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Spatial quantile predictions for elliptical random fields

Véronique Maume-Deschamps, Didier Rullière, Antoine Usseglio-Carle
1 Institut Camille Jordan, Lyon
2 Laboratoire de Sciences Actuarielle et Financière, Lyon

Introduction

Kriging (see Krige (1951)) aims at predicting the conditional mean of a random field \(Z(t)\) given the values \(Z(t_1), \ldots, Z(t_n)\) of the field at some points \(t_1, \ldots, t_n \in T\), where typically \(T \subset \mathbb{R}^d\). It seems natural to predict, in the same spirit as Kriging, other functionalities. In our study, we focus on quantiles for elliptical random fields.

Elliptical Distributions

Cambanis et al. (1981) give the representation: the random vector \(X \in \mathbb{R}^d\) is elliptical with parameters \(\mu \in \mathbb{R}^d\) and \(\Sigma \in \mathbb{R}^{d \times d}\), if and only if

\[ X = \mu + R \Lambda U \Sigma^{1/2}, \]

where \(\Lambda \Sigma = \Sigma\), \(U \in \mathbb{R}^{d \times d}\) is a \(d\)-dimensional random vector uniformly distributed on \(S^{d-1}\) (the unit disk of dimension \(d\)), and \(R\) is a non-negative random variable independent of \(U\). Furthermore, \(X\) is consistent if:

\[ R \not\perp X \quad \text{(2)} \]

Table 1: Some consistent distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>(\Sigma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>(\sigma_1^2)</td>
</tr>
<tr>
<td>Student, (\nu &gt; 0)</td>
<td>(\frac{\nu}{\nu + \nu})</td>
</tr>
<tr>
<td>Unimodal Gaussian Mixture</td>
<td>(\frac{\mu}{\sqrt{1 - \alpha}})</td>
</tr>
<tr>
<td>Laplace, (\lambda &gt; 0)</td>
<td>(\frac{\lambda}{\sqrt{1 - \alpha}})</td>
</tr>
<tr>
<td>Uniform Gaussian Mixture</td>
<td>(\mu \in [0, 1])</td>
</tr>
</tbody>
</table>

Now, we consider \(X = (X_1, X_2)^T\) to be a consistent \((R, d)\)-elliptical random vector with \(R \sim \mathcal{N}(0, 1)\) and \(\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_2 \end{pmatrix}\), \(\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\).

The conditional distribution \(X_2|X_1 = x_1\) has parameters:

\[ \begin{align*}
\mu_{2|1} &= \mu_2 + \Sigma_{21}(\Sigma_{11}^{-1})(x_1 - \mu_1) \\
\Sigma_{2|1} &= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}
\end{align*} \]

Furthermore, \(X_2|X_1 = x_1\) is elliptical, with radius \(R^*\) given by:

\[ R^* = \frac{x_1 - \mu_1}{\sqrt{\Sigma_{11}^{-1}}} \]

Quantile Regression

Quantile regression, introduced by Koenker and Bassett (1978), approximates the conditional quantile as follows:

\[ \Phi_q(x) = P(R \in \mathcal{U}^{(1)} \leq x) \]

where \(\beta^*\) and \(\beta^*_0\) are solutions of the following minimization problem

\[ \beta^* = \arg \min_{\beta \in R} E[\Phi_q(x - \beta X - \beta_0)] \]

and where the scoring function \(\Phi_q(x)\) is

\[ \Phi_q(x) = \frac{1}{x} \text{sign}(x) \text{sgn}(x) = 0 \]

We obtain such simple results for other elliptical distributions. It is why we propose, in what follows, two approaches.

Extremal quantiles

In this section, the aim is to establish a relation between \(\Phi_{q_1}\) and \(\Phi_{q_2}\) for extremal values of \(\alpha\). For that,

\[ \left\{ \begin{array}{l}
\frac{1}{x} - \Phi_{q_1}(x) = \xi \\
\frac{1}{x} - \Phi_{q_2}(x) = \xi
\end{array} \right. \]

Under this assumption, we can define Extremal Conditional Quantiles Predictors:

\[ \begin{align*}
\Phi_{q_1}(X_2|X_1 = x_1) &= \mu_2 + \xi \Sigma_{21}^{-1}(x_1 - \mu_1) \\
\Phi_{q_2}(X_2|X_1 = x_1) &= \mu_2 - \xi \Sigma_{21}^{-1}(x_1 - \mu_1)
\end{align*} \]

Table: Some examples

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<th>(\xi)</th>
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<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>(N + 1)</td>
<td>(N + 1)</td>
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<tr>
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<td>1</td>
<td>(N + 1)</td>
</tr>
</tbody>
</table>

Figure 2: Q-Q plots for Student example

Abstrack: We denote:

\[ R, d \]

Furthermore, \(X \in \mathbb{R}^d\), \(\Sigma \in \mathbb{R}^{d \times d}\), if and only if:

\[ X = \mu + R \Lambda U \Sigma^{1/2}, \]

where \(\Lambda \Sigma = \Sigma\), \(U \in \mathbb{R}^{d \times d}\) is a \(d\)-dimensional random vector uniformly distributed on \(S^{d-1}\) (the unit disk of dimension \(d\)), and \(R\) is a non-negative random variable independent of \(U\). Furthermore, \(X\) is consistent if:

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References


