu_i}} - \lambda_i \end{vmatrix} \neq 0. \tag{8}$$

where $\Delta_{i_j,1}$ denotes the determinant of the $(m - \mu_i)$-order principal submatrix obtained from $A^{(i_j)} - \lambda_i I_{\nu_i}$ by deleting its first row-column pair

is non null and we can take in (7) $l_1 = 1$.
In terms of matrices, the system (1) can be written

$$\frac{\partial u}{\partial t} - A\Delta u = f(t,x,u), \qquad \text{in } \mathbb{R}^+ \times \Omega,$$

with the boundary conditions analogous to (2)

$$\frac{\partial u}{\partial \eta} = g\left(x,u\right), \text{ on } \Gamma^1 \text{ and } \quad u(t,x) = \Theta\left(x\right), \quad \text{on } \Gamma^2 \ ,$$

and the initial data

$$u(0,x) = u^0(x), \quad \text{in } \Omega,$$

where $g = (g_1, ..., g_m)^t$ and $\Theta = (\theta_1, ..., \theta_m)^t$.
If we denote by $D = diag\left\{\lambda_1, ..., \lambda_m\right\}$ and make the change of variables $z = Xu$, this system is equivalent to the following diagonal system

$$\frac{\partial z}{\partial t} - D\Delta z = Xf(t,x,X^{-1}z), \qquad \text{in } \mathbb{R}^+ \times \Omega, \tag{9}$$

with the boundary conditions

$$\frac{\partial z}{\partial \eta} = Xg\left(x, X^{-1}z\right), \text{ on } \mathbb{R}^+ \times \Gamma^1,$$

and

$$z(t,x) = X\Theta\left(x\right), \quad \text{on } \mathbb{R}^+ \times \Gamma^2,$$

and initial data

$$z(0,x) = Xu^0(x), \quad \text{in } \Omega.$$

In order to see things more clearly, we shall write the diagonal system on its scalar form. But before this, let us collect here briefly some notations used frequently in the following sections. We shall denote for all $j = 1, ..., \mu_i$ the components of the jth eigenvector x_{i_j} associated to the eigenvalue λ_i of multiplicity μ_i by x_{i_j,l_k}, $k = 1, ..., \nu_i$, the others are zeros. When the eigenvalue is simple, for simplicity in notation, we shall consider it as of multiplicity $\mu_i = 1$, then $\nu_i = m$ and the corresponding components of its associate eigenvector are denoted in the sequel by the same way: x_{i_j,l_k}, $k = 1, ..., \nu_i$.

To each eigenvector x_{i_j}, $j = 1, ..., \mu_i$ associated to the eigenvalue λ_i of multiplicity μ_i, corresponds an equation

$$\frac{\partial z_{i_j}}{\partial t} - \lambda_i \Delta z_{i_j} = F_{i_j}\left(t,x,z\right), \qquad \text{in } \mathbb{R}^+ \times \Omega, \tag{10}$$

with the boundary conditions

$$\frac{\partial z_{i_j}(t,x)}{\partial \eta} = G_{i_j}(x,z) \quad \text{on } \mathbb{R}^+ \times \Gamma^1,$$
$$\text{and} \tag{11}$$
$$z_{i_j}(t,x) = \Theta_{i_j}(x) \quad \text{on } \mathbb{R}^+ \times \Gamma^2,$$

and the initial data

$$z_{i_j}(0,x) = z_{i_j}(0,x) = z^0_{i_j}(x), \quad \text{in } \Omega, \tag{12}$$

where

$$z_{i_j} = \sum_{k=1}^{\nu_i} x_{i_j,l_k} u_{l_k}, \tag{13}$$

$$F_{i_j}(t,x,z) = \sum_{k=1}^{\nu_i} x_{i_j,l_k} f_{l_k}\left(t,x,X^{-1}z\right), \tag{14}$$

$$G_{i_j}(x,z) = \sum_{k=1}^{\nu_i} x_{i_j,l_k} g_{l_k}\left(x,X^{-1}z\right), \tag{15}$$

and

$$\Theta_{i_j}(x) = \sum_{k=1}^{\nu_i} x_{i_j,l_k} \theta_{l_k}(x).$$

Let us define, for each simple eigenvalue, the linear form $\Phi_i(u)$ to be the determinant of the matrix obtained from $(A^t - \lambda_i I_m)$ by replacing the first row with the corresponding components of u. i.e.

$$\Phi_i(u) = \begin{vmatrix} u_1 & u_2 & \cdots & u_m \\ a_{12} & a_{22}-\lambda_i & \cdots & a_{m2} \\ \vdots & \cdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{mm}-\lambda_i \end{vmatrix}. \tag{16}$$

When the eigenvalue λ_i is of multiplicity μ_i, we shall prove the existence of μ_i linearly independent forms $\Phi_{i_j}(u)$, $j = 1, ..., \mu_i$ analogous to $\Phi_i(u)$, each one depends on ν_i components of u. Indeed, as λ_i is a simple eigenvalue of the ν_i-order submatrix $A^{(i_j)}$ given by (7), then analogously to the case when the eigenvalue is simple, for each $j = 1, ..., \mu_i$, we define the linear form $\Phi_{i_j}(u)$ to be the determinant of the ν_i-order matrix obtained from $A^{(i_j)} - \lambda_i I_{\nu_i}$ by replacing its first row with the corresponding components of u. It can be written as follows

$$\Phi_{i_j}(u) = \begin{vmatrix} u_{l_1} & u_{l_2} & \cdots & u_{l_{\nu_i}} \\ a_{l_1 l_2} & a_{l_2 l_2}-\lambda_i & \cdots & a_{l_{\nu_i} l_2} \\ \vdots & \cdots & \ddots & \vdots \\ a_{l_1 l_{\nu_i}} & a_{l_2 l_{\nu_i}} & \cdots & a_{l_{\nu_i} l_{\nu_i}}-\lambda_i \end{vmatrix}, \ j = 1, ..., \mu_i, \tag{17}$$

where the coefficient $\Delta_{i_j,1}$ of $u_1 =: u_{l_1}$ is given by (8). We note that in the case of simple eigenvalues the definitions (16) and (17) coincide. In other words, for each $j = 1, ..., \mu_i$, the linear form $\Phi_{i_j}(u)$ analogous to (16) is the determinant of the ν_i-order submatrix obtained from $(A^t - \lambda_i I_m)$ by replacing its first row by the corresponding components of u, then we delete the $(\mu_i - 1)$ row-column pairs $i_1, ..., i_{j-1}, i_{j+1}, ..., i_{\mu_i}$.

Remark 1 *Technically, the μ_i forms $\Phi_{i_j}(u)$, $j = 1, ..., \mu_i$, can be obtained directly from the expression (16) of $\Phi_i(u)$ by replacing λ_i with the multiple eigenvalue and deleting for each j, the $(\mu_i - 1)$ row-column pairs $i_1, ..., i_{j-1}, i_{j+1}, ..., i_{\mu_i}$.*

Remark 2 *We note that, from the Cauchy interlace Theorem for symmetrizable matrices, all the determinants of the $(m - \mu_i)$-order principal submatrices of $\left(A^{(i_j)} - \lambda_i I_{\nu_i}\right)$ have the same sign as Δ_{μ_i} given by (6) which is independent of j and of course, that given by (8).*

In order to clarify the situation for the reader, we give the following example

Example 3 *Take $m = 4$, $0 < \lambda_1 < \lambda_2 < \lambda_3 = \lambda_4 = \lambda$ and suppose that*

$$\begin{vmatrix} a_{11} - \lambda_i & a_{31} \\ a_{13} & a_{33} - \lambda_i \end{vmatrix} \neq 0, \quad i = 3, 4$$

then

$$\Phi_i(u) = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ a_{12} & a_{22} - \lambda_i & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} - \lambda_i & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} - \lambda_i \end{vmatrix}, \quad i = 1,\ 2.$$

The two remained characteristics associated to the double eigenvalue λ are obtained from the above expression of $\Phi_i(u)$ for $i = 3$: one time by deleting its fourth row-column pair to get the first characteristic $\Phi_3^2(u)$. Another time we delete its second row-column pair to obtain the second characteristic $\Phi_3^4(u)$:

$$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ a_{12} & a_{22} - \lambda & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} - \lambda & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} - \lambda \end{vmatrix} \rightarrow \begin{cases} \Phi_3^2(u) = \begin{vmatrix} u_1 & u_2 & u_3 \\ a_{12} & a_{22} - \lambda & a_{32} \\ a_{13} & a_{23} & a_{33} - \lambda \end{vmatrix}, \\ \\ \Phi_3^4(u) = \begin{vmatrix} u_1 & u_3 & u_4 \\ a_{13} & a_{33} - \lambda & a_{43} \\ a_{14} & a_{34} & a_{44} - \lambda \end{vmatrix} \end{cases}$$

Now we shall write the reactions (14) and the boundary functions (15) of the diagonal system (10) each one with respect to its corresponding linear form given by (17).
As each eigenvector $x_{i_j},\ j=1,...,\mu_i$, associated to the eigenvalue λ_i of multiplicity μ_i (even if $\mu_i = 1$) is parallel to the fixed eigenvector $\left(x_{i_j}\right)_0$ with components $\Delta_{i_j,l_k},\ k=1,...,\nu_i$, given by

$$\Delta_{i_j,l_k} = (-1)^{l_k+1} \det\left((a_{l_s l_r})_{s\neq k,\ r\neq 1} - \lambda_i I_{m-\mu_i} \right), \tag{18}$$

and since the first component of any eigenvector in different from zero, then $\Delta_{i_j,1} \neq 0$. Consequently the new reaction (14) and the boundary functions (15) can be written as follows

$$F_{i_j}(t,x,z) = \frac{x_{i_j,1}}{\Delta_{i_j,1}} \Phi_{i_j}(f(t,x,u)), \quad j=1,...,\mu_i, \tag{19}$$

and

$$G_{i_j}(x,z) = \frac{x_{i_j,1}}{\Delta_{i_j,1}} \Phi_{i_j}(g(x,u)), \quad j=1,...,\mu_i, \tag{20}$$

respectively, where $u = X^{-1}z$. Recall that all the determinants $\Delta_{i_j,1},\ j = 1,...,\mu_i$ have the same sign depending only on i and not on j. We should also remark that the coefficients of the u_{l_k}'s intervening in the expression of the $\Phi_{i_j}(u)$'s represent the components non null $\Delta_{i_j,l_k},\ k=1,...,\nu_i$ given by (18) of the fixed eigenvector $\left(x_{i_j}\right)_0$.

Definition 4 *A subset $\Sigma \subset (\mathbb{L}^\infty(\Omega))^m$ is called a positively invariant region (or more simply an invariant region) for system 1, if all solutions with initial data in Σ remain in Σ for all time in their interval of existence.*

If there exists a bounded invariant set Λ of system (10), then using the following well known existence result (see [4], [6], [16], [17], [19] and [21]), the solution of the system (1) is global whenever u_0 is in the bounded subset $\Sigma =: X^{-1}(\Lambda)$.

Proposition 5 *The problem (1)-(3) admits a unique classical solution $u(t,x)$ on an interval $[0,T_{\max}[$ and (i) Either $\|u(t,.)\|_\infty$ is bounded on $[0,T_{\max}[$ and the solution is global (i.e. $T_{\max} = +\infty$).*
(ii) Or $\lim\limits_{t\to T_{\max}} \|u(t,.)\|_\infty = +\infty$ and the solution is not global, we say that it blows up in finite time $T_{\max}$ or that it ceases existing.

3 Results and proofs

3.1 The homogeneous boundary conditions case

In this subsection we consider the case when the Neumann boundary conditions in (2) are homogenous (i.e. $g_i(x,u) \equiv 0$). The nonhomogeneous case will be deduced easily in the next subsection. We suppose that for $|u_1|$ sufficiently large, the reactions satisfy

- The dissipativity condition: Each of the functions $u_1 f_k(t,x,u)$, $k = 1,...,m$, does not change sign on $\mathbb{R}^+ \times \Omega \times \mathbb{R}^m$. That is

$$\sigma_k . u_1 f_k(t,x,u) \leq 0, \tag{21}$$

where $\sigma_k = \pm 1$ is a sign application from the subset of integers $\{1,2,...,m\}$ and takes its values in $\{-1,+1\}$.

- The balance law condition: The characteristics $\Phi_{i_j}(\sigma|f|)$ of the system are alternate functions with respect to i

$$(-1)^{i+1} \Phi_{i_j}(\sigma|f|) \geq 0, \quad j = 1,...,\mu_i, \quad i = 1,...,l, \tag{22}$$

where $\sigma|f| = (\sigma_1|f_1|,...,\sigma_m|f_m|)$.

When the eigenvalue λ_i is of multiplicity $\mu_i > 1$, the above condition means that the characteristics $\Phi_{i_j}(\sigma|f|)$, $j = 1,...,\mu_i$ don't change sign. For example, when $\sigma_k = (-1)^{k+1}$, the assumptions (21) and (22) become

$$u_1 f_1(t,x,u) \leq 0, \quad f_k(t,x,u).f_{k+1}(t,x,u) \leq 0, \quad k = 1,...,m-1,$$

and

$$(-1)^{i+1} .\Phi_{i_j}\left((-1)^{k+1}|f_k|\right) \geq 0, \quad i = 1,...,l,$$

respectively.
Let us define the set

$$\Lambda = \bigcap_{p=1}^{l} \left(\bigcap_{q=1}^{\mu_p} \Lambda_{p_q} \right), \tag{23}$$

where Λ_{p_q} represents the rectangle

$$\Lambda_{p_q} = \left\{ z \in \mathbb{R}^m : \alpha_{p_q} \leq z_{p_q} \leq \beta_{p_q} \right\}, \tag{24}$$

$p = 1,...,l,\ q = 1,...,\mu_p$, with edges

$$\Lambda_{i_j}\left(\alpha_{i_j}\right) = \left\{ z \in \Lambda_{p_q} : z_{i_j} = \alpha_{i_j} \right\}, \ p_q \neq i_j, \tag{25}$$

and

$$\Lambda_{i_j}\left(\beta_{i_j}\right)=\left\{z\in\Lambda_{p_q}:z_{i_j}=\beta_{i_j}\right\},\ p_q\neq i_j, \tag{26}$$

$i=1,...,l,\ j=1,...,\mu_i$.

Our main result on the invariant regions and global existence of system (1) is the following

Theorem 6 *Suppose that the diffusion matrix is symmetrizable, then under conditions (21) and (22), the region* $\Sigma=X^{-1}(\Lambda)$ *with* Λ *defined by (23) is invariant for system (1). Moreover the solution is global and uniformly bounded on* Ω *for any initial data in* $L_\infty(\Omega)$. *Furthermore when the conditions are satisfied for all* $u_1\in\mathbb{R}$, *then*

$$\|u\|_\infty\leq C\left\|u^0\right\|_\infty \tag{27}$$

where C *is a positive constant depending only on the diffusion matrix* A *and equal to the unity when* A *is symmetric.*

Proof. Since the matrix A is diagonalizable with positive eigenvalues, the two systems (1) and (9) are both parabolic and equivalent. Then if the diagonal system has an invariant rectangle Λ, the original has an invariant region $X^{-1}(\Lambda)$. But, thanks to the invariant region's method (see H. j. Kuiper [11] and [12] and J. A. Smoller [23]), the region Λ given by (23) is invariant for system (9) under the following conditions

$$\begin{gathered}F_{i_j}(t,x,z)\geq 0,\ \text{, for all } z\in\Lambda_{i_j}\left(\alpha_{i_j}\right),\\ \text{and}\\ F_{i_j}(t,x,z)\leq 0,\ \text{for all } z\in\Lambda_{i_j}\left(\beta_{i_j}\right),\end{gathered} \tag{28}$$

for all $i=1,...,l,\quad j=1,...,\mu_i$. Since A is symmetrizable, then the system can be transformed to an equivalent system with symmetric diffusion matrix. Therefore, we can suppose that A is symmetric and then the matrix X formed with the rows x_{i_j} can be orthogonalized (i.e. chosen such that $X^{-1}=X^t$). Consequently, for all $i=1,...,l,\ j=1,...,\mu_i$, the formula

$$z_{i_j}=\sum_{k=1}^{\nu_i}x_{i_j,l_k}u_{l_k},$$

gives

$$u_{i_j}=\sum_{p=1}^{l}\left(\sum_{q=1}^{\mu_p}x_{p,i_j}^{p_q}z_{p_q}\right) \tag{29}$$

Particularly, the first component of the unknown u can be written as follows

$$u_1 = x_{i_j,1} z_{i_j} + \sum_{p=1}^{l} \left(\sum_{q=1,\ p_q \neq i_j}^{\mu_p} x_{p_q,1} z_{p_q} \right) \tag{30}$$

where from (8) $x_{i_j,1} \neq 0$. Let us verify that the rectangle given by (24) is invariant under conditions (21) and (22). For this purpose, we should verify (28) for all $i = 1, ..., l,\ j = 1, ..., \mu_i$. To prove this, we need to verify two inequalities along each of the m edges of the rectangle given by (24):

For $z \in \Lambda_{i_j} \left(\alpha_{i_j} \right)$, we have $z_{i_j} = \alpha_{i_j}$. Taking α_{i_j} negative and sufficiently large in absolute value, then the summation $\sum_{p=1}^{l} \left(\sum_{q=1,\ p_q \neq i_j}^{\mu_p} x_{p_q,1} z_{p_q} \right)$, in the above expression (30) of u_1 is bounded ($\alpha_{p_q} \leq z_{p_q} \leq \beta_{p_q}\ ,\ p_q \neq i_j$). This gives u_1 is also large in absolute value, but its sign depends on that of $x_{i_j,1}$ and so we have two cases:

When $x_{i_j,1} > 0$, then u_1 is negative which gives from (21) $f_k = \sigma_k |f_k|\,,\ k = 1, ...m$. Consequently, from (19) $F_{i_j}(t, x, z) \geq 0$ under the following condition

$$\frac{\Phi_{i_j}(\sigma |f|)}{\Delta_{i_j,1}} \geq 0. \tag{31}$$

As A is symmetric, then from the well known Cauchy's interlace Theorem, the eigenvalues of A and those of each of its principal submatrices of order $m-1$ interlace. That is

$$(-1)^{i+1} \Delta_{i_j,1} \geq 0, \quad i = 1, ..., l. \tag{32}$$

Thus, (22) gives $F_{i_j}(t, x, z) \geq 0$ for all $z \in \Lambda_{i_j} \left(\alpha_{i_j} \right)$.

When $x_{i_j,1} < 0$, then u_1 is positive and again from (21), we have $f_k = -\sigma_k |f_k|\,,\ k = 1, ...m$. This gives

$$F_{i_j}(t, x, z) = -x_{i_j,1} \frac{\Phi_{i_j}(\sigma |f|)}{\Delta_{i_j,1}}. \tag{33}$$

Since $x_{i_j,1} < 0$, then using (32), we can conclude the positivity of $F_{i_j}(t, x, z)$ on $\Lambda_{i_j} \left(\alpha_{i_j} \right)$ under the condition (22).

Following the same reasoning for $z \in \Lambda_{i_j} \left(\beta_{i_j} \right)$ by choosing β_{i_j} positive and sufficiently large, we can deduce that u_1 is also large in absolute value. Analogously, we have two cases:

When $x_{i_j,1} < 0$, then u_1 is negative and this gives from (21) $f_k = \sigma_k |f_k|\,,\ k = 1, ...m$. Thus $F_{i_j}(t, x, z)$ can be written on the form (19). Using (31) and (32), we can say that (22) gives $F_{i_j}(t, x, z) \leq 0$ for all

$z\in\Lambda_{i_j}\left(\beta_{i_j}\right)$.
Finally when $x_{i_j,1}>0$, then u_1 is positive, also from (21), we have $f_k=-\sigma_k\left|f_k\right|,\ k=1,...m$ and then (33). Another time with (31) and (32) together, we can conclude that condition (22) gives $F_{i_j}\left(t,x,z\right)\leq 0$ for all $z\in\Lambda_{i_j}\left(\beta_{i_j}\right)$. This ends the proof of the first part of the Theorem when the diffusion matrix is symmetric.
When A is symmetrizable, the diffusion matrix of the new system becomes $B=S^{\frac{1}{2}}AS^{-\frac{1}{2}}$ which is symmetric and the reaction is $S^{\frac{1}{2}}f$. If we denote by $\Phi_{i_j}^{A}\left(u\right)$ the linear form $\Phi_{i_j}\left(u\right)$ given by (16) and associated to the diffusion matrix A, then $\Phi_{i_j}^{B}\left(\sigma\left|S^{\frac{1}{2}}f\right|\right)=\sqrt{s_1}\Phi_{i_j}^{A}\left(\sigma\left|f\right|\right)$ and $\Delta_{i_j,1},\ j=1,...,\mu_i$, remains the same, for all $i=1,...,l$. We follow the same steps of the proof by replacing u_1 and f by $\sqrt{s_1}u_1$ and $S^{\frac{1}{2}}f$ respectively, where $S=Diag(s_1,...,s_m)$.
Since Λ is invariant (i.e. $\|z\|\leq\|z_0\|$), the uniform bounds (27) of the solutions are a trivial consequence of the relations (13) which, when written on the matricial form $z^t=Xu^t$, give

$$\left\|S^{\frac{1}{2}}u\right\|\leq\left\|X^{-1}z\right\|=:\|z\|=\text{ and }\|z_0\|\leq\left\|XS^{\frac{1}{2}}u_0\right\|=:\left\|S^{\frac{1}{2}}u_0\right\|.$$

This ends the proof of the Theorem.
■

Remark 7 *When all the eigenvalues of the diffusion matrix are simple, then* $P\left(\lambda_i\right)=\Phi_i\left(\sigma\left|f\right|\right)$ *is a polynomial of degree* $m-1$ *with respect to* λ_i*, with leading coefficient* $\sigma_1\left|f_1\right|$*. But from (22)* $P\left(\lambda_i\right)$ *changes sign* m *times and* $P\left(\lambda_1\right)\geq 0$*. Consequently we should take* $\sigma_1=1$ *(i.e.* $u_1f_1\leq 0$ *on* $\mathbb{R}^+\times\Omega\times\mathbb{R}^m$*).*

Remark 8 *When the balance law is strict (i.e.* $\left|f_k\right|=\left|f_1\right|,\ k=1,...,m$*), the conditions (22) are independent of the reactions and become a simple conditions on the diffusion terms.*

3.2 The non-homogeneous boundary conditions case

When the Neumann boundary conditions are non-homogeneous, we consider, for $|u_1|$ is sufficiently large, the following hypothesis:

H1) The reaction terms satisfy (21) and the following conditions analogous to (22):

$$\frac{\Phi_{i_j}\left(\sigma\left|f\right|\right)}{\Delta_{i_j,1}}\geq-\lambda_i\nu\left(x\right),\quad i=1,...,l,\ j=1,...,\mu_i,\tag{34}$$

on $\mathbb{R}^+\times\Omega\times\mathbb{R}^m$, where $\nu\left(x\right)$ is a measurable function.

H2) The functions g_i satisfy a conditions analogous to (21) and (34):

$$\rho_k.u_1 g_k(x,u) \le 0, \quad k = 1, ..., m, \tag{35}$$

and

$$\frac{\Phi_{i_j}(\rho|g|)}{\Delta_{i_j,1}} \ge -\xi(x), \quad i = 1, ..., l, \ j = 1, ..., \mu_i, \tag{36}$$

on $\partial\Omega \times \mathbb{R}^m$, where $\xi(x)$ is a measurable function and ρ is a sign application from the subset of integers $\{1, 2, ..., m\}$ and takes its values in $\{-1, +1\}$, not necessary equal to σ.

In order to construct invariant regions in this case, we use a given positive and bounded $\varphi(x)$ on $\overline{\Omega}$. The edges of the set Λ, given by (23) are translated, each one by a vector function of $\mathbb{R}^m$ which components are equal to $\pm|x_{i_j,1}|\varphi(x)$, $i = 1, ..., l, \ j = 1, ..., \mu_i$. In other words we define the parallelepiped

$$\Lambda(\varphi) = \bigcap_{p=1}^{l}\left(\bigcap_{q=1}^{\mu_p}\Lambda_{p_q}(\varphi)\right), \tag{37}$$

where

$$\Lambda_{p_q}(\varphi) = \left\{z \in \mathbb{R}^m : \alpha_{p_q} - |x_{p_q,1}|\varphi \le z_{p_q} \le \beta_{p_q} + |x_{p_q,1}|\varphi\right\},$$

$p = 1, ..., l, \ q = 1, ..., \mu_i$, with edges the rectangles

$$\Lambda_{i_j}\left(\alpha_{i_j} - |x_{i_j,1}|\varphi\right) = \\ \left\{z \in \Lambda_{p_q} : z_{i_j} = \alpha_{i_j} - |x_{i_j,1}|\varphi, \quad p_q \ne i_j, \right\}$$

and

$$\Lambda_{i_j}\left(\beta_{i_j} + |x_{i_j,1}|\varphi\right) = \\ \left\{z \in \Lambda_{p_q} : z_{i_j} = \beta_{i_j} + |x_{i_j,1}|\varphi, \quad p_q \ne i_j, \right\}$$

$j = 1, ..., \mu_i, \ i = 1, ..., l$.
Following [12] in the case $m = 2$, we choose the function φ as a strong solution of the following linear elliptic problem

$$\begin{aligned} -\Delta\varphi &\ge \nu(x), \\ \frac{\partial\varphi}{\partial\eta} &\ge \xi(x). \end{aligned} \tag{38}$$

H3) The above elliptic problem has a solution φ two times continuously differentiable on Ω with first and second derivatives bounded on $\overline{\Omega}$. In the case of Neumann boundary conditions on all $\partial\Omega$, it is of course necessary to satisfy the compatibility condition

$$\int_\Omega \nu(x)\,dx \le -\int_{\partial\Omega}\xi(x)\,d\sigma.$$

In order to state our results in this case, we apply the following standard proposition which can be found in the wide bibliography cited at the beginning of our introduction.

Proposition 9 *Suppose that, for some bounded φ on $\overline{\Omega}$, we have, for all $i = 1, ..., l,\ j = 1, ..., \mu_i$,*

$$\begin{aligned}
-\lambda_i \Delta z_{i_j} &\leq F_{i_j}(t,x,z), \quad \forall z \in \Lambda_{i_j}\left(\alpha_{i_j} - \left|x_{i_j,1}\right| \varphi\right), \\
-\lambda_i \Delta z_{i_j} &\geq F_{i_j}(t,x,z), \quad \forall z \in \Lambda_{i_j}\left(\beta_i^{i_j} + \left|x_{i_j,1}\right| \varphi\right),
\end{aligned} \tag{39}$$

in $\mathbb{R}^+ \times \Omega$ and

$$\begin{aligned}
\frac{\partial z_{i_j}}{\partial \eta} &\leq G_{i_j}(x,z), \quad \forall z \in \Lambda_{i_j}\left(\alpha_{i_j} - \left|x_{i_j,1}\right| \varphi\right), \\
\frac{\partial z_{i_j}}{\partial \eta} &\geq G_{i_j}(x,z), \quad \forall z \in \Lambda_{i_j}\left(\beta_i^{i_j} + \left|x_{i_j,1}\right| \varphi\right),
\end{aligned} \tag{40}$$

in $\mathbb{R}^+ \times \partial\Omega$, then the region $\Lambda(\varphi)$ given by (37) is invariant for the diagonal system (10).

Our main result when the boundary conditions are non-homogeneous is the following

Theorem 10 *Suppose that hypotheses H_1-H_3 are satisfied, then the set $\Sigma = X^{-1}(\Lambda(\varphi))$, where $\Lambda(\varphi)$ is given by (37) is invariant for system (1) and for any bounded initial condition on Ω, there exists a unique global solution of problem (1)-(3) uniformly bounded on Ω. Furthermore when the conditions are satisfied for all $u_1 \in \mathbb{R}$, we have*

$$\|u\|_\infty \leq C\left(\left\|u^0\right\|_\infty + \|\varphi\|_\infty\right), \tag{41}$$

where C is a positive constant depending only on the diffusion matrix A and equal to the unity when A is symmetric.

Proof. The two systems (1) and (9) being equivalent, then by this, if we verify the two inequalities (39) for the reactions and two others (40) for the boundary along each of the m edges, the set $\Lambda(\varphi)$ is invariant and consequently, by a convenient choice of the constant α_{i_j} and $\beta_i^{i_j}$, it can contains any bounded initial data. So, using (38), it remains to verify, for all $i = 1, ..., l,\ j = 1, ..., \mu_i$, the following inequalities

$$\begin{gathered}
F_{i_j}(t,x,z) \geq -\lambda_i \left|x_{i_j,1}\right| \nu(x), \text{ for all } z \in \Lambda_{i_j}\left(\alpha_{i_j} - \left|x_{i_j,1}\right| \varphi\right), \\
\text{and} \\
F_{i_j}(t,x,z) \leq \lambda_i \left|x_{i_j,1}\right| \nu(x), \text{ for all } z \in \Lambda_{i_j}\left(\beta_{i_j} + \left|x_{i_j,1}\right| \varphi\right),
\end{gathered} \tag{42}$$

for the reaction terms and

$$G_{i_j}(x,z) \geq -\left|x_{i_j,1}\right| \xi(x), \text{ for all } z \in \Lambda_{i_j}\left(\alpha_{i_j} - \left|x_{i_j,1}\right| \varphi\right),$$

$$\text{and} \tag{43}$$

$$G_{i_j}(x,z) \leq \left|x_{i_j,1}\right| \xi(x), \text{ for all } z \in \Lambda_{i_j}\left(\beta_{i_j} + \left|x_{i_j,1}\right| \varphi\right),$$

for the boundary conditions. We follow the same reasoning as in the proof of Theorem 6 for the homogenous case:
Taking $z_{i_j} = \alpha_{i_j} - \left|x_{i_j,1}\right| \varphi$ with α_{i_j} negative and sufficiently large in absolute value, then from (30) u_1 is also large in absolute value, but its sign depends on that of $x_{i_j,1}$ and so we have two cases:
When $x_{i_j,1} > 0$, then $z_{i_j} = \alpha_{i_j} - x_{i_j,1}\varphi$ and u_1 are both negative which gives from (21) $f_k = \sigma_k |f_k|$, $k = 1, ...m$. Then $F_{i_j}(t,x,z)$ can be written on the form (19). Thus (34) gives, from our choice of φ the first inequality of (42). The needed inequality on the boundary condition follows analogously.
Next when $x_{i_j,1} < 0$, then $z_{i_j} = \alpha_{i_j} + x_{i_j,1}\varphi$ and u_1 are of opposite signs with u_1 positive and again from (21), we have $f_k = -\sigma_k |f_k|$, $k = 1, ...m$. Then $F_{i_j}(t,x,z)$ can be written on the form (33). Hence from the condition (34) we get another time the first inequality of (42).
Following the same reasoning for $z \in \Lambda_{i_j}\left(\beta_{i_j} + \left|x_{i_j,1}\right| \varphi\right)$ by choosing β_{i_j} positive and sufficiently large, then u_1 is also large in absolute value, but its sign depends on that of $x_{i_j,1}$. Analogously, we have two cases:
When $x_{i_j,1} < 0$, then $z_{i_j} = \beta_{i_j} - x_{i_j,1}\varphi$ and u_1 are of opposite signs with u_1 negative which gives from (21), $f_k = \sigma_k |f_k|$, $k = 1, ...m$. Then $F_{i_j}(t,x,z)$ has the form (19). Consequently the second inequality of (42) is deduced from the condition (34).
Finally when $x_{i_j,1} > 0$, then $z_{i_j} = \beta_{i_j} + x_{i_j,1}\varphi$ and u_1 is positive which gives from (21) $f_k = -\sigma_k$, $k = 1, ...m$. Then $F_{i_j}(t,x,z)$ can be written on the form (33). Thus from the condition (34) we get again the second inequality of (42). We have therefore proved the first part of the Theorem when the diffusion matrix is symmetric. When A is symmetrizable we follow the same reasoning as in the end of the proof for the homogenous case taking in the account that $x_{i_j,1}$ should be multiplied by $\sqrt{s_1}$.
The uniform bounds (41) of the solutions can obtained in the same way as in the homogenous case. This ends the proof of the Theorem. ■

Remark 11 *When $m = 2$, we take $x_i^0 = (a_{22} - \lambda_i, -a_{12})^t$ as fixed eigenvector associated to the simple eigenvalues λ_i, then for any other eigen-*

vector $x_i = (x_{i1}, x_{i1})^t$ *, we have*

$$z_i \; = x_{i1}u_1 + x_{i2}u_2 = \frac{x_{i1}}{a_{22} - \lambda_i}\left[(a_{22} - \lambda_i)\, u_1 - a_{12}u_2\right],$$

and

$$F_i\,(t,x,z) = \frac{x_{i1}}{a_{22} - \lambda_i}\left[(a_{22} - \lambda_i)\, f_1 - a_{12}f_2\right], \qquad i = 1,\ 2,$$

then

$$F_i\,(t,x,z) = x_{i1}\frac{\Phi_i\,(f_1, f_2)}{a_{22} - \lambda_i},$$

where Φ_i *are given by (16). Theorem 10 is applicable with* $\Delta_{i,1} = a_{22} - \lambda_i$ *and we have the same conclusions even the diffusion matrix is not symmetrizable.*

Remark 12 *Following [11] in the case of systems of two equations, Theorem 10 can be extended to the mixed boundary conditions or those of the type Dirichlet.*

Remark 13 *The operator* $-\Delta$ *can be replaced by any uniformly elliptic operator* L.

Remark 14 *Theorem 10 is applicable even when* $u_1 f_k$ *and* $u_1 g_k$ *change sign for* u_1 *small and this is not the case in the results obtained by authors until now. It is, for example the case of the two component reaction diffusion system modeling the two reversible chemical reaction with nonlinear terms* $f_1\,(u,v) = -\; f_2\,(u,v) = u^\alpha v^\beta - u^p v^q$*, where the exponents are positive integers representing the number of molecules of each reactant.*

4 Applications

4.1 Strongly coupled reaction diffusion equations

Let us consider the reaction-diffusion system (1) for m=2, with boundary conditions (2) and initial data (3). To get positive eigenvalues $0 < \lambda_1 < \lambda_2$, we should assume that

$$a_{11}a_{22} - a_{12}a_{21} > 0, \tag{44}$$

and

$$a_{11} + a_{22} > 0. \tag{45}$$

Suppose that

$$u_1 f_1 \leq 0, \;\; u_1 f_2 \geq 0, \text{ on } \mathbb{R}^+ \times \Omega \times \mathbb{R}^2$$

$$\text{and} \tag{46}$$

$$u_1 g_1 \leq 0, \; u_1 g_2 \geq 0, \;\text{ on } \partial\Omega \times \mathbb{R}^2,$$

with $|u_1|$ sufficiently large. This means that conditions (21) and (35) are satisfied with $\sigma_1 = \rho_1 = 1$ and $\sigma_2 = \rho_2 = -1$. By application of the remark 11, the conditions (34) and (36) become respectively

$$\frac{(a_{22} - \lambda_i)\,|f_1(t,x,u)| + a_{12}\,|f_2(t,x,u)|}{(a_{22} - \lambda_i)} \geq -\lambda_i \nu\,(x)\,, \qquad i = 1,2,$$

and

$$\frac{(a_{22} - \lambda_i)\,|g_1(x,u)| + a_{12}\,|g_2(x,u)|}{(a_{22} - \lambda_i)} \geq -\xi\,(x)\,, \qquad i = 1,2.$$

Using (44) and (45), we have have two cases

- When $a_{12}a_{21} > 0$ then $\lambda_1 < a_{22} < \lambda_2$ and the conditions can be written as follows

$$(a_{22} - \lambda_2)\,\lambda_2 \nu\,(x) + (a_{22} - \lambda_2)\,|f_1|$$

$$\leq -a_{12}\,|f_2| \leq (a_{22} - \lambda_1)\,|f_1| + (a_{22} - \lambda_1)\,\lambda_1 \nu\,(x)\,,$$

$$\text{and} \tag{47}$$

$$(a_{22} - \lambda_2)\,\xi\,(x) + (a_{22} - \lambda_2)\,|g_1|$$

$$\leq -a_{12}\,|g_2| \leq (a_{22} - \lambda_1)\,|g_1| + (a_{22} - \lambda_1)\,\xi\,(x)\,.$$

- When $a_{12}a_{21} < 0$ then $a_{11} < \lambda_1 < \lambda_2 < a_{22}$. The conditions (34) and (36) become respectively

$$|f_1(t,x,u)| + \tfrac{a_{12}}{(a_{22} - \lambda_i)}\,|f_2(t,x,u)| \geq -\lambda_i \nu\,(x)\,,$$

$$\text{and} \tag{48}$$

$$|g_1(t,x,u)| + \tfrac{a_{12}}{(a_{22} - \lambda_i)}\,|g_2(t,x,u)| \geq -\xi\,(x)\,, \quad i = 1,2.$$

We can state the following

Corollary 15 *Suppose that conditions (44), (45) and (46) are satisfied and the problem (38) has a bounded solution, then global existence of system (1)-(3) for bounded initial data occurs under conditions (47) when* $a_{12}a_{21} > 0$ *and (48) when* $a_{12}a_{21} < 0$.

Remark 16 *When* $|f_1| = |f_2|$ *and* $\nu(x) = \xi(x) = 0$*, then global existence for system (1)-(3) occurs under the conditions (46) on the reactions and the following condition on the diffusion terms analogous to that of [12] but without restriction on the sign of* a_{12}

$$(a_{22} - a_{11} + a_{12} - a_{21})\, a_{21} \leq 0.$$

4.2 Three components systems

Let us consider the reaction-diffusion system (1) for m=3, then the symmetrizer can be taken to be, for example

$$S = diag(1, \frac{a_{12}}{a_{21}}, \frac{a_{13}}{a_{31}}).$$

When the Neumann boundary conditions are homogenous, two cases should be considered.

4.2.1 The eigenvalues of the diffusion matrix are simple

The characteristics of the system are

$$\Phi_i(\sigma|f|) = \begin{vmatrix} \sigma_1|f_1| & \sigma_2|f_2| & \sigma_3|f_3| \\ a_{12} & a_{22}-\lambda_i & a_{32} \\ a_{13} & a_{23} & a_{33}-\lambda_i \end{vmatrix}, \quad i = 1, 2, 3. \tag{49}$$

By application of the Theorem 6, the conditions (22) become

$$\Phi_1(\sigma|f|), \quad \Phi_3(\sigma|f|) \ \geq 0 \geq \Phi_2(\sigma|f|). \tag{50}$$

Constrained by the above inequalities, we take $\sigma_1 = 1$. We see that $\Phi_i(\sigma|f|)$ is a second degree polynomial with respect to λ_i with positive leading coefficient and then condition (50) is satisfied by a large class of reactions f without conditions on their growth.

Corollary 17 *Suppose that the diffusion matrix is symmetrizable with positive and simple eigenvalues and that* $u_1 f_1(u) \leq 0$ *with* $u_1 f_2(u)$ *and* $u_1 f_3(u)$ *don't change sign for all* $|u_1|$ *sufficiently large, then global existence of solutions for system (1)-(3) with homogenous Neumann boundary conditions and bounded initial data, occurs under conditions (50).*

4.2.2 An eigenvalue of the diffusion matrix is double

When an eigenvalue of A is double (for example λ_1 is simple and $\lambda_1 < \lambda_2 = \lambda_3$ is double), we should be careful when we construct the $\Phi_i's$, but the situation becomes less difficult to deduce formulas analogous to (50). We define $\Phi_1$as above

$$\Phi_1(\sigma|f|) = \begin{vmatrix} \sigma_1|f_1| & \sigma_2|f_2| & \sigma_3|f_3| \\ a_{12} & a_{22}-\lambda_1 & a_{32} \\ a_{13} & a_{23} & a_{33}-\lambda_1 \end{vmatrix},$$

but for the two characteristics corresponding to the double eigenvalue, we use Remark 1: We replace λ_1 by λ_2 in the above expression of $\Phi_1(\sigma|f|)$, then by deleting the third row-column pair we obtain the first characteristic, denoted by $\Phi_2^1(\sigma|f|)$. The second one, denoted by $\Phi_2^2(\sigma|f|)$, is the second order determinant obtained from the expression of $\Phi_1(\sigma|f|)$ by deleting its second row-column pair with λ_1 replaced by λ_2. That is:

$$\begin{vmatrix} \sigma_1|f_1| & \sigma_2|f_2| & \sigma_3|f_3| \\ a_{12} & a_{22}-\lambda_1 & a_{32} \\ a_{13} & a_{23} & a_{33}-\lambda_1 \end{vmatrix} \rightarrow \begin{cases} \Phi_2^1 = \begin{vmatrix} \sigma_1|f_1| & \sigma_2|f_2| \\ a_{12} & a_{22}-\lambda_2 \end{vmatrix}, \\ \\ \Phi_2^2 = \begin{vmatrix} \sigma_1|f_1| & \sigma_3|f_3| \\ a_{13} & a_{33}-\lambda_3 \end{vmatrix}. \end{cases}$$

The conditions (22) become

$$\begin{aligned} &\Phi_1(\sigma|f|) \geq 0, \\ &(a_{22}-\lambda_2)\sigma_1|f_1| - a_{12}\sigma_2|f_2| \leq 0, \\ &(a_{33}-\lambda_3)\sigma_1|f_1| - a_{13}\sigma_3|f_3| \leq 0. \end{aligned} \tag{51}$$

Since the second order determinants of the matrix $A^t - \lambda_2 I$ are all null, then we can calculate explicitly the eigenvalues of A^t :

$$\lambda_2 = \lambda_3 = a_{11} - \frac{a_{21}a_{13}}{a_{23}} \text{ and } \lambda_1 = trA - 2\lambda_2,$$

which shows that the coefficients of the $|f_k|$'s in the inequalities (51) depend only on the diffusion matrix, especially in the case when the balance law is strict.

Corollary 18 *Suppose that the diffusion matrix has a double eigenvalue and that $u_1 f_k(u)$, $k = 1,2,3$ don't change sign for all $u \in \mathbb{R}^3$ with $|u_1|$ sufficiently large, then under conditions (51) we have global existence of solutions for system (1)-(3) with homogenous Neumann boundary conditions and bounded initial data.*

Acknowledgement 19 *The author gratefully acknowledge Qassim University, represented by the Deanship of Scientific Research, on the material support for this research under the number (3388) during the academic year 1436 AH / 2015 AD.*

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